

# Singular Levy processes and dispersive effects of generalized Schrödinger equations

Yannick Sire, Xueying Yu, Haitian Yue, and Zehua Zhao

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ABSTRACT. We introduce new models for Schrödinger-type equations, which generalize standard NLS and for which different dispersion occurs depending on the directions. Our purpose is to understand dispersive properties depending on the directions of propagation, in the spirit of waveguide manifolds, but where the diffusion is of different types. We mainly consider the standard Euclidean space and the waveguide case but our arguments extend easily to other types of manifolds (like product spaces). Our approach unifies in a natural way several previous results. Those models are also generalizations of some appearing in seminal works in mathematical physics, such as relativistic strings. In particular, we prove the large data scattering on waveguide manifolds  $\mathbb{R}^d \times \mathbb{T}$ ,  $d \geq 3$ . This result can be regarded as the analogue of [63, 65] in our setting and the waveguide analogue investigated in [28]. A key ingredient of the proof is a Morawetz-type estimate for the setting of this model.

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## 1. Introduction

Anomalous diffusion, also called fractional diffusion, appears naturally in the physics and mathematical physics literature in the study of relativistic matter and strings, see e.g. for the works [9, 18, 24, 25, 46, 47] and references therein. Similarly, a tentative description of some quantum mechanics has been undertaken in [44, 43, 45]. The classical relativistic operator is the multiplier  $\sqrt{|\xi|^2 + m^2} - m$  where  $m \geq 0$  is a constant. Fractional Schrödinger operators are a fundamental equation of fractional quantum mechanics, which was derived by Laskin [45] as a result of extending the Feynman path integral, from the Brownian-like to Lévy-like quantum mechanical paths. The corresponding physical realizations were made in condensed matter physics [54] and in nonlinear optics [48].

In the present contribution, we investigate dispersive properties of Schrödinger operators generalizing the previous operator. One possible generalization is to replace the square root with any power between 0 and 1 (see e.g. [9] for a similar generalization). Those operators have been investigated recently in different directions. The long-time behaviors (such as global well-posedness, scattering, blow-up, and the existence of invariant measures) of the solutions are interesting and widely studied. In [6], the blow-up with radial data in certain regimes was constructed by deriving a localized virial estimate for the fractional Schrödinger equation. In [28], the first author of this paper together with Guo, Wang and Zhao performs the Kenig-Merle's concentration-compactness-rigidity method [39] and obtains global well-posedness and scattering in the energy space in the defocusing case, and in the focusing case with energy below the ground state. We also refer the reader to [27, 15, 42, 37, 34, 57, 55, 52, 56, 19, 59, 58] and references therein for many other results on the long time behaviors for relativistic NLS. In all the previous works the operator under consideration is  $(-\Delta)^\sigma$  for  $\sigma \in (0, 1)$ , which is the multiplier  $|\xi|^\sigma$ . The ambient space is either the whole Euclidean space  $\mathbb{R}^d$  or the (rational) torus  $\mathbb{T}^d$ . The aim is two-fold: first, we consider the waveguide manifold, i.e.  $(-\Delta)^\sigma$  defined on  $\mathbb{R}^d \times \mathbb{T}$ ; more interestingly, we introduce a new model where the operator acts differently according to the spatial direction.

In the following, we consider only the case of the waveguide  $\mathbb{R}^d \times \mathbb{T}^n$ . To motivate the reason behind the equations we consider, we first state the Levy-Kintchine formula (see e.g. [5]): every (isotropic) Levy process with pure jumps in  $\mathbb{R}^d$  is given by the multiplier

$$\mathfrak{m}(\xi) = \int_{\mathbb{R}^d} (1 - e^{i\xi \cdot x} + i\xi \cdot x \chi_{|x|<1}) \Pi(dx)$$

where  $\Pi(dx)$  is the so-called Levy measure satisfying the integrability condition

$$\int_{\mathbb{R}^d} \max(1, |x|^2) \Pi(dx) < \infty.$$

The multiplier  $\mathfrak{m}(\xi) = |\xi|^{2\sigma}$  is obtained by choosing  $\Pi(dx) = \frac{1}{|x|^{d+2\sigma}} dx$ . On the other hand, one could consider singular Levy measures just as Dirac masses supported along some (or all) axis coordinates. In this paper, we consider the following model

$$(1.1) \quad \begin{cases} i\partial_t u + ((-\Delta_x)^\sigma + (-\partial_y^2)^\sigma) u = F, & (x, y) \in \mathbb{R}^d \times \mathbb{T}^n, \\ u(0) = u_0 \in H^\sigma(\mathbb{R}^d \times \mathbb{T}^n). \end{cases}$$

In view of the previous discussion, the operator  $(-\Delta_x)^\sigma + (-\partial_y^2)^\sigma$  corresponds to the multiplier  $\mathfrak{m}(\xi, \eta) = |\xi|^{2\sigma} + |\eta|^{2\sigma}$  where  $\xi$  is the Fourier variable corresponding to  $x \in \mathbb{R}^d$  and  $\eta$  the one to  $y \in \mathbb{T}^n$ . In particular, this is a Levy process with a singular Levy measure. To the best of our knowledge, dispersive and space-time estimates for such propagators have not been considered yet in the literature.

REMARK 1.1. It is important to notice that all the (dual) variables in  $\mathbb{R}^d \times \mathbb{T}^n$  have to be present in the multiplier  $\mathfrak{m}$ . Otherwise, the associated operator becomes subelliptic, and major dispersive issues arise as in the case of the Schrödinger propagator in the Heisenberg group [2] or the Szegö model of Gérard and Grellier (see e.g. [26] and subsequent articles).

The case of *regular* Levy measures has been considered for instance in [16, 28, 34]. We would like to comment further on the model under consideration: It is a well-known fact that on a compact manifold, there is no dispersion, and loss of derivatives occurs in the Strichartz estimates (see e.g. [8]). Similarly, on a waveguide manifold (or a product manifold), e.g.  $\mathbb{R}^d \times \mathbb{T}^n \subset \mathbb{R}^{d+n}$ , dispersion occurs but the fact that only part of the directions ( $d$  here) contributes to diffusion introduces several complications. Our model is another instance of such a phenomenon: it is by now well-known that anomalous diffusion of order  $2\sigma$  exhibits a loss of  $1 - \sigma$ . A variation on our model would be for example to consider the Euclidean factors  $\mathbb{R}^d \times \mathbb{R}$  but considering a different diffusion on each factor  $\mathbb{R}^d$  and  $\mathbb{R}$ . Product spaces  $\mathbb{R}^d \times \mathbb{T}^n$  are known as ‘waveguide manifolds’ and are of particular interest in nonlinear optics. We refer to [14, 13, 30, 32, 33, 35, 36, 40, 64, 66, 68, 69, 70] with regard to waveguide settings.

An interesting feature of the equation (gNLS) is its product structure. In particular, this allows us to get interaction Morawetz estimates using the tensorization argument in [17] and also use in a much better-streamlined way the vector-valued argument of Tzvetkov-Visciglia [62]. This latter operator satisfies the non-degeneracy assumptions in Schippa too [53]. One could also consider different geometries in the ambient space like the pure tori case  $\mathbb{T}^d \times \mathbb{T}$  or the pure Euclidean one  $\mathbb{R}^d \times \mathbb{R}$ . We chose to consider the waveguide case as a middle point between these two geometries. It is relatively straightforward to generalize our results to those two geometries. In the case of the classical fractional laplacian on pure tori, global well-posedness has been considered by Schippa in [53] using a decoupling approach. As far as nonlinear applications are concerned, we will be considering (unless otherwise stated we will always consider  $n = 1$ )

$$(gNLS) \quad \begin{cases} i\partial_t u + ((-\Delta_x)^\sigma + (-\partial_y^2)^\sigma) u = \mu |u|^p u, & (x, y) \in \mathbb{R}^d \times \mathbb{T}, \\ u(0) = u_0 \in H^\sigma(\mathbb{R}^d \times \mathbb{T}), \end{cases}$$

where  $\mu = \pm 1$  and  $\frac{4\sigma}{d} < p < \frac{4\sigma}{d+1-2\sigma}$ .

REMARK 1.2. We consider the exponent  $p$  to be in the subcritical range, for some technical reasons. The left endpoint indicates mass-critical if we ignore the torus direction; the right endpoint indicates energy-critical if we regard the torus direction as Euclidean direction. So essentially, the problem is energy-subcritical and mass-supercritical.

The well-posedness theory and the long-time dynamics for generalized Schrödinger operators on waveguide manifolds are understudied. It is our goal here to fill this

gap in the literature by exhibiting a wealth of different techniques previously used to deal with some models in the case here of waveguides. For the extremal cases of tori or Euclidean spaces, we refer the reader to [16, 20, 28, 34] and references therein.

The operators under consideration are well-designed to generalize Tzvetkov-Visciglia's results in [62, 63] thanks to the product structure of the operator. Our main result is

**THEOREM 1.3.** *Let  $\sigma > \frac{1}{2}$  in (gNLS) and assume the spatial variable is radial in  $\mathbb{R}^d$ . Then we have:*

- (i) for any initial datum  $u_0 \in H_{x,y}^\sigma$ , the IVP (gNLS) has a unique local solution  $u(t, x, y) \in \mathcal{C}((-T, T); H_{x,y}^\sigma)$  where  $T = T(\|u_0\|_{H_{x,y}^\sigma}) > 0$ ;
- (ii) Moreover, when  $\mu = -1$  (defocusing case), the solution  $u(t, x, y)$  can be extended globally in time. Moreover, the solution scatters in the following sense: there exist  $f^\pm \in H_{x,y}^\sigma(\mathbb{R}^d \times \mathbb{T})$  such that

$$\lim_{t \rightarrow \pm\infty} \|u(t, x, y) - e^{it((-\Delta_x)^\sigma + (-\partial_y^2)^\sigma)} f^\pm\|_{H_{x,y}^\sigma(\mathbb{R}^d \times \mathbb{T})} = 0.$$

We comment on the previous results:

- (1) The radially assumption is to avoid loss of derivatives in the Strichartz estimates. For our range of powers, this is a technical assumption that can be removed.
- (2) The assumption  $\sigma > \frac{1}{2}$  is also due to some technical reasons if one wants to modify the method in Tzvetkov-Visciglia [63]. Roughly speaking, it is because the Sobolev embedding exponent in 1D is  $\frac{1}{2}+$ .
- (3) We provide two different proofs of the well-posedness result: one using decoupling and one using vector-valued analysis.

**REMARK 1.4.** Since NLS with a harmonic trapping potential has similar properties/behaviors as NLS on tori, heuristically one may compare NLS with a partial harmonic potential with NLS on waveguides. (See [1, 11, 31] and the references therein.) Thus one may conjecture that it is possible to obtain the analogue of Theorem 1.3 with a partial harmonic potential.

The paper is organized as follows: In section 2, we discuss preliminaries including notations and some useful estimates; in Section 3, we discuss well-posedness theory using vector-valued argument; in Section 4, we discuss well-posedness theory using decoupling argument; in Section 5, we establish a Morawetz-type estimate for our equation, which is the crucial step for obtaining the decay property of solutions of (gNLS); in Section 6, we give the proof for the large data scattering; in Section 7, we give a few more remarks on the research line of ‘dispersive equations on waveguide manifolds’.

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## 2. Preliminaries

In this section, we briefly discuss notations and some basic estimates.

**2.1. Notations.** We write  $A \lesssim B$  to say that there is a constant  $C$  such that  $A \leq CB$ . We use  $A \simeq B$  when  $A \lesssim B \lesssim A$ . Particularly, we write  $A \lesssim_u B$  to express that  $A \leq C(u)B$  for some constant  $C(u)$  depending on  $u$ .

Then we give some more preliminaries on the setting of the waveguide manifold. The tori case can be defined similarly. In fact, it is included since it is a special case. Throughout this paper, we regularly refer to the spacetime norms

$$\|u\|_{L_t^p L_z^q(I_t \times \mathbb{R}^m \times \mathbb{T}^n)} = \left( \int_{I_t} \left( \int_{\mathbb{R}^m \times \mathbb{T}^n} |u(t, z)|^q dz \right)^{\frac{p}{q}} dt \right)^{\frac{1}{p}}.$$

Similarly, we can define the composition of three  $L^p$ -type norms like  $L_t^p L_x^q L_y^2$ . Moreover, we turn to the Fourier transformation and Littlewood-Paley theory. We define the Fourier transform on  $\mathbb{R}^m \times \mathbb{T}^n$  as follows:

$$(\mathcal{F}f)(\xi) = \int_{\mathbb{R}^m \times \mathbb{T}^n} f(z) e^{-iz \cdot \xi} dz,$$

where  $\xi = (\xi_1, \xi_2, \dots, \xi_d) \in \mathbb{R}^m \times \mathbb{Z}^n$  and  $d = m + n$ . We also note the Fourier inversion formula

$$f(z) = c \sum_{(\xi_{m+1}, \dots, \xi_d) \in \mathbb{Z}^n} \int_{(\xi_1, \dots, \xi_m) \in \mathbb{R}^m} (\mathcal{F}f)(\xi) e^{iz \cdot \xi} d\xi_1 \dots d\xi_m.$$

For convenience, we may consider the discrete sum to be an integral with the discrete measure so we can combine the above integrals together and treat them to be one integral. Moreover, we define the Schrödinger propagator  $e^{it\Delta}$  by

$$(\mathcal{F}e^{it\Delta} f)(\xi) = e^{-it|\xi|^2} (\mathcal{F}f)(\xi).$$

Similarly, for more general operator  $e^{it\phi(\nabla/i)}$ ,

$$\left( \mathcal{F}e^{it\phi(\nabla/i)} f \right)(\xi) = e^{-it\phi(\xi)} (\mathcal{F}f)(\xi).$$

We are now ready to define the Littlewood-Paley projections. First, we fix  $\eta_1 : \mathbb{R} \rightarrow [0, 1]$ , a smooth even function satisfying

$$\eta_1(\xi) = \begin{cases} 1, & |\xi| \leq 1, \\ 0, & |\xi| \geq 2, \end{cases}$$

and  $N = 2^j$  a dyadic integer. Let  $\eta^d = \mathbb{R}^d \rightarrow [0, 1]$ ,  $\eta^d(\xi) = \eta_1(\xi_1)\eta_1(\xi_2)\eta_1(\xi_3)\dots\eta_1(\xi_d)$ . We define the Littlewood-Paley projectors  $P_{\leq N}$  and  $P_N$  by

$$\mathcal{F}(P_{\leq N} f)(\xi) := \eta^d \left( \frac{\xi}{N} \right) \mathcal{F}(f)(\xi), \quad \xi \in \mathbb{R}^m \times \mathbb{Z}^n,$$

and

$$P_N f = P_{\leq N} f - P_{\leq \frac{N}{2}} f.$$

For any  $N \in (0, \infty)$ , we define

$$P_{\leq N} := \sum_{M \leq N} P_M, \quad P_{> N} := \sum_{M > N} P_M.$$

**2.2. Some useful estimates.** In this subsection, we discuss some useful estimates to prove Theorem 1.3. We note that there is a symmetric assumption for the initial data in Theorem 1.3, but for the sake of completeness, we also discuss the nonradial Strichartz estimate. We refer to Tao [60] for the standard Strichartz estimate for NLS and refer to Guo-Wang [29] and the references therein for general radial Strichartz estimates.

We say that  $(p, q)$  is admissible if

$$\frac{2}{p} + \frac{d}{q} = \frac{d}{2}, \quad 2 \leq p, q \leq \infty \quad (p, q, d) \neq (2, \infty, 2).$$

For the nonradial case, we define the following Strichartz norm

$$\|u\|_{S_{p,q}^s} := \|\ |\nabla|^{-d(1-\sigma)(\frac{1}{2}-\frac{1}{q})} u\|_{L_t^p L_x^{s,q}}$$

where  $I = [0, T)$ . We use  $u_0$  for the initial data and  $F$  for the nonlinearity. Consider

$$(2.1) \quad (i\partial_t + (-\Delta_x)^\sigma)u = F, \quad u(0) = u_0 \in H^\sigma(\mathbb{R}^d).$$

Then we have the Strichartz estimate (see [16, 34]),

LEMMA 2.1 (Strichartz estimate). *Fix  $\sigma \in (0, 1)$  and  $s > 0$ . For an admissible pair  $(p, q)$  and  $(a, b)$ , we have*

$$\|e^{it(-\Delta)^\sigma} u_0\|_{S_{p,q}^s} \lesssim \|u_0\|_{H_x^s}$$

and

$$\left\| \int_0^t e^{i(t-s)(-\Delta)^\sigma} F(s) ds \right\|_{S_{p,q}^s} \lesssim \|\ |\nabla|^s F\|_{L_t^{a'} L_x^{b'}}$$

COROLLARY 2.2. *Consider  $(\tilde{p}, \tilde{q})$  satisfying*

$$\frac{2}{\tilde{p}} + \frac{d}{\tilde{q}} = \frac{d}{2} - s + d(1 - \sigma)\left(\frac{1}{2} - \frac{1}{q}\right),$$

where  $s \geq d(1 - \sigma)\left(\frac{1}{2} - \frac{1}{q}\right)$ . Consider  $(a, b)$  to be admissible pair. Then

$$\|e^{it(-\Delta)^\sigma} u_0\|_{L_t^{\tilde{p}} L_x^{\tilde{q}}} \lesssim \|u_0\|_{H_x^s}$$

and

$$\left\| \int_0^t e^{i(t-s)(-\Delta)^\sigma} F(s) ds \right\|_{L_t^{\tilde{p}} L_x^{\tilde{q}}} \lesssim \|\ |\nabla_x|^s F\|_{L_t^{a'} L_x^{b'}}$$

For proving Theorem 1.3, we introduce the radial Strichartz estimate for  $\sigma$ -admissible pairs (see Lemma 2.1 of [28]). Consider  $\sigma > \frac{1}{2}$  and  $u$  solving (2.1), then the following estimate holds

$$\|u\|_{L_t^p L_x^q} \lesssim \|u_0\|_{\dot{H}^\gamma} + \|F\|_{L_t^{p'} L_x^{q'}},$$

where the pairs satisfy

$$(2.2) \quad \frac{2\sigma}{p} + \frac{d}{q} = \frac{d}{2} - \gamma, \quad \frac{2\sigma}{\tilde{p}} + \frac{d}{\tilde{q}} = \frac{d}{2} + \gamma.$$

In the exponent relation (2.2), we call  $(p, q)$  is  $\sigma$ -admissible with  $\gamma$  regularity; we call  $(p, q)$   $\sigma$ -admissible if  $\gamma = 0$ .

The previous estimates hold on Euclidean spaces of dimension  $d$ . On the waveguide  $\mathbb{R}^d \times \mathbb{T}$ , one can adopt the strategy in [62] (Proposition 2.1 for the NLS case) based on mixed norms (or in other words vector-valued norms). We discuss this case now. We prove

LEMMA 2.3. Consider (1.1) and pairs  $(p, q), (\tilde{p}, \tilde{q})$  satisfying the exponent relations (2.2), then

$$\|u\|_{L_t^p L_x^q L_y^2} \lesssim \|u_0\|_{\dot{H}^\gamma} + \|F\|_{L_t^{\tilde{p}'} L_x^{\tilde{q}'} H_y^\gamma},$$

and

$$\|u\|_{L_t^p L_x^q H_y^\gamma} \lesssim \|u_0\|_{\dot{H}^\gamma} + \|F\|_{L_t^{\tilde{p}'} L_x^{\tilde{q}'} H_y^\gamma}.$$

PROOF OF LEMMA 2.3. The proof of Lemma 2.3 is very similar to Proposition 2.1 of [62] (NLS case) so we just explain the difference from the NLS case here. For the NLS case, the main idea is decomposing the functions with respect to the orthonormal basis of  $L^2(\mathbb{T})$  given by the eigenfunctions  $\{\phi_j\}_j$  of  $-\Delta_y$ . For the FNLS case, we consider  $\phi_j = \phi_j(y)$  then

$$(-\Delta_x - \Delta_y)\phi_j = \lambda_j^2 \phi_j, \quad \lambda_j > 0.$$

So we can write  $u(x, y)$  by

$$u(x, y) = \sum_j u_j(x) \phi_j(y),$$

and  $u_j(x)$  in Fourier satisfies

$$i\partial_t \hat{u}_j + (|\xi|^{2\sigma} + \lambda_j^{2\sigma}) \hat{u}_j = \hat{F}_j.$$

Hence we are reduced to the case of Guo-Wang [29] for the symbol  $|\xi|^{2\sigma} + \lambda_j^{2\sigma}$  and the result follows.  $\square$

For the record, we state below the nonradial case, taking the  $y$ -direction into consideration:

LEMMA 2.4. Consider (1.1). For  $(\tilde{p}, \tilde{q})$  and  $(a, b)$  as in Lemma 2.1, we have

$$\|e^{it(-\Delta)^\sigma} u_0\|_{S_{\tilde{p}, \tilde{q}}^s H_y^\gamma} \lesssim \|u_0\|_{H_x^a H_y^\gamma}$$

and

$$\left\| \int_0^t e^{i(t-s)(-\Delta)^\sigma} F(s) ds \right\|_{S_{\tilde{p}, \tilde{q}}^s H_y^\gamma} \lesssim \| |\nabla_x|^s F \|_{L_t^{a'} L_x^{b'} H_y^\gamma}.$$

At last, we recall the following useful lemma (see [63])

LEMMA 2.5. For every  $0 < s < 1$ ,  $p > 0$  there exists  $C = C(p, s) > 0$  such that

$$\|u|u|^p\|_{\dot{H}_y^s} \leq C \|u\|_{\dot{H}_y^s} \|u\|_{L_y^\infty}^p.$$

### 3. Well-posedness theory for (gNLS) with vector-valued arguments

In this section, we establish well-posedness theory for (gNLS) in Theorem 1.3 by the standard contraction mapping method together with the conservation law (from local to global). The main work is to construct suitable function spaces and to show the natural Duhamel mapping is a contraction mapping. It is tightly based on the Strichartz estimate for (gNLS) on  $\mathbb{R}^d \times \mathbb{T}$  and careful choices of the exponents. We refer to Section 4 of [63] for the NLS analogue and Section 3 of [65] (the fourth-order NLS case).

We also note that the analysis in this section covers the standard NLS case (when  $\sigma = 1$ ), which is consistent with [63]. It is essential to assume  $\sigma > \frac{1}{2}$  as we can see from the proof shortly.

First, we introduce the integral operator by Duhamel formula,

$$\Phi_{u_0}(u) = e^{it((-\Delta_x)^\sigma + (-\partial_y^2)^\sigma)} u_0 - i \int_0^t e^{i(t-s)((-\Delta_x)^\sigma + (-\partial_y^2)^\sigma)} (|u|^p u) ds.$$

Then we can construct function space and show  $\Phi_{u_0}$  is a contraction mapping. For the defocusing case, conservation law allows us to extend local well-posedness to global well-posedness.

We define three norms,

$$\|u\|_{X_T} = \|u\|_{L_t^q L_x^r H_y^{\frac{1}{2}+\delta}([-T, T] \times \mathbb{R}^d \times \mathbb{T})},$$

$$\|u\|_{Y_T^1} = \sum_{k=0,1} \|\ |\nabla_x|^{k\sigma} u \|_{L_t^l L_x^m L_y^2([-T, T] \times \mathbb{R}^d \times \mathbb{T})},$$

and

$$\|u\|_{Y_T^2} = \sum_{k=0,1} \|\ |\partial_y|^{k\sigma} u \|_{L_t^l L_x^m L_y^2([-T, T] \times \mathbb{R}^d \times \mathbb{T})},$$

where  $(l, m)$  is  $\sigma$ -admissible and  $(q, r)$  is  $\sigma$ -admissible with  $s$  regularity  $(s + \frac{1}{2} + \delta \leq \sigma)$ . We will give the precise restrictions for the indices shortly, i.e. (3.1), (3.2) and . Here we **note** that  $\sigma > \frac{1}{2}$  such that it is ok to find a  $s > 0$ . Combining the above three norms together, we define

$$\|u\|_{Z_T} = \|u\|_{X_T} + \|u\|_{Y_T^1} + \|u\|_{Y_T^2}.$$

Now we prove a contraction mapping for  $\Phi_{u_0}$ .

**Step 1.** ( $\Phi_{u_0}$  is from  $Z_T$  to  $Z_T$ ) Consider  $X_T$  norm first. By Strichartz estimate, Lemma 2.5 and the Hölder,

$$\begin{aligned} \| |u|^p u \|_{L_t^{\tilde{q}'} L_x^{\tilde{r}'} H_y^{\frac{1}{2}+\delta}} &\lesssim \| |u|^{p+1} \|_{H_y^{\frac{1}{2}+\delta}} \|_{L_t^{\tilde{q}'} L_x^{\tilde{r}'}} \\ &\lesssim T^{\alpha(p)} \| |u|^{p+1} \|_{L_t^q L_x^r H_y^{\frac{1}{2}+\delta}}. \end{aligned}$$

with  $\alpha(p) > 0$ . Here we choose the indices such that,

$$(3.1) \quad \frac{1}{\tilde{r}'} = \frac{p+1}{r}, \quad \frac{1}{\tilde{q}'} > \frac{p+1}{q}.$$

It is manageable since the problem is subcritical, which is similar to the NLS case.

Then consider  $Y_T^1$  and  $Y_T^2$  norms. By Strichartz and the Hölder, for  $k = 0, 1$ ,

$$\begin{aligned} \| D^k u |u|^p \|_{L_t^{l'} L_x^{m'} L_y^2} &\lesssim \| |D^k u| \|_{L_y^2} \| |u|^p \|_{L_t^{l'} L_x^{m'}} \\ &\lesssim \| |D^k u| \|_{L_y^2} \| |u|^p \|_{H_y^{\frac{1}{2}+\delta}} \|_{L_t^{l'} L_x^{m'}} \\ &\lesssim T^{\alpha(p)} \| |D^k u| \|_{L_t^l L_x^m L_y^2} \| |u|^p \|_{L_t^q L_x^r H_y^{\frac{1}{2}+\delta}}, \end{aligned}$$

with  $\alpha(p) > 0$ , where  $D$  stands for  $|\nabla_x|^\sigma, |\partial_y|^\sigma$ . (We **note** that we have used the fractional rule, i.e. Lemma A4 in Kato [38]. See also Lemma 2.6 of Dinh [20].)

Here we choose the indices such that,

$$(3.2) \quad \frac{1}{m'} = \frac{1}{m} + \frac{p}{r}, \quad \frac{1}{l'} > \frac{1}{l} + \frac{p}{q}.$$

It is also manageable since the problem is subcritical, which is similar to the NLS case.



Thus, taking the above estimates into consideration, we can take the proper  $T = T(\|u_0\|_{H_{x,\alpha}^\sigma})$  and  $R = R(\|u_0\|_{H_{x,\alpha}^\sigma})$  such that  $\Phi_{u_0}(B_{Z_{T'}}) \subset B_{Z_{T'}}$

**Step 2. ( $\Phi_{u_0}$  is a contraction)** In this step, we show the contraction for  $\Phi_{u_0}$ . Let  $T, R > 0$  be as in **Step 1**. Then there exist  $\bar{T} = \bar{T}(\|u_0\|_{H_{x,y}^\sigma}) < T$  such that  $\Phi_{u_0}$  is a contraction on  $B_{Z_{\bar{T}}}(0, R)$ , equipped with the norm  $L_T^q L_x^r L_y^2$ .

Using Strichartz estimate, the Hölder and the Sobolev inequality,

$$\begin{aligned} & \|\Phi_{u_0}(v_1) - \Phi_{u_0}(v_2)\|_{L_t^q L_x^r L_y^2} \lesssim \|v_1|v_1|^p - v_2|v_2|^p\|_{L_t^{\bar{q}'} L_x^{\bar{r}'} L_y^2} \\ & \lesssim \| |v_1 - v_2| L_y^2 (\|v_1\|_{L_y^\infty}^p + \|v_2\|_{L_y^\infty}^p) \|_{L_t^{\bar{q}'} L_x^{\bar{r}'}} \\ & \lesssim T^{\alpha(p)} \|v_1 - v_2\|_{L_t^q L_x^r L_y^2} (\|v_1\|_{L_t^q L_x^r H_y^{\frac{1}{2}+\delta}}^p + \|v_2\|_{L_t^q L_x^r H_y^{\frac{1}{2}+\delta}}^p). \end{aligned}$$

with  $\alpha(p) > 0$ . Thus we conclude by taking  $T$  small sufficiently.

**Step 3. (Uniqueness and Existence in  $Z$ )**

It is the same as the analogue in Section 4 in [63]. We just use the contraction mapping argument so we skip it.

**Step 4.  $u \in C((-T, T); H_{x,y}^\sigma)$**

It is the same as the NLS analogue in Section 4 of [63] so we omit it. We just use Strichartz estimates again as in Step 1 to guarantee that  $u(t, x, y) \in C((-T, T); H_{x,y}^\sigma)$ .

**Step 5. (Unconditional uniqueness)** We prove that for

$$u_1, u_2 \in C((-T, T); H_{x,y}^\sigma)$$

are fixed points of  $\Phi_{u_0}$ , then  $u_1 = u_2$ .

Considering the difference of the integral equations satisfied by  $u_1$  and  $u_2$  and using Strichartz estimate,

$$\begin{aligned} \|u_1 - u_2\|_{L_t^l L_x^m L_y^2} & \lesssim \| |u_1|u_1|^p - |u_2|u_2|^p \|_{L_t^{l'} L_x^{m'} L_y^2} \\ & \lesssim \|u_1 - u_2\|_{L_t^l L_x^m L_y^2} \left( \|u_1\|_{L^{l-\frac{1}{p}} L_x^{\frac{mp}{m-2}} L_y^2}^p + \|u_2\|_{L^{l-\frac{1}{p}} L_x^{\frac{mp}{m-2}} L_y^2}^p \right) \\ & \lesssim \|u_1 - u_2\|_{L_t^l L_x^m L_y^2} T^{\alpha(p)} \left( \|u_1\|_{L^\infty L_x^{\frac{mp}{m-2}} L_y^2}^p + \|u_2\|_{L^\infty L_x^{\frac{mp}{m-2}} L_y^2}^p \right). \end{aligned}$$

with  $\alpha(p) > 0$ . It is now like the NLS case. We can let  $T$  be small enough to ensure uniqueness. (For Sobolev inequality reason) We note that

$$2 < \frac{mp}{m-2} < \frac{2d}{d+1-2\sigma},$$

is required. We note again we still need  $\sigma > \frac{1}{2}$  in this estimate.

The proof for the global well-posedness part in Theorem 1.3 is now complete using vector-valued arguments.

#### 4. Well-posedness theory for (gNLS) with decoupling arguments

This section is devoted to the proof of the well-posedness aspects of Theorem 1.3 using decoupling arguments, in the spirit of the strategy designed by Schippa [53]. To be more specific, we consider  $p = 3$  in (gNLS) and assume for sake of generalization that the index of the regularity of the initial data  $u_0$  is  $s$ . We then have

**THEOREM 4.1.** *There exists  $s_0(d, \sigma)$  such that the initial value problem (gNLS) with  $p = 2$  is locally well-posed for  $s > s_0(d, \sigma)$ .*

Since the dimension of the  $y$  component here is easier to handle than in the previous section, we choose to consider directly the general waveguide  $\mathbb{R}^{d-n} \times \mathbb{T}^n$ . The argument is based on the approximation of the Euclidean component as in [4] together with the decoupling theorem of Bourgain and Demeter [7]. We will even here consider the case of a multiplier  $\mathbf{m}(\xi, \eta) = |\xi|^{2\sigma} + |\eta|^2$ , which means that the diffusion is Euclidean in tori directions. The adaptation to an anomalous diffusion is straightforward.

On a dyadic interval  $I_j = (2^{-j}, 2^{-j+1})$ , the multiplier  $\mathbf{m}$  behaves at *low frequencies* like  $|\xi|^{2\sigma}$  and at *high frequencies* like  $|\eta|^2$ . An important tool in decoupling arguments is the behavior of the hessian of the phase function. The hessian  $D^2\mathbf{m}(\xi, \eta)$  is block-diagonal with a first  $(d-n) \times (d-n)$ -matrix corresponding to the hessian of the function  $\xi \rightarrow |\xi|^{2\sigma}$  and a second block which is an  $n \times n$ -matrix which is a multiple of the identity. Therefore, the degeneracy of the phase  $\mathbf{m}$  is only dictated by the behavior of the eigenvalues at a given  $\xi$  of the hessian of  $|\xi|^{2\sigma}$  which are comparable on a dyadic interval to  $2\sigma(2\sigma-1)N^{2(\sigma-1)}$  for a dyadic integer  $N$ . Notice that the convexity of the phase changes according to  $\sigma$  w.r.t. the value  $\sigma = 1/2$ . Finally notice that for  $N$  dyadic  $\nabla|\xi|^{2\sigma} \sim N^{2\sigma-1}$ . We denote by  $\psi(N) := N^{2(\sigma-1)} + N^2$  the phase function describing the behavior at dyadic scale  $N$  of the eigenvalues of the Hessian of  $\mathbf{m}$ . We also denote by  $k$  the minimum between the number of negative eigenvalues and positive eigenvalues of  $D^2\mathbf{m}$ , so that actually  $k = 0$  whenever  $\sigma > 1/2$ .

We start by recalling the following decoupling-type lemma.

LEMMA 4.2. *Suppose  $g, g_l$  are Schwartz functions on  $\mathbb{R}^{d-n} \times \mathbb{T}^n$  with  $g = P_{\leq N}g$ , such that*

$$g(x, y) = \sum_{l \in \mathbb{Z}^n, |l| \leq N} \int_{[-N, N]^{d-n}} \hat{g}_l(\xi) e^{2\pi i(x \cdot \xi + y \cdot l)} d\xi.$$

Cover  $[-N, N]^{d-n}$  by finitely-overlapping cubes  $Q_k$  of side-length  $\sim 1$ , let  $\{\phi_k\}$  be a partition of unity adapted to the  $Q_k$ , and define  $g_{\theta_{m,k}} = e^{2\pi i y \cdot m} \mathcal{F}_x^{-1}(\hat{g}_m \phi_k)$ . Then for  $p \geq \frac{2(d+2-k)}{d-k}$  and any time interval  $I$  of length  $\sim 1$  we have

$$(4.1) \quad \begin{aligned} & \|e^{it\mathbf{m}(\nabla/i)}g\|_{L^p(I \times \mathbb{R}^{d-n} \times \mathbb{T}^n)} \\ & \lesssim_\epsilon \frac{N^{\epsilon + \frac{d}{2} - \frac{d+2}{p}}}{(\min\{\mathbf{m}(N), 1\})^{\frac{1}{p}}} \left( \sum_{m,k} \|e^{\frac{it\mathbf{m}(N\nabla/i)}{N^2\mathbf{m}(N)}} g_{\theta_{m,k}} w_I\|_{L^p(\mathbb{R} \times \mathbb{R}^{d-n} \times \mathbb{T}^n)}^2 \right)^{\frac{1}{2}}, \end{aligned}$$

where  $w_I$  is a bump function adapted to  $I$ .

PROOF. The proof of Lemma 4.2 is similar to the proof of the discrete restriction theorem in Bourgain-Demeter [7]. We approximate functions on product space  $\mathbb{R}^{d-n} \times \mathbb{T}^n$  by functions on  $\mathbb{R}^d$ , apply the decoupling theorem, and then take limits. We refer to Lemma 3.2 of Barron [4] for the standard Strichartz case, which has similar spirits. There are two differences from Barron [4] that we need to be careful about. First, the range for  $p$  is different (more narrow). Also, we need to make discussions regarding the value of  $\mathbf{m}(N)$ , compared to 1. Let  $B_l \in \mathbb{R}^{d-n}$  be a fixed ball of radius  $N$  if  $\mathbf{m}(N) \ll 1$  (radius  $N\mathbf{m}(N)$  if  $\mathbf{m}(N) \gtrsim 1$ ), and let  $w_l$  be a smooth weight adapted to  $B_l \times [-1, 1]$ . We observe that to prove Lemma 4.2 it will suffice

to show that,

$$(4.2) \quad \|e^{itm(\nabla/i)}g\|_{L^p(I \times B_l \times \mathbb{T}^n)} \lesssim_\epsilon \frac{N^{\epsilon + \frac{d}{2} - \frac{d+2}{p}}}{(\min\{\mathfrak{m}(N), 1\})^{\frac{1}{p}}} \left( \sum_{m,k} \|e^{\frac{itm(N\nabla/i)}{N^2\mathfrak{m}(N)}} g_{\theta_{m,k}} w_I\|_{L^p(w_I)}^2 \right)^{\frac{1}{2}}.$$

Then to prove the full estimate (4.1) on  $\mathbb{R}^{d-n} \times \mathbb{T}^n$ , we choose a finitely-overlapping collection of balls  $B_l$  that cover  $\mathbb{R}^{d-n}$ , and then apply (4.2) in each  $B_l$  and use Minkowski's inequality to sum that:

$$\begin{aligned} \|e^{itm(\nabla/i)}g\|_{L^p(I \times \mathbb{R}^{d-n} \times \mathbb{T}^n)}^p &\lesssim_\epsilon \sum_l \|e^{itm(\nabla/i)}g\|_{L^p(I \times B_l \times \mathbb{T}^n)}^p \\ &\lesssim_\epsilon \frac{N^{\epsilon + \frac{d}{2} - \frac{d+2}{p}}}{(\min\{\mathfrak{m}(N), 1\})^{\frac{1}{p}}} \sum_l \left( \sum_{m,k} \|e^{\frac{itm(N\nabla/i)}{N^2\mathfrak{m}(N)}} g_{\theta_{m,k}}\|_{L^p(w_I)}^2 \right)^{\frac{p}{2}} \\ &\lesssim_\epsilon \frac{N^{\epsilon + \frac{d}{2} - \frac{d+2}{p}}}{(\min\{\mathfrak{m}(N), 1\})^{\frac{1}{p}}} \left( \sum_{m,k} \|e^{\frac{itm(N\nabla/i)}{N^2\mathfrak{m}(N)}} g_{\theta_{m,k}}\|_{L^p(w)}^2 \right)^{\frac{p}{2}}, \end{aligned}$$

which completes the proof of Proposition 4.2. Thus it suffices to prove (4.2). This reduction process is standard. We note that we have to discuss two situations: i.e.  $\mathfrak{m}(N) \ll 1$  and  $\mathfrak{m}(N) \gtrsim 1$  respectively. The two cases are essentially similar and the main difference is the integral range. Let's consider the first case as an example. For the other case, we can modify the arguments accordingly.

We start with rescaling  $u_0$  to have frequency support in  $[-1, 1]^d$ . We let

$$u(x, y, t) = e^{itm(\nabla/i)}g.$$

Note that

$$\begin{aligned} &u(N^{-1}x, N^{-1}y, N^{-2}\mathfrak{m}(N)^{-1}t) \\ &= N^{d-n} \int_{B_1^{d-n}} \sum_{m \in N^{-1}\mathbb{Z}^n \cap B_1^n} \hat{g}(N\xi, Nm) e^{i(x \cdot \xi + y \cdot m + t(\frac{\mathfrak{m}(N\xi)}{N^2\mathfrak{m}(N)} + \frac{\mathfrak{m}(Nm)}{N^2\mathfrak{m}(N)})})} d\xi. \end{aligned}$$

For convenience, we denote the set  $\Lambda_N = N^{-1}\mathbb{Z}^n \cap B_1^n$ . Similarly to Barron [4], we then let  $Ef$  denote the extension operator as follows,

$$Ef = \int_{B_1^{d-n}} \sum_{m \in N^{-1}\mathbb{Z}^n \cap B_1^n} f(\xi, m) e^{i(x \cdot \xi + y \cdot m + t(\frac{\mathfrak{m}(N\xi)}{N^2\mathfrak{m}(N)} + \frac{\mathfrak{m}(Nm)}{N^2\mathfrak{m}(N)})})} d\xi,$$

where  $f(\xi, m) = \hat{g}(N\xi, Nm)$ . After applying a change of variables on the spatial side and using periodicity in the  $y$  variable, we see that

$$\begin{aligned} \|u\|_{L^p(B_N \times \mathbb{T}^n \times [0,1])} &= N^{d-n} N^{-\frac{(d+2)}{p}} \mathfrak{m}(N)^{-\frac{1}{p}} \|Ef\|_{L^p(B_{N^2} \times N\mathbb{T}^n \times [0, N^2\mathfrak{m}(N)])} \\ &= N^{d-n} N^{-\frac{(d+2)}{p}} \mathfrak{m}(N)^{-\frac{1}{p}} N^{-\frac{n}{p}} \|Ef\|_{L^p(B_{N^2} \times N^2\mathbb{T}^n \times [0, N^2\mathfrak{m}(N)])} \\ &\leq N^{d-n} N^{-\frac{(d+2)}{p}} \mathfrak{m}(N)^{-\frac{1}{p}} N^{-\frac{n}{p}} \|Ef\|_{L^p(B_{N^2} \times N^2\mathbb{T}^n \times [0, N^2])}. \end{aligned}$$

Then following Barron [4], we introduce the operator  $\tilde{E}$  defined by

$$\tilde{E}f = \int_{B_1^{d-n}} \int_{B_1^n} f(\xi_1, \xi_2) e^{i(x \cdot \xi_1 + y \cdot \xi_2 + t(\frac{\mathfrak{m}(N\xi_1)}{N^2\mathfrak{m}(N)} + \frac{\mathfrak{m}(N\xi_2)}{N^2\mathfrak{m}(N)})})} d\xi_1 d\xi_2.$$

Given a function  $f$  on  $[-1, 1]^{d-n} \times \Lambda_N$ , let

$$f^\delta(\xi_1, \xi_2) = \sum_{m \in \Lambda_N} c_d \delta^{-d} 1_{\{|\xi_2 - m| \leq \delta\}} f(\xi_1, m), \quad \delta < \frac{1}{N},$$

where  $c_d$  is a dimensional constant chosen for normalization. Then by Lebesgue differentiation and Fatou lemma, we have

$$(4.3) \quad \|Ef\|_{L^p(B_{N^2} \times N^2\mathbb{T}^n \times [0, N^2])} \leq \liminf_{\delta \rightarrow 0} \|\tilde{E}f^\delta\|_{L^p(B_{N^2} \times N^2\mathbb{T}^n \times [0, N^2])}.$$

We will begin by estimating  $\tilde{E}f$  for arbitrary  $f$  on  $B_1^d$  before specializing to  $f^\delta$  and passing to the limit later in the argument. The above process reduces the waveguide case to the Euclidean case. Then we can use Bourgain-Demeter’s decoupling result [7] and the limiting argument (4.3) to obtain,

$$\|Ef\|_{L^{p^*}(B_{N^2} \times N^2\mathbb{T}^n \times [0, N^2])} \leq \liminf_{\delta \rightarrow 0} N^{\epsilon + \alpha_d} \left( \sum_{m, k} \|\tilde{E}f_{m, k}^\delta\|_{L^{p^*}(w_{N^2})}^2 \right).$$

The rest of the proof follows as in Barron [4]. □

We prove now the crucial (localized) Strichartz estimates whose proof has a similar spirit with Proposition 3.4 [4]. The tori analogue of Lemma 4.3 is proved in Schippa [53].

LEMMA 4.3. *Let the interval  $I$  be compact. Then, we have the following estimates, holding up to any  $\epsilon > 0$ ,*

$$\|P_N e^{itm(\nabla/i)} f\|_{L^p(I \times \mathbb{R}^{d-n} \times \mathbb{T}^n)} \lesssim_{\epsilon, |I|} \frac{N^{\frac{d}{2} - \frac{d+2}{p} + \epsilon}}{(\min\{\mathfrak{m}(N), 1\})^{\frac{1}{p}}} \|f\|_{L^2(\mathbb{R}^{d-n} \times \mathbb{T}^n)},$$

where  $p \geq \frac{2(d+2-k)}{d-k}$ . The number  $k$  has been introduced before.

PROOF. Suppose  $f = P_{\leq N} f$ . By interpolating with  $p = \infty$  (via Bernstein’s inequality) it suffices to prove the endpoint case  $p = \frac{2(d+2-k)}{d-k}$ . By Lemma 4.2,

$$\begin{aligned} & \|e^{itm(\nabla/i)} f\|_{L^p(I \times \mathbb{R}^{d-n} \times \mathbb{T}^n)} \\ & \lesssim_\epsilon \frac{N^{\epsilon + \frac{d}{2} - \frac{d+2}{p}}}{(\min\{\mathfrak{m}(N), 1\})^{\frac{1}{p}}} \left( \sum_{m, k} \|e^{\frac{itm(N\nabla/i)}{N^2\mathfrak{m}(N)}} f_{\theta_{m, k}} w_I\|_{L^p(\mathbb{R} \times \mathbb{R}^{d-n} \times \mathbb{T}^n)}^2 \right)^{\frac{1}{2}}. \end{aligned}$$

By Plancherel theorem it suffices to prove the desired estimate when  $f = P_\theta f$  and  $\theta = \theta_{m, k}$ . In this case, we apply Hölder’s inequality in time to get

$$\|e^{\frac{itm(N\nabla/i)}{N^2\mathfrak{m}(N)}} f_\theta w_I\|_{L^p(\mathbb{R} \times \mathbb{R}^{d-n} \times \mathbb{T}^n)} \lesssim \|e^{\frac{itm(N\nabla/i)}{N^2\mathfrak{m}(N)}} f_\theta\|_{L_t^q L_x^p(\mathbb{R} \times \mathbb{R}^{d-n})}$$

where  $q = \frac{4p}{(d-n)(p-2)}$  is the admissible time exponent for the Strichartz estimate on  $\mathbb{R}^{d-n}$ . Applying the Strichartz estimate for fractional Schrödinger operator (see [29, 16, 20] and the reference therein),

$$\|e^{\frac{itm(N\nabla/i)}{N^2\mathfrak{m}(N)}} f_\theta\|_{L_t^q L_x^p(\mathbb{R} \times \mathbb{R}^{d-n})} \lesssim \|f_\theta\|_{L_x^2(\mathbb{R} \times \mathbb{R}^{d-n})}.$$

This completes the proof. □

REMARK 4.4. By interpolation with the trivial bound  $L_t^\infty L_x^2$  or using the Bernstein inequality, a  $L_t^p L_x^q$ -version Strichartz estimate can be obtained as below

$$\|P_N e^{itm(\nabla/i)} u_0\|_{L_t^p L_x^q(I \times \mathbb{R}^{d-n} \times \mathbb{T}^n)} \lesssim_{\epsilon, |I|} \frac{N^{\frac{d}{2} - \frac{2}{p} - \frac{d}{q} + \epsilon}}{(\min\{\mathbf{m}(N), 1\})^{\frac{1}{p}}},$$

where  $p, q \geq \frac{2(d+2-k)}{d-k}$ .

One can then prove the following bilinear estimates which give the desired well-posedness result for the cubic generalized NLS model in the setting of waveguide manifolds, i.e. the statement in Theorem 4.1.

PROPOSITION 4.5. *Let  $I$  be a compact interval. Then, there exists  $s(d, k)$  such that we have the estimate,*

$$\|P_N e^{itm(\nabla/i)} u_0 P_K e^{itm(\nabla/i)} v_0\|_{L_{t,x}^2(I \times \mathbb{R}^{d-n} \times \mathbb{T}^n)} \lesssim_{C_s, |I|} K^{2s} \|P_N u_0\|_{L^2} \|P_K v_0\|_{L^2}$$

to hold for  $s > s(d, k)$ .

PROOF. We note that the proof is based on the Strichartz Lemma 4.3. We refer to Proposition 1.3 of [53] for the tori version.

Let  $P_N = \sum_{K_1} R_{K_1}$ , where  $R_K$  projects to cubes of side-length  $K$ . Then, by means of almost orthogonality

$$\begin{aligned} & \|P_N e^{itm(\nabla/i)} u_0 P_K e^{itm(\nabla/i)} v_0\|_{L_{t,x}^2(I \times \mathbb{R}^{d-n} \times \mathbb{T}^n)}^2 \\ & \lesssim \sum_{K_1} \|P_{K_1} e^{itm(\nabla/i)} u_0 P_K e^{itm(\nabla/i)} v_0\|_{L_{t,x}^2(I \times \mathbb{R}^{d-n} \times \mathbb{T}^n)}^2 \end{aligned}$$

In viewing of Hölder’s inequality we are left with estimating two  $L_{t,x}^4$ -norms. Clearly, by Lemma 4.3,

$$\|P_K e^{it\phi(\nabla/i)} v_0\|_{L_{t,x}^4(I \times \mathbb{R}^{d-n} \times \mathbb{T}^n)} \lesssim K^s \|P_K v_0\|_{L^2}.$$

Then it suffices to treat the other term. The rest of the proof follows line to line from Proposition 1.3 of [53] so we omit.  $\square$

### 5. Morawetz estimates on waveguides

In this section, we establish a Morawetz estimate for solutions to (gNLS) on  $\mathbb{R}^d \times \mathbb{T}$ . This step is crucial to obtain the decay property for solutions of (gNLS).

We first define the following Morawetz action on the waveguide  $\mathbb{R}^d \times \mathbb{T}$ :

$$M_\phi[u(t)] := 2 \operatorname{Im} \int_{\mathbb{R}^d \times \mathbb{T}} \bar{u}(t, x, y) \nabla_x \phi(x) \cdot \nabla_x u(t, x, y) \, dx dy.$$

Note that, employing the similar idea in [51], the weight function  $\nabla_x \phi(x)$  that we chose here depends on only  $x$ , and not on  $y$ .

Then we present the main result in this section. For a ready-to-use Morawetz estimate, see Corollary 5.2.

LEMMA 5.1. *If  $u$  solves (gNLS), then the Morawetz action satisfies the identity*

$$\begin{aligned} \frac{d}{dt} M_\phi[u(t)] &= \int_0^\infty m^s \int_{\mathbb{R}^d \times \mathbb{T}} \left( 4 \overline{\partial_{x_k} u_m} (\partial_{x_k x_l} \phi) \partial_{x_l} u_m - \Delta_x^2 \phi |u_m|^2 \right) \, dx dy dm \\ &- \frac{2p\mu}{p+2} \int_{\mathbb{R}^d \times \mathbb{T}} \Delta_x \phi |u|^{p+1} \, dx dy. \end{aligned}$$

PROOF OF LEMMA 5.1. Following the strategy in [6], we define

$$\Gamma_\phi := i(\nabla_x \cdot \nabla_x \phi + \nabla_x \phi \cdot \nabla_x),$$

that is

$$\Gamma_\phi f := i[\nabla_x \cdot ((\nabla_x \phi)f) + \nabla_x \phi \cdot \nabla_x f].$$

Under this notation, we claim that

$$(5.1) \quad \langle u(t), \Gamma_\phi(t) \rangle = -M_\phi[u(t)].$$

Note that  $\langle f, g \rangle = \text{Re} \int_{\mathbb{R}^d \times \mathbb{T}} \bar{f}g \, dx dy$ .

In fact,

$$\begin{aligned} \langle u(t), \Gamma_\phi(t) \rangle &= \langle u(t), i[\nabla_x \cdot ((\nabla_x \phi)u) + \nabla_x \phi \cdot \nabla_x u] \rangle \\ &= \langle u, i\nabla_x \cdot ((\nabla_x \phi)u) \rangle + \langle u, i\nabla_x \phi \cdot \nabla_x u \rangle. \end{aligned}$$

We then compute the two inner products separately:

$$\begin{aligned} \langle u, i\nabla_x \phi \cdot \nabla_x u \rangle &= \text{Re} \int_{\mathbb{R}^d \times \mathbb{T}} \bar{u}(i\nabla_x \phi \cdot \nabla_x u) \, dx dy \\ &= -\text{Im} \int_{\mathbb{R}^d \times \mathbb{T}} \bar{u} \nabla_x \phi \cdot \nabla_x u \, dx dy = -\frac{1}{2}M_\phi; \end{aligned}$$

and

$$\begin{aligned} &\langle u, i\nabla_x \cdot ((\nabla_x \phi)u) \rangle \\ &= \text{Re} \int_{\mathbb{R}^d \times \mathbb{T}} \bar{u} i \nabla_x \cdot ((\nabla_x \phi)u) \, dx dy = -\text{Im} \int_{\mathbb{R}^d \times \mathbb{T}} \bar{u} \partial_{x_l} (\partial_{x_l} \phi u) \, dx dy \\ &= -\text{Im} \int_{\mathbb{R}^d \times \mathbb{T}} \bar{u} \partial_{x_l x_l} \phi u \, dx dy - \text{Im} \int_{\mathbb{R}^d \times \mathbb{T}} \bar{u} \partial_{x_l} \phi \partial_{x_l} u \, dx dy \\ &= -\text{Im} \int_{\mathbb{R}^d \times \mathbb{T}} \bar{u} \nabla_x \phi \cdot \nabla_x u \, dx dy = -\frac{1}{2}M_\phi. \end{aligned}$$

Therefore, by combining these two terms, we conclude the claim (5.1)

$$\langle u(t), \Gamma_\phi(t) \rangle = -M_\phi[u(t)].$$

Next, we compute the derivative of  $M_\phi[u(t)]$  with respect to time  $t$ . Using (gNLS)

$$\partial_t u = i((-\Delta_x)^\sigma + (-\partial_y^2)^\sigma)u - i\mu|u|^p u$$

and Plancherel theorem, we write

$$\begin{aligned}
& \frac{d}{dt} M_\phi[u(t)] \\
&= \left\langle \frac{d}{dt} u(t), \Gamma_\phi u(t) \right\rangle + \left\langle u(t), \frac{d}{dt} \Gamma_\phi u(t) \right\rangle \\
&= \langle i((-\Delta_x)^\sigma + (-\partial_y^2)^\sigma) u - i\mu |u|^p u, \Gamma_\phi u(t) \rangle + \left\langle u(t), \Gamma_\phi \frac{d}{dt} u(t) \right\rangle \\
&= \langle i((-\Delta_x)^\sigma + (-\partial_y^2)^\sigma) u, \Gamma_\phi u(t) \rangle - \langle i\mu |u|^p u, \Gamma_\phi u(t) \rangle \\
&\quad + \langle u(t), i\Gamma_\phi((-\Delta_x)^\sigma + (-\partial_y^2)^\sigma) u - \mu |u|^p u \rangle \\
&= -\langle u(t), ((-\Delta_x)^\sigma + (-\partial_y^2)^\sigma) i\Gamma_\phi u(t) \rangle + \langle u(t), i\Gamma_\phi((-\Delta_x)^\sigma + (-\partial_y^2)^\sigma) u(t) \rangle \\
&\quad + \langle u(t), \mu |u|^p i\Gamma_\phi u(t) \rangle - \langle u(t), i\Gamma_\phi(\mu |u|^p u) \rangle \\
&= -\langle u(t), [(-\Delta_x)^\sigma + (-\partial_y^2)^\sigma, i\Gamma_\phi] u(t) \rangle + \langle u(t), [\mu |u|^p, i\Gamma_\phi] u(t) \rangle,
\end{aligned}$$

where we used the commutator notation  $[A, B] = AB - BA$ .

Noticing that

$$[(-\partial_y^2)^\sigma, i\Gamma_\phi] = 0,$$

we then have

$$\frac{d}{dt} M_\phi[u(t)] = \underbrace{-\langle u(t), [(-\Delta_x)^\sigma, i\Gamma_\phi] u(t) \rangle}_I + \underbrace{\langle u(t), [\mu |u|^p, i\Gamma_\phi] u(t) \rangle}_{II} =: I + II.$$

In the rest of the proof, we will work on the linear term  $I$  and the nonlinear term  $II$  separately.

First, we consider the linear term  $I$ . In order to deal with the  $[(-\Delta_x)^\sigma, i\Gamma_\phi]$  term inside  $I$ , we will employ the following Balakrishnan's representation formula for  $(-\Delta_x)^\sigma$  introduced in [3] for  $\sigma \in (0, 1)$ ,

$$(5.2) \quad (-\Delta_x)^\sigma = \frac{\sin(\pi\sigma)}{\pi} \int_0^\infty m^{\sigma-1} \frac{-\Delta_x}{-\Delta_x + m} dm.$$

In general, for  $A \geq 0$ ,  $m > 0$ , the following commutator has the form of

$$\left[ \frac{A}{A+m}, B \right] = m \frac{1}{A+m} [A, B] \frac{1}{A+m}.$$

In particular, if taking  $A = -\Delta_x$  and combining with (5.2), we write

$$(5.3) \quad [(-\Delta_x)^\sigma, B] = \frac{\sin(\pi\sigma)}{\pi} \int_0^\infty m^\sigma \frac{1}{-\Delta_x + m} [-\Delta_x, B] \frac{1}{-\Delta_x + m} dm.$$

Then taking  $B = i\Gamma_\phi$  in (5.3), we have

$$(5.4) \quad [(-\Delta_x)^\sigma, i\Gamma_\phi] = \frac{\sin(\pi\sigma)}{\pi} \int_0^\infty m^\sigma \frac{1}{-\Delta_x + m} [-\Delta_x, i\Gamma_\phi] \frac{1}{-\Delta_x + m} dm.$$

Now we claim that

$$(5.5) \quad [-\Delta_x, i\Gamma_\phi] = 4\partial_{x_k}(\partial_{x_k x_l} \phi) \partial_{x_l} + \Delta_x^2 \phi,$$

where to emphasize that  $\Delta_x$  takes derivative only in the  $\mathbb{R}^d$  direction, we put  $x$  in its subscript. Similarly, in the following calculations,  $\partial_{x_l}$  and  $\partial_{x_k}$  are differential operators in  $\mathbb{R}^d$  directions, while  $\partial_y$  is the  $\mathbb{T}$  direction derivative.

In fact, recalling

$$(5.6) \quad i\Gamma_\phi f := -[\nabla_x \cdot ((\nabla_x \phi)f) + \nabla_x \phi \cdot \nabla_x f],$$

we write

$$(5.7) \quad \begin{aligned} & [-\Delta_x, i\Gamma_\phi]f \\ &= -\Delta_x(i\Gamma_\phi)f + i\Gamma_\phi(\Delta_x f) \\ &= \Delta_x \nabla_x \cdot ((\nabla_x \phi)f) + \Delta_x(\nabla_x \phi \cdot \nabla_x f) - \nabla_x((\nabla_x \phi)\Delta_x f) - \nabla_x f \cdot \nabla_x(\Delta_x f) \\ &= \partial_{x_k x_k} \partial_{x_l} (\partial_{x_l} \phi f) + \partial_{x_k x_k} (\partial_{x_l} \phi \partial_{x_l} f) - \partial_{x_l} (\partial_{x_l} \phi \partial_{x_k x_k} f) - \partial_{x_l} \partial_{x_l x_k x_k} f. \end{aligned}$$

Then using the product rule, we continue from (5.7)

$$\begin{aligned} & [-\Delta_x, i\Gamma_\phi]f \\ &= \partial_{x_k x_k} (\partial_{x_l x_l} \phi f) + 2\partial_{x_k x_k} (\partial_{x_l} \phi \partial_{x_l} f) - \partial_{x_l x_l} \phi \partial_{x_k x_k} f \\ &\quad - \partial_{x_l} \phi \partial_{x_k x_k x_l} f - \partial_{x_l} f \partial_{x_k x_k x_l} f \\ &= \partial_{x_k x_k x_l x_l} \phi f + 2\partial_{x_k x_l x_l} \phi \partial_{x_k} f + \partial_{x_l x_l} \phi \partial_{x_k x_k} f + 2\partial_{x_k x_k x_l} \phi \partial_{x_l} f \\ &\quad + 4\partial_{x_k x_l} \phi \partial_{x_k x_l} f + 2\partial_{x_l} \phi \partial_{x_k x_k x_l} f \\ &\quad - \partial_{x_l x_l} \phi \partial_{x_k x_k} f - \partial_{x_l} \partial_{x_k x_k x_l} f - \partial_{x_l} \phi \partial_{x_k x_k x_l} f \\ &= \Delta_x^2 \phi f + 4\partial_{x_k x_l x_l} \phi \partial_{x_k} f + 4\partial_{x_k x_l} \phi \partial_{x_k x_l} f \\ &= 4\partial_{x_k} (\partial_{x_k x_l} \phi) \partial_{x_l} f + \Delta_x^2 \phi f. \end{aligned}$$

This proves the claim (5.5).

At this point, we are in a good position to compute the term  $I$ . First, combining (5.5) and (5.4), we write

$$\begin{aligned} [(-\Delta_x)^\sigma, i\Gamma_\phi] &= \frac{\sin(\pi\sigma)}{\pi} \int_0^\infty m^\sigma \frac{1}{-\Delta_x + m} [(-\Delta_x, i\Gamma_\phi)] \frac{1}{-\Delta_x + m} dm \\ &= \frac{\sin(\pi\sigma)}{\pi} \int_0^\infty m^\sigma \frac{1}{-\Delta_x + m} [4\partial_{x_k} (\partial_{x_k x_l} \phi) \partial_{x_l} + \Delta_x^2 \phi] \frac{1}{-\Delta_x + m} dm. \end{aligned}$$

Therefore

$$\begin{aligned} I &= -\langle u(t), [(-\Delta_x)^\sigma, i\Gamma_\phi]u \rangle \\ &= \int_{\mathbb{R}^d \times \mathbb{T}} \bar{u} \frac{\sin(\pi\sigma)}{\pi} \int_0^\infty m^\sigma \frac{1}{-\Delta_x + m} [-4\partial_{x_k} (\partial_{x_k x_l} \phi) \partial_{x_l} - \Delta_x^2 \phi] \frac{1}{-\Delta_x + m} u \, dm dx dy. \end{aligned}$$

For  $m > 0$ , we define

$$u_m(t) := c_\sigma \frac{1}{-\Delta_x + m} u(t) = c_\sigma \mathcal{F}^{-1} \left( \frac{\widehat{u}(t, \xi)}{|\xi|^2 + m} \right),$$

where

$$c_\sigma := \sqrt{\frac{\sin(\pi\sigma)}{\pi}}.$$



Under such change of variables and with Fubini's theorem, Plancherel theorem, and integration by parts, we obtain

$$\begin{aligned}
 I &= \int_0^\infty \int_{\mathbb{R}^d \times \mathbb{T}} \bar{u} c_\sigma m^\sigma \frac{1}{-\Delta_x + m} [-4\partial_{x_k}(\partial_{x_k x_l} \phi) \partial_{x_l} - \Delta_x^2 \phi] u_m \, dx dy dm \\
 &= \int_0^\infty m^\sigma \int_{\mathbb{R}^d \times \mathbb{T}} \frac{c_\sigma}{-\Delta_x + m} \bar{u} [-4\partial_{x_k}(\partial_{x_k x_l} \phi) \partial_{x_l} - \Delta_x^2 \phi] u_m \, dx dy dm \\
 &= \int_0^\infty m^\sigma \int_{\mathbb{R}^d \times \mathbb{T}} \overline{u_m} [-4\partial_{x_k}(\partial_{x_k x_l} \phi) \partial_{x_l} - \Delta_x^2 \phi] u_m \, dx dy dm \\
 &= \int_0^\infty m^\sigma \int_{\mathbb{R}^d \times \mathbb{T}} 4\overline{\partial_{x_k} u_m} (\partial_{x_k x_l} \phi) \partial_{x_l} u_m - \Delta_x^2 \phi |u_m|^2 \, dx dy dm.
 \end{aligned}$$

Then we consider the nonlinear term  $II$ . Noticing that

$$\nabla(|u|^{p+2}) = \frac{p+2}{p} \nabla(|u|^p) |u|^2,$$

and using (5.6) again, we obtain

$$\begin{aligned}
 II &= \langle u(t), [\mu |u|^p, i\Gamma_\phi] u(t) \rangle \\
 &= -\mu \langle u(t), [|u|^p, \nabla_x \cdot \nabla_x \phi + \nabla_x \phi \cdot \nabla_x] u \rangle \\
 &= -\mu \langle u(t), |u|^p \nabla_x \cdot ((\nabla_x \phi) u) \rangle - \mu \langle u(t), |u|^p \nabla_x \phi \cdot \nabla_x u \rangle \\
 &= -\frac{2p\mu}{p+2} \int_{\mathbb{R}^d \times \mathbb{T}} \Delta_x \phi |u|^{p+2} \, dx dy.
 \end{aligned}$$

Therefore, together with the computation on terms  $I$  and  $II$ , we get

$$\begin{aligned}
 \frac{d}{dt} M_\phi[u(t)] &= \int_0^\infty m^s \int_{\mathbb{R}^d \times \mathbb{T}} \left( 4\overline{\partial_{x_k} u_m} (\partial_{x_k x_l} \phi) \partial_{x_l} u_m - \Delta_x^2 \phi |u_m|^2 \right) \, dx dy dm \\
 &\quad - \frac{2p\mu}{p+2} \int_{\mathbb{R}^d \times \mathbb{T}} \Delta_x \phi |u|^{p+2} \, dx dy,
 \end{aligned}$$

which implies Lemma 5.1.  $\square$

**COROLLARY 5.2.** *Assume  $u$  to be a smooth solution to the initial value problem (gNLS) with  $d \geq 3$ , then we have the following Morawetz inequality*

$$\int_{\mathbb{R}} \int_{\mathbb{R}^d \times \mathbb{T}} \frac{|u(t, x, y)|^{p+2}}{|x|} \, dx dy dt \lesssim \sup_{t \in \mathbb{R}} \|u(t)\|_{\dot{H}^{\frac{1}{2}}}^2 \lesssim \sup_{t \in \mathbb{R}} \|u(t)\|_{H^\sigma}^2.$$

**PROOF OF COROLLARY 5.2.** We take  $\phi(x) = |x|$  (independent on  $y$ ) in Lemma 5.1. Hence

$$\begin{aligned}
 \nabla_x \phi &= \frac{x}{|x|}, \\
 \Delta_x \phi &= \frac{d-1}{|x|}, \\
 \partial_{x_k x_l} \phi &= \frac{\delta_{x_k x_l}}{|x|} - \frac{x_k x_l}{|x|^3} \\
 \Delta_x^2 \phi &= \begin{cases} -\pi \delta(x), & d = 3, \\ -(d-1)(d-3) |x|^{-3}, & d \geq 4. \end{cases}
 \end{aligned}$$

Under such choice of  $\phi$ , we claim that

$$(5.8) \quad \int_0^\infty m^s \int_{\mathbb{R}^d \times \mathbb{T}} \left( 4\overline{\partial_{x_k} u_m} (\partial_{x_k x_l} \phi) \partial_{x_l} u_m - \Delta_x^2 \phi |u_m|^2 \right) dx dy dm \geq 0.$$

Assuming (5.8), we obtain for the defocusing equation ( $\mu = -1$  in (gNLS))

$$(5.9) \quad \begin{aligned} \frac{d}{dt} M_{|x|}[u(t)] &\geq \frac{2p}{p+2} \int_{\mathbb{R}^d \times \mathbb{T}} \Delta_x(|x|) |u(t, x, y)|^{p+2} dx dy \\ &= \frac{2p}{p+2} \int_{\mathbb{R}^d \times \mathbb{T}} \frac{d-1}{|x|} |u(t, x, y)|^{p+2} dx dy, \end{aligned}$$

which gives Corollary 5.2 by combining the following upper bound of  $M_{|x|}$  in [17]

$$|M_{|x|}[u(t)]| \lesssim \sup_t \|u(t)\|_{\dot{H}^{\frac{1}{2}}}^2 \lesssim \sup_t \|u(t)\|_{H^\sigma}^2,$$

and integrating in  $t$  using the fundamental theorem of calculus.

To see (5.8), we write (when  $d \geq 4$ )

$$(5.10) \quad \begin{aligned} &\int_{\mathbb{R}^d} \left( 4\overline{\partial_{x_k} u_m} (\partial_{x_k x_l} \phi) \partial_{x_l} u_m - \Delta_x^2 \phi |u_m|^2 \right) dx \\ &= \int_{\mathbb{R}^d} \left( 4\overline{\partial_{x_k} u_m} \left( \frac{\delta_{x_k x_l}}{|x|} - \frac{x_k x_l}{|x|^3} \right) \partial_{x_l} u_m + \frac{(d-1)(d-3)}{|x|^3} |u_m|^2 \right) dx. \end{aligned}$$

The  $d = 3$  case can be handled similarly, hence omitted.

Using the notation  $\nabla_{\vec{e}} u = (\vec{e} \cdot \nabla) \frac{\vec{e}}{|\vec{e}|}$  and  $\nabla_{\vec{e}}^\perp u = \nabla u - \nabla_{\vec{e}} u$  with  $\vec{e} = \vec{x}$ , we can decompose  $\nabla u$  orthogonally. Then we have

$$\begin{aligned} \overline{\partial_{x_k} u_m} x_k x_l \partial_{x_l} u_m &\leq \frac{1}{2} |\partial_{x_k} u_m x_k|^2 + \frac{1}{2} |\partial_{x_l} u_m x_l|^2 \\ &\leq \frac{1}{2} |\nabla_{\vec{e}} u_m|^2 |x|^2 + \frac{1}{2} |\nabla_{\vec{e}}^\perp u_m|^2 |x|^2 \\ &= |\nabla_x u_m|^2 |x|^2. \end{aligned}$$

Then continuing from (5.10), we obtain

$$(5.10) \geq \int_{\mathbb{R}^d} \left( 4 \frac{|\nabla_x u_m|^2}{|x|} - 4 \frac{|\nabla_{\vec{e}} u_m|^2 |x|^2}{|x|^3} + \frac{(d-1)(d-3)}{|x|^3} |u_m|^2 \right) dx \\ = \int_{\mathbb{R}^d} \left( 4 \frac{|\nabla_x u_m|^2 - |\nabla_{\vec{e}} u_m|^2 |x|^2}{|x|} + \frac{(d-1)(d-3)}{|x|^3} |u_m|^2 \right) dx \\ = \int_{\mathbb{R}^d} \left( 4 \frac{|\nabla_{\vec{e}}^\perp u_m|^2}{|x|} + \frac{(d-1)(d-3)}{|x|^3} |u_m|^2 \right) dx \geq 0,$$

then (5.8) follows by integrating (5.9) in both  $y$  and  $m$ . This completes the proof of Corollary 5.2.  $\square$

## 6. Proof of the scattering result

In this section, we give the proof for the scattering result in Theorem 1.3. There are four steps and we will discuss them step by step. The strategy has similar spirit with [63, 65].

**6.1. Step 1: the Morawetz bound.** Recall the Morawetz estimate (Corollary 5.2) established in Section 5, we have

$$(6.1) \quad \int_{\mathbb{R}} \int_{\mathbb{R}^d \times \mathbb{T}} \frac{|u(t, x, y)|^{2+p}}{|x|} dt dx dy \lesssim_{\text{data}} 1.$$

**6.2. Step 2: Proof of the decay property.** Based on the Morawetz bound above, we aim to show the decay property of (gNLS), i.e.

$$\lim_{t \rightarrow \infty} \|u(t, x, y)\|_{L_{x,y}^q(\mathbb{R}^d \times \mathbb{T})} = 0,$$

where  $2 < q \leq 2 + r$  (for any  $r < \frac{2+4\sigma+dp+p+2p\sigma}{2d}$ ). This decay property is essential for us to obtain the scattering result.

REMARK 6.1. One may also consider the (stronger) pointwise type decay which describes the decay rate of nonlinear solutions quantitatively. See [23, 67] for recent results and the references therein.

In viewing of interpolation with the mass conservation law, it suffices to show the endpoint case, that is,

$$(6.2) \quad \lim_{t \rightarrow \infty} \|u(t, x, y)\|_{L_{x,y}^{2+r}(\mathbb{R}^d \times \mathbb{T})} = 0.$$

We will prove it by contradiction. Before starting with the proof, we recall a radial Sobolev embedding as follows,

LEMMA 6.2 (Radial Sobolev Embeddings in  $\mathbb{R}^d$  in [61]). *Let  $d \geq 1$ ,  $1 \leq q \leq \infty$ ,  $0 < s < d$  and  $\beta \in \mathbb{R}$  obey the conditions*

$$\beta > -\frac{d}{q}, \quad 0 \leq \frac{1}{p} - \frac{1}{q} \leq s$$

*and the scaling condition*

$$\beta + s = \frac{d}{p} - \frac{d}{q}$$

*with at most one of the equalities*

$$p = 1, \quad p = \infty, \quad q = 1, \quad q = \infty, \quad \frac{1}{p} - \frac{1}{q} = s$$

*holding. Then for any spherically symmetric function  $f \in \dot{W}^{s,p}(\mathbb{R}^d)$ , we have*

$$\left\| |x|^\beta f \right\|_{L^q(\mathbb{R}^d)} \lesssim \left\| |\nabla|^s f \right\|_{L^p(\mathbb{R}^d)}.$$

Let  $\beta$  satisfies  $(2+p)\beta + 1 = 2 + r$ .

Via Lemma 6.2, Hölder inequality and Sobolev embedding, we have

$$\begin{aligned}
& \|u(t, x, y)\|_{L_{x,y}^{2+r}(\mathbb{R}^d \times \mathbb{T})} \\
&= \left( \int |u(t, x, y)|^{2+r} dx dy \right)^{\frac{1}{2+r}} \\
&= \left( \int \frac{|u(t, x, y)|^{(2+p)\beta}}{|x|^\beta} \cdot |x|^\beta |u| \right)^{\frac{1}{2+r}} \\
&\lesssim \left( \int_{\mathbb{R}^d \times \mathbb{T}} \frac{|u(t, x, y)|^{2+p}}{|x|} dx dy \right)^{\frac{\beta}{2+r}} \cdot \left( \int (|x|^\beta |u|)^{\frac{1}{1-\beta}} dx dy \right)^{\frac{1-\beta}{2+r}} \\
&= \left( \int_{\mathbb{R}^d \times \mathbb{T}} \frac{|u(t, x, y)|^{2+p}}{|x|} dx dy \right)^{\frac{\beta}{2+r}} \cdot \left( \| |x|^\beta |u| \|_{L_{x,y}^{\frac{1}{1-\beta}}} \right)^{\frac{1}{2+r}} \\
&\lesssim \left( \int_{\mathbb{R}^d \times \mathbb{T}} \frac{|u(t, x, y)|^{2+p}}{|x|} dx dy \right)^{\frac{\beta}{2+r}} \cdot \left( \| |\nabla|^s u \|_{L_x^2 L_y^{\frac{1}{1-\beta}}} \right)^{\frac{1}{2+r}} \\
&\lesssim \left( \int_{\mathbb{R}^d \times \mathbb{T}} \frac{|u(t, x, y)|^{2+p}}{|x|} dx dy \right)^{\frac{\beta}{2+r}} \cdot \left( \| |\nabla_x|^s |\nabla_y|^\tau u \|_{L_x^2 L_y^2} \right)^{\frac{1}{2+r}} \\
&\lesssim \left( \int_{\mathbb{R}^d \times \mathbb{T}} \frac{|u(t, x, y)|^{2+p}}{|x|} dx dy \right)^{\frac{\beta}{2+r}}.
\end{aligned}$$

We require the indices satisfy:

$$s + \tau \leq \sigma \quad (\text{regularity requirement from the energy conservation}),$$

$$\left(\beta - \frac{1}{2}\right)_+ = \tau \quad (\text{Sobolev embedding in 1D}),$$

$$\beta + s = \frac{d}{2} - d(1 - \beta) \quad (\text{radial Sobolev embedding}),$$

$$(2 + p)\beta + 1 = 2 + r \quad (\text{the relation between } \beta \text{ and } r).$$

We need to choose  $\beta$  satisfies  $\beta < \frac{1}{2} + \frac{\frac{1}{2} + \sigma}{d}$ . Correspondingly,  $r < \frac{2+4\sigma+d p+p+2p\sigma}{2d}$ . That is the exponent requirement in the decay estimate (6.2).

We are now ready to prove (6.2) by contradiction argument. If (6.2) does not hold, using the estimate above, we deduce the existence of a sequence  $\{t_n\}_n \rightarrow \infty$  and  $\epsilon_0 > 0$  such that

$$\left( \int_{\mathbb{R}^d \times \mathbb{T}} \frac{|u(t_n, x, y)|^{2+p}}{|x|} dx dy \right)^{\frac{\beta}{2+r}} \geq \|u(t_n, x, y)\|_{L_{x,y}^{2+r}(\mathbb{R}^d \times \mathbb{T})} \geq \epsilon_0 > 0.$$

Without loss of generality, we consider  $t_n \rightarrow +\infty$ . Similar as in [63], we get the existence of  $T > 0$  such that

$$\inf_n \inf_{t \in (t_n, t_n + T)} \left( \int_{\mathbb{R}^d \times \mathbb{T}} \frac{|u(t, x, y)|^{2+p}}{|x|} dx dy \right)^{\frac{\beta}{2+r}} \geq \frac{\epsilon_0}{2}.$$

Notice that since  $\{t_n\}_n \rightarrow +\infty$  then we can assume (modulo subsequence) that the intervals  $(t_n, t_n + T)$  are disjoint. In particular we have

$$\begin{aligned}
\sum_n T \left(\frac{\epsilon_0}{2}\right)^{\frac{2+r}{\beta}} &\leq \sum_n \int_{t_n}^{t_n+T} \left( \int_{\mathbb{R}^d \times \mathbb{T}} \frac{|u(t, x, y)|^{2+p}}{|x|} dx dy \right) dt \\
&\leq \int \left( \int_{\mathbb{R}^d \times \mathbb{T}} \frac{|u(t, x, y)|^{2+p}}{|x|} dx dy \right) dt
\end{aligned}$$

and hence we get a contradiction since the left hand side is divergent and the right hand side is bounded by (6.1).

**6.3. Step 3: Proof of the spacetime bound.** We aim to show,

$$(6.3) \quad u \in L_t^{q_\theta} L_x^{r_\theta} H_y^{\frac{1}{2}+\delta}(\mathbb{R}_t \times \mathbb{R}^d \times \mathbb{T})$$

and

$$\begin{aligned} & \lim_{t_1, t_2 \rightarrow \infty} (\|u\|_{L_t^l L_x^{m'} L_y^2([t_1, t_2] \times \mathbb{R}^d \times \mathbb{T})} + \|\nabla_x |^\sigma(u)\|_{L_t^l L_x^m L_y^2([t_1, t_2] \times \mathbb{R}^d \times \mathbb{T})} \\ & + \|\partial_y |^\sigma(u)\|_{L_t^l L_x^m L_y^2([t_1, t_2] \times \mathbb{R}^d \times \mathbb{T})}) < \infty. \end{aligned}$$

The above spacetime bounds are sufficient to show the scattering for (gNLS). In this step, all spacetime norms are over  $\mathbb{R}_t \times \mathbb{R}^d \times \mathbb{T}$  unless indicated otherwise. For example, we define

$$\|f(t)\|_{L_{t>t_0}^p} := \left( \int_{t_0}^{\infty} |f(t)|^p dt \right)^{\frac{1}{p}}$$

for any given time-dependent function  $f(t)$ , and similarly we can define  $\|f(t)\|_{L_{t<t_0}^p}$ .

We note that we will apply an  $H_y^{\frac{1}{2}+\delta}$  valued version of the critical analysis of [10].

PROOF. Using Strichartz estimates and the Hölder inequality,

$$\begin{aligned} \|u\|_{L_{t>t_0}^{q_\theta} L_x^{r_\theta} H_y^{\frac{1}{2}+\delta}} & \lesssim \|u_0\|_{H^\sigma(\mathbb{R}^d \times \mathbb{T})} + \| |u|^p u \|_{L_{t>t_0}^{\tilde{q}_\theta'} L_x^{\tilde{r}_\theta'} H_y^{\frac{1}{2}+\delta}} \\ & \lesssim \|u_0\|_{H^\sigma(\mathbb{R}^d \times \mathbb{T})} + \|u\|_{L_{t>t_0}^{(1+p)\tilde{q}_\theta'} L_x^{(1+p)\tilde{r}_\theta'} H_y^{\frac{1}{2}+\delta}}^{1+p} \\ & \lesssim \|u_0\|_{H^\sigma(\mathbb{R}^d \times \mathbb{T})} + \|u\|_{L_{t>t_0}^{q_\theta} L_x^{r_\theta} H_y^{\frac{1}{2}+\delta}}^{(1+p)\theta} \|u\|_{L_{t>t_0}^\infty L_x^{\frac{p d}{2}} H_y^{\frac{1}{2}+\delta}}^{(1+p)(1-\theta)}. \end{aligned}$$

Similar to Lemma 2.5 in [63] (this lemma is an analysis result which does not involve the nonlinear PDE structure so we can use it directly), based on the decay property (6.2), we can further obtain

$$(6.4) \quad \|u\|_{L_x^{\frac{p d}{2}} H_y^{\frac{1}{2}+\delta}} = o(1).$$

Using the decay property (6.4), we see for every  $\epsilon > 0$  there exists  $t_0 = t_0(\epsilon) > 0$  such that

$$\|u\|_{L_{t>t_0}^{q_\theta} L_x^{r_\theta} H_y^{\frac{1}{2}+\delta}} \leq C \|u_0\|_{H^2(\mathbb{R}^d \times \mathbb{T})} + \epsilon \|u\|_{L_{t>t_0}^{q_\theta} L_x^{r_\theta} H_y^{\frac{1}{2}+\delta}}.$$

We can now use the continuity argument to obtain

$$\|u\|_{L_{t>0}^{q_\theta} L_x^{r_\theta} H_y^{\frac{1}{2}+\delta}} < \infty.$$

Similarly, we obtain  $\|u\|_{L_{t<0}^{q_\theta} L_x^{r_\theta} H_y^{\frac{1}{2}+\delta}} < \infty$ .

Now we consider the second estimate. We show  $\|\partial_y |^\sigma(u)\|_{L_t^l L_x^m L_y^2}$ , the other estimates are similar. Using Strichartz estimate and the Hölder inequality,

$$\begin{aligned} \|\partial_y |^\sigma(u)\|_{L_{t>t_0}^l L_x^m L_y^2} & \lesssim \|u_0\|_{H^\sigma(\mathbb{R}^d \times \mathbb{T})} + \|\partial_y |^\sigma(|u|^p u)\|_{L_{t>t_0}^l L_x^{m'} L_y^2} \\ & \lesssim \|u_0\|_{H^\sigma(\mathbb{R}^d \times \mathbb{T})} + \|\partial_y |^\sigma(u)\|_{L_{t>t_0}^l L_x^m L_y^2} \|u\|_{L_{t>t_0}^{q_\theta} L_x^{r_\theta} L_y^\infty}^p \\ & \lesssim \|u_0\|_{H^\sigma(\mathbb{R}^d \times \mathbb{T})} + \|\partial_y |^\sigma(u)\|_{L_{t>t_0}^l L_x^m L_y^2} \|u\|_{L_{t>t_0}^{q_\theta} L_x^{r_\theta} H_y^{\frac{1}{2}+\delta}}^p. \end{aligned}$$

We conclude by choosing  $t_0$  large enough and by recalling (6.3).

For FNLS, due to the Strichartz estimates and Sobolev embedding, we choose the indices satisfying,

$$s + \frac{1}{2} + \delta \leq \sigma,$$

(regularity requirement from the energy conservation)

$$\frac{2\sigma}{q_\theta} + \frac{d}{r_\theta} = \frac{d}{2} - s, \quad \frac{2\sigma}{q_\theta} + \frac{d}{\tilde{r}_\theta} + \frac{2\sigma}{\tilde{q}_\theta} + \frac{d}{r_\theta} = d,$$

(Strichartz exponent relations)

$$\frac{1}{(p+1)\tilde{q}_\theta'} = \frac{\theta}{q_\theta}, \quad \frac{1}{(p+1)\tilde{r}_\theta'} = \frac{\theta}{r_\theta} + \frac{2(1-\theta)}{pd},$$

(the Hölder inequality, or say, interpolation) and

$$\frac{2\sigma}{l} + \frac{d}{m} = \frac{d}{2}, \quad \frac{1}{m'} = \frac{1}{m} + \frac{p}{r_\theta}, \quad \frac{1}{l'} = \frac{1}{l} + \frac{p}{q_\theta}$$

(Strichartz exponent relations and the Hölder inequality). □

**6.4. Step 4: Proof of the scattering asymptotics.** In fact by using the integral equation, it is sufficient to prove that

$$\lim_{t_1, t_2 \rightarrow \infty} \left\| \int_{t_1}^{t_2} e^{-is((-\Delta_x)^\sigma - \partial_y^{2\sigma})} (|u|^p u) ds \right\|_{H_{x,y}^\sigma(\mathbb{R}^d \times \mathbb{T})} = 0.$$

Moreover, using Strichartz estimates, we only need to show,

$$\lim_{t_1, t_2 \rightarrow \infty} \left( \| |u|^p u \|_{L_t^{l'} L_x^{m'} L_y^2([t_1, t_2] \times \mathbb{R}^d \times \mathbb{T})} + \| |\nabla_x|^\sigma (|u|^p u) \|_{L_t^{l'} L_x^{m'} L_y^2([t_1, t_2] \times \mathbb{R}^d \times \mathbb{T})} + \| |\partial_y|^\sigma (|u|^p u) \|_{L_t^{l'} L_x^{m'} L_y^2([t_1, t_2] \times \mathbb{R}^d \times \mathbb{T})} \right) = 0.$$

Noticing the two established estimates, the above limit follows in a straightforward way. Thus we proved scattering in the energy space.

### 7. Further remarks

In this section, we make a few more remarks on this research line, i.e. *long time dynamics for dispersive equations on waveguide manifolds*. As mentioned in the introduction, this area has been developed a lot in recent decades. The authors are interested in this research line for several years. Though many theories/tools/results have been established, there are still many interesting open questions left. We list some interesting related problems in this line for interested readers.

1. *The critical regime.* The cases we are considering in this paper are of a ‘double subcritical’ nature (gNLS). In fact, it is also quite interesting to consider the scattering theory for the critical regime. For example,

$$(i\partial_t + (-\Delta_x)^\sigma + (-\partial_y^2)^\sigma)u = \mu |u|^{\frac{4\sigma}{d}} u, \quad u(0, x, y) = u_0(x, y) \in H^\sigma(\mathbb{R}^d \times \mathbb{T}),$$

and

$$(i\partial_t + (-\Delta_x)^\sigma + (-\partial_y^2)^\sigma)u = \mu |u|^{\frac{4\sigma}{d+1-2\sigma}} u, \quad u(0, x, y) = u_0(x, y) \in H^\sigma(\mathbb{R}^d \times \mathbb{T}).$$

The first one is of mass-critical nature and the second one is of energy-critical nature. New techniques are needed including function spaces, profile decomposition, profile approximations and even resonant systems. See [13, 30, 68, 69] for the NLS case.

2. *Improvements/generalizations for Theorem 1.3.* One may also try to remove the radial assumption in Theorem 1.3 or consider (gNLS) on more general waveguide manifolds  $\mathbb{R}^n \times \mathbb{T}^m$ . However, when  $m \geq 2$ , one can at most show the global well-posedness since the scattering is not expected to hold. Another point is to consider the cases  $d = 1, 2$ . Some more techniques are required, such as the Morawetz-type estimate in 1D and 2D (see Section 5).

One may consider other problems for (gNLS) such as growth of Sobolev norms (weak turbulence) or low regularity type results (see [70] and the references therein).

3. *Scattering for focusing NLS/4NLS/gNLS on waveguide manifolds.* The results discussed in this paper concern mainly the defocusing case. In general, large data scattering for the focusing NLS (or other dispersive equations) on waveguides is comparably less understood than the defocusing case. Threshold assumptions are necessary and new ingredients are needed to handle this type of problems. See [66] for a recent global well-posedness result, see [21, 22, 39, 41] for the Euclidean results and see [12, 49] for some very recent scattering result.

4. *Critical NLS on higher dimensional waveguide manifolds.* For critical NLS (or other dispersive models) on waveguide manifolds, most of the models are lower dimensional (with no higher than four whole dimensions), which leads to quintic or cubic nonlinearity. This gives one advantage to applying function spaces to deal with nonlinearity. In general, the difficulty of the critical NLS problem on  $\mathbb{R}^m \times \mathbb{T}^n$  increases if the dimension  $m + n$  is increased or if the number  $m$  of copies of  $\mathbb{R}$  is decreased (which is concluded in [36]). There are no large data global results for critical NLS on waveguide manifolds with at least 5 whole dimensions, to the best knowledge of the authors. Moreover, the Hartree analogues are also less understood.

5. *NLS on other product spaces.* Instead of waveguide manifolds, one may consider dispersive equations on other types of product spaces, for example,  $\mathbb{R}^d \times \mathbb{S}^n$  where  $\mathbb{S}^n$  are  $n$ -dimensional spheres ( $\mathbb{S}^n$  can be replaced by other manifolds). See [50] for a global well-posedness result of NLS on pure spheres. In this regime, NLS may be a good model to start with. One can also replace  $\mathbb{S}^n$  by other manifolds.

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DEPARTMENT OF MATHEMATICS, JOHNS HOPKINS UNIVERSITY, BALTIMORE, MD 21218, USA  
*Email address:* `ysire1@jhu.edu`

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF WASHINGTON, SEATTLE, WA 98195, USA  
*Email address:* `xueying@uw.edu`

INSTITUTE OF MATHEMATICAL SCIENCES, SHANGHAI TECH UNIVERSITY, SHANGHAI, CHINA  
*Email address:* `yuehaitian@shanghaitech.edu.cn`

DEPARTMENT OF MATHEMATICS AND STATISTICS, BEIJING INSTITUTE OF TECHNOLOGY, BEIJING, CHINA, ALSO MIIT KEY LABORATORY OF MATHEMATICAL THEORY AND COMPUTATION IN INFORMATION SECURITY, BEIJING, CHINA  
*Email address:* `zzh@bit.edu.cn`