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Local well-posedness and regularity criterion for nonhomogeneous magneto-micropolar fluid equations without angular viscosity

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Abstract. We study an initial-boundary-value problem for three-dimensional nonhomogeneous magneto-micropolar fluid equations without angular viscosity. Using linearization and Banach's fixed point theorem, we prove the local existence and uniqueness of strong solutions. Moreover, a regularity criterion is also obtained.

Contents

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1. Introduction

Let $\Omega \subset \mathbb{R}^3$ be a bounded smooth domain, we consider nonhomogeneous magneto-micropolar system (see [**15**]) in $\Omega \times (0,T)$:

$$\rho_t + u \cdot \nabla \rho = 0, \tag{1.1}$$

$$\rho u_t + \rho u \cdot \nabla u + \nabla \pi - (\mu + \zeta)\Delta u = 2\zeta \operatorname{rot} w + \operatorname{rot} b \times b, \tag{1.2}$$

$$\rho w_t + \rho u \cdot \nabla w - \tilde{\mu}\Delta w - (\tilde{\lambda} + \tilde{\mu})\nabla \operatorname{div} w + 4\zeta w = 2\zeta \operatorname{rot} u, \tag{1.3}$$

$$b_t + \operatorname{rot}(b \times u) = \Delta b, \tag{1.4}$$

$$\operatorname{div} u = \operatorname{div} b = 0 \text{ in } \Omega \times (0,T), \tag{1.5}$$

$$u = 0, w = 0, b \cdot n = 0, \operatorname{rot} b \times n = 0 \text{ on } \partial\Omega \times (0,T), \tag{1.6}$$

$$(\rho, u, w, b)(\cdot, 0) = (\rho_0, u_0, w_0, b_0)(\cdot) \text{ in } \Omega. \tag{1.7}$$

Here the unknowns ρ, u, π, w, and b stand for the density, velocity, pressure, micro-rational velocity, and magnetic field, respectively. The viscosity coefficients $\mu, \zeta, \tilde{\mu}$, and $\tilde{\lambda}$ are constants satisfying

$$\mu, \zeta > 0, \ \tilde{\mu} \geq 0, \mu - 2\zeta > 0, \text{ and } 2\tilde{\mu} + 3\tilde{\lambda} \geq 0.$$

n is the unit outward normal vector to $\partial\Omega$.

In the 1970s, Ahmadi and Shahinpoor [**2**] introduced a mathematical model for the motion of an electrically conducting micropolar fluids in the presence of an arbitrary magnetic field. The magneto-micropolar fluid equations can be used to describe the motion of aggregates of small solid ferromagnetic particles relative to viscous magnetic fluids under the action of magnetic fields, such as salt water, ester, and fluorocarbon, which is of great importance in practical and mathematical applications (see [**9, 10, 15**]).

First, let us give a short survey for the study of nonhomogeneous micropolar fluid equations, that is, (1.1)–(1.5) with $b = 0$. When the initial density is strictly away from vacuum (i.e., the initial density is strictly positive), the authors [**4**] proved some existence and uniqueness results for strong solutions. Meanwhile, Braz e Silva et al. [**5**] investigated the global existence and uniqueness of solutions for 3D Cauchy problem through a Lagrangian approach. On the other hand, for the initial density allowing vacuum states, Łukaszewicz [**14**] (see also [**15**, Chapter 3]) obtained the short-time existence of weak solutions provided that the initial functions u_0 and w_0 are in H_0^1 and that the initial density ρ_0 is uniformly bounded and satisfies $\|\rho_0^{-1}\|_{L^3} < \infty$, while Braz e Silva and Santos [**8**] established the global existence of weak solutions. In [**7**], under smallness assumptions on the initial data, weak solutions with improved regularity were obtained. Recently, imposing a compatibility condition on the initial data, Zhang and Zhu [**20**] showed the global existence of strong solution with nonnegative density in $\mathbb{R}^3$ under some smallness condition. Later on, Ye [**19**] improved their result by removing the compatibility condition and furthermore obtained exponential decay of strong solutions.

As a couple system, (1.1)–(1.5) contain much richer structures than the nonhomogeneous micropolar fluid equations. It is not merely a combination of micropolar fluid equations and magnetic field equations but an interactive system. Their distinctive features make analytical studies a great challenge but offer new opportunities. Tang and Sun [**17**] obtained the local well-posedness of strong solutions for (1.1)–(1.7). This result was recently extended to be a global one by Zhong [**21**] under some smallness condition. At the same time, Yang and Zhong [**18**] proved the global existence and decay estimates of strong solutions for the 3D Cauchy problem. The aim of this paper is to prove the local well-posedness of strong solutions to the problem (1.1)–(1.7) without angular viscosity (i.e., $\tilde{\mu} = \tilde{\lambda} = 0$).

Our main result states as follows.

THEOREM 1.1. *Let $\tilde{\mu} = \tilde{\lambda} = 0$ and assume that the initial data (ρ_0, u_0, b_0, w_0) satisfies*

$$\begin{cases} 0 < \inf \rho_0 \le \rho_0 \le \sup \rho_0 < \infty, \ \rho_0 \in H^3, \ (u_0, w_0) \in H_0^1 \cap H^2, \ b_0 \in H^2, \\ \operatorname{div} u_0 = \operatorname{div} b_0 = 0, \ b_0 \cdot n|_{\partial\Omega} = 0, \ \operatorname{rot} b_0 \times n|_{\partial\Omega} = 0, \end{cases} \tag{1.8}$$

then there exists a positive time $T > 0$ such that the problem (1.1)–(1.7) *has a unique local strong solution (ρ, u, w, b) on $\Omega \times (0, T]$ satisfying*

$$\begin{cases} \inf \rho_0 \le \rho \le \sup \rho_0, \rho \in C([0,T]; H^3), \rho_t \in C([0,T]; H^2), \\ u \in C([0,T]; H_0^1 \cap H^2) \cap L^2(0,T; H^3), \\ u_t \in C([0,T]; L^2) \cap L^2(0,T; H^1), \\ w \in C([0,T]; H^2), w_t \in C([0,T]; H^1), \\ b \in C([0,T]; H^2) \cap L^2(0,T; H^3), \\ b_t \in C([0,T]; L^2) \cap L^2(0,T; H^1). \end{cases} \tag{1.9}$$

Furthermore, if

$$u \in L^2(0,T; L^\infty) \quad \text{and} \quad \nabla u \in L^1(0,T; L^\infty) \tag{1.10}$$

hold true with some finite $0 < T < \infty$, then the solution (ρ, u, w, b) can be extended beyond T.

2. Preliminaries

Before we present the proof of Theorem 1.1, we need some technical results.

The first one is the bilinear commutator and the product estimates.

LEMMA 2.1. *([**12**]). (i) If $f \in W^{s,p}(\Omega) \cap C^1(\Omega)$ and $g \in W^{s-1,p}(\Omega) \cap C(\Omega)$, then, for $|\alpha| \le s$,*

$$\|D^\alpha(fg) - fD^\alpha g\|_{L^p} \le C(\|f\|_{W^{s,p_1}} \|g\|_{L^{q_1}} + \|\nabla f\|_{L^{p_2}} \|g\|_{W^{s-1,q_2}}). \tag{2.1}$$

(ii) If $f, g \in W^{s,p}(\Omega) \cap C(\Omega)$, then, for integer $s \ge 1$ and $|\alpha| \le s$,

$$\|D^\alpha(fg)\|_{L^p} \le C(\|f\|_{W^{s,p_1}} \|g\|_{L^{q_1}} + \|f\|_{L^{p_2}} \|g\|_{W^{s,q_2}}) \tag{2.2}$$

with $\dfrac{1}{p} = \dfrac{1}{p_1} + \dfrac{1}{q_1} = \dfrac{1}{p_2} + \dfrac{1}{q_2}$, and $1 < p < \infty$.

We will also use the following two lemmas.

LEMMA 2.2. *([**3**]). Let $\Omega \subset \mathbb{R}^3$ be a smooth bounded domain and let $b : \Omega \to \mathbb{R}^3$ be a smooth vector field. Then, for $1 < p < \infty$,*

$$\begin{aligned} -\int |b|^{p-2} b \cdot \Delta b \mathrm{d}x &= \int |b|^{p-2} |\nabla b|^2 \mathrm{d}x + 4\frac{p-2}{p^2} \int |\nabla |b|^{\frac{p}{2}}|^2 \mathrm{d}x \\ &\quad - \int_{\partial\Omega} |b|^{p-2} (b \cdot \nabla) b \cdot n \mathrm{d}S - \int_{\partial\Omega} |b|^{p-2} (\operatorname{rot} b \times n) \cdot b \mathrm{d}S. \end{aligned} \tag{2.3}$$

LEMMA 2.3. *([**1, 16**]). Let Ω be a bounded smooth domain. Then, for all $b \in W^{1,p}(\Omega)$ with $1 < p < \infty$, we have that*

$$\|b\|_{L^p(\partial\Omega)} \leq C\|b\|_{L^p}^{1-\frac{1}{p}} \|b\|_{W^{1,p}}^{\frac{1}{p}}. \tag{2.4}$$

For $\operatorname{div} u = \operatorname{div} b = 0$, we have the well-known vector identities

$$\operatorname{rot}(b \times u) = u \cdot \nabla b - b \cdot \nabla u, \tag{2.5}$$

$$\operatorname{rot} b \times b = b \cdot \nabla b - \frac{1}{2}\nabla|b|^2. \tag{2.6}$$

Finally, when b satisfies $b \cdot n = 0$ on $\partial\Omega$, we will also use the identity

$$(b \cdot \nabla)b \cdot n = -(b \cdot \nabla)n \cdot b \quad \text{on} \quad \partial\Omega, \tag{2.7}$$

for any smooth vector field b.

3. Proof of Theorem 1.1

3.1. Local well-posedness. In this subsection, we will use Banach's fixed point theorem to prove the local well-posedness. To this end, we define the nonlinear map $\mathcal{F}$ as follows

$$\mathcal{F} : v \in \mathbf{X} \to u \in \mathbf{X},$$

where

$$\begin{aligned} \mathbf{X} := \Big\{ v \in \mathbf{X} : &\|v\|_{C([0,T];H_0^1 \cap H^2)} + \|v\|_{L^2(0,T;H^3)} \\ &+ \|v_t\|_{C([0,T];L^2)} + \|v_t\|_{L^2(0,T;H^1)} \leq R, \\ &\operatorname{div} v = 0 \quad \text{in} \quad \Omega \times (0,T) \quad \text{and} \quad v(\cdot, 0) = u_0 \Big\}, \end{aligned} \tag{3.1}$$

and R is a positive constant to be determined later. We will consider the following four problems:

$$b_t + \operatorname{rot}(b \times v) = \Delta b, \operatorname{div} b = 0 \quad \text{in} \quad \Omega \times (0,T), \tag{3.2a}$$

$$b \cdot n = 0, \operatorname{rot} b \times n = 0 \quad \text{on} \quad \partial\Omega \times (0,T), \tag{3.2b}$$

$$b(\cdot, 0) = b_0 \quad \text{in} \quad \Omega. \tag{3.2c}$$

$$\rho_t + v \cdot \nabla\rho = 0 \quad \text{in} \quad \Omega \times (0,T), \tag{3.3a}$$

$$\rho(\cdot, 0) = \rho_0(\cdot) \quad \text{in} \quad \Omega. \tag{3.3b}$$

$$\rho w_t + \rho v \cdot \nabla w + 4\zeta w = 2\zeta \operatorname{rot} v \quad \text{in} \quad \Omega \times (0,T), \tag{3.4a}$$

$$w(\cdot, 0) = w_0(\cdot) \quad \text{in} \quad \Omega. \tag{3.4b}$$

$$\rho u_t + \rho v \cdot \nabla u + \nabla\pi - (\mu + \zeta)\Delta u = 2\zeta \operatorname{rot} w + \operatorname{rot} b \times b \quad \text{in} \quad \Omega \times (0,T), \tag{3.5a}$$

$$\operatorname{div} u = 0 \quad \text{in} \quad \Omega \times (0,T), \tag{3.5b}$$

$$u = 0 \quad \text{on} \quad \partial\Omega \times (0,T), \tag{3.5c}$$

$$u(\cdot, 0) = u_0(\cdot) \quad \text{in} \quad \Omega. \tag{3.5d}$$

We comment that $u := \mathcal{F}(v)$ is defined as the solution of the above four problems.

LEMMA 3.1. *Given $v \in \mathbf{X}$, then the problem* (3.2a)–(3.2c) *has a unique solution b satisfying*

$$\|b\|_{L^\infty(0,T;H^2)} + \|b\|_{L^2(0,T;H^3)} \le C, \tag{3.6}$$

$$\|b_t\|_{L^\infty(0,T;L^2)} + \|b_t\|_{L^2(0,T;H^1)} \le C, \tag{3.7}$$

provided that T is small enough. Here and in what follows, C will be a constant independent of $R > 0$.

Proof. Since the equation (3.2a) is linear, we only need to show a priori estimates.

Testing (3.2a) by b, we see that

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int |b|^2 \mathrm{d}x + \int |\operatorname{rot} b|^2 \mathrm{d}x &= \int (v \times b) \operatorname{rot} b \mathrm{d}x \\ &\le \|v\|_{L^\infty}\|b\|_{L^2}\|\operatorname{rot} b\|_{L^2} \\ &\le \frac{1}{2}\|\operatorname{rot} b\|_{L^2}^2 + CR^2\|b\|_{L^2}^2,\end{aligned}$$

which gives that

$$\|b\|_{L^\infty(0,T;L^2)} \le \|b_0\|_{L^2}\exp(CR^2T) \le C \tag{3.8}$$

if $R^2 T \le 1$.

Testing (3.2a) by $-\Delta b$, we get that

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int |\operatorname{rot} b|^2 \mathrm{d}x + \int |\Delta b|^2 \mathrm{d}x &= \int (v\cdot\nabla b - b\cdot\nabla v)\Delta b \mathrm{d}x \\ &\le (\|v\|_{L^\infty}\|\nabla b\|_{L^2} + \|b\|_{L^6}\|\nabla v\|_{L^3})\|\Delta b\|_{L^2} \\ &\le CR\|\nabla b\|_{L^2}\|\Delta b\|_{L^2} \\ &\le \frac{1}{2}\|\Delta b\|_{L^2}^2 + CR^2\|\operatorname{rot} b\|_{L^2}^2,\end{aligned}$$

which yields that

$$\|\nabla b\|_{L^2} \le C\|\operatorname{rot} b\|_{L^2} \le C\|\nabla b_0\|_{L^2}\exp(CR^2T) \le C, \tag{3.9}$$

as long as $R^2 T \le 1$.

Applying ∂_t to (3.2a) and testing by b_t, we arrive at

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int |b_t|^2 \mathrm{d}x + \int |\operatorname{rot} b_t|^2 \mathrm{d}x &= \int (b_t \times v + b \times v_t) \operatorname{rot} b_t \mathrm{d}x \\ &\le (\|v\|_{L^\infty}\|b_t\|_{L^2} + \|b\|_{L^6}\|v_t\|_{L^3})\|\operatorname{rot} b_t\|_{L^2} \\ &\le \left(CR\|b_t\|_{L^2} + C\|v_t\|_{L^2}^{\frac{1}{2}}\|\nabla v_t\|_{L^2}^{\frac{1}{2}}\right)\|\operatorname{rot} b_t\|_{L^2} \\ &\le \frac{1}{2}\|\operatorname{rot} b_t\|_{L^2}^2 + CR^2\|\partial_t b\|_{L^2}^2 + CR\|\nabla v_t\|_{L^2},\end{aligned}$$

which implies that

$$\begin{aligned}\|b_t\|_{L^2}^2 &\le \left(\|\partial_t b_0\|_{L^2}^2 + \int_0^T CR\|\nabla v_t\|_{L^2}\mathrm{d}t\right)\exp(CR^2T) \\ &\le (C_0 + CR\sqrt{T})\exp(CR^2T) \le C, \qquad (3.10)\end{aligned}$$

and

$$\int_0^T \|\operatorname{rot} b_t\|_{L^2}^2 \mathrm{d}t \le C \tag{3.11}$$

if $R^2T \le 1$.

It is clear that $\nabla v(\cdot,t) = \nabla v_0 + \int_0^t \partial_s(\nabla v)\mathrm{d}s$, thus we get that

$$\|\nabla v\|_{L^2} \le \|\nabla v_0\|_{L^2} + \int_0^T \|\nabla v_s\|_{L^2}\mathrm{d}s \le \|\nabla v_0\|_{L^2} + R\sqrt{T} \le C \tag{3.12}$$

if $R^2T \le 1$.

Using the H^2-theory of Poisson equation (see [**11**]):

$$\Delta b = f := b_t + \mathrm{rot}(b\times v) \tag{3.13}$$

with the Navier boundary condition (3.2b), one has

$$\begin{aligned}\|b\|_{H^2} &\le C\|f\|_{L^2} + C\|b\|_{L^2}\\ &\le C\|b_t\|_{L^2} + C\|v\cdot\nabla b - b\cdot\nabla v\|_{L^2} + C\\ &\le C + C(\|v\|_{L^6}\|\nabla b\|_{L^3} + \|\nabla v\|_{L^2}\|b\|_{L^\infty})\\ &\le C + C\|\nabla b\|_{L^2}^{\frac12}\|b\|_{H^2}^{\frac12}\\ &\le \frac12\|b\|_{H^2} + C,\end{aligned}$$

which leads to

$$\|b\|_{H^2} \le C. \tag{3.14}$$

Similarly, we have

$$\begin{aligned}\|b\|_{H^3} &\le C\|f\|_{H^1} + C\|b\|_{L^2}\\ &\le C + C\|\nabla f\|_{L^2}\\ &\le C + C\|\nabla b_t\|_{L^2} + C\|\nabla v\|_{L^3}\|\nabla b\|_{L^6} + C\|b\|_{L^\infty}\|v\|_{H^2} + C\|v\|_{L^\infty}\|b\|_{H^2}\\ &\le CR + C\|\nabla b_t\|_{L^2} \le CR,\end{aligned}$$

thus

$$\|b\|_{L^2(0,T;H^3)} \le C \tag{3.15}$$

if $R^2T \le 1$. □

LEMMA 3.2. *Given $v \in \mathbf{X}$, then the problem* (3.3a) *and* (3.3b) *has a unique solution ρ satisfying*

$$\|\rho\|_{C([0,T];H^3)} + \|\rho_t\|_{C([0,T];H^1)} \le C, \tag{3.16}$$

$$\|\rho_t\|_{C([0,T];H^2)} \le CR \tag{3.17}$$

if T is small enough.

Proof. We only need to establish *a priori* estimates.

First, it follows from (3.3a) that

$$\inf \rho_0 \le \rho \le \sup \rho_0. \tag{3.18}$$

Taking ∇ to (3.3a) and testing the resultant by $|\nabla\rho|^{q-2}\nabla\rho$ $(2 < q < \infty)$, we deduce that

$$\frac{1}{q}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\rho\|_{L^q}^q \le \|\nabla v\|_{L^\infty}\|\nabla\rho\|_{L^q}^q,$$

which combined with Gronwall's inequality and the definition of $\mathbf{X}$ yields that

$$\begin{aligned}\|\nabla\rho\|_{L^q} &\le \|\nabla\rho_0\|_{L^q}\exp\left(\int_0^T \|\nabla v\|_{L^\infty}\mathrm{d}t\right)\\ &\le \|\nabla\rho_0\|_{L^q}\exp\left(C\int_0^T \|v\|_{H^3}\mathrm{d}t\right)\\ &\le \|\nabla\rho_0\|_{L^q}\exp(CR\sqrt{T}).\end{aligned}$$

Taking $q\to\infty$, one has

$$\|\nabla\rho\|_{L^\infty}\le C \tag{3.19}$$

if $R^2T\le 1$.

Applying D^α ($|\alpha|=3$) to (3.3a), testing the resultant by $D^\alpha\rho$, and summing up over α, we obtain from (2.1) that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\|D^3\rho\|_{L^2}^2 &\le C\|D^3(v\cdot\nabla\rho)-v\cdot\nabla D^3\rho\|_{L^2}\|D^3\rho\|_{L^2}\\ &\le C(\|v\|_{H^3}\|\nabla\rho\|_{L^\infty}+C\|\nabla v\|_{L^\infty}\|\rho\|_{H^3})\|D^3\rho\|_{L^2}\\ &\le (C\|v\|_{H^3}+C\|v\|_{H^3}\|D^3\rho\|_{L^2})\|D^3\rho\|_{L^2},\end{aligned}$$

which implies that

$$\|D^3\rho\|_{L^2}\le\left(\|D^3\rho_0\|_{L^2}+\int_0^T C\|v\|_{H^3}\mathrm{d}t\right)\exp\left(C\int_0^T\|v\|_{H^3}\mathrm{d}t\right)\le C \tag{3.20}$$

if $R^2T\le 1$.

Now it is obvious that

$$\|\nabla\rho_t\|_{L^2}\le\|\nabla v\|_{L^2}\|\nabla\rho\|_{L^\infty}+\|v\|_{L^3}\|\nabla^2\rho\|_{L^6}\le C, \tag{3.21}$$

and

$$\|\rho_t\|_{H^2}\le\|v\cdot\nabla\rho\|_{H^2}\le C\|v\|_{H^2}\|\rho\|_{H^3}\le CR. \tag{3.22}$$

LEMMA 3.3. *Given $v\in\mathbf{X}$, then the problem* (3.4a) *and* (3.4b) *has a unique solution w satisfying*

$$\|w\|_{L^\infty(0,T;H^2)}+\|w_t\|_{L^\infty(0,T;L^2)}\le C, \tag{3.23}$$

$$\|w_t\|_{L^\infty(0,T;H^1)}\le CR+C \tag{3.24}$$

if T is small enough.

Proof. We only need to prove the *a priori* estimates.

Testing (3.4a) by w and using (3.3a), we find that

$$\begin{aligned}\frac12\frac{\mathrm{d}}{\mathrm{d}t}\int\rho|w|^2\mathrm{d}x+4\zeta\int|w|^2\mathrm{d}x &= 2\zeta\int w\,\mathrm{rot}\,v\mathrm{d}x\le 2\zeta\|w\|_{L^2}\|\,\mathrm{rot}\,v\|_{L^2}\\ &\le CR\|w\|_{L^2}\le\zeta\|w\|_{L^2}^2+CR^2,\end{aligned}$$

which gives that

$$\|w\|_{L^2}\le C \tag{3.25}$$

if $R^2T\le 1$.

(3.4a) can be rewritten as

$$w_t + v \cdot \nabla w + \frac{4\zeta}{\rho} w = \frac{2\zeta}{\rho} \operatorname{rot} v. \tag{3.26}$$

Applying D^α to (3.26), testing the resultant by $D^\alpha w$, and summing over α ($|\alpha| = 2$), we get from (2.1) and (2.2) that

$$\begin{aligned}
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|D^2 w\|_{L^2}^2 &= -\sum_\alpha \int (D^\alpha(v\cdot\nabla w) - v\cdot\nabla D^\alpha w) D^\alpha w \mathrm{d}x \\
&\quad - \sum_\alpha \int D^\alpha\left(\frac{4\zeta}{\rho} w\right) D^\alpha w \mathrm{d}x + \sum_\alpha \int D^\alpha \left(\frac{2\zeta}{\rho}\operatorname{rot} v\right) D^\alpha w \mathrm{d}x \\
&\le C(\|\nabla v\|_{L^\infty}\|D^2 w\|_{L^2} + C\|\nabla w\|_{L^3}\|D^2 v\|_{L^6})\|D^2 w\|_{L^2} \\
&\quad + C\left\|D^2 \frac{w}{\rho}\right\|_{L^2} \|D^2 w\|_{L^2} + C\left\|D^2 \frac{\operatorname{rot} v}{\rho}\right\|_{L^2} \|D^2 w\|_{L^2} \\
&\le C\|v\|_{H^3}\|D^2 w\|_{L^2}^2 + C\|D^2 w\|_{L^2}^2 + C\|v\|_{H^3}\|D^2 w\|_{L^2},
\end{aligned}$$

which leads to

$$\|D^2 w\|_{L^2} \le C \tag{3.27}$$

if $T \le 1$ and $R^2 T \le 1$.

It is obvious that

$$\|w_t\|_{L^2} \le C\|\operatorname{rot} v\|_{L^2} + C\|w\|_{L^2} + C\|v\|_{L^6}\|\nabla w\|_{L^3} \le C \tag{3.28}$$

and

$$\begin{aligned}
\|\nabla w_t\|_{L^2} &\le C\|\nabla \operatorname{rot} v\|_{L^2} + C\|\nabla\rho\|_{L^\infty}\|\operatorname{rot} v\|_{L^2} + C\|\nabla w\|_{L^2} \\
&\quad + C\|\nabla\rho\|_{L^\infty}\|w\|_{L^2} + C\|v\|_{L^\infty}\|\nabla^2 w\|_{L^2} + C\|\nabla v\|_{L^3}\|\nabla w\|_{L^6} \\
&\le C\|v\|_{H^2} + C \le CR + C.
\end{aligned} \tag{3.29}$$

LEMMA 3.4. *Given $v \in \mathbf{X}$, then the problem* (3.5a)–(3.5d) *has a unique solution u satisfying*

$$\|u\|_{L^\infty(0,T;H^2)} + \|u\|_{L^2(0,T;H^3)} + \|u_t\|_{L^\infty(0,T;L^2)} + \|u_t\|_{L^2(0,T;H^1)} \le 10C_1 \tag{3.30}$$

if T is small enough.

Proof. By the very similar procedures as that in [**13**], one can prove the well-posedness, we omit the detail here. We only need to show the *a priori* estimates.

Applying ∂_t to (3.5a), testing by u_t, using (3.3a), Lemmas 3.1, 3.2 and 3.3, we reach

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int \rho|u_t|^2\mathrm{d}x + (\mu+\zeta)\int |\nabla u_t|^2\mathrm{d}x \\
&= -\int \rho_t|u_t|^2\mathrm{d}x - \int \rho_t v\cdot\nabla u\cdot u_t\mathrm{d}x - \int \rho v_t\cdot\nabla u\cdot u_t\mathrm{d}x \\
&\quad + \int 2\zeta w_t \operatorname{rot} u_t\mathrm{d}x - \int \partial_t(b\otimes b) : \nabla u_t\mathrm{d}x \\
&\le \|\rho_t\|_{L^6}\|u_t\|_{L^2}\|u_t\|_{L^3} + \|\rho_t\|_{L^6}\|v\|_{L^\infty}\|\nabla u\|_{L^2}\|u_t\|_{L^3} \\
&\quad + \|\rho\|_{L^\infty}\|v_t\|_{L^3}\|\nabla u\|_{L^2}\|u_t\|_{L^6} \\
&\quad + 2\zeta\|w_t\|_{L^2}\|\operatorname{rot} u_t\|_{L^2} + C\|b\|_{L^\infty}\|b_t\|_{L^2}\|\nabla u_t\|_{L^2} \\
&\le \frac{\mu+\zeta}{2}\|\nabla u_t\|_{L^2}^2 + C\|u_t\|_{L^2}^2 + CR^2 + CR\|\nabla v_t\|_{L^2} + C,
\end{aligned}
$$

which gives that

$$\|u_t\|_{L^\infty(0,T;L^2)} + \|u_t\|_{L^2(0,T;H^1)} \le C_1 \tag{3.31}$$

if $T\le 1, R^2T\le 1$ and $R^2\sqrt{T}\le 1$.

(3.5a) can be written as

$$\nabla\pi - (\mu+\zeta)\Delta u = g := 2\zeta\operatorname{rot} w + \operatorname{rot} b\times b - \rho\partial_t u - \rho v\cdot\nabla u. \tag{3.32}$$

Using the H^2-theory of Stokes system, we arrive at

$$
\begin{aligned}
\|u\|_{H^2} &\le C\|g\|_{L^2} \\
&\le C\|\operatorname{rot} w\|_{L^2} + C\|b\|_{L^\infty}\|\operatorname{rot} b\|_{L^2} + C\|u_t\|_{L^2} + C\|v\|_{L^6}\|\nabla u\|_{L^3} \\
&\le C + C\|\nabla u\|_{L^3} \le C + C\|\nabla u\|_{L^2}^{\frac{1}{2}}\|u\|_{H^2}^{\frac{1}{2}} \\
&\le C + \frac{1}{2}\|u\|_{H^2},
\end{aligned}
$$

which implies that

$$\|u\|_{H^2} \le C_2. \tag{3.33}$$

Similarly, one has

$$
\begin{aligned}
\|u\|_{H^3} &\le C\|g\|_{H^1} \\
&\le C\|g\|_{L^2} + C\|\nabla g\|_{L^2} \\
&\le C + C\|w\|_{H^2} + C\|b\|_{L^\infty}\|b\|_{H^2} + C\|u_t\|_{H^1} + C\|\nabla\rho\|_{L^\infty}\|u_t\|_{L^2} \\
&\quad + C\|\nabla\rho\|_{L^\infty}\|v\|_{L^6}\|\nabla u\|_{L^3} + C\|\nabla v\|_{L^2}\|\nabla u\|_{L^\infty} + C\|v\|_{L^6}\|\Delta u\|_{L^3} \\
&\le C + C\|u_t\|_{H^1} + C\|\nabla u\|_{L^\infty} + C\|\Delta u\|_{L^3} \\
&\le C + C\|u_t\|_{H^1} + C\|\nabla^2 u\|_{L^2}^{\frac{1}{2}}\|u\|_{H^3}^{\frac{1}{2}} \\
&\le \frac{1}{2}\|u\|_{H^3} + C + C\|u_t\|_{H^1},
\end{aligned}
$$

which yields

$$\|u\|_{H^3} \le C + C\|u_t\|_{H^1},$$

whence

$$\|u\|_{L^2(0,T;H^3)} \le C_3. \tag{3.34}$$

We can take $R := 10C_3$. This proves the following result.

LEMMA 3.5. *If T is small enough, then $\mathcal{F}$ maps $v \in \mathbf{X}$ into $u \in \mathbf{X}$.*

Now we want to show that $\mathcal{F}$ is a contraction mapping in low norm if T is small enough. To prove it, we let $v_1, v_2 \in \mathbf{X}$ and denote the corresponding solutions by $(\rho_1, u_1, \pi_1, w_1, b_1)$ and $(\rho_2, u_2, \pi_2, w_2, b_2)$.

It is clear that

$$\partial_t(b_1 - b_2) + \operatorname{rot}(b_1 \times v_1 - b_2 \times v_2) = \Delta(b_1 - b_2), \tag{3.35}$$

$$\partial_t(\rho_1 - \rho_2) + v_1 \cdot \nabla(\rho_1 - \rho_2) + (v_1 - v_2)\nabla\rho_2 = 0, \tag{3.36}$$

$$\begin{aligned} &\rho_1\partial_t(w_1 - w_2) + \rho_1 v_1 \cdot \nabla(w_1 - w_2) + 4\zeta(w_1 - w_2) \\ &= 2\zeta \operatorname{rot}(v_1 - v_2) - (\rho_1 - \rho_2)\partial_t w_2 - (\rho_1 v_1 - \rho_2 v_2) \cdot \nabla w_2, \end{aligned} \tag{3.37}$$

$$\begin{aligned} &\rho_1\partial_t(u_1 - u_2) + \rho_1 v_1 \cdot \nabla(u_1 - u_2) + \nabla\Big(\pi_1 - \pi_2 + \frac{1}{2}|b_1|^2 - \frac{1}{2}|b_2|^2\Big) \\ &\quad - (\mu + \zeta)\Delta(u_1 - u_2) \\ &= 2\zeta \operatorname{rot}(w_1 - w_2) + b_1 \cdot \nabla b_1 - b_2 \cdot \nabla b_2 - (\rho_1 - \rho_2)\partial_t u_2 \\ &\quad - (\rho_1 v_1 - \rho_2 v_2) \cdot \nabla u_2. \end{aligned} \tag{3.38}$$

Testing (3.35) by $b_1 - b_2$, we observe that

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int |b_1 - b_2|^2 \mathrm{d}x + \int |\operatorname{rot}(b_1 - b_2)|^2 \mathrm{d}x \\ &= \int (v_1 \times b_1 - v_2 \times b_2) \operatorname{rot}(b_1 - b_2)\mathrm{d}x \\ &= \int (v_1 \times (b_1 - b_2) + (v_1 - v_2) \times b_2) \operatorname{rot}(b_1 - b_2)\mathrm{d}x \\ &\le (\|v_1\|_{L^\infty}\|b_1 - b_2\|_{L^2} + \|v_1 - v_2\|_{L^2}\|b_2\|_{L^\infty})\|\operatorname{rot}(b_1 - b_2)\|_{L^2} \\ &\le \frac{1}{8}\|\operatorname{rot}(b_1 - b_2)\|_{L^2}^2 + C\|b_1 - b_2\|_{L^2}^2 + C\|v_1 - v_2\|_{L^2}^2. \end{aligned} \tag{3.39}$$

Testing (3.36) by $|\rho_1 - \rho_2|(\rho_1 - \rho_2)$, we have

$$\frac{1}{3}\frac{\mathrm{d}}{\mathrm{d}t}\int |\rho_1 - \rho_2|^3 \mathrm{d}x \le \|\nabla\rho_2\|_{L^\infty}\|v_1 - v_2\|_{L^3}\|\rho_1 - \rho_2\|_{L^3}^2,$$

whence

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\rho_1 - \rho_2\|_{L^3}^2 \le C\|v_1 - v_2\|_{L^3}^2 + C\|\rho_1 - \rho_2\|_{L^3}^2. \tag{3.40}$$

Testing (3.37) by $w_1 - w_2$ and using $\partial_t\rho_1 + \operatorname{div}(\rho_1 v_1) = 0$, we have

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int \rho_1|w_1 - w_2|^2 \mathrm{d}x + 4\zeta\int |w_1 - w_2|^2 \mathrm{d}x \\ &\le C\|\operatorname{rot}(v_1 - v_2)\|_{L^2}\|w_1 - w_2\|_{L^2} + \|\rho_1 - \rho_2\|_{L^3}\|\partial_t w_2\|_{L^6}\|w_1 - w_2\|_{L^2} \\ &\quad + \|\rho_1\|_{L^\infty}\|v_1 - v_2\|_{L^3}\|\nabla w_2\|_{L^6}\|w_1 - w_2\|_{L^2} \\ &\quad + \|\rho_1 - \rho_2\|_{L^3}\|v_2\|_{L^\infty}\|\nabla w_2\|_{L^6}\|w_1 - w_2\|_{L^2} \\ &\le \frac{\mu + \zeta}{10000}\|\nabla(v_1 - v_2)\|_{L^2}^2 + C\|\rho_1 - \rho_2\|_{L^3}^2 \\ &\quad + C\|v_1 - v_2\|_{L^2}^2 + C\|w_1 - w_2\|_{L^2}^2. \end{aligned} \tag{3.41}$$

Testing (3.38) by $u_1 - u_2$ and using $\partial_t \rho_1 + \operatorname{div}(\rho_1 v_1) = 0$, we compute

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int \rho_1 |u_1 - u_2|^2 \mathrm{d}x + (\mu+\zeta)\int |\nabla(u_1-u_2)|^2 \mathrm{d}x \\
&\le 2\zeta \int (w_1 - w_2)\operatorname{rot}(u_1-u_2)\mathrm{d}x \\
&\quad + C(\|b_1\|_{L^\infty} + \|b_2\|_{L^\infty})\|b_1 - b_2\|_{L^2}\|\nabla(u_1-u_2)\|_{L^2} \\
&\quad + \|\rho_1 - \rho_2\|_{L^3}\|\partial_t u_2\|_{L^2}\|u_1-u_2\|_{L^6} \\
&\quad + \|\rho_1\|_{L^\infty}\|v_1 - v_2\|_{L^3}\|\nabla u_2\|_{L^6}\|u_1-u_2\|_{L^2} \\
&\quad + \|\rho_1 - \rho_2\|_{L^3}\|v_2\|_{L^\infty}\|\nabla u_2\|_{L^6}\|u_1-u_2\|_{L^2} \\
&\le \frac{\mu+\zeta}{2}\|\nabla(u_1-u_2)\|_{L^2}^2 + C\|w_1-w_2\|_{L^2}^2 + C\|b_1-b_2\|_{L^2}^2 + C\|u_1-u_2\|_{L^2}^2 \\
&\quad + C\|\rho_1-\rho_2\|_{L^3}^2 + C\|v_1-v_2\|_{L^2}^2 + \frac{\mu+\zeta}{10000}\|\nabla(v_1-v_2)\|_{L^2}^2.
\end{aligned}$$

This along with (3.39), (3.40), and (3.41) gives that

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}(\|b_1-b_2\|_{L^2}^2 + \|\rho_1-\rho_2\|_{L^3}^2 + \|w_1-w_2\|_{L^2}^2 + \|u_1-u_2\|_{L^2}^2) \\
&\quad + (\mu+\zeta)\|\nabla(u_1-u_2)\|_{L^2}^2 \\
&\le C\|b_1-b_2\|_{L^2}^2 + C\|\rho_1-\rho_2\|_{L^3}^2 + C\|v_1-v_2\|_{L^2}^2 + C\|w_1-w_2\|_{L^2}^2 \\
&\quad + C\|u_1-u_2\|_{L^2}^2 + \frac{\mu+\zeta}{1000}\|\nabla(v_1-v_2)\|_{L^2}^2,
\end{aligned}$$

which implies that

$$\begin{aligned}
&\|u_1-u_2\|_{L^\infty(0,T;L^2)}^2 + \int_0^T \|\nabla(u_1-u_2)\|_{L^2}^2 \mathrm{d}t \\
&\le CT\|v_1-v_2\|_{L^\infty(0,T;L^2)}^2 + \frac{1}{1000}\int_0^T \|\nabla(v_1-v_2)\|_{L^2}^2\mathrm{d}t \\
&\le \frac{1}{2}\left(\|v_1-v_2\|_{L^\infty(0,T;L^2)}^2 + \int_0^T \|\nabla(v_1-v_2)\|_{L^2}^2\mathrm{d}t\right),
\end{aligned} \tag{3.42}$$

provided that T is suitably small. This completes the proof of local well-posedness. □

3.2. Regularity criterion. We only need to show *a priori* estimates.

First, we still have (3.18) and (3.19).

Testing (1.2), (1.3) and (1.4) by u, w and b, respectively, summing up the results, we get that

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int (\rho|u|^2 + \rho|w|^2 + |b|^2)\mathrm{d}x + (\mu+\zeta)\int |\nabla u|^2\mathrm{d}x + \int |\operatorname{rot} b|^2 \mathrm{d}x + 4\zeta\int |w|^2\mathrm{d}x \\
&= 4\zeta\int w\operatorname{rot} u\mathrm{d}x \le 4\zeta\int |w|^2\mathrm{d}x + \zeta\int |\operatorname{rot} u|^2\mathrm{d}x,
\end{aligned}$$

whence

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int (\rho|u|^2 + \rho|w|^2 + |b|^2)\mathrm{d}x + \mu\int |\nabla u|^2\mathrm{d}x + \int |\operatorname{rot} b|^2\mathrm{d}x \le 0. \tag{3.43}$$

Testing (1.4) by $|b|^4 b$, we derive from (1.5), (2.3), (2.4), and (2.7) that

$$\begin{aligned}
&\frac{1}{6}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |b|^6\mathrm{d}x + \frac{1}{2}\int_\Omega |b|^4|\nabla b|^2\mathrm{d}x + \frac{4}{9}\int_\Omega |\nabla |b|^3|^2\mathrm{d}x \\
&= -\int_{\partial\Omega} |b|^4 b\cdot\nabla n\cdot b\mathrm{d}S + \int_\Omega (b\cdot\nabla)u\cdot |b|^4 b\mathrm{d}x \\
&\le \|\nabla n\|_{L^\infty}\int_{\partial\Omega}|b|^6\mathrm{d}S + \|\nabla u\|_{L^\infty}\int_\Omega |b|^6\mathrm{d}x,
\end{aligned}$$

which implies, by denoting $\phi := |b|^3$, that

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \phi^2\mathrm{d}x + \int_\Omega |\nabla\phi|^2\mathrm{d}x &\le C\int_{\partial\Omega}\phi^2\mathrm{d}S + C\|\nabla u\|_{L^\infty}\int_\Omega \phi^2\mathrm{d}x \\
&\le C\|\phi\|_{L^2(\Omega)}\|\phi\|_{H^1(\Omega)} + C\|\nabla u\|_{L^\infty}\int_\Omega \phi^2\mathrm{d}x \\
&\le \frac{1}{2}\|\nabla\phi\|^2_{L^2(\Omega)} + C\|\phi\|^2_{L^2(\Omega)} + C\|\nabla u\|_{L^\infty}\int_\Omega \phi^2\mathrm{d}x,
\end{aligned}$$

from which and Gronwall's inequality gives that

$$\|b\|_{L^6} \le C.$$

Testing (1.4) by $-\Delta b$, we write

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int |\operatorname{rot} b|^2\mathrm{d}x + \int |\Delta b|^2\mathrm{d}x \\
&= \int (u\cdot\nabla b - b\cdot\nabla u)\Delta b\mathrm{d}x \\
&\le (\|u\|_{L^\infty}\|\nabla b\|_{L^2} + \|b\|_{L^\infty}\|\nabla u\|_{L^2})\|\Delta b\|_{L^2} \\
&\le C(\|u\|_{L^\infty}\|\nabla b\|_{L^2} + \|b\|^{\frac{1}{2}}_{L^6}\|b\|^{\frac{1}{2}}_{H^2}\|\nabla u\|_{L^2})\|\Delta b\|_{L^2} \\
&\le \frac{1}{4}\|\Delta b\|^2_{L^2} + C\|u\|^2_{L^\infty}\|\nabla b\|^2_{L^2} + C\|\nabla u\|^4_{L^2}. \qquad (3.44)
\end{aligned}$$

Equation (1.3) can be written as

$$w_t + u\cdot\nabla w + \frac{4\zeta}{\rho}w = \frac{2\zeta}{\rho}\operatorname{rot} u. \qquad (3.45)$$

Applying ∇ to (3.45) and testing the resultant by ∇w, we get from (1.5) that

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int |\nabla w|^2\mathrm{d}x + \int \frac{4\zeta}{\rho}|\nabla w|^2\mathrm{d}x \\
&\le \|\nabla u\|_{L^\infty}\int |\nabla w|^2\mathrm{d}x + C\|\nabla\rho\|_{L^\infty}\|w\|_{L^2}\|\nabla w\|_{L^2} \\
&\quad + C\|\nabla \operatorname{rot} u\|_{L^2}\|\nabla w\|_{L^2} + C\|\nabla\rho\|_{L^\infty}\|\nabla u\|_{L^2}\|\nabla w\|_{L^2} \\
&\le \frac{(\mu+\zeta)^2}{4C}\|\nabla^2 u\|^2_{L^2} + C\|\nabla u\|_{L^\infty}\int |\nabla w|^2\mathrm{d}x + C\|\nabla w\|^2_{L^2} \\
&\quad + C\|\nabla u\|^2_{L^2} + C. \qquad (3.46)
\end{aligned}$$

Equation (1.2) can be written as

$$u_t + u\cdot\nabla u + \frac{1}{\rho}(\nabla\pi - (\mu+\zeta)\Delta u) = \frac{2\zeta}{\rho}\operatorname{rot} w + \frac{\operatorname{rot} b}{\rho}\times b. \qquad (3.47)$$

Testing (3.47) by $\nabla\pi-(\mu+\zeta)\Delta u$ and using (1.5), we have

$$\begin{aligned}&\frac{\mu+\zeta}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int|\nabla u|^2\mathrm{d}x+\int\frac{1}{\rho}|\nabla\pi-(\mu+\zeta)\Delta u|^2\mathrm{d}x\\&\le(\|u\|_{L^\infty}\|\nabla u\|_{L^2}+C\|\nabla w\|_{L^2}+C\|\nabla b\|_{L^3}\|b\|_{L^6})\|\nabla\pi-(\mu+\zeta)\Delta u\|_{L^2}\\&\le\frac{1}{2}\int\frac{1}{\rho}|\nabla\pi-(\mu+\zeta)\Delta u|^2\mathrm{d}x+C\|u\|_{L^\infty}^2\|\nabla u\|_{L^2}^2+C\|\nabla w\|_{L^2}^2\\&\quad+\frac{1}{4}\|\nabla^2 b\|_{L^2}^2+C\|\nabla b\|_{L^2}^2,\end{aligned}$$

which combined with (3.44), (3.46), and Gronwall's inequality leads to

$$\|(u,w,b)\|_{L^\infty(0,T;H^1)}+\|(u,b)\|_{L^2(0,T;H^2)}\le C.$$

It is clear that

$$\begin{aligned}\|w_t\|_{L^2}&=\left\|\frac{2\zeta}{\rho}\operatorname{rot}u-\frac{4\zeta}{\rho}w-u\cdot\nabla w\right\|_{L^2}\\&\le C\|\operatorname{rot}u\|_{L^2}+C\|w\|_{L^2}+\|u\|_{L^\infty}\|\nabla w\|_{L^2}\\&\le C+C\|u\|_{L^\infty},\end{aligned}$$

thus

$$\|w_t\|_{L^\infty(0,T;L^2)}\le C.$$

Applying ∂_t to (1.2) and testing the resultant by $\partial_t u$, we infer from (1.5), (1.1), (3.18), and (3.19) that

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int\rho|\partial_t u|^2\mathrm{d}x+(\mu+\zeta)\int|\nabla\partial_t u|^2\mathrm{d}x\\&=-\int\partial_t\rho|\partial_t u|^2\mathrm{d}x-\int\partial_t\rho u\cdot\nabla u\cdot\partial_t u\mathrm{d}x-\int\rho\partial_t u\cdot\nabla u\cdot\partial_t u\mathrm{d}x\\&\quad+2\zeta\int\partial_t w\cdot\operatorname{rot}\partial_t u\mathrm{d}x-\int\partial_t(b\otimes b):\nabla\partial_t u\mathrm{d}x\\&\le\|u\|_{L^\infty}\|\nabla\rho\|_{L^\infty}\int|\partial_t u|^2\mathrm{d}x+\|\nabla\rho\|_{L^\infty}\|u\|_{L^\infty}^2\|\nabla u\|_{L^2}\|\partial_t u\|_{L^2}\\&\quad+\|\rho\|_{L^\infty}\|\nabla u\|_{L^\infty}\int|\partial_t u|^2\mathrm{d}x+2\zeta\|\partial_t w\|_{L^2}\|\operatorname{rot}\partial_t u\|_{L^2}\\&\quad+C\|b\|_{L^\infty}\|\partial_t b\|_{L^2}\|\nabla\partial_t u\|_{L^2}\\&\le\frac{\mu+\zeta}{4}\|\nabla\partial_t u\|_{L^2}^2+C\|u\|_{L^\infty}\int|\partial_t u|^2\mathrm{d}x+C\|u\|_{L^\infty}^2\|\partial_t u\|_{L^2}\\&\quad+C\|\nabla u\|_{L^\infty}\int|\partial_t u|^2\mathrm{d}x+C\|\partial_t w\|_{L^2}^2+C\|b\|_{L^\infty}^2\|\partial_t b\|_{L^2}^2.\end{aligned}\tag{3.48}$$

Applying ∂_t to (1.4) and testing the resultant by b_t, we have

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int|b_t|^2\mathrm{d}x+\int|\operatorname{rot}b_t|^2\mathrm{d}x&=\int(u_t\times b+u\times b_t)\operatorname{rot}b_t\mathrm{d}x\\&\le(\|b\|_{L^\infty}\|u_t\|_{L^2}+\|u\|_{L^\infty}\|b_t\|_{L^2})\|\operatorname{rot}b_t\|_{L^2}\\&\le\frac{1}{4}\|\operatorname{rot}b_t\|_{L^2}^2+C\|b\|_{L^\infty}^2\|u_t\|_{L^2}^2+C\|u\|_{L^\infty}^2\|b_t\|_{L^2}^2,\end{aligned}$$

which together with (3.48) and Gronwall's inequality yields that

$$\|(u_t,b_t)\|_{L^\infty(0,T;L^2)}+\|(u_t,b_t)\|_{L^2(0,T;H^1)}\le C.$$

We still have (3.14), (3.33), and (3.15).

Similarly to (3.34), we note

$$\|u\|_{H^3} \le C + C\|w\|_{H^2} + C\|u_t\|_{H^1}.$$

Applying D^α ($|\alpha| = 2$) to (1.1), testing the resultant by $D^\alpha\rho$, and summing over α, we deduce from (1.5) and (2.1) that

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|D^2\rho\|_{L^2}^2 &= -\sum_\alpha \int (D^\alpha(u\cdot\nabla\rho) - u\cdot\nabla D^\alpha\rho)D^\alpha\rho\mathrm{d}x \\ &\le C\|\nabla u\|_{L^\infty}\|D^2\rho\|_{L^2}^2 + C\|\nabla\rho\|_{L^\infty}\|D^2u\|_{L^2}\|D^2\rho\|_{L^2},\end{aligned}$$

which leads to

$$\|\rho\|_{L^\infty(0,T;H^2)} \le C.$$

Similarly to (3.33), we arrive at

$$\|u\|_{W^{2,4}} \le C\|\nabla w\|_{L^4} + C\|u_t\|_{L^4} + C.$$

Similarly to (3.46), we have

$$\begin{aligned}&\frac{1}{4}\frac{\mathrm{d}}{\mathrm{d}t}\int|\nabla w|^4\mathrm{d}x + \int\frac{4\zeta}{\rho}|\nabla w|^4\mathrm{d}x \\ &\le \|\nabla u\|_{L^\infty}\int|\nabla w|^4\mathrm{d}x + C\|\nabla\rho\|_{L^\infty}\|w\|_{L^4}\|\nabla w\|_{L^4}^3 \\ &\quad + C\|\nabla^2u\|_{L^4}\|\nabla w\|_{L^4}^3 + C\|\nabla\rho\|_{L^\infty}\|\nabla u\|_{L^4}\|\nabla w\|_{L^4}^3,\end{aligned}$$

which implies that

$$\|\nabla w\|_{L^4} \le C.$$

Applying D^α ($|\alpha| = 2$) to (3.45), testing the resultant by $D^\alpha w$, and summing over α, we derive from (1.5), (2.1), and (2.2) that

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int|D^2w|^2\mathrm{d}x &= -\sum_\alpha\int(D^\alpha(u\cdot\nabla w) - u\cdot\nabla D^\alpha w)D^\alpha w\mathrm{d}x \\ &\quad + \sum_\alpha\int D^\alpha\left(\frac{2\zeta}{\rho}u\right)D^\alpha w\mathrm{d}x - \sum_\alpha\int D^\alpha\left(\frac{4\zeta}{\rho}w\right)D^\alpha w\mathrm{d}x \\ &\le C\|\nabla u\|_{L^\infty}\|D^2w\|_{L^2}^2 + C\|\nabla w\|_{L^3}\|D^2u\|_{L^6}\|D^2w\|_{L^2} \\ &\quad + C\|u\|_{H^3}\|w\|_{H^2} + C\|\nabla u\|_{L^\infty}\|\rho\|_{H^2}\|w\|_{H^2} + C\|w\|_{H^2}^2 \\ &\quad + C\|\rho\|_{H^2}\|w\|_{L^\infty}\|w\|_{H^2} \\ &\le C\|\nabla u\|_{L^\infty}\|w\|_{H^2}^2 + C\|u\|_{H^3}\|w\|_{H^2} + C\|\nabla u\|_{L^\infty}\|w\|_{H^2} \\ &\quad + C\|w\|_{H^2}^2 + C,\end{aligned}$$

which gives that

$$\|w\|_{L^\infty(0,T;H^2)} \le C.$$

We note that

$$\|u\|_{L^2(0,T;H^3)} \le C.$$

Now it is easy to deduce that

$$\|\rho\|_{L^\infty(0,T;H^3)} \le C.$$

This completes the proof. □

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