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Explicit solutions of atmospheric Ekman flows for some eddy viscosities in ellipsoidal coordinates

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Abstract. In ellipsoidal coordinates, we study the motion of the wind in the steady atmospheric Ekman layer for the height-dependent eddy viscosities in the form of some quadratic, fourth and rational power functions. We construct the explicit solutions for these forms of the eddy viscosities by using suitable boundary conditions. Furthermore, we write down a formula of the angle between the wind vector and the geostrophic wind vector at any height.

Contents

1. Introduction

The Ekman layer is a boundary layer where there is a balance between the viscous friction force and the Coriolis force at non-equatorial latitudes. The classical Ekman theory needs to ignore the nonlinear effects and achieve a balance of three forces, namely the Coriolis force, the pressure gradient force and the friction force [**1, 2**]. But in equatorial regions, the Coriolis effect disappears, so the nonlinear effects must be considered [**3, 4, 5, 6**]. The Ekman layer theory is applicable to

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many fields such as the bottom of the atmosphere, the bottom of the ocean, and surface waters.

Under the assumption of constant vertical eddy viscosity, the first mathematical model to describe the behavior of wind-generated steady surface streams was formulated by Ekman [**7**]. He further analyzed the model, derived the Ekman flow and gave three predictions from it. Two of the predictions have been confirmed by some data from non-equatorial regions. However, there is a big difference between the prediction of the deflection angle of the surface flow from the wind direction and the actual measurement data [**8, 9, 10**]. This difference is caused by the assumption of the constant eddy viscosity. Therefore, the authors in [**11, 12, 13, 14, 15, 16, 17, 18, 19**] further considered the problem of atmospheric flow with the eddy viscosity varying with the height in the Ekman layer. It is worth noting that Ionescu-Kruse in [**13**] constructed the explicit solutions for the cases where the eddy viscosities are some quadratic or rational power functions, and Guan et al. in [**19**] gave the explicit solutions for the cases where the eddy viscosity is a quadratic or piecewise function.

The above works were done in spherical coordinates and local Cartesian coordinates. However, for the study of the atmosperic flow, the spherical coordinates and the local Cartesian coordinates cannot fully preserve the details of the Earth's curved-space geometry. Fortunately, a more accurate simulation of the atmospheric movement was given. Constantin and Johnson in [**20**] used the ellipsoid approximation of the Earth to model the large-scale atmospheric flow, that is, an ellipsoid coordinates was established on the Earth that could describe the atmospheric flow more accurately. The general governing equations for the steady motion of the viscous compressible atmosphere in ellipsoidal coordinates were further presented, and how the new system recovers the classical Ekman equations was shown. Further, Constantin and Johnson in [**21**] discussed the propagation of waves in the atmosphere in ellipsoidal coordinates. In addition, using asymptotic methods, Constantin and Johnson in [**22**] investigated the propagation of nonlinear waves in the atmosphere. Based on the the thin-shell approximation, Johnson in [**23**] was concerned with the mathematical fluid dynamics of the atmospheric Walker circulation. Ionescu-Kruse in [**24**] considered the viscous compressible zonal flows in a neighbourhood of the Equator. Here, we highlight that Constantin and Johnson proposed a new extended Ekman model in [**20, 21**]. Then, Yang et al. presented the dynamical properties [**25**] and some explicit solutions [**26**] for the atmospheric Ekman flows in ellipsoidal coordinates.

Based on the above papers, we discuss the atmospheric Ekman flows with the eddy viscosities in the form of quadratic, fourth and rational power functions in ellipsoidal coordinates. After choosing the suitable boundary conditions, we construct the explicit solutions for these cases. Then, we write down a formula of the angle between the wind vector and the geostrophic wind vector at any height.

2. Governing equations

In atmospheric science, the shape of Earth's sea-level geopotential surface is usually approximated by an ellipsoid obtained by rotating an ellipse around its semi-minor (polar) axis (of length $\tilde{d}_p \approx 6357$ km), and its semi-major (equatorial) axis of length $\tilde{d}_E \approx 6378$ km. (We use prime numbers to denote physical [dimension] variables; these will be removed when we introduce a suitable non-dimensionalisation.)

The coordinate system $(\varphi, \theta, \tilde{z})$, $\tilde{z}$ being the vertical distance up from the surface of the ellipsoid, is associated with the ellipsoid which is rotating about its polar axis with (constant) angular speed $\tilde{\Omega} \approx 7.29 \times 10^{-5}$ rad $\cdot$ s^{-1}. In this system, the unit tangent vectors at the surface of the ellipsoid are $(\mathbf{e}_\varphi, \mathbf{e}_\beta, \mathbf{e}_z)$; $\mathbf{e}_\varphi$ points from West to East along the geodetic parallel, $\mathbf{e}_\beta$ from South to North along the geodetic meridian, and $\mathbf{e}_z$ points upwards. The spherical coordinates, the hybrid spherical-geopotential rotating coordinates and the ellipsoidal coordinates are shown in Figure 1. The introduction of the ellipsoidal coordinates and the connection between the three coordinates are described in [**20**].

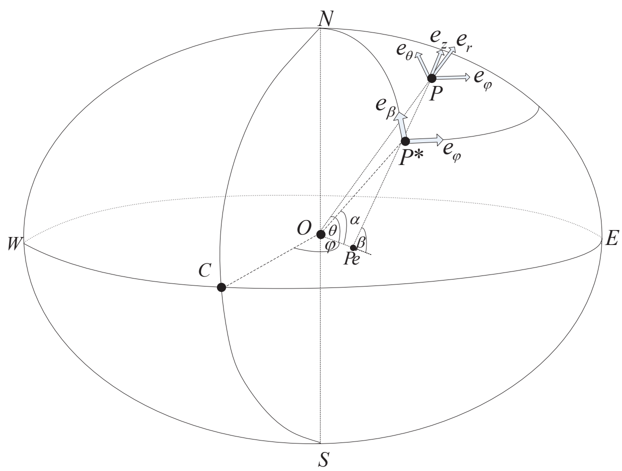

Figure 1. Away from the polar axis, we represent a point P in the atmosphere using the hybrid spherical-geopotential rotating coordinate system $(\varphi, \theta, \tilde{z})$, obtained from the spherical system $(\mathbf{e}_\varphi, \mathbf{e}_\theta, \mathbf{e}_r)$ and the geopotential system $(\mathbf{e}_\varphi, \mathbf{e}_\beta, \mathbf{e}_z)$. Here, φ and θ are the longitude and geocentric latitude of P, respectively, β and α are the geodetic and geocentric latitude of the projection P^* of P on the ellipsoidal geoid, respectively, and $\mathbf{e}_z$ points upwards along the normal P^*P to the geoid (which intersects the equatorial plane in the point P_e). The unit vectors $(\mathbf{e}_\theta, \mathbf{e}_r)$ are obtained by rotating the unit vectors $(\mathbf{e}_\beta, \mathbf{e}_z)$ by the angle $(\beta - \theta)$, in the plane of fixed longitude φ.

Here, we consider the governing equation of the atmosphere is the Navier-Stokes equation for a compressible fluid with the dynamic eddy viscosity which depends on only $\tilde{z}$. By finding the transformation formula between the spherical coordinates and the ellipsoidal coordinates, and introducing the appropriate non-dimensionalisation, the approximate parameter of thin-shell ε and the parameter $\delta = \mathfrak{e}^2$ ($\mathfrak{e}$ is the eccentricity) which measures the effects of small deviations of the ellipsoid from the spherical, we derive the governing equation of the viscous compressible atmosphere in ellipsoidal coordinates from the governing equation of the viscous compressible atmosphere in spherical coordinates (see [**20**]), as follows,

$$-2\varepsilon\rho v\sin\theta = -\frac{(1-\varepsilon z+\delta\Delta)}{\cos\theta}\frac{\partial p}{\partial\varphi} + \frac{\varepsilon}{R_e}\frac{\partial}{\partial z}\left(m\frac{\partial u}{\partial z}\right) + O(\varepsilon^2,\varepsilon\delta,\delta^2), \tag{2.1}$$

$$\begin{aligned} 2\varepsilon^2\rho u\sin\theta + {}&\varepsilon\rho(1+\varepsilon z-\delta\Delta)\sin\theta\cos\theta \\ &= -(1-\varepsilon z+\delta\Delta)\left[\varepsilon\frac{\partial p}{\partial\theta} + (2\delta\Delta+\delta^2 D)\frac{\partial p}{\partial z}\right] \\ &\quad - 2\delta\Delta\rho g[1+\delta(\sin\theta+\cos\theta)\cos\theta - 3\varepsilon z] \\ &\quad + \frac{\varepsilon^2}{R_e}\frac{\partial}{\partial z}\left(m\frac{\partial v}{\partial z}\right) + O(\varepsilon^3,\varepsilon^2\delta,\varepsilon\delta^2), \end{aligned} \tag{2.2}$$

$$-\varepsilon\rho\cos^2\theta = -\frac{\partial p}{\partial z} - \rho g(1-2\varepsilon z+2\delta\Delta) + O(\varepsilon^2,\varepsilon\delta,\delta^2), \tag{2.3}$$

where $(u(\varphi,\theta,z), v(\varphi,\theta,z), w(\varphi,\theta,z))$ is the velocity, $\rho(\varphi,\theta,z)$ is the density, $p(\varphi,\theta,z)$ is the pressure, $m(z)$ is the dynamic eddy viscosity, $g \approx 9.81\ \mathrm{m\cdot s^{-2}}$ is the acceleration of gravity, R_e is the Reynolds number, $\Delta(\theta) = \frac{1}{2}\sin\theta\cos\theta$, and $D(\theta) = (1-\frac{3}{2}\sin^2\theta)\sin\theta\cos\theta$.

We find that (2.1)-(2.3) describe a purely horizontal flow. For purely horizontal flow the atmosphere behaves as though it were an incompressible fluid [**1**]. Furthermore, the basic state (standard atmosphere) density varies across the lowest kilometer of the atmosphere by only about 10%, and the fluctuating component of density deviates from the basic state by only a few percentage points [**1**]. So, the density $\rho(\varphi,\theta,z)$ can be replaced by a constant mean value ρ. Then the classical Ekman equations can be recovered from the above equations (2.1)-(2.3), as follows (see [**20**]),

$$\begin{cases} 2\rho v_0\sin\theta + \dfrac{1}{R_e}\dfrac{\partial}{\partial z}\left(m\dfrac{\partial u_0}{\partial z}\right) = 0, \\ 2\rho u_0\sin\theta - \dfrac{1}{R_e}\dfrac{\partial}{\partial z}\left(m\dfrac{\partial v_0}{\partial z}\right) = 0. \end{cases} \tag{2.4}$$

We take the following boundary conditions for the system (2.4)

$$u_0 = a, \quad v_0 = b, \quad \text{at} \quad z = z_0, \tag{2.5}$$

and

$$u_0 = u_g, \quad v_0 = v_g, \quad \text{at} \quad z = \frac{\pi\sqrt{\frac{m}{\Omega\sin\theta}}}{\tilde{H}}, \tag{2.6}$$

where a and b are nonzero constants, u_g and v_g are the nondimensionalize geostrophic velocities, and $\tilde{H}$ the maximum height of the troposphere.

REMARK 2.1. When $z = \frac{\pi\sqrt{\frac{m}{\Omega\sin\theta}}}{\tilde{H}}$ (a constant), the wind (u,v) is parallel to and nearly equal to the geostrophic value (u_g, v_g), it is conventional to designate this level as the top of the Ekman layer [**1**], so we change the condition (2.6) to

$$u_0 = u_g, \quad v_0 = v_g, \quad \text{at} \quad z = z_3 = \frac{h}{\tilde{H}}, \tag{2.7}$$

where h is a constant (observations indicate that $h \approx 1$km, see [**1**]).

3. Main results

Let $\phi = u_0 + iv_0$, and from (2.4), we obtain

$$m\frac{\partial^2\phi}{\partial z^2} + m'\frac{\partial\phi}{\partial z} - i2\rho\phi\sin\theta R_e = 0, \tag{3.1}$$

the boundary conditions (2.5) and (2.7) become

$$\phi = \phi_0 = a + ib, \quad \text{at} \quad z = z_0, \tag{3.2}$$

and

$$\phi = \phi_1 = u_g + iv_g, \quad \text{at} \quad z = z_3. \tag{3.3}$$

If the eddy viscosity $m =$ constant, then (3.1) will be transform into

$$\frac{\partial^2\phi}{\partial z^2} - i\frac{2R_e\rho\phi\sin\theta}{m} = 0, \tag{3.4}$$

then, we get the solution of the equation (3.4) is

$$\phi = Ae^{\sqrt{\frac{R_e\rho\sin\theta}{m}}(1+i)z} + Be^{-\sqrt{\frac{R_e\rho\sin\theta}{m}}(1+i)z},$$

with A, B constants. We determine the constants A and B that satisfy the boundary conditions (3.2) and (3.3). Imposing the boundary conditions (3.2) and (3.3), we get

$$\begin{cases} Ae^{\sqrt{\frac{R_e\rho\sin\theta}{m}}(1+i)z_0} + Be^{-\sqrt{\frac{R_e\rho\sin\theta}{m}}(1+i)z_0} = \phi_0, \\ Ae^{\sqrt{\frac{R_e\rho\sin\theta}{m}}(1+i)z_3} + Be^{-\sqrt{\frac{R_e\rho\sin\theta}{m}}(1+i)z_3} = \phi_1. \end{cases} \tag{3.5}$$

By (3.5), we obtain

$$A = \frac{\phi_0 e^{-\sqrt{\frac{R_e\rho\sin\theta}{m}}(1+i)z_3} - \phi_1 e^{-\sqrt{\frac{R_e\rho\sin\theta}{m}}(1+i)z_0}}{e^{\sqrt{\frac{R_e\rho\sin\theta}{m}}(1+i)(z_0-z_3)} - e^{\sqrt{\frac{\rho\sin\theta R_e}{m}}(1+i)(z_3-z_0)}},$$

$$B = \frac{\phi_1 e^{\sqrt{\frac{R_e\rho\sin\theta}{m}}(1+i)z_0} - \phi_0 e^{\sqrt{\frac{R_e\rho\sin\theta}{m}}(1+i)z_3}}{e^{\sqrt{\frac{R_e\rho\sin\theta}{m}}(1+i)(z_0-z_3)} - e^{\sqrt{\frac{R_e\rho\sin\theta}{m}}(1+i)(z_3-z_0)}}.$$

So, the solution of the equation (3.4) is

$$\begin{aligned} \phi =& \phi_0 \frac{e^{\sqrt{\frac{R_e\rho\sin\theta}{m}}(1+i)(z-z_3)} - e^{\sqrt{\frac{R_e\rho\sin\theta}{m}}(1+i)(z_3-z)}}{e^{\sqrt{\frac{R_e\rho\sin\theta}{m}}(1+i)(z_0-z_3)} - e^{\sqrt{\frac{R_e\rho\sin\theta}{m}}(1+i)(z_3-z_0)}} \\ &+ \phi_1 \frac{e^{\sqrt{\frac{R_e\rho\sin\theta}{m}}(1+i)(z_0-z)} - e^{\sqrt{\frac{R_e\rho\sin\theta}{m}}(1+i)(z-z_0)}}{e^{\sqrt{\frac{R_e\rho\sin\theta}{m}}(1+i)(z_0-z_3)} - e^{\sqrt{\frac{R_e\rho\sin\theta}{m}}(1+i)(z_3-z_0)}}. \end{aligned}$$

However, the eddy viscosity $m(z)$ always varies with height [**10**], here, we consider the following cases.

3.1. Case I. Here, we consider the case of an eddy viscosity given by

$$m(z) := \frac{2R_e\rho\sin\theta}{c}z^2,$$

where $c > 0$.

We write (3.1) as

$$(m(z)\phi')' - i \cdot 2R_e\rho\phi\sin\theta = 0, \tag{3.6}$$

we integrate the equation (3.6) and get

$$m(z)\phi' - i \cdot 2R_e\rho\sin\theta \int \phi dz = 0. \tag{3.7}$$

With the notation

$$\eta(z) = \int \phi dz, \tag{3.8}$$

the equation (3.7) will become

$$m(z)\eta''(z) - i \cdot 2R_e\rho\sin\theta\eta(z) = 0. \tag{3.9}$$

Note that $m(z) = \frac{2R_e\rho\sin\theta}{c}z^2$, so we have

$$\eta'' - i \cdot \frac{c}{z^2}\eta = 0. \tag{3.10}$$

By the transformation

$$\eta(z) = z\zeta(s), \quad s = \frac{1}{z}, \tag{3.11}$$

the equation (3.10) becomes

$$\zeta''(s) = i \cdot s^{-2}\zeta(s).$$

We look for solutions of the form

$$\zeta(s) = s^{\alpha}.$$

The equation that α has to satisfy is

$$\alpha^2 - \alpha - i \cdot c = 0.$$

Thus, there are two possible values for α:

$$\alpha = \frac{1 \pm \sqrt{1 + i \cdot 4c}}{2} = \frac{1}{2} \pm \left(\frac{c}{\sqrt{\frac{\sqrt{16c^2+1}-1}{2}}} + i \cdot \frac{1}{2}\sqrt{\frac{\sqrt{16c^2+1}-1}{2}} \right).$$

Then, the general solution of the equation in ζ is

$$\zeta(s) = A's^{\frac{1}{2}+c\sqrt{\frac{2}{\sqrt{16c^2+1}-1}}+i\cdot\frac{1}{2}\sqrt{\frac{\sqrt{16c^2+1}-1}{2}}} + B's^{\frac{1}{2}-c\sqrt{\frac{2}{\sqrt{16c^2+1}-1}}-i\cdot\frac{1}{2}\sqrt{\frac{\sqrt{16c^2+1}-1}{2}}},$$

where A', B' are constants. So, by (3.11),

$$\eta(z) = A'z^{\frac{1}{2}-c\sqrt{\frac{2}{\sqrt{16c^2+1}-1}}-i\cdot\frac{1}{2}\sqrt{\frac{\sqrt{16c^2+1}-1}{2}}} + B'z^{\frac{1}{2}+c\sqrt{\frac{2}{\sqrt{16c^2+1}-1}}+i\cdot\frac{1}{2}\sqrt{\frac{\sqrt{16c^2+1}-1}{2}}},$$

and by (3.8), the general solution of the equation (3.1) is

$$\phi = Az^{-\frac{1}{2}-c\sqrt{\frac{2}{\sqrt{16c^2+1}-1}}-i\cdot\frac{1}{2}\sqrt{\frac{\sqrt{16c^2+1}-1}{2}}} + Bz^{-\frac{1}{2}+c\sqrt{\frac{2}{\sqrt{16c^2+1}-1}}+i\cdot\frac{1}{2}\sqrt{\frac{\sqrt{16c^2+1}-1}{2}}},$$

with A, B constants. Imposing the boundary conditions (3.2) and (3.3), we obtain (3.12)

$$\begin{cases} Az_0^{-\frac{1}{2}-c\sqrt{\frac{2}{\sqrt{16c^2+1}-1}}-i\cdot\frac{1}{2}\sqrt{\frac{\sqrt{16c^2+1}-1}{2}}} + Bz_0^{-\frac{1}{2}+c\sqrt{\frac{2}{\sqrt{16c^2+1}-1}}+i\cdot\frac{1}{2}\sqrt{\frac{\sqrt{16c^2+1}-1}{2}}} = \phi_0, \\ Az_3^{-\frac{1}{2}-c\sqrt{\frac{2}{\sqrt{16c^2+1}-1}}-i\cdot\frac{1}{2}\sqrt{\frac{\sqrt{16c^2+1}-1}{2}}} + Bz_3^{-\frac{1}{2}+c\sqrt{\frac{2}{\sqrt{16c^2+1}-1}}+i\cdot\frac{1}{2}\sqrt{\frac{\sqrt{16c^2+1}-1}{2}}} = \phi_1. \end{cases}$$

We can rewrite the formula (3.12) as

$$
\left\{
\begin{aligned}
& Az_0^{-\frac{1}{2}-c\sqrt{\frac{2}{\sqrt{16c^2+1}-1}}}e^{-i\cdot\frac{\ln z_0}{2}\sqrt{\frac{\sqrt{16c^2+1}-1}{2}}} \\
& +Bz_0^{-\frac{1}{2}+c\sqrt{\frac{2}{\sqrt{16c^2+1}-1}}}e^{i\cdot\frac{\ln z_0}{2}\sqrt{\frac{\sqrt{16c^2+1}-1}{2}}} = \phi_0, \\
& Az_3^{-\frac{1}{2}-c\sqrt{\frac{2}{\sqrt{16c^2+1}-1}}}e^{-i\cdot\frac{\ln z_3}{2}\sqrt{\frac{\sqrt{16c^2+1}-1}{2}}} \\
& +Bz_3^{-\frac{1}{2}+c\sqrt{\frac{2}{\sqrt{16c^2+1}-1}}}e^{i\cdot\frac{\ln z_3}{2}\sqrt{\frac{\sqrt{16c^2+1}-1}{2}}} = \phi_1.
\end{aligned}
\right. \tag{3.13}
$$

Let $A = A_1 + iA_2$, $B = B_1 + iB_2$ and $q = \sqrt{\frac{2}{\sqrt{16c^2+1}-1}}$, and expanding the fomula (3.13) into real and imaginary parts, we have

$$
\left\{
\begin{aligned}
& A_1 z_0^{-\frac{1}{2}-cq}\cos\left(\frac{q\ln z_0}{2}\right) + A_2 z_0^{\frac{1}{2}-cq}\sin\left(\frac{q\ln z_0}{2}\right) \\
& +B_1 z_0^{-\frac{1}{2}+cq}\cos\left(\frac{q\ln z_0}{2}\right) - B_2 z_0^{-\frac{1}{2}+cq}\sin\left(\frac{q\ln z_0}{2}\right) = a, \\
& A_2 z_0^{\frac{1}{2}-cq}\cos\left(\frac{q\ln z_0}{2}\right) - A_1 z_0^{-\frac{1}{2}-cq}\sin\left(\frac{q\ln z_0}{2}\right) \\
& +B_1 z_0^{-\frac{1}{2}+cq}\sin\left(\frac{q\ln z_0}{2}\right) + B_2 z_0^{-\frac{1}{2}+cq}\cos\left(\frac{q\ln z_0}{2}\right) = b, \\
& A_1 z_3^{-\frac{1}{2}-cq}\cos\left(\frac{q\ln z_3}{2}\right) + A_2 z_3^{\frac{1}{2}-cq}\sin\left(\frac{q\ln z_3}{2}\right) \\
& +B_1 z_3^{-\frac{1}{2}+cq}\cos\left(\frac{q\ln z_3}{2}\right) - B_2 z_3^{-\frac{1}{2}+cq}\sin\left(\frac{q\ln z_3}{2}\right) = u_g, \\
& A_2 z_3^{\frac{1}{2}-cq}\cos\left(\frac{q\ln z_3}{2}\right) - A_1 z_3^{-\frac{1}{2}-cq}\sin\left(\frac{q\ln z_3}{2}\right) \\
& +B_1 z_3^{-\frac{1}{2}+cq}\sin\left(\frac{q\ln z_3}{2}\right) + B_2 z_3^{-\frac{1}{2}+cq}\cos\left(\frac{q\ln z_3}{2}\right) = v_g.
\end{aligned}
\right.
$$

Thus,

$$
\begin{bmatrix}
z_0^{-\frac{1}{2}-cq}\cos\left(\frac{q\ln z_0}{2}\right) & z_0^{\frac{1}{2}-cq}\sin\left(\frac{q\ln z_0}{2}\right) & z_0^{-\frac{1}{2}+cq}\cos\left(\frac{q\ln z_0}{2}\right) & -z_0^{-\frac{1}{2}+cq}\sin\left(\frac{q\ln z_0}{2}\right) \\
-z_0^{-\frac{1}{2}-cq}\sin\left(\frac{q\ln z_0}{2}\right) & z_0^{\frac{1}{2}-cq}\cos\left(\frac{q\ln z_0}{2}\right) & z_0^{-\frac{1}{2}+cq}\sin\left(\frac{q\ln z_0}{2}\right) & z_0^{-\frac{1}{2}+cq}\cos\left(\frac{q\ln z_0}{2}\right) \\
z_3^{-\frac{1}{2}-cq}\cos\left(\frac{q\ln z_3}{2}\right) & z_3^{\frac{1}{2}-cq}\sin\left(\frac{q\ln z_3}{2}\right) & z_3^{-\frac{1}{2}+cq}\cos\left(\frac{q\ln z_3}{2}\right) & -z_3^{-\frac{1}{2}+cq}\sin\left(\frac{q\ln z_3}{2}\right) \\
-z_3^{-\frac{1}{2}-cq}\sin\left(\frac{q\ln z_3}{2}\right) & z_3^{\frac{1}{2}-cq}\cos\left(\frac{q\ln z_3}{2}\right) & z_3^{-\frac{1}{2}+cq}\sin\left(\frac{q\ln z_3}{2}\right) & z_3^{-\frac{1}{2}+cq}\cos\left(\frac{q\ln z_3}{2}\right)
\end{bmatrix}
$$

$$
\times \begin{bmatrix} A_1 \\ A_2 \\ B_1 \\ B_2 \end{bmatrix} = \begin{bmatrix} a \\ b \\ u_g \\ v_g \end{bmatrix},
$$

i.e.,

$$
d\begin{bmatrix} A_1 \\ A_2 \\ B_1 \\ B_2 \end{bmatrix} = \begin{bmatrix} a \\ b \\ u_g \\ v_g \end{bmatrix}. \tag{3.14}
$$

Obviously,

$$\begin{aligned}\det(d) &= z_0^{-1-2cq} z_3^{-1-2cq} \left[z_0^{4cq} + z_3^{4cq} - 2 z_0^{2cq} z_3^{2cq} \cos\left(\frac{\ln(z_0) - \ln(z_3)}{q} \right) \right] \\ &\geq z_0^{-1-2cq} z_3^{-1-2cq} \left[z_0^{4cq} + z_3^{4cq} - 2 z_0^{2cq} z_3^{2cq} \right] \\ &= z_0^{-1-2cq} z_3^{-1-2cq} \left[z_3^{2cq} - z_2^{2cq} \right]^2 > 0.\end{aligned}$$

So according to Cramer's Rule, we obtain an unique solution of the equation (3.14),

$$A_1 = \frac{\det(d_1)}{\det(d)},\ A_2 = \frac{\det(d_2)}{\det(d)},\ B_1 = \frac{\det(d_3)}{\det(d)},\ B_2 = \frac{\det(d_4)}{\det(d)}, \tag{3.15}$$

where

$$d_1 = \begin{bmatrix} a & z_0^{\frac{1}{2}-cq} \sin\left(\frac{q \ln z_0}{2}\right) & z_0^{-\frac{1}{2}+cq} \cos\left(\frac{q \ln z_0}{2}\right) & -z_0^{-\frac{1}{2}+cq} \sin\left(\frac{q \ln z_0}{2}\right) \\ b & z_0^{\frac{1}{2}-cq} \cos\left(\frac{q \ln z_0}{2}\right) & z_0^{-\frac{1}{2}+cq} \sin\left(\frac{q \ln z_0}{2}\right) & z_0^{-\frac{1}{2}+cq} \cos\left(\frac{q \ln z_0}{2}\right) \\ u_g & z_3^{\frac{1}{2}-cq} \sin\left(\frac{q \ln z_3}{2}\right) & z_3^{-\frac{1}{2}+cq} \cos\left(\frac{q \ln z_3}{2}\right) & -z_3^{-\frac{1}{2}+cq} \sin\left(\frac{q \ln z_3}{2}\right) \\ v_g & z_3^{\frac{1}{2}-cq} \cos\left(\frac{q \ln z_3}{2}\right) & z_3^{-\frac{1}{2}+cq} \sin\left(\frac{q \ln z_3}{2}\right) & z_3^{-\frac{1}{2}+cq} \cos\left(\frac{q \ln z_3}{2}\right) \end{bmatrix},$$

$$d_2 = \begin{bmatrix} z_0^{-\frac{1}{2}-cq} \cos\left(\frac{q \ln z_0}{2}\right) & a & z_0^{-\frac{1}{2}+cq} \cos\left(\frac{q \ln z_0}{2}\right) & -z_0^{-\frac{1}{2}+cq} \sin\left(\frac{q \ln z_0}{2}\right) \\ -z_0^{-\frac{1}{2}-cq} \sin\left(\frac{q \ln z_0}{2}\right) & b & z_0^{-\frac{1}{2}+cq} \sin\left(\frac{q \ln z_0}{2}\right) & z_0^{-\frac{1}{2}+cq} \cos\left(\frac{q \ln z_0}{2}\right) \\ z_3^{-\frac{1}{2}-cq} \cos\left(\frac{q \ln z_3}{2}\right) & u_g & z_3^{-\frac{1}{2}+cq} \cos\left(\frac{q \ln z_3}{2}\right) & -z_3^{-\frac{1}{2}+cq} \sin\left(\frac{q \ln z_3}{2}\right) \\ -z_3^{-\frac{1}{2}-cq} \sin\left(\frac{q \ln z_3}{2}\right) & v_g & z_3^{-\frac{1}{2}+cq} \sin\left(\frac{q \ln z_3}{2}\right) & z_3^{-\frac{1}{2}+cq} \cos\left(\frac{q \ln z_3}{2}\right) \end{bmatrix},$$

$$d_3 = \begin{bmatrix} z_0^{-\frac{1}{2}-cq} \cos\left(\frac{q \ln z_0}{2}\right) & z_0^{\frac{1}{2}-cq} \sin\left(\frac{q \ln z_0}{2}\right) & a & -z_0^{-\frac{1}{2}+cq} \sin\left(\frac{q \ln z_0}{2}\right) \\ -z_0^{-\frac{1}{2}-cq} \sin\left(\frac{q \ln z_0}{2}\right) & z_0^{\frac{1}{2}-cq} \cos\left(\frac{q \ln z_0}{2}\right) & b & z_0^{-\frac{1}{2}+cq} \cos\left(\frac{q \ln z_0}{2}\right) \\ z_3^{-\frac{1}{2}-cq} \cos\left(\frac{q \ln z_3}{2}\right) & z_3^{\frac{1}{2}-cq} \sin\left(\frac{q \ln z_3}{2}\right) & u_g & -z_3^{-\frac{1}{2}+cq} \sin\left(\frac{q \ln z_3}{2}\right) \\ -z_3^{-\frac{1}{2}-cq} \sin\left(\frac{q \ln z_3}{2}\right) & z_3^{\frac{1}{2}-cq} \cos\left(\frac{q \ln z_3}{2}\right) & v_g & z_3^{-\frac{1}{2}+cq} \cos\left(\frac{q \ln z_3}{2}\right) \end{bmatrix},$$

$$d_4 = \begin{bmatrix} z_0^{-\frac{1}{2}-cq} \cos\left(\frac{q \ln z_0}{2}\right) & z_0^{\frac{1}{2}-cq} \sin\left(\frac{q \ln z_0}{2}\right) & z_0^{-\frac{1}{2}+cq} \cos\left(\frac{q \ln z_0}{2}\right) & a \\ -z_0^{-\frac{1}{2}-cq} \sin\left(\frac{q \ln z_0}{2}\right) & z_0^{\frac{1}{2}-cq} \cos\left(\frac{q \ln z_0}{2}\right) & z_0^{-\frac{1}{2}+cq} \sin\left(\frac{q \ln z_0}{2}\right) & b \\ z_3^{-\frac{1}{2}-cq} \cos\left(\frac{q \ln z_3}{2}\right) & z_3^{\frac{1}{2}-cq} \sin\left(\frac{q \ln z_3}{2}\right) & z_3^{-\frac{1}{2}+cq} \cos\left(\frac{q \ln z_3}{2}\right) & u_g \\ -z_3^{-\frac{1}{2}-cq} \sin\left(\frac{q \ln z_3}{2}\right) & z_3^{\frac{1}{2}-cq} \cos\left(\frac{q \ln z_3}{2}\right) & z_3^{-\frac{1}{2}+cq} \sin\left(\frac{q \ln z_3}{2}\right) & v_g \end{bmatrix}.$$

From the above discussion, we get the following result.

THEOREM 3.1. *The solution of the problem* (3.1) *with the boundary conditions* (3.2) *and* (3.3) *is*

$$
\begin{cases}
u_0 = & A_1 z^{-\frac{1}{2}-c\sqrt{\frac{2}{\sqrt{16c^2+1}-1}}}\cos\left(\frac{\ln z}{2}\sqrt{\frac{2}{\sqrt{16c^2+1}-1}}\right) \\
& +A_2 z^{\frac{1}{2}-c\sqrt{\frac{2}{\sqrt{16c^2+1}-1}}}\sin\left(\frac{\ln z}{2}\sqrt{\frac{2}{\sqrt{16c^2+1}-1}}\right) \\
& +B_1 z^{-\frac{1}{2}+c\sqrt{\frac{2}{\sqrt{16c^2+1}-1}}}\cos\left(\frac{\ln z}{2}\sqrt{\frac{2}{\sqrt{16c^2+1}-1}}\right) \\
& -B_2 z^{-\frac{1}{2}+c\sqrt{\frac{2}{\sqrt{16c^2+1}-1}}}\sin\left(\frac{\ln z}{2}\sqrt{\frac{2}{\sqrt{16c^2+1}-1}}\right), \\
v_0 = & A_2 z^{\frac{1}{2}-c\sqrt{\frac{2}{\sqrt{16c^2+1}-1}}}\cos\left(\frac{\ln z}{2}\sqrt{\frac{2}{\sqrt{16c^2+1}-1}}\right) \\
& -A_1 z^{-\frac{1}{2}-c\sqrt{\frac{2}{\sqrt{16c^2+1}-1}}}\sin\left(\frac{\ln z}{2}\sqrt{\frac{2}{\sqrt{16c^2+1}-1}}\right) \\
& +B_1 z^{-\frac{1}{2}+c\sqrt{\frac{2}{\sqrt{16c^2+1}-1}}}\sin\left(\frac{\ln z}{2}\sqrt{\frac{2}{\sqrt{16c^2+1}-1}}\right) \\
& +B_2 z^{-\frac{1}{2}+c\sqrt{\frac{2}{\sqrt{16c^2+1}-1}}}\cos\left(\frac{\ln z}{2}\sqrt{\frac{2}{\sqrt{16c^2+1}-1}}\right),
\end{cases}
\tag{3.16}
$$

where $A_1,\ A_2,\ B_1$ *and* B_2 *are given by* (3.15).

We denote the angle between the wind vector at any height and the geostrophic vector by $\tau(z)$, that is

$$\tau(z) = \arg \frac{u_0 + iv_0}{u_g + iv_g} = \arg \frac{u_g u_0 + v_0 v_g + i(v_0 u_g - u_0 u_g)}{u_g^2 + v_g^2} = \arctan \frac{v_0 u_g - u_0 v_g}{u_0 u_g + v_0 v_g}.$$

We assume $v_g = 0$ for simplicity, together with (3.16), we get

$$
\begin{aligned}
\tau(z) &= \arctan \frac{v_0(z)}{u_0(z)} \\
&= \arctan \frac{-A_1 \tan\left(\frac{q\ln z}{2}\right) + A_2 z + B_2 z^{2cq}\tan\left(\frac{q\ln z}{2}\right) + B_2 z^{2cq}}{A_1 + A_2 z \tan\left(\frac{q\ln z}{2}\right) + B_1 z^{2cq} - B_2 z^{2cq}\tan\left(\frac{q\ln z}{2}\right)},
\end{aligned}
$$

with $q = \sqrt{\frac{2}{\sqrt{16c^2+1}-1}}$. Then, we find that

$$\tau(z_0) = \arctan \frac{b}{a} \quad \text{and} \quad \tau(z_3) = \arctan \frac{v_g}{u_g} = 0,$$

which means that the value of the angle between the wind vector and the geostrophic vector at the boundary is determined by the value of the velocity components at the boundary. In other words, as the height of the boundary changes, the velocity of the atmosphere at the boundary will also change, resulting in a change in the value of the angle between the wind vector and the geostrophic vector. Furthermore, the angle between the wind vector and the geostrophic vector becomes zero when the height reaches z_3 (the velocity components $u_0 = u_g$ and $v_0 = v_g$).

REMARK 3.2. For the top of the Ekman layer $z = z_3$, the angle $\tau(z_3) = 0$ which means the wind vector becomes the geostrophic vector, this is consistent with the result in the paper [**19**]. For the bottom of the Ekman layer, the angle $\tau(z_0) = \arctan \frac{b}{a}$. If we consider $a > 0$ and $b > 0$, then we obtain that the angle

$\tau(z_0) \in (0, \frac{\pi}{2})$, this is consistent with the result in the paper [**13**]. If we consider $a > b > 0$, then the angle $\tau(z_0) \in (0, \frac{\pi}{4})$, this is consistent with the result in the paper [**19**].

3.2. Case II. We consider the case of an eddy viscosity given by

$$m(z) := 2R_e\rho \sin\theta z^4.$$

Then, the equation (3.9) becomes

$$\eta'' = i \cdot z^{-4}\eta. \tag{3.17}$$

THEOREM 3.3. *The solution of* (3.17) *can be expressed by the following formula*

$$\eta(z) = Aze^{-\frac{\sqrt{i}}{z}} + Bze^{\frac{\sqrt{i}}{z}}. \tag{3.18}$$

And the solution of (3.1) *with the boundary conditions* (3.2) *and* (3.3) *is*

$$\phi(z) = A\left(1+\frac{\sqrt{i}}{z}\right)e^{-\frac{\sqrt{i}}{z}} + B\left(1-\frac{\sqrt{i}}{z}\right)e^{\frac{\sqrt{i}}{z}}, \tag{3.19}$$

where

$$A = \frac{\phi_0\left(1-\frac{\sqrt{i}}{z_3}\right)\left(1+\frac{\sqrt{i}}{z_0}\right)e^{\frac{\sqrt{i}}{z_3}-\frac{\sqrt{i}}{z_0}} - \phi_1\left(1-\frac{i}{z_0^2}\right)}{\left(1-\frac{\sqrt{i}}{z_3}\right)\left(1+\frac{\sqrt{i}}{z_0}\right)^2 e^{\frac{\sqrt{i}}{z_3}-\frac{2\sqrt{i}}{z_0}} - \left(1-\frac{i}{z_0^2}\right)\left(1+\frac{\sqrt{i}}{z_3}\right)e^{-\frac{\sqrt{i}}{z_3}}},$$

$$B = \frac{\phi_1\left(1+\frac{\sqrt{i}}{z_0}\right)e^{-\frac{\sqrt{i}}{z_0}} - \phi_0\left(1+\frac{\sqrt{i}}{z_3}\right)e^{-\frac{\sqrt{i}}{z_3}}}{\left(1-\frac{\sqrt{i}}{z_3}\right)\left(1+\frac{\sqrt{i}}{z_0}\right)e^{\frac{\sqrt{i}}{z_3}-\frac{\sqrt{i}}{z_0}} - \left(1-\frac{\sqrt{i}}{z_0}\right)\left(1+\frac{\sqrt{i}}{z_3}\right)e^{\frac{\sqrt{i}}{z_0}-\frac{\sqrt{i}}{z_3}}}.$$

PROOF. From (3.18), we have

$$\eta'(z) = A\left(1+\frac{\sqrt{i}}{z}\right)e^{-\frac{\sqrt{i}}{z}} + B\left(1-\frac{\sqrt{i}}{z}\right)e^{\frac{\sqrt{i}}{z}},$$

then

$$\eta''(z) = A\frac{i}{z^3}e^{-\frac{\sqrt{i}}{z}} + B\frac{i}{z^3}e^{\frac{\sqrt{i}}{z}},$$

therefore,

$$\begin{aligned} i \cdot z^{-4}\eta(z) &= i \cdot z^{-4}\left(Aze^{-\frac{\sqrt{i}}{z}} + Bze^{\frac{\sqrt{i}}{z}}\right) \\ &= A\frac{i}{z^3}e^{-\frac{\sqrt{i}}{z}} + B\frac{i}{z^3}e^{\frac{\sqrt{i}}{z}} = \eta''(z), \end{aligned}$$

which implies (3.18) is the solution of (3.17).

By (3.8), we obtain the solution of (3.1) is

$$\phi(z) = A\left(1+\frac{\sqrt{i}}{z}\right)e^{-\frac{\sqrt{i}}{z}} + B\left(1-\frac{\sqrt{i}}{z}\right)e^{\frac{\sqrt{i}}{z}}.$$

Considering the boundary conditions (3.2) and (3.3), we get

$$\phi_0 = A\left(1+\frac{\sqrt{i}}{z_0}\right)e^{-\frac{\sqrt{i}}{z_0}} + B\left(1-\frac{\sqrt{i}}{z_0}\right)e^{\frac{\sqrt{i}}{z_0}}, \tag{3.20}$$

$$\phi_1 = A\left(1+\frac{\sqrt{i}}{z_3}\right)e^{-\frac{\sqrt{i}}{z_3}} + B\left(1-\frac{\sqrt{i}}{z_3}\right)e^{\frac{\sqrt{i}}{z_3}}. \tag{3.21}$$

Combining (3.20) and (3.21), we have

$$A = \frac{\phi_0 \left(1 - \frac{\sqrt{i}}{z_3}\right)\left(1 + \frac{\sqrt{i}}{z_0}\right) e^{\frac{\sqrt{i}}{z_3} - \frac{\sqrt{i}}{z_0}} - \phi_1 \left(1 - \frac{i}{z_0^2}\right)}{\left(1 - \frac{\sqrt{i}}{z_3}\right)\left(1 + \frac{\sqrt{i}}{z_0}\right)^2 e^{\frac{\sqrt{i}}{z_3} - \frac{2\sqrt{i}}{z_0}} - \left(1 - \frac{i}{z_0^2}\right)\left(1 + \frac{\sqrt{i}}{z_3}\right) e^{-\frac{\sqrt{i}}{z_3}}},$$

$$B = \frac{\phi_1 \left(1 + \frac{\sqrt{i}}{z_0}\right) e^{-\frac{\sqrt{i}}{z_0}} - \phi_0 \left(1 + \frac{\sqrt{i}}{z_3}\right) e^{-\frac{\sqrt{i}}{z_3}}}{\left(1 - \frac{\sqrt{i}}{z_3}\right)\left(1 + \frac{\sqrt{i}}{z_0}\right) e^{\frac{\sqrt{i}}{z_3} - \frac{\sqrt{i}}{z_0}} - \left(1 - \frac{\sqrt{i}}{z_0}\right)\left(1 + \frac{\sqrt{i}}{z_3}\right) e^{\frac{\sqrt{i}}{z_0} - \frac{\sqrt{i}}{z_3}}}.$$

This proof is complete. □

We introduce the angle between the wind vector at any height and the geostrophic vector:

$$\tau(z) = \arg \frac{u_0 + iv_0}{u_g + iv_g} = \arg \frac{\phi(z)}{\phi_1},$$

which, by taking into account (3.19), is equal to:

$$\begin{aligned}
&\tau(z)\\
&= \arg \frac{A\left(1 + \frac{\sqrt{i}}{z}\right) e^{-\frac{\sqrt{i}}{z}} + B\left(1 - \frac{\sqrt{i}}{z}\right) e^{\frac{\sqrt{i}}{z}}}{\phi_1}\\
&= \arg \left(\frac{\frac{\phi_0}{\phi_1}\left(1 - \frac{\sqrt{i}}{z_3}\right)\left(1 + \frac{\sqrt{i}}{z_0}\right)\left(1 + \frac{\sqrt{i}}{z}\right) e^{\frac{2\sqrt{i}}{z_3} + \frac{\sqrt{i}}{z_0}} - \left(1 - \frac{i}{z_0^2}\right)\left(1 + \frac{\sqrt{i}}{z}\right) e^{\frac{\sqrt{i}}{z_3} + \frac{2\sqrt{i}}{z_0}}}{\left(1 - \frac{\sqrt{i}}{z_3}\right)\left(1 + \frac{\sqrt{i}}{z_0}\right)^2 e^{\frac{2\sqrt{i}}{z_3} + \frac{\sqrt{i}}{z}} - \left(1 - \frac{i}{z_0^2}\right)\left(1 + \frac{\sqrt{i}}{z_3}\right) e^{\frac{2\sqrt{i}}{z_0} + \frac{\sqrt{i}}{z}}} \right.\\
&\qquad \left. + \frac{\left(1 + \frac{\sqrt{i}}{z_0}\right)\left(1 - \frac{\sqrt{i}}{z}\right) e^{\frac{\sqrt{i}}{z} + \frac{\sqrt{i}}{z_3}} - \frac{\phi_0}{\phi_1}\left(1 + \frac{\sqrt{i}}{z_3}\right)\left(1 + \frac{\sqrt{i}}{z}\right) e^{\frac{\sqrt{i}}{z_0} + \frac{\sqrt{i}}{z}}}{\left(1 - \frac{\sqrt{i}}{z_3}\right)\left(1 + \frac{\sqrt{i}}{z_0}\right) e^{\frac{2\sqrt{i}}{z_3}} - \left(1 - \frac{\sqrt{i}}{z_0}\right)\left(1 + \frac{\sqrt{i}}{z_3}\right) e^{\frac{2\sqrt{i}}{z_0}}} \right).
\end{aligned}$$

Then, we get the angles at the bottom of the layer and the top of the layer are $\tau(z_0) = \arctan \frac{bu_g - av_g}{au_g + bv_g}$ and $\tau(z_3) = \arg \frac{\phi(z_3)}{\phi_1} = \arg 1 = 0$, respectively.

3.3. Case III. We consider the case of an eddy viscosity given by

$$m(z) := 2R_e \rho \sin\theta z^{\frac{12}{7}}.$$

Then, the equation (3.9) becomes

$$\eta'' = i \cdot z^{-\frac{12}{7}} \eta. \tag{3.22}$$

For the exponent $\alpha = -\frac{12}{7}$, and the constant $c = i$, we let $q := \frac{1}{2}\alpha + 1 = \frac{1}{2}(-\frac{12}{7}) + 1 = \frac{1}{7}$. Due to $\frac{1}{q} = 7$ is an odd number, by following [**27**], the example 2.14, we obtain the general solution to the equation (3.22) is the Cayley solution given by

$$\begin{aligned}
\eta(z) =& A\left(1 - 7\sqrt{i}z^{\frac{1}{7}} + \frac{98}{5}iz^{\frac{2}{7}} - \frac{686}{15}i\sqrt{i}z^{\frac{3}{7}}\right) e^{7\sqrt{i}z^{\frac{1}{7}}}\\
&+ B\left(1 + 7\sqrt{i}z^{\frac{1}{7}} + \frac{98}{5}iz^{\frac{2}{7}} + \frac{686}{15}i\sqrt{i}z^{\frac{3}{7}}\right) e^{-7\sqrt{i}z^{\frac{1}{7}}},
\end{aligned}$$

with A, B constants. By (3.8), the general solution of the equation (3.1) is

$$\phi(z) = A\left(-\frac{7}{5}iz^{-\frac{7}{5}} + \frac{686}{15}z^{-\frac{3}{7}}\right) e^{7\sqrt{i}z^{\frac{1}{7}}} + B\left(-\frac{7}{5}iz^{-\frac{7}{5}} + \frac{686}{15}z^{-\frac{3}{7}}\right) e^{-7\sqrt{i}z^{\frac{1}{7}}}.$$

Let $A = A_1 + iA_2$, $B = B_1 + iB_2$, and expanding this into real and imaginary parts, we have

$$\begin{aligned} u_0 =& A_1\left(\frac{7}{5}z^{-\frac{5}{7}}\sin\left(\frac{7}{\sqrt{2}}z^{\frac{1}{7}}\right)+\frac{686}{15}z^{-\frac{3}{7}}\cos\left(\frac{7}{\sqrt{2}}z^{\frac{1}{7}}\right)\right)e^{\frac{7}{\sqrt{2}}z^{\frac{1}{7}}} \\ &+ A_2\left(\frac{7}{5}z^{-\frac{5}{7}}\cos\left(\frac{7}{\sqrt{2}}z^{\frac{1}{7}}\right)-\frac{686}{15}z^{-\frac{3}{7}}\sin\left(\frac{7}{\sqrt{2}}z^{\frac{1}{7}}\right)\right)e^{\frac{7}{\sqrt{2}}z^{\frac{1}{7}}} \\ &+ B_1\left(-\frac{7}{5}z^{-\frac{5}{7}}\sin\left(\frac{7}{\sqrt{2}}z^{\frac{1}{7}}\right)+\frac{686}{15}z^{-\frac{3}{7}}\cos\left(\frac{7}{\sqrt{2}}z^{\frac{1}{7}}\right)\right)e^{-\frac{7}{\sqrt{2}}z^{\frac{1}{7}}} \\ &+ B_2\left(\frac{7}{5}z^{-\frac{5}{7}}\cos\left(\frac{7}{\sqrt{2}}z^{\frac{1}{7}}\right)+\frac{686}{15}z^{-\frac{3}{7}}\sin\left(\frac{7}{\sqrt{2}}z^{\frac{1}{7}}\right)\right)e^{-\frac{7}{\sqrt{2}}z^{\frac{1}{7}}}, \end{aligned} \tag{3.23}$$

$$\begin{aligned} v_0 =& A_1\left(-\frac{7}{5}z^{-\frac{5}{7}}\cos\left(\frac{7}{\sqrt{2}}z^{\frac{1}{7}}\right)+\frac{686}{15}z^{-\frac{3}{7}}\sin\left(\frac{7}{\sqrt{2}}z^{\frac{1}{7}}\right)\right)e^{\frac{7}{\sqrt{2}}z^{\frac{1}{7}}} \\ &+ A_2\left(\frac{7}{5}z^{-\frac{5}{7}}\sin\left(\frac{7}{\sqrt{2}}z^{\frac{1}{7}}\right)+\frac{686}{15}z^{-\frac{3}{7}}\cos\left(\frac{7}{\sqrt{2}}z^{\frac{1}{7}}\right)\right)e^{\frac{7}{\sqrt{2}}z^{\frac{1}{7}}} \\ &+ B_1\left(-\frac{7}{5}z^{-\frac{5}{7}}\cos\left(\frac{7}{\sqrt{2}}z^{\frac{1}{7}}\right)-\frac{686}{15}z^{-\frac{3}{7}}\sin\left(\frac{7}{\sqrt{2}}z^{\frac{1}{7}}\right)\right)e^{-\frac{7}{\sqrt{2}}z^{\frac{1}{7}}} \\ &+ B_2\left(-\frac{7}{5}z^{-\frac{5}{7}}\sin\left(\frac{7}{\sqrt{2}}z^{\frac{1}{7}}\right)+\frac{686}{15}z^{-\frac{3}{7}}\cos\left(\frac{7}{\sqrt{2}}z^{\frac{1}{7}}\right)\right)e^{-\frac{7}{\sqrt{2}}z^{\frac{1}{7}}}. \end{aligned} \tag{3.24}$$

To determine the constants A_1, A_2, B_1 and B_2, imposing the boundary conditions (3.2) and (3.3), we get

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \\ B_1 \\ B_2 \end{bmatrix} = \begin{bmatrix} a \\ b \\ u_g \\ v_g \end{bmatrix}, \tag{3.25}$$

where

$$\begin{aligned} & a_{11} = (f_1(z_0)g_1(z_0) + f_1(z_0)g_2(z_0))c_1, a_{12} = (f_1(z_0)g_2(z_0) - f_2(z_0)g_1(z_0))c_1, \\ & a_{13} = \frac{-f_1(z_0)g_1(z_0) + f_1(z_0)g_2(z_0)}{c_1}, a_{14} = \frac{f_1(z_0)g_2(z_0) + f_2(z_0)g_1(z_0)}{c_1}, \\ & a_{21} = (-f_1(z_0)g_2(z_0) + f_2(z_0)g_1(z_0))c_1, a_{22} = (f_1(z_0)g_1(z_0) + f_2(z_0)g_2(z_0))c_1, \\ & a_{23} = \frac{-f_1(z_0)g_2(z_0) - f_2(z_0)g_1(z_0)}{c_1}, a_{24} = \frac{-f_1(z_0)g_1(z_0) + f_2(z_0)g_2(z_0)}{c_1}, \\ & a_{31} = (f_1(z_3)g_1(z_3) + f_1(z_3)g_2(z_3))c_2, a_{32} = (f_1(z_3)g_2(z_3) - f_2(z_3)g_1(z_3))c_2, \\ & a_{33} = \frac{-f_1(z_3)g_1(z_3) + f_1(z_3)g_2(z_3)}{c_2}, a_{34} = \frac{f_1(z_3)g_2(z_3) + f_2(z_3)g_1(z_3)}{c_2}, \\ & a_{41} = (-f_1(z_3)g_2(z_3) + f_2(z_3)g_1(z_3))c_2, a_{42} = (f_1(z_3)g_1(z_3) + f_2(z_3)g_2(z_3))c_2, \\ & a_{43} = \frac{-f_1(z_3)g_2(z_3) - f_2(z_3)g_1(z_3)}{c_2}, a_{44} = \frac{-f_1(z_3)g_1(z_3) + f_2(z_3)g_2(z_3)}{c_2}, \end{aligned}$$

$f_1(z) = \frac{7}{5}z^{-\frac{5}{7}}$, $f_2(z) = \frac{686}{15}z^{-\frac{3}{7}}$, $g_1(z) = \sin\left(\frac{7}{\sqrt{2}}z^{\frac{1}{7}}\right)$, $g_2(z) = \cos\left(\frac{7}{\sqrt{2}}z^{\frac{1}{7}}\right)$, $c_1 = e^{\frac{7}{\sqrt{2}}z_0^{\frac{1}{7}}}$ and $c_2 = e^{\frac{7}{\sqrt{2}}z_3^{\frac{1}{7}}}$.

The first matrix of formula (3.25) on the left side is denoted by d. Clearly,

$$\begin{aligned}&\det(d)\\ =&\frac{4802(9+9604z_0^{\frac{4}{7}})(9+9604z_3^{\frac{4}{7}})\left[\cosh\left(7\sqrt{2}\left(z_0^{\frac{1}{7}}-z_3^{\frac{1}{7}}\right)\right)-\cos\left(7\sqrt{2}\left(z_0^{\frac{1}{7}}-z_3^{\frac{1}{7}}\right)\right)\right]}{50625z_0^{\frac{10}{7}}z_3^{\frac{10}{7}}}\\ >&0.\end{aligned}$$

By the Cramer's Rule, we determine that the constants A_1, A_2, B_1 and B_2 are expressed as follows

$$A_1=\frac{\det(d_1)}{\det(d)},\ A_2=\frac{\det(d_2)}{\det(d)},\ B_1=\frac{\det(d_3)}{\det(d)},\ B_2=\frac{\det(d_4)}{\det(d)},$$

where

$$d_1=\begin{bmatrix}a & a_{12} & a_{13} & a_{14}\\ b & a_{22} & a_{23} & a_{24}\\ u_g & a_{32} & a_{33} & a_{34}\\ v_g & a_{42} & a_{43} & a_{44}\end{bmatrix},\quad d_2=\begin{bmatrix}a_{11} & a & a_{13} & a_{14}\\ a_{21} & b & a_{23} & a_{24}\\ a_{31} & u_g & a_{33} & a_{34}\\ a_{41} & v_g & a_{43} & a_{44}\end{bmatrix},$$
$$d_3=\begin{bmatrix}a_{11} & a_{12} & a & a_{14}\\ a_{21} & a_{22} & b & a_{24}\\ a_{31} & a_{32} & u_g & a_{34}\\ a_{41} & a_{42} & v_g & a_{44}\end{bmatrix},\quad d_4=\begin{bmatrix}a_{11} & a_{12} & a_{13} & a\\ a_{21} & a_{22} & a_{23} & b\\ a_{31} & a_{32} & a_{33} & u_g\\ a_{41} & a_{42} & a_{43} & v_g\end{bmatrix}.$$

We introduce the angle between the wind vector at any height and the geostrophic vector:

$$\tau(z)=\arctan\frac{v_0}{u_0},$$

which, by taking into account (3.23) and (3.24), is equal to:

$$\tau(z)=\arctan\frac{E_1+F_1}{E_2+F_2},\ \text{for } z\in[z_0,z_3],$$

where

$$\begin{aligned}E_1=&A_1\left(\frac{686}{15}z^{-\frac{3}{7}}\tan\left(\frac{7}{\sqrt{2}}z^{\frac{1}{7}}\right)-\frac{7}{5}z^{-\frac{5}{7}}\right)e^{\frac{7}{\sqrt{2}}z^{\frac{1}{7}}}\\ &+A_2\left(\frac{7}{5}z^{-\frac{5}{7}}\tan\left(\frac{7}{\sqrt{2}}z^{\frac{1}{7}}\right)+\frac{686}{15}z^{-\frac{3}{7}}\right)e^{\frac{7}{\sqrt{2}}z^{\frac{1}{7}}},\\ F_1=&B_2\left(\frac{686}{15}z^{-\frac{3}{7}}-\frac{7}{5}z^{-\frac{5}{7}}\tan\left(\frac{7}{\sqrt{2}}z^{\frac{1}{7}}\right)\right)e^{-\frac{7}{\sqrt{2}}z^{\frac{1}{7}}}\\ &-B_1\left(\frac{7}{5}z^{-\frac{5}{7}}+\frac{686}{15}z^{-\frac{3}{7}}\tan\left(\frac{7}{\sqrt{2}}z^{\frac{1}{7}}\right)\right)e^{-\frac{7}{\sqrt{2}}z^{\frac{1}{7}}},\end{aligned}$$

$$\begin{aligned}E_2 =& A_1\left(\frac{7}{5}z^{-\frac{5}{7}}\tan\left(\frac{7}{\sqrt{2}}z^{\frac{1}{7}}\right)+\frac{686}{15}z^{-\frac{3}{7}}\right)e^{\frac{7}{\sqrt{2}}z^{\frac{1}{7}}}\\&+A_2\left(\frac{7}{5}z^{-\frac{5}{7}}-\frac{686}{15}z^{-\frac{3}{7}}\tan\left(\frac{7}{\sqrt{2}}z^{\frac{1}{7}}\right)\right)e^{\frac{7}{\sqrt{2}}z^{\frac{1}{7}}},\\F_2 =& B_1\left(\frac{686}{15}z^{-\frac{3}{7}}-\frac{7}{5}z^{-\frac{5}{7}}\tan\left(\frac{7}{\sqrt{2}}z^{\frac{1}{7}}\right)\right)e^{-\frac{7}{\sqrt{2}}z^{\frac{1}{7}}}\\&+B_2\left(\frac{7}{5}z^{-\frac{5}{7}}+\frac{686}{15}z^{-\frac{3}{7}}\tan\left(\frac{7}{\sqrt{2}}z^{\frac{1}{7}}\right)\right)e^{-\frac{7}{\sqrt{2}}z^{\frac{1}{7}}}.\end{aligned}$$

Then, we get the angles at the bottom of the layer and the top of the layer are $\tau(z_0)=\arctan\frac{b}{a}$ and $\tau(z_3)=0$, respectively.

3.4. Case IV. Here, we consider the case of an eddy viscosity given by

$$m(z) := 2R_e\rho\sin\theta z^{\frac{8}{3}}.$$

Then, the equation (3.9) becomes

$$\eta''(z)-i\cdot z^{-\frac{8}{3}}\eta=0. \tag{3.26}$$

THEOREM 3.4. *The solution of* (3.26) *can be expressed by the following formula*

$$\eta(z)=A\left(z+3\sqrt{i}z^{\frac{2}{3}}\right)e^{-3\sqrt{i}z^{-\frac{1}{3}}}+B\left(z-3\sqrt{i}z^{\frac{2}{3}}\right)e^{3\sqrt{i}z^{-\frac{1}{3}}}. \tag{3.27}$$

And the solution of (3.1) *with the boundary conditions* (3.2) *and* (3.3) *is*

$$\begin{aligned}\phi(z) =& A\left(1+3\sqrt{i}z^{-\frac{1}{3}}+3iz^{-\frac{2}{3}}\right)e^{-3\sqrt{i}z^{-\frac{1}{3}}}\\&+B\left(1-3\sqrt{i}z^{-\frac{1}{3}}+3iz^{-\frac{2}{3}}\right)e^{3\sqrt{i}z^{-\frac{1}{3}}},\end{aligned} \tag{3.28}$$

where

$$A=\frac{\phi_0 f(z_0)g(z_3)e^{3\sqrt{i}\left(z_3^{-\frac{1}{3}}-z_0^{-\frac{1}{3}}\right)}-\phi_1 f(z_0)g(z_0)}{f^2(z_0)g(z_3)e^{3\sqrt{i}\left(z_3^{-\frac{1}{3}}-2z_0^{-\frac{1}{3}}\right)}-f(z_0)f(z_3)g(z_0)e^{-3\sqrt{i}z_3^{-\frac{1}{3}}}},$$

$$B=\frac{\phi_1 f(z_0)e^{-3\sqrt{i}z_0^{-\frac{1}{3}}}-\phi_0 f(z_3)e^{-3\sqrt{i}z_3^{-\frac{1}{3}}}}{f(z_0)g(z_3)e^{3\sqrt{i}\left(z_3^{\frac{1}{3}}-z_0^{-\frac{1}{3}}\right)}-f(z_3)g(z_0)e^{3\sqrt{i}\left(z_0^{-\frac{1}{3}}-z_3^{-\frac{1}{3}}\right)}},$$

and $f(z)=\left(1+3\sqrt{i}z^{-\frac{1}{3}}+3iz^{-\frac{2}{3}}\right)$, $g(z)=\left(1-3\sqrt{i}z^{-\frac{1}{3}}+3iz^{-\frac{2}{3}}\right)$.

PROOF. From (3.27), we get

$$\eta'(z)=A\left(1+3\sqrt{i}z^{-\frac{1}{3}}+3iz^{-\frac{2}{3}}\right)e^{-3\sqrt{i}z^{-\frac{1}{3}}}+B\left(1-3\sqrt{i}z^{-\frac{1}{3}}+3iz^{-\frac{2}{3}}\right)e^{3\sqrt{i}z^{-\frac{1}{3}}},$$

then

$$\eta''(z)=A\left(1+3\sqrt{i}z^{-\frac{1}{3}}\right)iz^{-\frac{5}{3}}e^{-3\sqrt{i}z^{-\frac{1}{3}}}+B\left(1+3\sqrt{i}z^{-\frac{1}{3}}\right)iz^{-\frac{5}{3}}e^{3\sqrt{i}z^{-\frac{1}{3}}},$$

therefore,

$$\begin{aligned} i \cdot z^{-\frac{8}{3}}\eta(z) &= i \cdot z^{-\frac{8}{3}} \left(A\left(z + 3\sqrt{i}z^{\frac{2}{3}}\right) e^{-3\sqrt{i}z^{-\frac{1}{3}}} + B\left(z - 3\sqrt{i}z^{\frac{2}{3}}\right) e^{3\sqrt{i}z^{-\frac{1}{3}}} \right) \\ &= A\left(1 + 3\sqrt{i}z^{-\frac{1}{3}}\right) iz^{-\frac{5}{3}} e^{-3\sqrt{i}z^{-\frac{1}{3}}} + B\left(1 + 3\sqrt{i}z^{-\frac{1}{3}}\right) iz^{-\frac{5}{3}} e^{3\sqrt{i}z^{-\frac{1}{3}}} \\ &= \eta''(z), \end{aligned}$$

which means (3.27) is the solution of (3.26).

By (3.8), we obtain the solution of (3.1) is

$$\phi(z) = A\left(1 + 3\sqrt{i}z^{-\frac{1}{3}} + 3iz^{-\frac{2}{3}}\right) e^{-3\sqrt{i}z^{-\frac{1}{3}}} + B\left(1 - 3\sqrt{i}z^{-\frac{1}{3}} + 3iz^{-\frac{2}{3}}\right) e^{3\sqrt{i}z^{-\frac{1}{3}}},$$

Imposing the boundary conditions (3.2) and (3.3), we get

$$\begin{aligned} \phi_0 = {} & A\left(1 + 3\sqrt{i}z_0^{-\frac{1}{3}} + 3iz_0^{-\frac{2}{3}}\right) e^{-3\sqrt{i}z_0^{-\frac{1}{3}}} \\ & + B\left(1 - 3\sqrt{i}z_0^{-\frac{1}{3}} + 3iz_0^{-\frac{2}{3}}\right) e^{3\sqrt{i}z_0^{-\frac{1}{3}}}, \end{aligned} \tag{3.29}$$

$$\begin{aligned} \phi_1 = {} & A\left(1 + 3\sqrt{i}z_3^{-\frac{1}{3}} + 3iz_3^{-\frac{2}{3}}\right) e^{-3\sqrt{i}z_3^{-\frac{1}{3}}} \\ & + B\left(1 - 3\sqrt{i}z_3^{-\frac{1}{3}} + 3iz_3^{-\frac{2}{3}}\right) e^{3\sqrt{i}z_3^{-\frac{1}{3}}}. \end{aligned} \tag{3.30}$$

Let $f(z) = 1 + 3\sqrt{i}z^{-\frac{1}{3}} + 3iz^{-\frac{2}{3}}$, $g(z) = 1 - 3\sqrt{i}z^{-\frac{1}{3}} + 3iz^{-\frac{2}{3}}$, combining (3.29) and (3.30), we obtain

$$\begin{aligned} A &= \frac{\phi_0 f(z_0)g(z_3)e^{3\sqrt{i}\left(z_3^{-\frac{1}{3}} - z_0^{-\frac{1}{3}}\right)} - \phi_1 f(z_0)g(z_0)}{f^2(z_0)g(z_3)e^{3\sqrt{i}\left(z_3^{-\frac{1}{3}} - 2z_0^{-\frac{1}{3}}\right)} - f(z_0)f(z_3)g(z_0)e^{-3\sqrt{i}z_3^{-\frac{1}{3}}}}, \\ B &= \frac{\phi_1 f(z_0)e^{-3\sqrt{i}z_0^{-\frac{1}{3}}} - \phi_0 f(z_3)e^{-3\sqrt{i}z_3^{-\frac{1}{3}}}}{f(z_0)g(z_3)e^{3\sqrt{i}\left(z_3^{\frac{1}{3}} - z_0^{-\frac{1}{3}}\right)} - f(z_3)g(z_0)e^{3\sqrt{i}\left(z_0^{-\frac{1}{3}} - z_3^{-\frac{1}{3}}\right)}}. \end{aligned}$$

□

We introduce the angle between the wind vector at any height and the geostrophic vector:

$$\tau(z) = \arg\frac{u_0 + iv_0}{u_g + iv_g} = \arg\frac{\phi(z)}{\phi_1},$$

which, by taking into account (3.28), is equal to:

$$\begin{aligned} &\tau(z) \\ &= \arg\frac{A\left(1 + 3\sqrt{i}z^{-\frac{1}{3}} + 3iz^{-\frac{2}{3}}\right) e^{-3\sqrt{i}z^{-\frac{1}{3}}} + B\left(1 - 3\sqrt{i}z^{-\frac{1}{3}} + 3iz^{-\frac{2}{3}}\right) e^{3\sqrt{i}z^{-\frac{1}{3}}}}{\phi_1} \\ &= \arg\left(\frac{\frac{\phi_0}{\phi_1} f(z)f(z_0)g(z_3)e^{3\sqrt{i}\left(z_3^{-\frac{1}{3}} - z_0^{-\frac{1}{3}} - z^{-\frac{1}{3}}\right)} - f(z)f(z_0)g(z_0)e^{-z^{-\frac{1}{3}}}}{f^2(z_0)g(z_3)e^{3\sqrt{i}\left(z_3^{-\frac{1}{3}} - 2z_0^{-\frac{1}{3}}\right)} - f(z_0)f(z_3)g(z_0)e^{-3\sqrt{i}z_3^{-\frac{1}{3}}}} \right. \end{aligned}$$

$$+\frac{g(z)f(z_0)e^{3\sqrt{i}\left(z^{-\frac{1}{3}}-z_0^{-\frac{1}{3}}\right)}-\frac{\phi_0}{\phi_1}g(z)f(z_3)e^{3\sqrt{i}\left(z^{-\frac{1}{3}}-z_3^{-\frac{1}{3}}\right)}}{f(z_0)g(z_3)e^{3\sqrt{i}\left(z_3^{\frac{1}{3}}-z_0^{-\frac{1}{3}}\right)}-f(z_3)g(z_0)e^{3\sqrt{i}\left(z_0^{-\frac{1}{3}}-z_3^{-\frac{1}{3}}\right)}}\Bigg).$$

Then, we obtain the angles at the bottom of the layer and the top of the layer are $\tau(z_0) = \arctan\frac{bu_g - av_g}{au_g + bv_g}$ and $\tau(z_3) = \arg\frac{\phi(z_3)}{\phi_1} = \arg 1 = 0$, respectively.

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