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Convergence to steady states of parabolic sine-Gordon

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Abstract. Based on the recent surprising work on the symmetry breaking phenomenon of the Allen-Cahn equation [**11, 12**], we consider the one-dimensional parabolic sine-Gordon equation with periodic boundary conditions. Particularly, we derive a strong dependence of the non-trivial steady states on the diffusion coefficient κ and provide some description on them for $0 < \kappa < 1$. To further investigate the property of energy associated to the steady states, we give a complete classification and prove the monotonicity of the ground state energy with respect to the diffusion constant κ. Finally, we identify the exact decay rate of the solution to the parabolic equation together with the explicit leading term for $\kappa \geq 1$.

Contents

1. Introduction

In this paper, we consider the following one-dimensional parabolic sine-Gordon equation on the domain $\mathbb{T} = [-\pi, \pi]$

$$(1.1) \qquad \begin{cases} \partial_t u = \kappa^2 \partial_{xx} u - f(u), \quad (t,x) \in (0,\infty) \times \mathbb{T}, \\ u|_{t=0} = u_0, \end{cases}$$

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where κ is the diffusion constant, $f(u) = -\sin u = F'(u)$, and $F(u) = \cos u + 1$ is the usual double-well potential. The unknown function $u : \mathbb{T} \to \mathbb{R}$ represents the concentration difference in the phase field context and typically takes the value in $[-\pi, \pi]$.

The sine-Gordon model was first discovered in the 1860s when studying surfaces with constant negative curvature [**2**]. Since then, it has been widely investigated in both physics and geometry. Due to its complete integrability, the sine-Gordon model occupies a particular position in other well-known nonlinear scalar field theories. One can see [**4, 15, 16**] and the references therein for the background in details.

The classical one-dimensional sine-Gordon equation

$$\frac{\partial^2 \phi}{\partial x^2} - \frac{\partial^2 \phi}{\partial t^2} = \sin \phi \tag{1.2}$$

is used to approximate the differential equation, which describes the propagation of a slip in an infinite chain of elastically bound atoms lying over a fixed lower chain of similar atoms (cf. [**6**]). Concerning equation (1.2), G.L. Lamb [**9**] proved that it admits a solution of the following form

$$\phi = 4 \arctan\left(X(x)/T(t)\right).$$

Based on the expression, we can easily find out that $X(x)$ and $T(t)$ must satisfy

$$\begin{aligned} (X'(x))^2 &= kX^4(x) + mX^2(x) + n, \\ (T'(t))^2 &= -kT^4(t) + (m-1)T^2(t) - n, \end{aligned}$$

where k, m, n are arbitrary constants. For a special case, $k = 0$ and $n = m$, we can get

$$X(x) = \sinh \sqrt{m} x, \quad T(t) = \frac{m}{m-1} \cosh \sqrt{(m-1)} t,$$

and

$$\phi = 4 \arctan \left(\frac{u^2 \sinh \frac{x}{\sqrt{1-u^2}}}{\cosh \frac{ut}{\sqrt{1-u^2}}} \right),$$

where

$$u = \sqrt{1 - 1/m}.$$

This solution has been already discovered, which was used to describe a collision between two kinks in a centre-of-mass coordinate system, see [**14**] and the reference therein.

It is natural to extend (1.2) in high spatial dimensions. Particularly, in the two spatial dimension, the equation reads as

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} - \frac{\partial^2 \phi}{\partial t^2} = \sin \phi, \tag{1.3}$$

which is invariant under the Lorentz transformation. Barone-Esposito-Magee [**1**] concluded that particle-like solutions of (1.3) occur only if the boundary conditions at the origin are able to provide the gradient of

$$\nabla \phi \approx -\alpha \frac{\bar{u}_r}{r},$$

where $\bar{u}_r$ is a unit vector in a radial direction. We refer the readers to [**4, 5, 10, 13**] for more background.

Our parabolic sine-Gordon equation (1.1) can be viewed as the parabolic version of the usual wave-type sine-Gordon equation in the phase-field context. Recently, in the work of Li-Quan-Tang-Yang [**11**], they observe the symmetry breaking phenomenon in the typical high-precision computation of the Allen-Cahn equation with very smooth initial data. It happens due to the manifestation of the gradual accumulation of non-negligible machine round off errors over time. In fact, this phenomenon also happens for the other equations with double well potential. In order to solve this issue, they proposed a new Fourier filter method which works successfully for a class of initial data with certain symmetry and band-gap properties. Furthermore, they obtained several interesting results concerning the one-dimensional Allen-Cahn equation with periodic boundary conditions, including the complete classification of steady states, the monotonicity property of the associated energy to the diffusion constant and the asymptotic behavior of the profiles in [**12**]. In the current paper, we would like to consider the same problem for the sine-Gordon equation, where the nonlinearity term behaves similarly as the Allen-Cahn equation. Concerning the steady state of the sine-Gordon equation, Cheng-Li-Quan-Yang [**3**] established the maximum principle on the torus for all dimensions and classified all bounded steady states in the one-dimensional for (1.1), this is a starting point of our work in this paper.

To get a better understanding of the steady state, we shall consider the following elliptic equation on the real line, i.e.

$$\kappa^2 u'' + \sin u = 0 \quad \text{in } \mathbb{R}. \tag{1.4}$$

The first result of this paper is:

THEOREM 1.1. *Let $0 < \kappa < 1$ and m_κ be the largest positive integer such that $m_\kappa < 1/\kappa$, then the equation (1.4) admits exactly m_κ non-constant 2π periodic solutions up to some translation and odd reflection.*

In [**3**], Cheng-Li-Quan-Yang showed that any periodic solution of (1.4) with $|u(0)| \le \pi$ is bounded, i.e. $|u| \le \pi$. Except for the trivial solutions $u = \pm\pi$, all the other nontrivial solutions satisfy $|u| < \pi$ and our interests are focused on these solutions.

For equation (1.4), we introduce the associated energy

$$E(u) = \int_{\mathbb{R}} \left(\frac{\kappa^2}{2} |u'|^2 + \cos u + 1 \right) dx. \tag{1.5}$$

Based on the (1.5), we further define the energy of the periodic functions

$$E_\kappa(u) = \int_{\mathbb{T}} \left(\frac{\kappa^2}{2} |u'|^2 + \cos u + 1 \right) dx. \tag{1.6}$$

Let

$$E_\kappa = \inf_{u \in \mathcal{F}} E_\kappa(u), \tag{1.7}$$

where

$$\mathcal{F} = \left\{ \phi \in H^1(\mathbb{T}) \middle| \phi(x) = \phi(x + 2\pi) \text{ and } |\phi| < \pi, x \in \mathbb{R} \right\}.$$

Usually, the 2π-periodic solution u_κ of (1.4) is called *ground state* if u_κ is odd and achieves E_κ. For a fixed $\kappa \in (0, 1)$, it can be obtained from [**3**, Proposition 3.2] that the ground state solution is unique after translation and reflection if necessary. For convenience, we introduce the notion of *odd zero-up solution*, which is odd and the first derivative at zero is positive. Based on it, we also introduce the notion of *odd zero-up ground states* U_κ, if a ground state is odd zero-up solution (see Definition 3.1). For any $0 < \kappa < 1$, defining m_κ to be the unique integer such that

$$\frac{1}{m_\kappa + 1} \le \kappa < \frac{1}{m_\kappa},$$

and each $j = 1, \cdots, m_\kappa$,

$$\tilde{u}_{\kappa,j}(x) = U_{j\kappa}(jx), \tag{1.8}$$

then $\{\tilde{u}_{\kappa,j}\}_{j=1}^{m_\kappa}$ are all the possible odd zero-up solutions to (2.1). In fact, the energy of $\tilde{u}_{\kappa,j}(x)$ satisfies

$$E_\kappa(\tilde{u}_{\kappa,j}) = \int_{\mathbb{T}} \left(\frac{\kappa^2}{2} (\partial_x \tilde{u}_{\kappa,j})^2 + \cos \tilde{u}_{\kappa,j} + 1 \right) dx = E_{j\kappa}(U_{j\kappa}).$$

Now we state the following theorem on the energy functional $E_\kappa(u)$ of the 2π-periodic solutions.

THEOREM 1.2. *Let E_κ be defined in* (1.7). *Then it can be achieved for any $\kappa > 0$. Furthermore, the following hold:*

(a) *$E_\kappa = 4\pi$ for $\kappa > 1$ and it is only achieved by trivial solution $u = 0$.*
(b) *E_κ is achieved by U_κ whenever $\kappa \in (0, 1)$.*
(c) *If $0 < \kappa_1 < \kappa_2 < 1$, then there is strict monotonicity property $E_{\kappa_1} < E_{\kappa_2}$.*
(d) *The odd zero-up ground state U_κ satisfies*

$$\left| U_\kappa(x) - 2 \arcsin \tanh \left(\frac{x}{\kappa} \right) \right| \le C e^{-\frac{d}{\kappa}},$$

where $C > 0, d > 0$, and

$$\lim_{\kappa \to 0} \frac{E_\kappa}{\kappa} = 16 > 0.$$

(e) *For $0 < \kappa < 1$, the 2π-periodic solutions of* (1.4) *have the following property: any 2π-periodic solution of* (1.4) *which is not identically $\pm\pi$ or 0 must coincide with $\tilde{u}_{\kappa,j}$, for an integral $j < 1/\kappa$, where $\tilde{u}_{\kappa,j}$ is defined in* (1.8). *Furthermore $E_\kappa(u) = E_{m\kappa}(U_{m\kappa})$.*

In the second part of this paper, we consider the parabolic equation (1.1), proving the convergence results of the solution and identifying the explicit profiles. Precisely:

THEOREM 1.3. *Let $0 < \kappa < 1$. Assume the initial data $u_0 : \mathbb{R} \to \mathbb{R}$ is 2π-periodic, odd, $|u_0| \le \pi$ and non-negative in $[0, \pi]$. Suppose u is the solution to* (1.1) *corresponding to the initial data u_0. Then we have $u(x, t) \to U_\kappa$ or 0 as $t \to \infty$. Moreover, if $u_0 \ne 0$ and $E_\kappa(u_0) \le 4\pi$, then $u(x, t) \to U_\kappa$ as $t \to \infty$ and the rate of convergence is exponential in time.*

THEOREM 1.4. *(Vanishing as $t \to \infty$). Let $\kappa \ge 1$. Assume u_0 is 2π periodic, odd and $|u_0| \le \pi$ and non-negative in $[0, \pi]$. Suppose u is the solution to* (1.1) *corresponding to the initial data u_0. If $\kappa > 1$, we have*

$$u(x, t) = e^{-(\kappa^2 - 1)t} \alpha_* \sin x + r(t), \quad \forall\ t \ge 1, \tag{1.9}$$

where the constant α_* depends on (u_0, κ), and $\|r(t)\|_{H^2(\mathbb{T})} = o(e^{-(\kappa^2-1)t})$ as $t \to +\infty$.

For $\kappa = 1$, we have

$$u(x,t) = t^{-\frac{1}{2}}\beta_* \sin x + r_1(t), \tag{1.10}$$

where the constant β_* depends on u_0, and $\|r_1(t)\|_{H^2(\mathbb{T})} = o(t^{-\frac{1}{2}})$ as $t \to \infty$.

The rest of this paper is organized as follows. In Section 2, we provide some description of steady state for $0 < \kappa < 1$. The Section 3 is devoted to the proof of Theorem 1.1 and 1.2. In addition, we give a complete classification of the steady state and rigorously prove the energy monotonicity. In Section 4, we analyze the convergence of the sine-Gordon equation and prove Theorems 1.3 and 1.4. Some concluding remarks are given in the last section.

2. Property of steady state

We call $u_{\kappa,\infty}$ the *steady state* if it satisfies

$$\kappa^2 u_\kappa'' + \sin u_\kappa = 0, \quad \text{on } \mathbb{T}. \tag{2.1}$$

It is worth noting that the solutions to (1.1) are rigid [**3**, Proposition 3.1]. Noted that for $u_0(x) = \sin x$, we have

$$u_\kappa(x,t) = \sum_{m\geq 1:\ m \text{ is odd}} c_m(t) \sin mx.$$

In particular, it follows that the corresponding steady state $u_{\kappa,\infty}$ is odd. If 2π is the minimal period, we have $u_{\kappa,\infty}(0) = u'_{\kappa,\infty}(\frac{\pi}{2}) = 0$. We want to find a steady state such that it is monotonically increasing on $[0, \frac{\pi}{2}]$ with

$$u_{\kappa,\infty}(0) = u'_{\kappa,\infty}\left(\frac{\pi}{2}\right) = 0. \tag{2.2}$$

For simplicity, we denote $u_{\kappa,\infty}$ as u_κ to represent desired steady state. We let $u_\kappa(\frac{\pi}{2}) = N_\kappa < \pi$ when $\kappa \in (0,1)$, and observed that we should have $N_\kappa \to \pi$ as $\kappa \to 0$. Multiplying (2.1) by u_κ' and using (2.2), we obtain

$$(u_\kappa')^2 = \frac{2}{\kappa^2}(\cos u_\kappa - \cos N_\kappa). \tag{2.3}$$

If u_κ is monotonically increasing in $\left(0, \frac{\pi}{2}\right)$, (2.3) is equivalent to

$$u_\kappa'(x) = \frac{\sqrt{2}}{\kappa}\sqrt{\cos u_\kappa - \cos N_\kappa}, \tag{2.4}$$

with $u_\kappa(0) = 0, u_\kappa(\frac{\pi}{2}) = N_\kappa$, which gives that

$$\begin{aligned} g(N_\kappa) &:= \int_0^{\frac{\pi}{2}} \frac{u_\kappa'}{\sqrt{\cos u_\kappa - \cos N_\kappa}} dx = \int_0^{N_\kappa} \frac{1}{\sqrt{\cos u_\kappa - \cos N_\kappa}} du_\kappa \\ &\xlongequal{u_\kappa = N_\kappa t} \int_0^1 \frac{N_\kappa}{\sqrt{\cos(N_\kappa t) - \cos N_\kappa}} dt = \frac{\pi}{\sqrt{2}\kappa}. \end{aligned} \tag{2.5}$$

For each fixed $\kappa \in (0,1)$, there exists a unique $N_\kappa \in (0,\pi)$ such that (2.5) holds. Here we noted that $g(N_\kappa)$ is monotonically increasing for $\kappa \in (0,1)$. On the other hand, if $\kappa \geq 1$, we obtain the equation (2.1) only has trivial solution $u \equiv 0$. While if $0 < \kappa \ll 1$, one can obtain

$$\pi - N_\kappa = O(e^{-\frac{c}{\kappa}}).$$

In summarize, we have the following results:

Proposition 2.1. *The following holds:*

(1)The function g defined in (2.5) is monotonically increasing on $(0,\pi)$, and $g(N_\kappa)\to\infty$ as $N_\kappa\to\pi$.

(2)For any $0<\kappa<1$, there exists a unique $0<N_\kappa<\pi$ such that

$$g(N_\kappa)=\frac{\pi}{\sqrt{2}\kappa}. \tag{2.6}$$

Furthermore we have

$$c_1e^{-\frac{c_2}{\kappa}}<\pi-N_\kappa<c_3e^{-\frac{c_4}{\kappa}}, \tag{2.7}$$

where $c_i>0, i=1,\cdots,4$ are positive constants.

(3)For any $0<\kappa<1$, there exists a unique 2π-periodic C^∞ odd function u_κ such that

- *u_κ is a steady state, i.e. $\kappa^2u_\kappa''+\sin u_\kappa=0$.*
- *$u_\kappa(0)=u_\kappa'(\frac{\pi}{2})=0, u_\kappa(\frac{\pi}{2})=N_\kappa$, and u_κ is monotonically increasing on $[0,\frac{\pi}{2}]$.*
- *$u_\kappa(\pi-x)=u_\kappa(x)$ for $\frac{\pi}{2}\le x\le\pi$.*

Moreover for $0<\kappa\ll 1$, we have

$$0\le 2\arcsin\tanh\left(\frac{x}{\kappa}\right)-u_\kappa(x)\le c_5\exp\left(-\frac{c_6}{\kappa}\right), \tag{2.8}$$

where c_5,c_6 are positive constants.

Proof. We shall prove the Proposition point by point.

1. For $N_\kappa\in[0,\pi)$, taking the derivative of $g(N_\kappa)$ with respect to N_κ, then we have

$$g'(N_\kappa)=\int_0^1\frac{[2\cos(N_\kappa t)+N_\kappa t\sin(N_\kappa t)]-[2\cos N_\kappa+N_\kappa\sin N_\kappa]}{2(\cos(N_\kappa t)-\cos N_\kappa)^{\frac{3}{2}}}dt.$$

It is clear that the denominator is positive in $(0,1)$, it remains to show the numerator is also positive in $(0,1)$. Considering the function $f(x)=2\cos x+x\sin x,\ x\in(0,\pi)$, with its derivative

$$f'(x)=-\sin x+x\cos x=\cos x(-\tan x+x),\quad x\in(0,\pi).$$

By the simple properties of trigonometric functions, it is not difficult to obtain that $f'(x)$ is always non-positive in $(0,\pi)$. It follows that $f(x)$ is monotonically decreasing in $(0,\pi)$. As a consequence, for $t\in(0,1)$, we derive

$$(2\cos(N_\kappa t)+N_\kappa t\sin(N_\kappa t))-(2\cos N_\kappa+N_\kappa\sin N_\kappa)=f(N_\kappa t)-f(N_\kappa)>0.$$

Hence, we have shown that $g'(N_\kappa)>0$ and it indicates that $g(N_\kappa)$ is monotonically increasing for $N_\kappa\in(0,\pi)$.

While as N_κ is close to π, we define $\delta = \pi - N_\kappa$ and introduce $t^* = \frac{\pi-\delta}{\pi+\delta}$, then we have $\delta \le \frac{\pi}{5}$ and $N_\kappa \ge \frac{4\pi}{5}$ when $t^* \ge \frac{2}{3}$. In addition,

(2.9)

$$\begin{aligned}
g(N_\kappa) &= \int_0^1 \frac{N_\kappa}{\sqrt{2\sin\frac{N_\kappa(t+1)}{2}\sin\frac{N_\kappa(1-t)}{2}}}dt = \int_0^1 \frac{N_\kappa}{\sqrt{2\sin\frac{\delta(t+1)+(1-t)\pi}{2}\sin\frac{N_\kappa(1-t)}{2}}}dt \\
&> \int_{\frac{2}{3}}^{t^*} \frac{N_\kappa}{\sqrt{2\sin\frac{\delta(t+1)+(1-t)\pi}{2}\sin\frac{N_\kappa(1-t)}{2}}}dt > \int_{\frac{2}{3}}^{t^*} \frac{N_\kappa}{\sqrt{2\sin\left(\pi(1-t)\right)\sin\frac{N_\kappa(1-t)}{2}}}dt \\
&> \int_{\frac{2}{3}}^{t^*} \frac{N_\kappa}{\sqrt{\pi N_\kappa(1-t)^2}}dt = \sqrt{\frac{N_\kappa}{\pi}}\ln\frac{\pi+\delta}{6\delta}.
\end{aligned}$$

Then it is easy to see that $g(N_\kappa) \to \infty$ as $N_\kappa \to \pi$.

2. The existence and uniqueness of N_κ follows from the behavior of the function $g(\cdot)$. Now we shall derive the upper and lower bounds on N_κ. From $u_\kappa\left(\frac{\pi}{2}\right) = N_\kappa$ and (2.1), we obtain that $N_\kappa \to \pi$ as $\kappa \to 0$. Therefore, when $N_\kappa \le \frac{4}{5}\pi$, κ is bounded away from zero by a positive constant, and (2.7) holds in this case. Thus it remains to consider the situation of $N_\kappa \in \left[\frac{4\pi}{5}, \pi\right)$. From (2.9), we have

$$\frac{\pi}{\sqrt{2}\kappa} = g(N_\kappa) > \sqrt{\frac{N_\kappa}{\pi}}\ln\frac{\pi+\delta}{6\delta} > \sqrt{\frac{N_\kappa}{\pi}}\ln\frac{\pi}{6\delta},$$

which directly yields

$$\pi - N_\kappa > \frac{\pi}{6}e^{\sqrt{\frac{\pi}{N_\kappa}}\left(-\frac{\pi}{\sqrt{2}\kappa}\right)} > \frac{\pi}{6}e^{-\sqrt{\frac{5}{8}}\left(\frac{\pi}{\kappa}\right)} = c_1 e^{-\frac{c_2}{\kappa}}. \tag{2.10}$$

On the other hand, we have

$$\begin{aligned}
g(N_\kappa) =& \int_0^1 \frac{N_\kappa}{\sqrt{\cos(N_\kappa t) - \cos N_\kappa}}dt = \int_0^{\frac{2}{3}} \frac{N_\kappa}{\sqrt{\cos(N_\kappa t) - \cos N_\kappa}}dt \\
&+ \left(\int_{\frac{2}{3}}^{t^*} + \int_{t^*}^1\right) \frac{N_\kappa}{\sqrt{2\sin\frac{\delta(t+1)+(1-t)\pi}{2}\sin\frac{N_\kappa(1-t)}{2}}}dt \\
:=& \hat{I}_1 + \hat{I}_2 + \hat{I}_3.
\end{aligned}$$

We shall estimate the right hand of above equality respectively as follows:

(i) For $\hat{I}_1$, using $\sin x > \frac{2}{\pi}x$, for $x \in \left(0, \frac{\pi}{2}\right)$, we get

$$\begin{aligned}
\hat{I}_1 &= \int_0^{\frac{2}{3}} \frac{N_\kappa}{\sqrt{\cos(N_\kappa t) - \cos N_\kappa}}dt < \int_0^{\frac{2}{3}} \frac{N_\kappa}{\sqrt{\cos\frac{2N_\kappa}{3} - \cos N_\kappa}}dt \\
&= \int_0^{\frac{2}{3}} \frac{N_\kappa}{\sqrt{2\sin\frac{5N_\kappa}{6}\sin\frac{N_\kappa}{6}}}dt < \int_0^{\frac{2}{3}} \frac{N_\kappa}{\sqrt{2\sin\left(\frac{5\pi}{6}\right)\frac{N_\kappa}{3\pi}}}dt = \sqrt{\frac{4}{3}\pi N_\kappa} < 2\pi.
\end{aligned}$$

(ii) For $\hat{I}_2$:

$$\begin{aligned}\hat{I}_2 &= \int_{\frac{2}{3}}^{t^*} \frac{N_\kappa}{\sqrt{2\sin\frac{\delta(t+1)+(1-t)\pi}{2}\sin\frac{N_\kappa(1-t)}{2}}}dt \\ &< \int_{\frac{2}{3}}^{t^*} \frac{N_\kappa}{\sqrt{\frac{2}{\pi^2}\delta N_\kappa(1+t)(1-t)+\frac{2N_\kappa}{\pi}(1-t)^2}}dt < \int_{\frac{2}{3}}^{t^*} \frac{N_\kappa}{\sqrt{\frac{2N_\kappa}{\pi}(1-t)^2}}dt \\ &= \sqrt{\frac{\pi N_\kappa}{2}}\int_{\frac{2}{3}}^{t^*}\frac{1}{1-t}dt < -\sqrt{\frac{\pi N_\kappa}{2}}\ln\frac{2\delta}{\pi+\delta} < -\frac{N_\kappa}{2}\ln\frac{5\delta}{3\pi},\end{aligned}$$

where we used $\delta \le \frac{\pi}{5}$.

(iii) Finally, for $\hat{I}_3$:

$$\begin{aligned}\hat{I}_3 &= \int_{t^*}^{1} \frac{N_\kappa}{\sqrt{2\sin\frac{\delta(t+1)+(1-t)\pi}{2}\sin\frac{N_\kappa(1-t)}{2}}}dt < \int_{t^*}^{1} \frac{N_\kappa}{\sqrt{\frac{2}{\pi}\left(\delta(t+1)+(1-t)\pi\right)\frac{N_\kappa(1-t)}{\pi}}}dt \\ &< \int_{t^*}^{1}\frac{N_\kappa}{\sqrt{\frac{2N_\kappa\delta}{\pi^2}(1-t)}}dt = 2\sqrt{\frac{\pi^2 N_\kappa}{\pi+\delta}} < 2\pi.\end{aligned}$$

Combining above inequalities, we have

$$\frac{\pi}{\sqrt{2}\kappa} = g(N_\kappa) < 4\pi - \frac{N_\kappa}{2}\ln\frac{5\delta}{3\pi}, \tag{2.11}$$

which yields

$$\pi - N_\kappa < \left(\frac{3\pi}{5}e^{10}\right)e^{-\frac{\sqrt{2}}{\kappa}} = c_3 e^{-\frac{c_4}{\kappa}}. \tag{2.12}$$

The desired result then follows by collecting the estimates (2.10) and (2.12).

3. Fixed $0<\kappa<1$ and consider the function

$$h(u_\kappa) = \int_0^{u_\kappa}\frac{\kappa}{\sqrt{2(\cos y - \cos N_\kappa)}}dy, \quad 0<u_\kappa<N_\kappa.$$

Clearly $h : [0, N_\kappa] \to [0, \frac{\pi}{2}]$ is strictly monotonically increasing and bijective. The inverse map of $h(u_\kappa)$ then defines the desired function u_κ in the interval $[0, \frac{\pi}{2}]$. It is known that $u_\kappa(\frac{\pi}{2}) = N_\kappa$ and $u'_\kappa(\frac{\pi}{2}) = 0$. In addition, according to [**3**, Proposition 3.1], we derive that u_κ is symmetry with respect to $x = \frac{\pi}{2}$, i.e., $u_\kappa(x) = u_\kappa(\pi - x)$ for $x \in (0, \frac{\pi}{2})$.

Finally to show (2.8), we denote $\theta(x) = 2\arcsin\tanh(\frac{x}{\kappa})$. Clearly

$$\begin{cases} \dfrac{du_\kappa}{dx} = \dfrac{\sqrt{2}}{\kappa}\sqrt{\cos u_\kappa - \cos N_\kappa}, \\ \dfrac{d\theta}{dx} = \dfrac{2}{\kappa}\cos\dfrac{\theta}{2}, \\ u_\kappa(0) = \theta(0) = 0. \end{cases}$$

Observe that

$$\frac{du_\kappa}{dx} < \frac{\sqrt{2}}{\kappa}\sqrt{\cos u_\kappa + 1} = \frac{2}{\kappa}\cos\frac{u_\kappa}{2}.$$

Denote $\eta(x) = u_\kappa(x) - \theta(x)$. We obtain $\eta'(0) < 0$ follows from

$$\eta'(x) < \frac{2}{\kappa}\cos\frac{u_\kappa}{2} - \frac{2}{\kappa}\cos\frac{\theta}{2}. \tag{2.13}$$

Here we claim that

$$\eta(x) < 0, \quad \text{for } 0 < x < \frac{\pi}{2}.$$

Otherwise there exists $\xi \in (0, \frac{\pi}{2})$, such that $\eta(\xi) = 0$. At this moment, $u_\kappa(\xi) = \theta(\xi)$ which implies that $\eta'(\xi) < 0$ by (2.13). On the other hand, $\eta(\xi) = 0$ and $\eta(y) < 0$ for $y \in (0, \xi)$, which means $\eta'(\xi) \geq 0$. Thus we arrive at a contradiction and it proves that $\eta(x) < 0$. Hence for $0 \leq x \leq \frac{\pi}{2}$,

$$u_\kappa(x) - \theta(x) \leq 0. \tag{2.14}$$

We shall prove the lower bound of $\eta(x)$ by dividing the discussion into the following two cases, for simplicity, we use c denotes a generic positive constant, which may change from line to line.

(1) $\eta(x)$ is monotonically decreasing on $\left[0, \frac{\pi}{2}\right]$ or $\eta(x)$ obtains the minimal value at $\frac{\pi}{2}$. Then

$$0 \geq \eta(x) \geq \eta\left(\frac{\pi}{2}\right) = N_\kappa - \theta\left(\frac{\pi}{2}\right) \geq N_\kappa - \pi \geq -\frac{3\pi}{5}e^{10-\frac{\sqrt{2}}{\kappa}} = -ce^{-\frac{c}{\kappa}}, \tag{2.15}$$

where we used (2.12) in the last inequality.

(2) $\eta(x)$ takes the global minimal value at $\xi \in \left(0, \frac{\pi}{2}\right)$. It suffices to consider the critical point which satisfies $\eta(\xi) \geq -ce^{-\frac{\kappa}{c}}$. Noted that

$$\sin\frac{u_\kappa + \theta}{4} < \sin\frac{N_\kappa + \pi}{4} = \sin\frac{2\pi - \delta}{4} < \frac{2\pi - \delta}{4}.$$

Using (2.13) and the fact that η is non-positive for $\left[0, \frac{\pi}{2}\right]$, we have

$$\frac{d\eta}{dx} < \frac{2}{\kappa}\cos\frac{u_\kappa}{2} - \frac{2}{\kappa}\cos\frac{\theta}{2} = -\frac{4}{\kappa}\sin\frac{u_\kappa + \theta}{4}\sin\frac{\eta}{4} < \frac{\delta - 2\pi}{4\kappa}\eta.$$

It follows that

$$\frac{d}{dx}\left(e^{\frac{(2\pi-\delta)x}{4\kappa}}\eta(x)\right) < 0.$$

Then we have

$$0 > e^{\frac{(2\pi-\delta)x}{4\kappa}}\eta(x) > e^{\frac{2\pi-\delta}{4\kappa}\cdot\frac{\pi}{2}}\eta\left(\frac{\pi}{2}\right) > -ce^{\frac{1}{\kappa}\left(\frac{\pi(2\pi-\delta)}{8} - \sqrt{2}\right)},$$

which implies

$$0 > \eta(x) > -ce^{\frac{1}{\kappa}\left(\frac{\pi(2\pi-\delta)}{8} - \sqrt{2} - \frac{(2\pi-\delta)x}{4}\right)}.$$

For $x > \frac{\pi}{2} - \frac{4\sqrt{2}}{2\pi-\delta} + \frac{1}{10}$, we obtain

$$0 > \eta(x) > -ce^{\frac{1}{\kappa}\left(\frac{\pi(2\pi-\delta)}{8} - \sqrt{2} - \frac{(2\pi-\delta)x}{4}\right)} > -ce^{-\frac{c}{\kappa}}.$$

While $x \in \left[0, \frac{\pi}{2} - \frac{4\sqrt{2}}{2\pi-\delta} + \frac{1}{10}\right]$. Using $\eta'(\xi) = 0$ due to ξ is the interior point such that $\eta(x)$ achieves the minimal value, we have

$$\cos u_\kappa(\xi) - \cos\theta(\xi) = \cos N_\kappa + 1 = 2ce^{-\frac{c}{\kappa}},$$

which is equivalent to

$$\sin\frac{u_\kappa(\xi) + \theta(\xi)}{2}\sin\frac{\theta(\xi) - u_\kappa(\xi)}{2} = ce^{-\frac{c}{\kappa}}. \tag{2.16}$$

then by computing directly, it is not difficulty to see that

$$\eta(\xi) \geq -ce^{-\frac{c}{\kappa}}. \tag{2.17}$$

Combining (2.15) and (2.17), we obtain

$$u_\kappa(x) - \theta(x) \geq -ce^{-\frac{c}{\kappa}}. \tag{2.18}$$

□

3. Classification of steady state energy

In this section, we consider the energy $E_\kappa(u)$ (see (1.6)) of solutions to (1.1). Combining the discussion of Section 2 with [**3**, Proposition 3.2], we see that the nontrivial bounded steady state solution is a periodic function of 2π. Hence, we are concerned about the following problem

$$\begin{cases} \kappa^2 u'' + \sin u = 0, & x \in \mathbb{R}, \\ u(x) = u(x + 2\pi). \end{cases} \tag{3.1}$$

For simplicity of presentation, we introduce the following definition.

Definition 3.1. (1) (Odd zero-up solution) We shall say that u is an odd zero-up solution to (3.1) provided that the solution u is odd and $u'(0) > 0$.
(2) (Odd zero-up ground states) For each $0 < \kappa < 1$, we define $U_\kappa = u_\kappa$, where u_κ is obtained in Proposition 2.1 as the odd zero-up ground state solution to (3.1). We also define the odd zero-up ground state energies $E_\kappa^{(0)}$ as

$$\begin{aligned} E_\kappa^{(0)} &= \int_{\mathbb{T}} \left(\frac{\kappa^2}{2} (U_\kappa'(x))^2 + \cos U_\kappa(x) + 1 \right) dx \\ &= \int_{\mathbb{T}} (2\cos U_\kappa - \cos N_\kappa + 1)\, dx, \end{aligned} \tag{3.2}$$

where we recall $0 < N_\kappa < \pi$ is the unique number satisfying:

$$\int_0^1 \frac{N_\kappa}{\sqrt{\cos(N_\kappa t) - \cos N_\kappa}} dt = \frac{\pi}{\sqrt{2}\kappa}. \tag{3.3}$$

For any solution of (3.1), we assume that its minimal period is $\frac{2\pi}{m}$ for some suitable positive integer m. From the [**3**, Proposition 3.2], it is clear to see that $u(x)$ has $2m$ zero points in $[x, x + 2\pi]$ for any $x \in \mathbb{R}$ and $u(x)$ has odd symmetry with respect to any zero point. Consequently, we may assume $u(0) = 0$ and u is odd after a suitable shift. Hence we may assume that $u'(0) > 0$ after reflection if necessary. Therefore, in this section, we shall restrict our discussion on the odd zero-up solutions of equation (3.1). Concerning all the odd zero-up solutions to (3.1), we have the following classification results.

Theorem 3.2. *(Classification of odd zero-up solutions to* (3.1)*). For any* $0 < \kappa < 1$*, define* $m_\kappa \geq 1$ *as the unique integer such that*

$$\frac{1}{m_\kappa + 1} \leq \kappa < \frac{1}{m_\kappa}.$$

Then there are only m_κ *odd zero-up solutions to* (3.1)*. More precisely, the following holds:*

For each $j = 1, \cdots, m_\kappa$*, define (note below that* $j\kappa < 1$*)*

$$\tilde{u}_{\kappa,j}(x) = U_{j\kappa}(jx). \tag{3.4}$$

Then $\{\tilde{u}_{\kappa,j}\}_{j=1}^{m_\kappa}$ *are all the possible odd zero-up solutions to* (3.1). *Furthermore the energies of* $\tilde{u}_{\kappa,j}$ *are given by*

$$E_{\kappa,j} = \int_{\mathbb{T}} \left(\frac{\kappa^2}{2} (\partial_x \tilde{u}_{\kappa,j})^2 + \cos \tilde{u}_{\kappa,j} + 1 \right) dx = E^{(0)}_{j\kappa}, \tag{3.5}$$

where $E^{(0)}_{j\kappa}$ *was defined in* (3.2).

Proof. Suppose u is a possible odd zero-up solution to (3.1). The crucial observation is that we must have u achieves its first peak at $x = \frac{\pi}{2j}$ for some integer $j \geq 1$. Now make a change of variable

$$y = jx, \quad \tilde{u}_{\kappa,j}(y) = u_\kappa(x).$$

Then clearly

$$j^2 \kappa^2 \tilde{u}''_{\kappa,j}(y) + \sin \tilde{u}_{\kappa,j}(y) = 0,$$

with $\tilde{u}_{\kappa,j}(0) = 0, \tilde{u}'_{\kappa,j}(0) > 0$, and $\tilde{u}'_{\kappa,j}(\frac{\pi}{2}) = 0$. From the proof in Step 3 of Proposition 2.1, there exists a unique solution u_κ with $|u_\kappa| < \pi$ solving the equation

$$u'_\kappa = \frac{\sqrt{2}}{\kappa} \sqrt{\cos u_\kappa - \cos N_\kappa},$$

with $u_\kappa(0) = 0, u'_\kappa(\frac{\pi}{2}) = 0$. As a consequence, we obtain that $\tilde{u}_{\kappa,j} = U_{j\kappa}$. Now note that $j\kappa < 1$ and this gives the constraint $j \leq m_\kappa$. The characterization (3.5) follows from the fact that

$$\begin{aligned} E_{\kappa,j} &= \int_{\mathbb{T}} (2 \cos \tilde{u}_{\kappa,j}(x) - \cos N_{j\kappa} + 1)\, dx \\ &= \int_{\mathbb{T}} (2 \cos U_{j\kappa}(jx) - \cos N_{j\kappa} + 1)\, dx, \end{aligned}$$

and the fact that $U_{j\kappa}$ is 2π-periodic. $\square$

By Theorem 3.2 one can easily get Theorem 1.1.

Theorem 3.3. *(Monotonicity and asymptotic of odd zero-up ground state energies). For any* $\kappa > 0$, *define*

$$\tilde{E}_\kappa = \inf_{u_\kappa \in \mathcal{S}_O} \int_{\mathbb{T}} \left(\frac{1}{2} (\kappa \partial_x u_\kappa)^2 + \cos u_\kappa + 1 \right) dx, \tag{3.6}$$

where

$$\mathcal{S}_O = \left\{ \phi \middle| \phi : \mathbb{T} \to \mathbb{R} \textit{ is odd and } C^1, \phi'(0) > 0 \right\}.$$

Then we have

(a) $\tilde{E}_\kappa = 4\pi$ *for* $\kappa \geq 1$, *and it is obtained only at the zero function.*
(b) $\tilde{E}_\kappa = E^{(0)}_\kappa$ *for* $0 < \kappa < 1$. *Moreover, the infimum is only achieved by* U_κ.
(c) *If* $0 < \kappa_1 < \kappa_2 < 1$, *then* $\tilde{E}_{\kappa_1} < \tilde{E}_{\kappa_2}$.

Furthermore

$$\lim_{\kappa \to 0} \frac{E^{(0)}_\kappa}{\kappa} = \gamma_* = 16 > 0. \tag{3.7}$$

Remark 3.4. The constant 1 in the definition of energy seems extra, but we add it in order to $U_\kappa \to 0$ as $\kappa \to 0$, and it plays the same role as the constant term in the definition of the energy functional of the Allen-Cahn equation.

Our proof process shall use the following lemma.

LEMMA 3.5. *Let U_κ be the odd zero-up ground state solution of equation* (3.1)*, then we have*

$$U_{\kappa_1}(x) > U_{\kappa_2}(x), \quad x \in \left(0, \frac{\pi}{2}\right] \tag{3.8}$$

provided that $0 < \kappa_1 < \kappa_2 < 1$.

PROOF. At first, we notice that $U_\kappa(0) = 0$, U_κ is monotone increasing for $x \in \left(0, \frac{\pi}{2}\right)$. By (2.4) we have

$$U_\kappa'(x) = \frac{\sqrt{2(\cos U_\kappa - \cos N_\kappa)}}{\kappa}, \quad \text{for } x \in \left(0, \frac{\pi}{2}\right), \tag{3.9}$$

where N_κ is the maximum value of U_κ in $[0, \frac{\pi}{2}]$, i.e., $N_\kappa = U_\kappa(\frac{\pi}{2})$. By (3.9) we have

$$\int_0^{U_\kappa(x)} \frac{\kappa}{\sqrt{2(\cos w - \cos N_\kappa)}} dw = x, \quad x \in \left(0, \frac{\pi}{2}\right). \tag{3.10}$$

If $0 < \kappa_1 < \kappa_2 < 1$, by (2.6) and monotonicity of $g(N_\kappa)$ in Proposition 2.1, we have $0 < N_{\kappa_2} < N_{\kappa_1} < \pi$. This implies that

$$-\cos N_{\kappa_1} > -\cos N_{\kappa_2}.$$

Therefore, for any $w \in \min\{N_{\kappa_1}, N_{\kappa_2}\}$, we have

$$\frac{\kappa_1}{\sqrt{2(\cos w - \cos N_{\kappa_1})}} < \frac{\kappa_2}{\sqrt{2(\cos w - \cos N_{\kappa_2})}}.$$

Together with (3.10) we derive that $U_{\kappa_1}(x) > U_{\kappa_2}(x)$ for $x \in \left(0, \frac{\pi}{2}\right]$. □

PROOF OF THEOREM 3.3. We shall prove the Theorem 3.3 point by point.
(a) We notice that $u_\kappa \equiv 0$ is the only odd zero-up solution to (3.1) whenever $\kappa \geq 1$. Then it is easy to verify that $\tilde{E}_\kappa = 4\pi$ for $\kappa \geq 1$.
(b) For any 2π-periodic odd zero-up solution to (3.1) which is different by U_κ, we denote its minimal period by $\frac{2\pi}{m}$ and the solution by $u_m, m \geq 2$. Consider the function

$$v(y) = u_m(x), \quad y = mx.$$

Then it is not difficulty to verify that

$$v(x) = U_{m\kappa}(x) \quad \text{for } x \in \left(0, \frac{\pi}{2}\right].$$

By Lemma 3.5, for $m \geq 2$ we have

$$U_\kappa(x) > U_{m\kappa}(x) \quad \text{for } x \in \left(0, \frac{\pi}{2}\right]. \tag{3.11}$$

On the other hand, we noticed that

$$\begin{aligned} E_\kappa^{(0)} &= \int_0^{\frac{\pi}{2}} \left(2\kappa^2 (U_\kappa')^2 + 4\cos U_\kappa + 4\right) dx \\ &= 2\kappa^2 U_\kappa U_\kappa' \Big|_{x=0}^{x=\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}} \left(-2\kappa^2 U_\kappa U_\kappa'' + 4\cos U_\kappa + 4\right) dx \\ &= \int_0^{\frac{\pi}{2}} (2U_\kappa \sin U_\kappa + 4\cos U_\kappa + 4)\, dx. \end{aligned} \tag{3.12}$$

It is not difficulty to verify that function $h(U_\kappa) = U_\kappa \sin U_\kappa + 2\cos U_\kappa + 2$ is monotone decreasing for $U_\kappa \in [0, \pi)$. Using (3.11) we have

$$E_\kappa^{(0)} < E_{m\kappa}^{(0)}. \tag{3.13}$$

By equation (3.5) we have

$$E_{m\kappa}^{(0)} = E(u_m) = \int_{\mathbb{T}} \left(\frac{\kappa^2}{2}(u_m')^2 + \cos u_m + 1 \right) dx.$$

Together with (3.13) we obtained that

$$E_{\kappa}^{(0)} = \tilde{E}_{\kappa}.$$

(c) The point (c) follows from (b), monotonicity of U_κ with respect to κ and the equation (3.12). In the end, we shall show the asymptotic as $\kappa \to 0$. By Proposition 2.1, the main part of U_κ on $[0, \frac{\pi}{2}]$ is given by $2 \arcsin \tanh(\frac{x}{\kappa})$. The result (3.7) then follows from

$$\begin{aligned} E_{\kappa}^{(0)} &= \int_{-\pi}^{\pi} (2\cos U_\kappa - \cos N_\kappa + 1)\, dx \\ &= 8 \int_0^{\frac{\pi}{2}} \cos\left(2 \arcsin \tanh\left(\frac{x}{\kappa}\right)\right) dx - 2\pi \cos N_\kappa + 2\pi + O\left(e^{-\frac{c_6}{\kappa}}\right) \\ &= 8\kappa \int_0^{\frac{\pi}{2\kappa}} (1 - 2\tanh^2 y) dy - 2\pi \cos N_\kappa + 2\pi + O\left(e^{-\frac{c_6}{\kappa}}\right) \\ &= 16\kappa \int_0^{\frac{\pi}{2\kappa}} d(\tanh y) - 2\pi - 2\pi \cos N_\kappa + O\left(e^{-\frac{c_6}{\kappa}}\right) \\ &= 16\kappa \tanh\left(\frac{\pi}{2\kappa}\right) - 2\pi - 2\pi \cos N_\kappa + O\left(e^{-\frac{c_6}{\kappa}}\right), \end{aligned}$$

where $y = \frac{x}{\kappa}$ and the c_6 corresponds to equation (2.8). By Proposition 2.1 and Taylor expansion, we obtain $\cos N_\kappa = -1 + O\left(e^{-\frac{c}{\kappa}}\right)$. Consequently we have

$$\gamma_* = \lim_{\kappa \to 0} \frac{E_\kappa^{(0)}}{\kappa} = 16.$$

□

By Theorem 3.3, we can obtain (a)-(c) in Theorem 1.2. In addition, we notice that the C^0 estimate in the point of (d) of Theorem 1.2 follows easily by (2.8). While for the point (e), one can easily prove it by some direct computations. Hence, we complete the entire proof of Theorem 1.2.

4. Convergence to steady state

In this section, we investigate the convergence rate of the solution and characterize the detailed profiles as $t \to \infty$.

4.1. Case of $0 < \kappa < 1$. We start this subsection with the following result in the spectrum analysis. This is important to indicate that the rate of convergence is exponential.

LEMMA 4.1. *Let $0 < \kappa < 1$. Assume U_κ is the odd zero-up ground state. Then for any 2π-periodic odd function $\phi \in H^1(\mathbb{T})$ we have*

$$\int_{\mathbb{T}} \kappa^2 |\phi'|^2 dx - \int_{\mathbb{T}} |\phi|^2 \cos U_\kappa dx \geq C \|\phi\|_{H^1(\mathbb{T})}^2 \tag{4.1}$$

for some constant $C > 0$.

Proof. Firstly, since U_κ is the odd zero-up ground state, then for any $\phi \in H^1(\mathbb{T})$ we have

$$\left(\frac{d^2}{ds^2}E_\kappa(U_\kappa + s\phi)\right)\Bigg|_{s=0} = \int_{\mathbb{T}} \kappa^2|\phi'|^2 dx - \int_{\mathbb{T}} |\phi|^2 \cos U_\kappa dx \geq 0.$$

Therefore, $C \geq 0$ obviously holds.

Next, we prove that $C > 0$ by contradiction. Suppose that $C = 0$, then we can find a sequence of odd function ϕ_n (vanishes on the boundary of $\partial\mathbb{T}$) such that $\|\phi_n\|_{H^1(\mathbb{T})} = 1$ and

$$\int_{\mathbb{T}} \kappa^2|\phi_n'|^2 dx - \int_{\mathbb{T}} |\phi_n|^2 \cos U_\kappa dx \leq \frac{1}{n}.$$

Passing to a subsequence if necessary, we obtain there exists a nontrivial odd function $\phi_* \in H^1(\mathbb{T})$ such that

$$\phi_n \rightharpoonup \phi_* \quad \text{in} \quad H^1(\mathbb{T})$$

and

$$\kappa^2\phi_*'' + \phi_* \cos U_\kappa = 0 \quad \text{on } \mathbb{T}. \tag{4.2}$$

Since U_κ is the odd zero-up ground state, by directly computing we see that

$$\begin{aligned} E_\kappa(U_\kappa + c\phi_*) =& E_\kappa(U_\kappa) + \frac{c^2}{2}\int_{\mathbb{T}} \left(\kappa^2(\phi_*')^2 - \phi_*^2 \cos U_\kappa\right) dx \\ &+ \frac{c^3}{3!}\int_{\mathbb{T}} \left(\phi_*^3 \sin U_\kappa\right) dx + O(c^4), \end{aligned} \tag{4.3}$$

for any real number c. Using (4.2) we see that the second term on the right hand side of (4.3) vanishes, then together with $E_\kappa(U_\kappa + c\phi_*) \geq E_\kappa(U_\kappa)$ for any c, we have

$$\int_{\mathbb{T}} \phi_*^3 \sin U_\kappa dx = 0.$$

It implies that ϕ_* must possess a zero point in $(0, \pi)$, denoted by x_*. By the Strum Comparison Theorem [**8**, Theorem VI-I-I], we derive that any solution of the following equation must have a zero point in $(0, x_*)$,

$$\kappa^2 U_\kappa'' + \sin U_\kappa = 0. \tag{4.4}$$

However we noticed that U_κ is a solution of (4.4) and positive in $(0, \pi)$. Thus we arrive at a contradiction and the lemma is proved. □

Proof of Theorem 1.3. Without loss of generality we may assume $u_0 \in C^\infty$ by smoothing estimate. It is not hard to verify that $u(x,t)$ is a 2π-periodic odd function and also odd symmetric with respect to $x = \pi$. Therefore

$$u(0,t) = u(\pi,t) \equiv 0, \quad \forall t \geq 0.$$

Together with that $u_0(x)$ is non-negative in $[0, \pi]$, we conclude that $u(x,t) \geq 0$ for $x \in [0, \pi]$ by maximum principle [**7**, section 2, Lemma 5]. Similarly, we have $u(x,t) \leq 0$ for $x \in [-\pi, 0]$. On the other hand,

$$\frac{d}{dt}\left(\frac{\kappa^2}{2}\|\partial_x u\|_2^2 + \int_{\mathbb{T}} (\cos u + 1)\, dx\right) = -\|\partial_t u\|_2^2. \tag{4.5}$$

It follows that $\|\partial_t u\|_{L^1_t L^2_x} < \infty$ and one can extract a subsequence such that $\partial_t u(t_n) \to 0$ in L^2. By using higher uniform Sobolev estimate one can obtain convergence in higher norms. In particular we can obtain $u(t_n) \to u_\infty$ for some steady state of (3.1). In addition, u_∞ is a 2π-periodic odd function and non-negative for $x \in [0, \pi]$. By the proof of Theorem 1.1 we see that 0 and U_κ are the only steady states which are non-negative in $[0, \pi]$. As a consequence, we derive that u_∞ could be either U_κ or the trivial solution 0.

Moreover, if $u_0(x) \neq 0$ and $E_\kappa(u_0) \leq 4\pi$, using (4.5) we see that

$$E_\kappa(u_\infty) \leq E_\kappa(u_0) \leq 4\pi.$$

The inequality sign holds only $u_\infty = u_0$. While it is known that $E_\kappa(0) = 4\pi$ and $u_0 \neq 0$, then we get $u_\infty = U_\kappa$. To obtain exponential convergence, we can take t_n large enough such that $u(t_n)$ is sufficiently close to the steady state U_κ. Combined with Lemma 4.1 then we obtain exponential convergence. Hence, we finish the entire proof. □

4.2. Case of $\kappa \geq 1$. In the case of $\kappa \geq 1$, we consider the parabolic sine-Gordon equation

$$\begin{cases} \partial_t u = \kappa^2 \partial_{xx} u + \sin u, & (x,t) \in \mathbb{T} \times (0, \infty), \\ u|_{t=0} = u_0. \end{cases} \tag{4.6}$$

To state the following result, we introduce the Fourier projection operators $\Pi_1, \Pi_{\geq 2}$ such that for $f = \sum_{m \geq 1} f_m \sin(mx)$ (assume the series converges sufficiently fast), it is easy to see

$$\Pi_1 f = f_1 \sin x; \quad \Pi_{\geq 2} f = \sum_{m \geq 2} f_m \sin(mx). \tag{4.7}$$

In other words, Π_1 is the projection to the first sine-mode, and $\Pi_{\geq 2}$ simply removes the first Fourier mode in the sine series expansion.

Theorem 4.2. *Let $\kappa \geq 1$. Assume u_0 is 2π periodic, odd and $|u_0| \leq \pi$. Suppose u is the solution to (4.6) corresponding to the initial data u_0.*

- *If $\kappa > 1$, we have exponential decay*

$$\begin{cases} \|u(t,\cdot)\|_2 \leq \|u_0\|_2 e^{-(\kappa^2-1)t}, & \forall t \geq 0; \\ \|u(t,\cdot)\|_{H^2} \leq \beta_1 e^{-(\kappa^2-1)t}, & \forall t \geq \frac{1}{2}; \\ \|\Pi_{\geq 2} u(t,\cdot)\|_{H^2} \leq \beta_2 e^{-3(\kappa^2-1)t}, & \forall t \geq \frac{1}{2}, \end{cases} \tag{4.8}$$

where $\beta_1 > 0, \beta_2 > 0$ depend on (u_0, κ), and $\Pi_{\geq 2}$ is defined in (4.7).

- *For $\kappa = 1$, we have algebraic decay*

$$\begin{cases} \|u(t,\cdot)\|_2 \leq \dfrac{\sqrt{12\pi}\|u_0\|_2}{\sqrt{t\|u_0\|_2^2 + 12\pi}}, & \forall t \geq 0; \\ \|u(t,\cdot)\|_{H^2} \leq \beta_3 t^{-\frac{1}{2}}, & \forall t \geq \frac{1}{2}; \\ \|\Pi_{\geq 2} u(t,\cdot)\|_{H^2} \leq \beta_4 t^{-\frac{3}{2}}, & \forall t \geq \frac{1}{2}, \end{cases} \tag{4.9}$$

where $\beta_3 > 0, \beta_4 > 0$ depend on u_0.

Remark 4.3. For $\kappa > 1$, higher (i.e. $H^m, m > 2$) Sobolev norms of u also decay exponentially but we shall not dwell on this issue here. Note that we state the decay result for $t \geq 1/2$ to allow the smoothing effect to kick in. The number $1/2$ is for convenience only and it can be replaced by any other $t_0 > 0$ with suitable adjustment of the corresponding pre-factors in the estimate.

Proof of Theorem 4.2. First we note that for bounded initial data, local and global well-posedness is not an issue and we merely focus on the decay estimates. We shall divide the proof into three parts.

(1) For the L^2 decay estimates, first we assume u_0 is smooth, and in particular has a finite sine-series expansion. It follows that $u(t)$ must have a spectral gap. Directly computing gives

$$\frac{1}{2}\frac{d}{dt}\left(\|u\|_2^2\right) = -\kappa^2\|\partial_x u\|_2^2 + \|u\|_2^2 - \frac{1}{3!}\|u\|_4^4 + \frac{1}{5!}\|u\|_6^6 + W,$$

where

$$\begin{aligned} W &= \sum_{m=2}^{\infty} -\frac{1}{(4m-1)!}\int_{\mathbb{T}} u^{4m}dx + \frac{1}{(4m+1)!}\int_{\mathbb{T}} u^{4m+2}dx \\ &\leq \sum_{m=2}^{\infty} -\frac{1}{(4m-1)!}\left(1 - \frac{\pi^2}{4m(4m+1)}\right)\int_{\mathbb{T}} u^{4m}dx \leq 0. \end{aligned}$$

Using Poincaré inequality

$$\|u\|_{L^2(\mathbb{T})} \leq \|\partial_x u\|_{L^2(\mathbb{T})},$$

and the fact that $\kappa > 1$, we obtain

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}(\|u\|_2^2) &\leq -\kappa^2\|\partial_x u\|_2^2 + \|u\|_2^2 - \frac{1}{3!}\|u\|_4^4 + \frac{1}{5!}\|u\|_6^6 \\ &\leq -(\kappa^2-1)\|u\|_2^2 - \frac{1}{12}\|u\|_4^4 \\ &\leq -(\kappa^2-1)\|u\|_2^2 - \frac{1}{24\pi}\|u\|_2^4, \end{aligned}$$

where we employed the estimates

$$-\frac{1}{3!}\|u\|_4^4 + \frac{1}{5!}\|u\|_6^6 \leq -\frac{1}{3!}\|u\|_4^4\left(1 - \frac{\pi^2}{20}\right) \leq -\frac{1}{12}\|u\|_4^4,$$

and Hölder's inequality in the last inequality. Then we derive that in the case of $\kappa > 1$,

$$\|u(t,\cdot)\|_2 \leq \|u_0\|_2 e^{-(\kappa^2-1)t},$$

while in the case of $\kappa = 1$,

$$\|u(t,\cdot)\|_2 \leq \frac{\sqrt{12\pi}\|u_0\|_2}{\sqrt{t\|u_0\|_2^2 + 12\pi}}.$$

By a simple approximation argument, both estimates also hold under the assumption that $u_0 \in L^\infty$.

(2) we now show the second inequality in (4.8). First by smoothing estimates and interpolation, we have

$$\left\|\partial_x^2(u(t,\cdot))\right\|_2 \leq \alpha_1 e^{-\kappa_1 t}, \quad \forall t \geq \frac{1}{2},$$

where $\alpha_1 > 0$ depends on (u_0, κ), and $\kappa_1 > 0$ depends only on κ. In addition, by the Sobolev Embedding Theorem, we can obtain that $u \in L^\infty$. Then we have

$$\begin{aligned}
&\int_{\mathbb{T}} \partial_x^2 \left(u^{2m+1}(t,\cdot)\right) \partial_x^2 u dx \\
&= \int_{\mathbb{T}} \left[(2m+1)u^{2m}\partial_x^2 u + 2m(2m+1)u^{2m-1}(\partial_x u)^2\right] \partial_x^2 u dx \\
&\leq \int_{\mathbb{T}} \left(c_1(2m+1)e^{-2m\kappa_1 t}\left(\partial_x^2 u\right)^2 + 2m(2m+1)c_2 e^{-(2m-1)\kappa_1 t}\left(\partial_x u\right)^2 \partial_x^2 u\right) dx \\
&\leq \left[c_1(2m+1)e^{-2m\kappa_1 t} + 2m(2m+1)c_3 e^{-(2m-1)\kappa_1 t}\right] \|\partial_x^2 u\|_2^2,
\end{aligned}$$

where c_1, c_2, c_3 are uniform positive constants and depend on (u_0, κ), furthermore, we have used Hölder's inequality and Interpolation inequality in the above first inequality. We now compute for $t \geq \frac{1}{2}$,

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\left(\|\partial_x^2 u(t,\cdot)\|_2^2\right) &= \int_{\mathbb{T}} \left[\left(\kappa^2 \partial_{xx}(\partial_x^2 u) + \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)!}\partial_x^2(u^{2m+1})\right)\partial_x^2 u\right] dx \\
&\leq \int_{\mathbb{T}} -\kappa^2\left(\nabla(\partial_x^2 u)\right)^2 dx + \int_{\mathbb{T}} (\partial_x^2 u)^2 dx \\
&\quad + \sum_{m=1}^{\infty} \frac{c_1(2m+1)e^{-2m\kappa_1 t} + 2m(2m+1)c_3 e^{-(2m-1)\kappa_1 t}}{(2m+1)!}\|\partial_x^2 u\|_2^2 \\
&\leq \left(-(\kappa^2 - 1) + \alpha_2 e^{-\kappa_1 t}\right)\|\partial_x^2 u\|_2^2,
\end{aligned}$$

where $\alpha_2 > 0$ depends on (u_0, κ). Integrating in time then yields the second inequality in (4.8).

(3) The proof of the third inequality in (4.8) is similar. Note that for all $t \geq \frac{1}{2}$,

$$\sum_{m=1}^{\infty} \frac{\|\partial_x^2\left(\Pi_{\geq 2} u^{2m+1}(t,\cdot)\right)\|_2}{(2m+1)!} \leq \sum_{m=1}^{\infty} \frac{\|\partial_x^2\left(u^{2m+1}(t,\cdot)\right)\|_2}{(2m+1)!} \leq \alpha_3 e^{-3(\kappa^2-1)t},$$

where $\alpha_3 > 0$ depends on (u_0, κ). With this we compute

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(\|\partial_x^2 \Pi_{\geq 2} u(t,\cdot)\|_2^2) \\
&\leq -\kappa^2\|\nabla(\partial_x^2 \Pi_{\geq 2} u)\|_2^2 + \|\partial_x^2 \Pi_{\geq 2} u\|_2^2 + \sum_{m=1}^{\infty} \frac{\|\partial_x^2(u^{2m+1})\|_2}{(2m+1)!}\|\partial_x^2 \Pi_{\geq 2} u\|_2 \\
&\leq -(4\kappa^2 - 1)\|\partial_x^2 \Pi_{\geq 2} u\|_2^2 + \alpha_3 e^{-3(\kappa^2-1)t}\|\partial_x^2 \Pi_{\geq 2} u\|_2,
\end{aligned}$$

where we used

$$\|\nabla(\partial_x^2 \Pi_{\geq 2} u)\|_2^2 \geq 4\|\partial_x^2 \Pi_{\geq 2} u\|_2^2.$$

Then integrating in time, we obtain

$$\|\Pi_{\geq 2} u(t,\cdot)\|_{H^2} \leq \beta_2 e^{-3(\kappa^2-1)t},$$

where β_2 depends on (u_0, κ). Hence we obtain the third inequality in (4.8).
Finally the third inequality in (4.9) follows from working with the system

$$\partial_t \Pi_{\geq 2} u = -\kappa^2 \partial_{xx}(\Pi_{\geq 2} u) + \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)!}\Pi_{\geq 2}(u^{2m+1}),$$

and bootstrapping estimate using the first inequality in (4.9). The estimate of second in (4.9) is obvious. We omit the details. □

THEOREM 4.4. *(Profiles as $t \to \infty$). Let $\kappa \geq 1$. Assume u_0 is 2π periodic, odd and $|u_0| \leq \pi$. Assume $\|u_0\|_2 > 0$ so that u_0 is not identically zero. Suppose u is the solution to (4.6) corresponding to the initial data u_0. Then the following hold*

- *Case $\kappa > 1$. For all $t \geq 1$, we have*

$$u(x,t) = e^{-(\kappa^2-1)t}\alpha_* \sin x + r(t), \tag{4.10}$$

where the constant α_* depends on (u_0, κ). The remainder term has the estimate

$$\|r(t)\|_{H^2} \leq \tilde{\alpha} e^{-3(\kappa^2-1)t}, \quad \forall t \geq 1,$$

with $\tilde{\alpha} > 0$ depends only on (u_0, κ).

- *Case $\kappa = 1$. For all $t \geq 1$, we have*

$$u(x,t) = t^{-\frac{1}{2}}\beta_* \sin x + r_1(t), \tag{4.11}$$

where the constant β_* depends on u_0.
If $\beta_* = 0$, then the remainder term $r_1(t)$ has the estimate

$$\|r_1(t)\|_{H^2} \leq \tilde{\beta} t^{-1}\sqrt{\ln(t+2)}, \quad \forall t \geq 1,$$

with $\tilde{\beta} > 0$ depends only on u_0.
If $\beta_* \neq 0$, then the remainder term $r_1(t)$ has the estimate

$$\|r_1(t)\|_{H^2} \leq \tilde{\beta} t^{-\frac{3}{2}} \ln(t+2), \quad \forall t \geq 1,$$

with $\tilde{\beta} > 0$ depends only on u_0,.

PROOF. We first consider $\kappa > 1$. Write

$$u = \Pi_1 u + \Pi_{\geq 2} u,$$

where the operators $\Pi_1, \Pi_{\geq 2}$ are defined in (4.7). Furthermore, note that (4.6) is equivalent to

$$\begin{cases} \partial_t u = \kappa^2 \partial_{xx} u + u + (\sin u - u), & (x,t) \in \mathbb{T} \times (0,\infty), \\ u|_{t=0} = u_0. \end{cases}$$

By Theorem 4.2 the term $\Pi_{\geq 2} u$ has the desired decay for $t \geq 1$ and can be included in the remainder $r(t)$. Thus we only need to treat the single-mode part $\Pi_1 u$. Denote

$$\Pi_1 u(t) = a(t) \sin x, \quad a(t) = \frac{1}{\pi}\int_{\mathbb{T}} \Pi_1 u(t,x) \sin x dx;$$

$$\Pi_1(\sin u(t) - u(t)) = b(t) \sin x, \quad b(t) = \frac{1}{\pi}\int_{\mathbb{T}} \Pi_1 (\sin u(t,x) - u(t,x)) \sin x dx.$$

By Theorem 4.2, we have for some $\tilde{C} > 0$ depending only on (u_0, κ),

$$|b(t)| \leq \tilde{C} e^{-3(\kappa^2-1)t}.$$

Clearly we have

$$\frac{d}{dt}a(t) = -(\kappa^2-1)a(t) + b(t).$$

Solving above ODE, we have for $t \geq 1$,

$$a(t) = e^{-(\kappa^2-1)(t-\frac{1}{2})} a\left(\frac{1}{2}\right) + \int_{\frac{1}{2}}^{t} e^{-(\kappa^2-1)(t-s)} b(s) ds$$
$$= e^{-(\kappa^2-1)t}\left(e^{\frac{1}{2}(\kappa^2-1)} a\left(\frac{1}{2}\right) + \int_{\frac{1}{2}}^{\infty} e^{(\kappa^2-1)s} b(s) ds\right) + \tilde{r}(t),$$

where

$$|\tilde{r}(t)| \leq e^{-(\kappa^2-1)t} \int_{t}^{\infty} e^{(\kappa^2-1)s} |b(s)| ds = O(e^{-3(\kappa^2-1)t}).$$

Clearly then (4.10) follows.

The proof of (4.11) is slightly more intricate. We only need to treat the piece $\Pi_1 u$ since the part $\Pi_{\geq 2} u$ can be included in the remainder term $r_1(t)$. Observed that for $t \geq \frac{1}{2}$, by Theorem 4.2 we have

$$(u(t))^{2m+1} = (\Pi_1 u(t) + \Pi_{\geq 2} u(t))^{2m+1} = (\Pi_1 u)^{2m+1} + \tilde{r}(t), \quad m = 1, 2, \cdots,$$

where

$$\|\tilde{r}(t)\|_{H^2} = O(t^{-\frac{2m+3}{2}}), \quad \forall t \geq \frac{1}{2},\ m = 1, 2, \cdots.$$

Denote $\Pi_1 u(t) = a(t) \sin x$. It follows that

$$\Pi_1((\Pi_1 u(t))^3) = \frac{3}{4} a(t)^3 \sin x. \tag{4.12}$$

For $a(t)$, we have the ODE

$$\frac{d}{dt} a(t) = -\frac{1}{8} a(t)^3 + \bar{r}(t), \quad \forall t \geq \frac{1}{2}, \tag{4.13}$$

where $|\bar{r}(t)| = O(t^{-\frac{5}{2}})$. Denote $\theta(t) = a(t)^2$. It is not difficulty to obtain that

$$\frac{d}{dt}\theta(t) = -\frac{1}{4}\theta(t)^2 + O(t^{-3}).$$

By Proposition 4.1 below, we have for $t \geq 3$,

$$\theta(t) = \frac{\theta_*}{t} + O(t^{-2} \ln t).$$

Note that $\theta_* \geq 0$ due to $\theta(t)$ is always nonnegative. We can take $\beta_* = 0$ if $\theta_* = 0$, then the desired result follows easily. If $\theta_* > 0$, then $|a(t)| \sim t^{-1/2}$ for t large. By continuity it can only take one sign. Thus we obtain $\beta_* = \sqrt{\theta_*}$ or $\beta_* = -\sqrt{\theta_*}$. The estimate for the remainder term is trivial. We omit the details. □

Clearly, Theorem 1.4 follows from above result. At last, we give the proof of conclusion used in the Theorem 4.4.

PROPOSITION 4.1. *Assume $T \geq 3$. Suppose $\theta : [T, \infty) \to [0, \infty)$ is continuously differentiable and satisfies*

$$\sup_{t \geq T_0} t\theta(t) < \infty; \quad \theta'(t) = -\frac{1}{4}\theta^2(t) + F(t), \quad \forall t \geq T_0 > T,$$

where for some $K_0 > 0$

$$|F(t)| \leq K_0 t^{-3}, \quad \forall t \geq T.$$

Then there exists $\theta_* \in \mathbb{R}$, such that

$$\theta(t) = \frac{\theta_*}{t} + O\left(\frac{\ln t}{t^2}\right),$$

and

$$\sup_{t \geq T} \frac{\left|\theta(t) - \frac{\theta_*}{t}\right|}{\frac{\ln t}{t^2}} < \infty.$$

PROOF. Note that we only need to investigate the regime $t \gg 1$. We shall discuss two cases:

Case 1. $0 < \limsup_{t\to\infty} t\theta(t) < \infty$. In this case we first claim that

$$\liminf_{t\to\infty} t\theta(t) > 0. \tag{4.14}$$

Denote

$$4\theta_0 = \limsup_{t\to\infty} t\theta(t) > 0. \tag{4.15}$$

Let

$$\Omega(t) = \theta(t) - \eta_0 t^{-1},$$

where $\eta_0 > 0$ satisfies

$$\eta_0 \leq \min\left\{\frac{1}{4}\theta_0, \frac{1}{10}\right\}.$$

By (4.15), we can choose $t_0 > 0$ sufficiently large such that

$$\theta(t_0) \geq \frac{\theta_0}{t_0}, \quad \frac{\eta_0}{4t_0^2} - K_0 t_0^{-3} > 0.$$

Note that the first condition above guarantees that

$$\Omega(t_0) \geq 0,$$

and second condition guarantees that

$$\frac{\eta_0}{4t^2} - K_0 t^{-3} > 0, \quad \forall t \geq t_0.$$

Now consider $\Omega(t)$ on the time interval $[t_0, \infty)$. If $\Omega(t) > 0$ for all $t \geq t_0$ we are done. Otherwise there exists some time $t_1 > t_0$ such that $\Omega(t_1) = 0$. Then clearly

$$\theta(t_1) = \frac{\eta_0}{t_1}; \quad \Omega'(t_1) \geq -\frac{1}{4}\frac{\eta_0^2}{t_1^2} + \frac{\eta_0}{t_1^2} - K_0 t_1^{-3} \geq \frac{\eta_0}{4t_1^2} - K_0 t_1^{-3} > 0.$$

Thus $\Omega(t)$ continues to be positive a little bit past t_1. This argument then guarantees that $\Omega(t) \geq 0$ for all $t \geq t_0$. Thus (4.14) is proved. Then for T large enough we have

$$\theta(t)t \sim 1, \quad \forall t \geq T.$$

From the ODE of $\theta(t)$ we obtain

$$\frac{d}{dt}\left(\frac{1}{\theta}\right) = \frac{1}{4} + O(t^{-1}).$$

It follows that for T' sufficiently large and all $t \geq T' + 2$,

$$\theta(t) = \frac{1}{d_1 + d_2(t - T') + O(\ln(t - T'))},$$

where $d_1, d_2 > 0$ are positive constants. The desired asymptotics then follows easily.
Case 2. $\limsup_{t\to\infty} t\theta(t) = 0$. In this case we make a change of variable:

$$t = N\tau, \quad \theta(t) = \gamma\Theta(\tau), \quad \frac{N}{\gamma}F(t) = \tilde{F}(\tau). \tag{4.16}$$

Clearly

$$\frac{d}{d\tau}\Theta(\tau) = -\frac{1}{4}\gamma N\Theta^2(\tau) + \tilde{F}(\tau), \quad |\tilde{F}(\tau)| \le K_0\frac{1}{\gamma N^2}\tau^{-3}.$$

Thus, if we take $\gamma = \frac{2}{N}$ and N sufficiently large such that $\frac{K_0}{2N} \le 1$, we obtain

$$\Theta(\tau) \le \frac{1}{2}\int_\tau^\infty \Theta^2(s)ds + \frac{1}{2\tau^2}, \quad \forall\tau \ge \tau_0, \tag{4.17}$$

where τ_0 is sufficiently large. Based on (4.17), we claim that there exists a constant $C_1 > 0$ depending on τ_0 such that

$$\Theta(\tau) \le \frac{C_1}{\tau^2}, \quad \forall\tau \ge \tau_0. \tag{4.18}$$

It is easy to see that $\limsup_{\tau\to\infty} \tau\Theta(\tau) = 0$ and we may assume that for some $\alpha_0 \in (0, 1)$ that

$$\Theta(\tau) \le \frac{\alpha_0}{\tau}, \quad \forall\tau \ge \tau_0.$$

Next, we run the following iteration argument. If we have

$$\Theta(\tau) \le \frac{\alpha}{\tau}, \quad \forall\tau \ge \max\left\{\tau_0, \frac{1}{\alpha}\right\},$$

then by (4.17) we obtain

$$\Theta(\tau) \le \frac{\alpha^2}{2\tau} + \frac{1}{2\tau^2} \le \frac{\alpha^2}{\tau}, \quad \forall\tau \ge \max\left\{\tau_0, \frac{1}{\alpha^2}\right\}.$$

Now define $\alpha_{k+1} = \alpha_k^2$, we get

$$\alpha_k = \alpha_0^{2^k}.$$

Obviously it holds that

$$\Theta(\tau) \le \frac{\alpha_k}{\tau}, \quad \forall\tau \ge \max\left\{\tau_0, \frac{1}{\alpha_k}\right\}.$$

Consider $\tau \in \left[\frac{1}{\alpha_k}, \frac{1}{\alpha_k^2}\right]$, it is easy to see $\alpha_k \le \tau^{-\frac{1}{2}}$. Then we have for all $\tau \in \left[\frac{1}{\alpha_k}, \frac{1}{\alpha_k^2}\right]$ with $\frac{1}{\alpha_k} \ge \tau_0$, it holds that $\Theta(\tau) \le \tau^{-\frac{3}{2}}$. As a consequence,

$$\Theta(\tau) \le \tau^{-\frac{3}{2}}, \quad \forall\tau \ge \tau_0.$$

Substituting it into (4.17) we obtain (4.18). Together with (4.16) we derive that

$$\theta(t) = O(t^{-2}).$$

Thus, we could also get desired estimate in this case by taking $\theta_* = 0$. □

5. Concluding remarks

In this work, we start by describing the steady states for $0 < \kappa < 1$. Then we give a full classification of steady state energies and confirm the monotonicity of the odd zero up ground state energies with respect to κ. In the second part, we analyze the convergence of the steady state. We obtain that the steady state is exponentially decaying to 0 for $0 < \kappa < 1$, and present the accurate asymptotic behavior for the case $\kappa \geq 1$ up to the second term.

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