Dynamics of PDE, Vol.20, No.3, 249-262, 2023

Analyticity of the semigroup corresponding to a strongly damped wave equation with a Ventcel boundary condition

Mehdi Badra and Takéo Takahashi

Communicated by Armen Shirikyan, received September 23, 2022.

Abstract. We consider a wave equation with a structural damping coupled with an undamped wave equation located at its boundary. We prove that, due to the coupling, the full system is parabolic. In order to show that the underlying operator generates an analytical semigroup, we study in particular the effect of the damping of the "interior" wave equation on the "boundary" wave equation and show that it generates a structural damping.

Contents

1.	Introduction	249
2.	Definition and properties of some operators	252
3.	Estimation of V_λ^{-1}	257
4.	Proof of the main result	259
Appendix A.	Formal derivation of Ventcel boundary conditions	261
References		262

1. Introduction

Assume Ω is a smooth domain of $\mathbb{R}^d$ with a boundary $\partial\Omega$ that is the union of two connected components Γ_0 and Γ_1. We consider the coupling between two wave equations:

$$(1.1)\qquad \left\{\begin{aligned} \partial_{tt}u - \nu\Delta\partial_t u - \Delta u = 0 \quad & \text{in } (0,\infty)\times\Omega,\\ u = 0 \quad & \text{on } (0,\infty)\times\Gamma_0,\\ u = \eta \quad & \text{on } (0,\infty)\times\Gamma_1,\\ \partial_{tt}\eta - \kappa\Delta_b\eta - \mu\Delta_b\partial_t\eta = -\nu\partial_n\partial_t u - \partial_n u \quad & \text{in } (0,\infty)\times\Gamma_1. \end{aligned}\right.$$

2020 *Mathematics Subject Classification.* 35B65, 35L05, 35K90.

Key words and phrases. Analytic semigroups, structurally damped wave equation, Ventcel boundary condition.

The authors were partially supported by the ANR Grant "TRECOS" (ANR-20-CE40-0009).

The constants of the above system are nonnegative: $\nu, \kappa, \mu \in \mathbb{R}^+$. The equations on Γ_1 can be seen as Ventcel boundary conditions and can model a thin layer structure at the boundary, see [**6**, Appendix A]. For sake of completeness, we present in Appendix A a formal way to see how such a system can be obtained. The operator Δ_b denotes the Laplace–Beltrami operator. We have also denoted by n the unit exterior normal of Ω and by ∂_n the normal derivative.

If we take $\nu = \mu = 0$ and $\kappa > 0$, then (1.1) corresponds to an hyperbolic system and the energy of the system is constant in time. By adding the dampings $-\nu\Delta\partial_t u$ and $-\mu\Delta_b\partial_t\eta$ (with $\nu, \mu \in \mathbb{R}_+^*$), we modify the nature of the equations. These terms correspond to a Kelvin-Voigt model for a viscoelastic material: the corresponding stress depends not only on the strain but also on its time derivative (see, for instance, [**12**] and references therein). In particular, if $\nu > 0$ and $\mu > 0$ (and $\kappa > 0$), then the underlying semigroup is analytic (see [**8**, Theorem 3.3]). The same result holds for $\kappa = 0$ and $\mu = 0$ (see [**8**, Theorem 3.4]).

Note that these damping terms have also an effect on the asymptotic behavior in time of the system. Since $\nu, \kappa, \mu \in \mathbb{R}^+$, the underlying semigroup of (1.1) is contractive, and if $\nu > 0$ or $\mu > 0$ then the semigroup is strictly contractive (see [**8**, Theorem 3.1]). Finally, if $\nu > 0$, $\kappa \geqslant 0$ and $\mu \geqslant 0$ then the semigroup is exponentially stable (see [**8**, Theorem 3.2]).

In this article, we consider the case

$$\nu > 0, \quad \kappa > 0 \quad \text{and} \quad \mu = 0. \tag{1.2}$$

This means that the wave equation in Ω has a structural damping whereas there is no damping in the wave equation on Γ_1. For this case, it is proved in [**9**] that the corresponding semigroup is of Gevrey type. Our aim is to improve this result by showing that this semigroup is analytic. Let us introduce some notation in order to state our main result. We set

$$\mathcal{D}(A_1) \overset{\text{def}}{=} H^2(\Gamma_1), \quad A_1 \overset{\text{def}}{=} -\kappa\Delta_b\eta : \mathcal{D}(A_1) \to L^2(\Gamma_1). \tag{1.3}$$

Note that (see [**4**, Prop. 6.1 pp. 171] and [**10**]) for $\alpha \geqslant 0$,

$$\mathcal{D}(A_1^\alpha) = H^{2\alpha}(\Gamma_1), \quad \mathcal{D}(A_1^\alpha)' = H^{-2\alpha}(\Gamma_1),$$

where $H^s(\Gamma_1)$ denotes the Sobolev space of order s. Note that since Γ_1 is without boundary, the dual of $H^s(\Gamma_1)$ with respect to $L^2(\Gamma_1)$ is $H^{-s}(\Gamma_1)$. Using this notation, we can write (1.1) with the condition (1.2) under the following form

$$\begin{cases} \partial_{tt}u - \nu\Delta\partial_t u - \Delta u = 0 & \text{in } (0,\infty)\times\Omega, \\ u = 0 & \text{on } (0,\infty)\times\Gamma_0, \\ u = \eta & \text{on } (0,\infty)\times\Gamma_1, \\ \partial_{tt}\eta + A_1\eta = -\nu\partial_n\partial_t u - \partial_n u & \text{in } (0,\infty). \end{cases} \tag{1.4}$$

We can also write the above system as

$$\frac{d}{dt}\begin{bmatrix} u \\ \partial_t u \\ \eta \\ \partial_t\eta \end{bmatrix} = \mathcal{A}\begin{bmatrix} u \\ \partial_t u \\ \eta \\ \partial_t\eta \end{bmatrix} \tag{1.5}$$

by introducing

$$\mathcal{H} \overset{\text{def}}{=} \left\{ (u_1, u_2, \eta_1, \eta_2) \in H^2(\Omega) \times L^2(\Omega) \times \mathcal{D}\left(A_1^{3/4}\right) \times \mathcal{D}\left(A_1^{1/4}\right) \right. \\ \left. ; \ u_1 = \eta_1 \quad \text{on } \Gamma_1, \quad u_1 = 0 \quad \text{on } \Gamma_0 \right\}, \tag{1.6}$$

$$\mathcal{D}(\mathcal{A}) \overset{\text{def}}{=} \left\{ (u_1, u_2, \eta_1, \eta_2) \in \mathcal{H} \cap \left(H^2(\Omega) \times H^2(\Omega) \times \mathcal{D}\left(A_1^{5/4}\right) \times \mathcal{D}\left(A_1^{3/4}\right) \right) \right. \\ \left. ; \ u_2 = \eta_2 \text{ on } \Gamma_1, \ u_2 = 0 \text{ on } \Gamma_0 \right\}, \tag{1.7}$$

$$\mathcal{A} \begin{bmatrix} u_1 \\ u_2 \\ \eta_1 \\ \eta_2 \end{bmatrix} \overset{\text{def}}{=} \begin{bmatrix} u_2 \\ \nu \Delta u_2 + \Delta u_1 \\ \eta_2 \\ -A_1 \eta_1 - \nu \partial_n u_2 - \partial_n u_1 \end{bmatrix}. \tag{1.8}$$

We are now in a position to state our main result:

THEOREM 1.1. *The operator $\mathcal{A}$ is the infinitesimal generator of an analytic and exponentially stable semigroup on $\mathcal{H}$. In particular,*

$$\sup_{\lambda \in \mathbb{C}^+} |\lambda| \left\| (\lambda I - \mathcal{A})^{-1} \right\|_{\mathcal{L}(\mathcal{H})} < \infty. \tag{1.9}$$

In the above statement we have used the notation

$$\mathbb{C}^+ \overset{\text{def}}{=} \{ \lambda \in \mathbb{C} \ ; \ \operatorname{Re}(\lambda) \geqslant 0 \}. \tag{1.10}$$

Remark 1.2. As explained above, Theorem 1.1 is related to [**9**] where they consider the same system and show that the semigroup generated by $\mathcal{A}$ is of Gevrey class. Here we improve this result by exploiting the damping in Ω and by showing that through the coupling, it leads to a damping in the wave equation on Γ_1 that is sufficient to obtain an analytic semigroup. We can also refer to the work [**7**] where the authors consider general damping terms for wave type equations and study the regularity of the corresponding semigroups.

Remark 1.3. It is worth noting that the approach considered here has been already used in other articles devoted to fluid-structure interaction systems. In that case, the system written in Ω corresponds to the Stokes equations whereas on the boundary Γ_1, one can consider the wave equation or the beam equation, see [**3**], [**1**], [**2**]. In that cases, we showed how the viscosity of the fluid affects the wave or the beam equation and obtain that the corresponding semigroups are analytic for a wave equation and of Gevrey class for a beam equation.

The outline of the article is as follows: in the next section (that is Section 2), we define and study several operators associated with (1.4) and more precisely on the resolvent equation associated with $\mathcal{A}$. We introduce in particular the operator V_λ corresponding to the wave equation on Γ_1 with the damping operator L_λ due to the coupling with the wave equation in Ω. We obtain some estimates of V_λ^{-1} in Section 3 by first introducing an approximation of V_λ. Then in Section 4, we use these estimates to show Theorem 1.1. In Appendix A, we give a formal derivation of our model.

2. Definition and properties of some operators

In this section, we define several operators associated with the resolvent equation. The first operators, U_λ and W_λ, correspond to the wave equation in Ω. The operator L_λ is a Dirichlet to Neumann operator that allows us to describe the influence of the wave in Ω on the wave at Γ_1. We decompose this operator with the help of L_0 and several operators $K_\lambda^{(1)}$, $K_\lambda^{(2)}$, K_λ and R_λ. The operator V_λ corresponds to the operator of the wave in Γ_1 where we include the action on the wave in Ω. We show that this operator is invertible and we estimate its inverse in the next section.

2.1. The operator U_λ. Let us consider the system

$$\begin{cases} \lambda^2 u_{\lambda,f} - (\nu\lambda+1)\Delta u_{\lambda,f} = f & \text{in } \Omega, \\ u_{\lambda,f} = 0 & \text{on } \Gamma_0 \cup \Gamma_1. \end{cases} \tag{2.1}$$

In the lemma below, we recall that this system is well-posed for $\lambda \in \mathbb{C}^+$ so that we can consider the operator U_λ defined by

$$U_\lambda f \stackrel{\text{def}}{=} u_{\lambda,f}.$$

Lemma 2.1. *Assume $\lambda \in \mathbb{C}^+$. Then, for any $f \in L^2(\Omega)$, the system* (2.1) *admits a unique solution $u_{\lambda,f} \in H^2(\Omega)$. Moreover, there exists $C > 0$ such that for any $\lambda \in \mathbb{C}^+$ and for any $f \in L^2(\Omega)$,*

$$|\lambda|^2 \|u_{\lambda,f}\|_{L^2(\Omega)} + (1+|\lambda|)\|u_{\lambda,f}\|_{H^2(\Omega)} \leqslant C\|f\|_{L^2(\Omega)}. \tag{2.2}$$

In particular,

$$U_\lambda \in \mathcal{L}(L^2(\Omega), H^2(\Omega)\cap H_0^1(\Omega)), \quad \|U_\lambda\|_{\mathcal{L}(L^2(\Omega), H^2(\Omega)\cap H_0^1(\Omega))} \leqslant \frac{C}{(1+|\lambda|)}.$$

Proof. The existence and uniqueness of $u_{\lambda,f} \in H^2(\Omega)$ for $f \in L^2(\Omega)$ and $\lambda \in \mathbb{C}^+$ follows from classical results on elliptic equations: the case $\lambda = 0$ reduces to the standard Laplace equation, whereas for $\lambda \neq 0$, we first use the Lax-Milgram lemma on the variational formula

$$\int_\Omega \left(\left(\nu + \frac{1}{\lambda} \right) \nabla u \cdot \nabla \overline{v} + \lambda u \overline{v} \right) dx = \frac{1}{\lambda} \int_\Omega f \overline{v}\, dx \quad (v \in H_0^1(\Omega)).$$

Using that $\operatorname{Re}\lambda \geqslant 0$, $\operatorname{Re}\lambda^{-1} \geqslant 0$, we deduce the existence and uniqueness of a solution $u_{\lambda,f} \in H_0^1(\Omega)$ of the above system and the H^2 regularity of $u_{\lambda,f}$ is a consequence of the ellipticity of the Laplace operator.

To obtain (2.2) we first take the inner product of (2.1) with $\lambda u_{\lambda,f}$ and the real part of the result yields

$$\operatorname{Re}\lambda \|\lambda u_{\lambda,f}\|_{L^2(\Omega)}^2 + \left(\nu |\lambda|^2 + \operatorname{Re}\lambda\right) \|\nabla u_{\lambda,f}\|_{L^2(\Omega)}^2 = \|f\|_{L^2(\Omega)} \|\lambda u_{\lambda,f}\|_{L^2(\Omega)}.$$

Combining this relation with the Poincaré inequality, we deduce that

$$|\lambda|\, \|\nabla u_{\lambda,f}\|_{L^2(\Omega)} \leqslant C\|f\|_{L^2(\Omega)}. \tag{2.3}$$

Then, taking the inner product of (2.1) with $-(\nu\lambda+1)\Delta u_{\lambda,f}$ and considering the real part of the result, we find

$$\nu \operatorname{Re}\lambda \|\lambda \nabla u_{\lambda,f}\|_{L^2(\Omega)}^2 + |\nu\lambda+1|^2 \|\Delta u_{\lambda,f}\|_{L^2(\Omega)}^2 \leqslant \|f\|_{L^2(\Omega)} |\nu\lambda+1| \|\Delta u_{\lambda,f}\|_{L^2(\Omega)} + |\lambda|^2 \|\nabla u_{\lambda,f}\|_{L^2(\Omega)}^2.$$

The above equation and (2.3) yield

$$|\nu\lambda + 1| \, \|\Delta u_{\lambda,f}\|_{L^2(\Omega)} \leqslant C\|f\|_{L^2(\Omega)}.$$

Hence, the required estimates of $\|u_{\lambda,f}\|_{H^2(\Omega)}$ follows from H^2-elliptic regularity of the Laplace operator and the estimate of $\|u_{\lambda,f}\|_{L^2(\Omega)}$ is then obtained from (2.1). □

2.2. The operators W_λ. Let us now consider the system:

$$\begin{cases} \lambda^2 w_{\lambda,\eta} - (\nu\lambda + 1)\Delta w_{\lambda,\eta} = 0 & \text{in } \Omega, \\ w_{\lambda,\eta} = 0 & \text{on } \Gamma_0, \\ w_{\lambda,\eta} = \eta & \text{on } \Gamma_1. \end{cases} \tag{2.4}$$

By using Lemma 2.1 and a standard transposition method, the above system is well-posed and the operator W_λ defined by

$$W_\lambda \eta \overset{\text{def}}{=} w_{\lambda,\eta}$$

satisfies

$$W_\lambda \in \mathcal{L}(H^{3/2}(\Gamma_1), H^2(\Omega)) \cap \mathcal{L}(H^{-1/2}(\Gamma_1), L^2(\Omega)) \cap \mathcal{L}(H^{1/2}(\Gamma_1), H^1(\Omega)). \tag{2.5}$$

In order to get estimates in the corresponding norms, we first note that

$$\begin{cases} -\Delta w_{0,\eta} = 0 & \text{in } \Omega, \\ w_{0,\eta} = 0 & \text{on } \Gamma_0, \\ w_{0,\eta} = \eta & \text{on } \Gamma_1, \end{cases} \tag{2.6}$$

and there exists a constant $C > 0$ such that

$$\|w_{0,\eta}\|_{H^2(\Omega)} \leqslant C \|\eta\|_{H^{3/2}(\Gamma_1)}, \quad \|w_{0,\eta}\|_{H^1(\Omega)} \leqslant C \|\eta\|_{H^{1/2}(\Gamma_1)}, \quad \|w_{0,\eta}\|_{L^2(\Omega)} \leqslant C \|\eta\|_{H^{-1/2}(\Gamma_1)}. \tag{2.7}$$

Then, we write

$$w_{\lambda,\eta} = w_{0,\eta} + \lambda z_{\lambda,\eta} \tag{2.8}$$

where

$$\begin{cases} \lambda^2 z_{\lambda,\eta} - (\nu\lambda + 1)\Delta z_{\lambda,\eta} = -\lambda w_{0,\eta} & \text{in } \Omega, \\ z_{\lambda,\eta} = 0 & \text{on } \Gamma_0 \cup \Gamma_1. \end{cases} \tag{2.9}$$

We can apply Lemma 2.1 with (2.7), and we deduce that for any $s \in [0, 2]$, there exists $C > 0$ such that

$$|\lambda|^{1-\frac{s}{2}} \|z_{\lambda,\eta}\|_{H^s(\Omega)} \leqslant C \|\eta\|_{H^{-1/2}(\Gamma_1)} \quad (\eta \in H^{-1/2}(\Gamma_1)). \tag{2.10}$$

In particular, combining the above relation with (2.7), there exists $C > 0$ such that for any $\lambda \in \mathbb{C}^+$,

$$\|W_\lambda\|_{\mathcal{L}(H^{-1/2}(\Gamma_1), L^2(\Omega))} \leqslant C, \tag{2.11}$$

and

$$\|W_\lambda \eta\|_{H^2(\Omega)} \leqslant C \left(\|\eta\|_{H^{3/2}(\Gamma_1)} + |\lambda| \, \|\eta\|_{H^{-1/2}(\Gamma_1)} \right). \tag{2.12}$$

2.3. The operators L_λ. We define

$$L_\lambda \eta \overset{\text{def}}{=} \partial_n w_{\lambda,\eta} = \partial_n W_\lambda \eta, \tag{2.13}$$

where $w_{\lambda,\eta}$ is the solution of (2.4). From (2.5), we have

$$L_\lambda \in \mathcal{L}(H^{3/2}(\Gamma_1), H^{1/2}(\Gamma_1)). \tag{2.14}$$

Taking the inner product of the first equation of (2.4) by $w_{\lambda,\widetilde{\eta}}$ and using (2.5), we obtain for $\widetilde{\eta} \in H^{1/2}(\Gamma_1)$

$$(\nu\lambda+1)\, \langle L_\lambda \eta, \widetilde{\eta}\rangle_{H^{-1/2}(\Gamma_1),H^{1/2}(\Gamma_1)} = \lambda^2 \int_\Omega w_{\lambda,\eta}\overline{w_{\lambda,\widetilde{\eta}}}\, dx + (\nu\lambda+1) \int_\Omega \nabla w_{\lambda,\eta}\cdot\nabla\overline{w_{\lambda,\widetilde{\eta}}}\, dx \tag{2.15}$$

and in particular, we can extend L_λ as

$$L_\lambda \in \mathcal{L}(H^{1/2}(\Gamma_1), H^{-1/2}(\Gamma_1)). \tag{2.16}$$

In the case $\lambda = 0$, we deduce from (2.15)

$$\langle L_0 \eta, \widetilde{\eta}\rangle_{H^{-1/2}(\Gamma_1),H^{1/2}(\Gamma_1)} = \int_\Omega \nabla w_{0,\eta} \cdot \nabla \overline{w_{0,\widetilde{\eta}}}\, dx$$

and the following result:

Proposition 2.2. The operator $L_0 \in \mathcal{L}(H^{1/2}(\Gamma_1), H^{-1/2}(\Gamma_1))$ is self-adjoint and there exists $\rho_1 > 0$ such that

$$\langle L_0 \eta, \eta\rangle_{H^{-1/2}(\Gamma_1),H^{1/2}(\Gamma_1)} \geqslant \rho_1 \|\eta\|^2_{H^{1/2}(\Gamma_1)} \quad (\eta \in H^{1/2}(\Gamma_1)). \tag{2.17}$$

Taking the inner product of the first equation of (2.6) by $z_{\lambda,\widetilde{\eta}}$ we deduce that

$$\int_\Omega \nabla w_{0,\eta} \cdot \nabla \overline{z_{\lambda,\widetilde{\eta}}}\, dx = 0. \tag{2.18}$$

Using the decomposition (2.8) into (2.15) and (2.18), we deduce

$$\begin{aligned}(\nu\lambda+1)\, \langle L_\lambda \eta, \widetilde{\eta}\rangle_{H^{-1/2}(\Gamma_1),H^{1/2}(\Gamma_1)} &= (\nu\lambda+1)\, \langle L_0 \eta, \widetilde{\eta}\rangle_{H^{-1/2}(\Gamma_1),H^{1/2}(\Gamma_1)} \\ &+ \lambda^2 \int_\Omega w_{\lambda,\eta}\overline{w_{\lambda,\widetilde{\eta}}}\, dx + (\nu\lambda+1)|\lambda|^2 \int_\Omega \nabla z_{\lambda,\eta}\cdot\nabla\overline{z_{\lambda,\widetilde{\eta}}}\, dx.\end{aligned} \tag{2.19}$$

2.4. The operators $K_\lambda^{(1)}$, $K_\lambda^{(2)}$, K_λ and R_λ. Next, using (2.5) and (2.10), we define for $\eta \in H^{-1/2}(\Gamma_1)$,

$$\langle K_\lambda^{(1)} \eta, \widetilde{\eta}\rangle_{H^{1/2}(\Gamma_1),H^{-1/2}(\Gamma_1)} \overset{\text{def}}{=} \int_\Omega w_{\lambda,\eta}\overline{w_{\lambda,\widetilde{\eta}}}\, dx, \tag{2.20}$$

$$\langle K_\lambda^{(2)} \eta, \widetilde{\eta}\rangle_{H^{1/2}(\Gamma_1),H^{-1/2}(\Gamma_1)} \overset{\text{def}}{=} \int_\Omega \nabla z_{\lambda,\eta}\cdot\nabla\overline{z_{\lambda,\widetilde{\eta}}}\, dx, \tag{2.21}$$

$$K_\lambda \overset{\text{def}}{=} K_\lambda^{(1)} + \nu\overline{\lambda} K_\lambda^{(2)}, \tag{2.22}$$

and

$$R_\lambda \overset{\text{def}}{=} L_0 + |\lambda|^2 K_\lambda^{(2)}. \tag{2.23}$$

Therefore (2.19) can be written as

$$(\nu\lambda+1)L_\lambda = (\nu\lambda+1)L_0 + \lambda^2 K_\lambda^{(1)} + (\nu\lambda+1)|\lambda|^2 K_\lambda^{(2)} = \nu\lambda L_0 + \lambda^2 K_\lambda + R_\lambda. \tag{2.24}$$

We have the following properties on K_λ:

Lemma 2.3. *For $\lambda \in \mathbb{C}^+$, $K_\lambda^{(1)}$, $K_\lambda^{(2)} \in \mathcal{L}(H^{-1/2}(\Gamma_1), H^{1/2}(\Gamma_1))$ and there exists a constant $C > 0$ such that for any $\lambda \in \mathbb{C}^+$,*

$$\left\| K_\lambda^{(1)} \right\|_{\mathcal{L}(H^{-1/2}(\Gamma_1),H^{1/2}(\Gamma_1))} + |\lambda| \left\| K_\lambda^{(2)} \right\|_{\mathcal{L}(H^{-1/2}(\Gamma_1),H^{1/2}(\Gamma_1))} \leqslant C$$

and

$$\begin{cases} \langle K_\lambda^{(1)} \eta, \eta \rangle_{H^{1/2}(\Gamma_1),H^{-1/2}(\Gamma_1)} \geqslant 0 \\ \langle K_\lambda^{(2)} \eta, \eta \rangle_{H^{1/2}(\Gamma_1),H^{-1/2}(\Gamma_1)} \geqslant 0 \end{cases} \qquad (\eta \in H^{-1/2}(\Gamma_1)).$$

Proof. The bound on $K_\lambda^{(1)}$ comes from (2.20) and (2.11) and the bound on $K_\lambda^{(2)}$ comes from (2.21) and (2.10). □

Lemma 2.4. *There exists a constant $C > 0$ such that the operator K_λ defined by (2.22) satisfies for any $\lambda \in \mathbb{C}^+$:*

$$\|(I + K_\lambda)\eta\|_{L^2(\Gamma_1)} \geqslant \|\eta\|_{L^2(\Gamma_1)} \quad (\eta \in L^2(\Gamma_1)), \tag{2.25}$$

$$\|K_\lambda \eta\|_{H^{1/2}(\Gamma_1)} \leqslant C\|\eta\|_{H^{-1/2}(\Gamma_1)} \quad (\eta \in H^{-1/2}(\Gamma_1)), \tag{2.26}$$

$$\|\eta\|_{H^{1/2}(\Gamma_1)} \leqslant C\|(I + K_\lambda)\eta\|_{H^{1/2}(\Gamma_1)} \quad (\eta \in H^{1/2}(\Gamma_1)), \tag{2.27}$$

$$\|(I + K_\lambda^*)\eta\|_{L^2(\Gamma_1)} \geqslant \|\eta\|_{L^2(\Gamma_1)} \quad (\eta \in L^2(\Gamma_1)), \tag{2.28}$$

$$\|K_\lambda^* \eta\|_{H^{1/2}(\Gamma_1)} \leqslant C\|\eta\|_{H^{-1/2}(\Gamma_1)} \quad (\eta \in H^{-1/2}(\Gamma_1)), \tag{2.29}$$

$$\|\eta\|_{H^{1/2}(\Gamma_1)} \leqslant C\|(I + K_\lambda^*)\eta\|_{H^{1/2}(\Gamma_1)} \quad (\eta \in H^{1/2}(\Gamma_1)). \tag{2.30}$$

Proof. We deduce relations (2.25), (2.26), (2.28), and (2.29) from Lemma 2.3. For (2.27), we use (2.25) and (2.26):

$$\begin{aligned} \|\eta\|_{H^{1/2}(\Gamma_1)} &\leqslant \|(I + K_\lambda)\eta\|_{H^{1/2}(\Gamma_1)} + \|K_\lambda \eta\|_{H^{1/2}(\Gamma_1)} \\ &\leqslant \|(I + K_\lambda)\eta\|_{H^{1/2}(\Gamma_1)} + C\,\|\eta\|_{H^{-1/2}(\Gamma_1)} \\ &\leqslant \|(I + K_\lambda)\eta\|_{H^{1/2}(\Gamma_1)} + C\,\|(I + K_\lambda)\eta\|_{L^2(\Gamma_1)} \leqslant C\,\|(I + K_\lambda)\eta\|_{H^{1/2}(\Gamma_1)}. \end{aligned}$$

We deduce (2.30) similarly. □

Lemma 2.5. *There exists a constant $C > 0$ such that the operator R_λ defined by (2.23) satisfies for any $\lambda \in \mathbb{C}^+$:*

$$\|R_\lambda \eta\|_{H^{1/2}(\Gamma_1)} \leqslant C\left(\|\eta\|_{H^{3/2}(\Gamma_1)} + |\lambda|\|\eta\|_{H^{-1/2}(\Gamma_1)}\right). \tag{2.31}$$

Proof. This is a consequence of (2.14) and Lemma 2.3. □

2.5. The operator V_λ. Let us define

$$V_\lambda \stackrel{\text{def}}{=} \lambda^2 I + A_1 + (\nu\lambda + 1)L_\lambda. \tag{2.32}$$

From (2.24), we have

$$V_\lambda = \lambda^2 (I + K_\lambda) + \nu\lambda L_0 + A_1 + R_\lambda. \tag{2.33}$$

First, we can show the following result

Lemma 2.6. *For any $\lambda \in \mathbb{C}^+$, the operators $V_\lambda : \mathcal{D}(A_1^{3/4}) \to \mathcal{D}(A_1^{1/4})'$ and $V_\lambda : \mathcal{D}(A_1^{5/4}) \to \mathcal{D}(A_1^{1/4})$ are isomorphisms.*

PROOF. We divide the proof into three cases:
Case 1: $\operatorname{Re}\lambda > 0$. For any $g \in \mathcal{D}(A_1^{1/4})'$, one has to show the existence and uniqueness of $\eta \in \mathcal{D}(A_1^{3/4})$ such that

$$\frac{V_\lambda \eta}{\lambda} = \left(\lambda I + \frac{1}{\lambda} A_1 + \left(\nu + \frac{1}{\lambda}\right) L_\lambda\right) \eta = g. \tag{2.34}$$

From (2.24), the above relation can be written as

$$\left(\lambda I + \frac{1}{\lambda} A_1 + \left(\nu + \frac{1}{\lambda}\right) L_0 + \lambda K_\lambda^{(1)} + \left(\nu + \frac{1}{\lambda}\right) |\lambda|^2 K_\lambda^{(2)}\right) \eta = g.$$

We then consider the variational formulation of the above relation

$$\begin{aligned} &\lambda \left(\eta, \widetilde{\eta}\right)_{L^2(\Gamma_1)} + \frac{1}{\lambda}\left(A_1^{1/2}\eta, A_1^{1/2}\widetilde{\eta}\right)_{L^2(\Gamma_1)} + \left(\nu + \frac{1}{\lambda}\right) \langle L_0 \eta, \widetilde{\eta}\rangle_{H^{-1/2}(\Gamma_1), H^{1/2}(\Gamma_1)} \\ &+ \lambda \left(K_\lambda^{(1)}\eta, \widetilde{\eta}\right)_{L^2(\Gamma_1)} + \left(\nu + \frac{1}{\lambda}\right) |\lambda|^2 \left(K_\lambda^{(2)}\eta, \widetilde{\eta}\right)_{L^2(\Gamma_1)} = \langle g, \widetilde{\eta}\rangle_{H^{-1/2}(\Gamma_1), H^{1/2}(\Gamma_1)} \\ &\qquad\qquad \left(\widetilde{\eta} \in \mathcal{D}(A_1^{1/2})\right). \end{aligned} \tag{2.35}$$

From Proposition 2.2 and Lemma 2.3 and using that $\operatorname{Re}\lambda > 0$, we can apply the Lax-Milgram lemma and we deduce the existence and uniqueness of $\eta \in \mathcal{D}(A_1^{1/2})$ satisfying (2.34). From (2.16), we deduce that $A_1\eta \in \mathcal{D}(A_1^{1/4})'$ and thus that $\eta \in \mathcal{D}(A_1^{3/4})$. If moreover, $g \in \mathcal{D}(A_1^{1/4})$, then we deduce from (2.14) that $A_1\eta \in \mathcal{D}(A_1^{1/4})$ and thus that $\eta \in \mathcal{D}(A_1^{5/4})$.
Case 2: $\operatorname{Re}\lambda = 0$, $\lambda \neq 0$. From Lemma 2.3 and Proposition 2.2, there exists $\varepsilon \in \left(0, \frac{\pi}{2}\right)$ (depending on λ) such that

$$\nu \langle L_0 \eta, \eta\rangle_{H^{-1/2}(\Gamma_1), H^{1/2}(\Gamma_1)} \geqslant |\lambda| \tan(\varepsilon) \left(\|\eta\|^2_{L^2(\Gamma_1)} + \left(K_\lambda^{(1)}\eta, \eta\right)_{L^2(\Gamma_1)}\right). \tag{2.36}$$

We define $\theta \stackrel{\text{def}}{=} \varepsilon$ if $\operatorname{Im}\lambda > 0$ and $\theta \stackrel{\text{def}}{=} -\varepsilon$ if $\operatorname{Im}\lambda < 0$ so that

$$\operatorname{Re}\left(\frac{e^{i\theta}}{\lambda}\right) = \frac{\sin\varepsilon}{|\lambda|} > 0, \quad \operatorname{Re}\left(\lambda e^{i\theta}\right) = -|\lambda| \sin\varepsilon < 0.$$

Now, we replace (2.34) by

$$e^{i\theta} \frac{V_\lambda \eta}{\lambda} = \left(\lambda e^{i\theta} I + \frac{e^{i\theta}}{\lambda} A_1 + \left(\nu e^{i\theta} + \frac{e^{i\theta}}{\lambda}\right) L_\lambda\right) \eta = g. \tag{2.37}$$

The corresponding variational formulation is

$$\begin{aligned} &\lambda e^{i\theta} \left(\eta, \widetilde{\eta}\right)_{L^2(\Gamma_1)} + \frac{e^{i\theta}}{\lambda} \left(A_1^{1/2}\eta, A_1^{1/2}\widetilde{\eta}\right)_{L^2(\Gamma_1)} \\ &\qquad + \left(\nu e^{i\theta} + \frac{e^{i\theta}}{\lambda}\right) \langle L_0\eta, \widetilde{\eta}\rangle_{H^{-1/2}(\Gamma_1), H^{1/2}(\Gamma_1)} + \lambda e^{i\theta} \left(K_\lambda^{(1)}\eta, \widetilde{\eta}\right)_{L^2(\Gamma_1)} \\ &+ \left(\nu e^{i\theta} + \frac{e^{i\theta}}{\lambda}\right) |\lambda|^2 \left(K_\lambda^{(2)}\eta, \widetilde{\eta}\right)_{L^2(\Gamma_1)} = \langle g, \widetilde{\eta}\rangle_{H^{-1/2}(\Gamma_1), H^{1/2}(\Gamma_1)} \quad \left(\widetilde{\eta} \in \mathcal{D}(A_1^{1/2})\right). \end{aligned}$$

We can apply the Lax-Milgram lemma since Lemma 2.3, Proposition 2.2 and (2.36) yield

$$\operatorname{Re}\left(\lambda e^{i\theta}\right)\|\eta\|^2_{L^2(\Gamma_1)} + \operatorname{Re}\left(\frac{e^{i\theta}}{\lambda}\right)\left\|A_1^{1/2}\eta\right\|^2_{L^2(\Gamma_1)}$$
$$+\left(\nu\cos\theta + \operatorname{Re}\left(\frac{e^{i\theta}}{\lambda}\right)\right)\langle L_0\eta,\eta\rangle_{H^{-1/2}(\Gamma_1),H^{1/2}(\Gamma_1)} + \operatorname{Re}\left(\lambda e^{i\theta}\right)\left(K_\lambda^{(1)}\eta,\eta\right)_{L^2(\Gamma_1)}$$
$$+\left(\nu\cos(\theta) + \operatorname{Re}\left(\frac{e^{i\theta}}{\lambda}\right)\right)|\lambda|^2\left(K_\lambda^{(2)}\eta,\eta\right)_{L^2(\Gamma_1)} \geqslant \frac{\sin\varepsilon}{|\lambda|}\left\|A_1^{1/2}\eta\right\|^2_{L^2(\Gamma_1)}$$

The proof follows then as in Case 1.

Case 3: $\lambda = 0$. For any $g \in \mathcal{D}(A_1^{1/4})'$, one has to show the existence and uniqueness of $\eta \in \mathcal{D}(A_1^{3/4})$ such that

$$V_0\eta = (A_1 + L_0)\,\eta = g.$$

Using Proposition 2.2, we can again apply the Lax-Milgram lemma and conclude as in Case 1. □

3. Estimation of V_λ^{-1}

In the previous section, we have defined V_λ by (2.32) (see also (2.33)) and we have shown that it is invertible. We now estimate its inverse. First, we introduce the notation

$$\mathbb{C}^+_\alpha \stackrel{\text{def}}{=} \left\{\lambda \in \mathbb{C}^+ \ ;\ |\lambda| > \alpha\right\}. \tag{3.1}$$

The main result of this section is the following:

Theorem 3.1. *There exists $\alpha > 0$ such that for $\lambda \in \mathbb{C}^+_\alpha$ and for $\theta, \beta \in [-1/4, 3/4]$ with $0 \leqslant \theta + \beta \leqslant 1$, the following estimate holds*

$$\sup_{\lambda\in\mathbb{C}^+_\alpha} |\lambda|^{2-2\theta-2\beta}\|A_1^\theta V_\lambda^{-1} A_1^\beta\|_{\mathcal{L}(L^2(\Gamma_1))} < +\infty. \tag{3.2}$$

In order to prove Theorem 3.1, we consider the following "approximation" of V_λ:

$$\widetilde{V}_\lambda \stackrel{\text{def}}{=} \lambda^2(I + K_\lambda) + 2\rho\lambda A_1^{1/2} + A_1 + R_\lambda. \tag{3.3}$$

Comparing (2.33) and the above relation, we have

$$V_\lambda - \widetilde{V}_\lambda = \lambda S, \tag{3.4}$$

where

$$S \stackrel{\text{def}}{=} \nu L_0 - 2\rho A_1^{1/2} : \mathcal{D}(A_1^{1/4}) \to \mathcal{D}(A_1^{1/4})'. \tag{3.5}$$

Using Proposition 2.2, there exists $\rho > 0$ small enough such that S is a positive self-adjoint operator. In what follows, we fix $\rho > 0$ so that it satisfies this property. We are going to estimate the inverse of $\widetilde{V}_\lambda$, see Theorem 3.3 to prove Theorem 3.1.

First, we recall the following result that can be found in [**3**, Lemma 3.4].

Lemma 3.2. *There exists a constant C_0 such that for all $\lambda \in \mathbb{C}^+$,*

$$\left\|(\lambda^2 I + 2\rho\lambda A_1^{1/2} + A_1)\eta\right\|_{L^2(\Gamma_1)} \geqslant C_0\left(|\lambda|^2\|\eta\|_{L^2(\Gamma_1)} + \|A_1\eta\|_{L^2(\Gamma_1)}\right) \qquad (\eta \in \mathcal{D}(A_1)). \tag{3.6}$$

THEOREM 3.3. *There exists $\alpha > 0$ such that for all $\lambda \in \mathbb{C}_\alpha^+$ the operator $\widetilde{V}_\lambda : \mathcal{D}(A_1^{5/4}) \to \mathcal{D}(A_1^{1/4})$ is an isomorphism and for $\theta, \beta \in [-1/4, 5/4]$ such that $0 \leqslant \theta + \beta \leqslant 1$, the following estimates hold*

$$\sup_{\lambda \in \mathbb{C}_\alpha^+} |\lambda|^{2-2\theta-2\beta} \|A_1^\theta \widetilde{V}_\lambda^{-1} A_1^\beta\|_{\mathcal{L}(L^2(\Gamma_1))} < +\infty, \tag{3.7}$$

$$\sup_{\lambda \in \mathbb{C}_\alpha^+} |\lambda|^{2-2\theta-2\beta} \left\| A_1^\theta \left(\widetilde{V}_\lambda^*\right)^{-1} A_1^\beta \right\|_{\mathcal{L}(L^2(\Gamma_1))} < +\infty. \tag{3.8}$$

PROOF. We write (3.3) as

$$\widetilde{V}_\lambda = (I + K_\lambda)\left[\lambda^2 + 2\rho\lambda A_1^{1/2} + A_1\right] - K_\lambda\left[2\rho\lambda A_1^{1/2} + A_1\right] + R_\lambda. \tag{3.9}$$

From Lemma 2.4 and Lemma 3.2, there exists a constant $c > 0$ such that

$$\left\|(I + K_\lambda)\left[\lambda^2\eta + 2\rho\lambda A_1^{1/2}\eta + A_1\eta\right]\right\|_{H^{1/2}(\Gamma_1)} \geqslant c\left(|\lambda|^2 \|A_1^{1/4}\eta\|_{L^2(\Gamma_1)} + \|A_1^{5/4}\eta\|_{L^2(\Gamma_1)}\right). \tag{3.10}$$

Combining Lemma 2.4 and Lemma 2.5, we obtain a constant $C > 0$ such that for $\lambda \in \mathbb{C}^+$,

$$\left\|K_\lambda\left[2\rho\lambda A_1^{1/2} + A_1\right]\eta\right\|_{H^{1/2}(\Gamma_1)} + \|R_\lambda\eta\|_{H^{1/2}(\Gamma_1)} \leqslant C\left(|\lambda|\left\|A_1^{1/4}\eta\right\|_{L^2(\Gamma_1)} + \left\|A_1^{3/4}\eta\right\|_{L^2(\Gamma_1)}\right).$$

Using an interpolation inequality and the Young inequality, we deduce from the above estimate that

$$\left\|K_\lambda\left[2\rho\lambda A_1^{1/2} + A_1\right]\eta\right\|_{H^{1/2}(\Gamma_1)} + \|R_\lambda\eta\|_{H^{1/2}(\Gamma_1)} \leqslant C|\lambda|^{-1}\left(|\lambda|^2\|A_1^{1/4}\eta\|_{L^2(\Gamma_1)} + \|A_1^{5/4}\eta\|_{L^2(\Gamma_1)}\right)$$

and thus, with (3.9) and (3.10), we deduce that for α large enough, and for $\lambda \in \mathbb{C}_\alpha^+$,

$$\left\|\widetilde{V}_\lambda\eta\right\|_{H^{1/2}(\Gamma_1)} \geqslant C\left(|\lambda|^2\|A_1^{1/4}\eta\|_{L^2(\Gamma_1)} + \|A_1^{5/4}\eta\|_{L^2(\Gamma_1)}\right). \tag{3.11}$$

Since

$$\widetilde{V}_\lambda^* = \overline{\lambda}^2(I + K_\lambda^*) + 2\rho\overline{\lambda}A_1^{1/2} + A_1 + R_\lambda\eta,$$

and since K_λ^* satisfies the same properties as K_λ, we also deduce that for α large enough, and for $\lambda \in \mathbb{C}_\alpha^+$,

$$\left\|\widetilde{V}_\lambda^*\eta\right\|_{H^{1/2}(\Gamma_1)} \geqslant C\left(|\lambda|^2\|A_1^{1/4}\eta\|_{L^2(\Gamma_1)} + \|A_1^{5/4}\eta\|_{L^2(\Gamma_1)}\right). \tag{3.12}$$

From (3.11), we deduce that $\widetilde{V}_\lambda : \mathcal{D}(A_1^{5/4}) \to \mathcal{D}(A_1^{1/4})$ is a closed operator and has a closed range: if $(\eta_n)_n$ is a sequence of $\mathcal{D}(A_1^{5/4})$ such that $\left(\widetilde{V}_\lambda\eta_n\right)$ is convergent, then (3.11) yields that $(\eta_n)_n$ is a Cauchy sequence of $\mathcal{D}(A_1^{5/4})$ and we deduce that $(\eta_n)_n$ is convergent in $\mathcal{D}(A_1^{5/4})$ which yields the result.

Using [**5**, Corollary II.17 (iv), p.28], we deduce from (3.11) and (3.12) that $\widetilde{V}_\lambda$ is invertible. Moreover, these relations also imply (3.7) and (3.8) for $(\theta, \beta) = (1/4, -1/4)$ and $(\theta, \beta) = (5/4, -1/4)$. By interpolation, this yields (3.7) and (3.8) for $(\theta,-1/4)$, $\theta\in[1/4,5/4]$. By a duality argument, we obtain (3.7) and (3.8) for $(-1/4,\beta)$, $\beta\in[1/4,5/4]$. Then, by interpolating (3.7) and (3.8) between $(1/4+\kappa,-1/4)$ and $(-1/4,1/4+\kappa)$ for $\kappa\in[0,1]$, we deduce the result. ☐

We are now in a position to prove Theorem 3.1.

Proof of Theorem 3.1. The proof is the same as the proof of Theorem 3.6 in [**3**]. The main idea to use (3.4) and (3.5) in order to compare V_λ^{-1} and $\widetilde{V}_\lambda^{-1}$. The key point in the proof of Theorem 3.6 in [**3**] is the relation $\operatorname{Re}\langle\lambda\zeta,\widetilde{V}_\lambda\zeta\rangle_{L^2(\Gamma_1)}\geqslant 0$. Here, we can show this relation by using (3.3), combined with (2.22) and (2.23): for any $\zeta\in\mathcal{D}(A_1)$,

$$\begin{aligned}\operatorname{Re}\langle\widetilde{V}_\lambda\zeta,\lambda\zeta\rangle_{L^2(\Gamma_1)}&=\operatorname{Re}\lambda\|\lambda\zeta\|^2_{L^2(\Gamma_1)}+|\lambda|^2\operatorname{Re}\lambda\langle\zeta,K_\lambda^{(1)}\zeta\rangle_{L^2(\Gamma_1)}\\&+\nu|\lambda|^4\langle\zeta,K_\lambda^{(2)}\zeta\rangle_{L^2(\Gamma_1)}+|\lambda|^2\operatorname{Re}\lambda\langle\zeta,K_\lambda^{(2)}\zeta\rangle_{L^2(\Gamma_1)}\\&+2\rho|\lambda|^2\|A_1^{1/4}\zeta\|^2_{L^2(\Gamma_1)}+\operatorname{Re}\lambda\|A_1^{1/2}\zeta\|^2_{L^2(\Gamma_1)}+\operatorname{Re}\lambda\langle\zeta,L_0\zeta\rangle_{L^2(\Gamma_1)}\end{aligned}\tag{3.13}$$

and we conclude by using Lemma 2.3 and Proposition 2.2. ☐

4. Proof of the main result

We are now in a position to prove Theorem 1.1. Assume

$$F=(f_1,f_2,g_1,g_2)\in H^2(\Omega)\times L^2(\Omega)\times\mathcal{D}\left(A_1^{3/4}\right)\times\mathcal{D}\left(A_1^{1/4}\right).\tag{4.1}$$

First, we show that we can solve the following equation for $\lambda\in\mathbb{C}^+$:

$$(\lambda I-\mathcal{A})\,U=F.\tag{4.2}$$

Writing

$$U=(u_1,u_2,\eta_1,\eta_2),$$

the above equation can be written as

$$\left\{\begin{aligned}\lambda u_1-u_2&=f_1\quad\text{in }\Omega,\\\lambda u_2-\nu\Delta u_2-\Delta u_1&=f_2\quad\text{in }\Omega,\\u_1=u_2&=0\quad\text{on }\Gamma_0,\\u_1=\eta_1,\ u_2&=\eta_2\quad\text{on }\Gamma_1,\\\lambda\eta_1-\eta_2&=g_1\quad\text{in }\Gamma_1,\\\lambda\eta_2+A_1\eta_1+\nu\partial_n u_2+\partial_n u_1&=g_2\quad\text{in }\Gamma_1.\end{aligned}\right.\tag{4.3}$$

Step 1. Here we show

$$\mathbb{C}^+\subset\rho(\mathcal{A}).\tag{4.4}$$

In order to do this, we deduce from the first two equations of (4.3) that

$$\lambda^2u_2-\lambda\nu\Delta u_2-\Delta u_2=\lambda f_2+\Delta f_1$$

and

$$\lambda^2u_1-\lambda\nu\Delta u_1-\Delta u_1=\lambda f_1-\nu\Delta f_1+f_2$$

so that

$$u_2=W_\lambda\eta_2+U_\lambda\left(\lambda f_2+\Delta f_1\right)\tag{4.5}$$

and

$$u_1=W_\lambda\eta_1+U_\lambda\left(\lambda f_1-\nu\Delta f_1+f_2\right).\tag{4.6}$$

Combining similarly the last two equations of (4.3), we find

$$\lambda^2\eta_2 + A_1\eta_2 + (\nu\lambda+1)\partial_n u_2 = \lambda g_2 - A_1 g_1 - \partial_n f_1$$

and

$$\lambda^2\eta_1 + A_1\eta_1 + (\nu\lambda+1)\partial_n u_1 = \lambda g_1 + g_2 - \nu\partial_n f_1.$$

Now, we use (4.5), (4.6), (2.13), and (2.32) to write the two previous equations as follows

$$V_\lambda\eta_2 = \lambda g_2 - A_1 g_1 - \partial_n f_1 - (\nu\lambda+1)\partial_n U_\lambda\left(\lambda f_2 + \Delta f_1\right), \tag{4.7}$$

$$V_\lambda\eta_1 = \lambda g_1 + g_2 - \nu\partial_n f_1 - (\nu\lambda+1)\partial_n U_\lambda\left(\lambda f_1 - \nu\Delta f_1 + f_2\right). \tag{4.8}$$

From (4.1) and Lemma 2.1, we deduce that the right-hand side of (4.7) and (4.8) are respectively in $\mathcal{D}(A_1^{1/4})'$ and $\mathcal{D}(A_1^{1/4})$. Thus from Lemma 2.6, there exists a unique solution $(\eta_1,\eta_2)\in\mathcal{D}(A_1^{5/4})\times\mathcal{D}(A_1^{3/4})$ of (4.7) and (4.8). Then (4.5), (4.6), combined with Lemma 2.1 and (2.5) yield $(u_1,u_2)\in H^2(\Omega)^2$. We have solved (4.2).

Step 2. We now show the estimate (1.9). Let us consider α from Theorem 3.1. From the above step, using the continuity of the resolvent, we already have

$$\sup_{\lambda\in\mathbb{C}^+,|\lambda|\leqslant\alpha}|\lambda|\left\|(\lambda I-\mathcal{A})^{-1}\right\|_{\mathcal{L}(\mathcal{H})}<\infty.$$

It is thus sufficient to show

$$\sup_{\lambda\in\mathbb{C}^+_\alpha}|\lambda|\left\|(\lambda I-\mathcal{A})^{-1}\right\|_{\mathcal{L}(\mathcal{H})}<\infty \tag{4.9}$$

to obtain (1.9). In order to do this, we use (4.7) to write

$$\eta_2 = -V_\lambda^{-1}\partial_n f_1 - (\nu\lambda+1)V_\lambda^{-1}\partial_n U_\lambda\left(\Delta f_1\right) - \lambda(\nu\lambda+1)V_\lambda^{-1}\partial_n U_\lambda f_2 - V_\lambda^{-1}A_1 g_1 + \lambda V_\lambda^{-1} g_2. \tag{4.10}$$

From Theorem 3.1 (with $(\theta,\beta)=(1/4,-1/4)$ or $(\theta,\beta)=(1/4,1/4)$) and Lemma 2.1, we have

$$\left\|\lambda A_1^{1/4}V_\lambda^{-1}\partial_n f_1 + \lambda(\nu\lambda+1)A_1^{1/4}V_\lambda^{-1}\partial_n U_\lambda\left(\Delta f_1\right)\right\|_{L^2(\Gamma_1)} \leqslant C\left\|f_1\right\|_{H^2(\Omega)}, \tag{4.11}$$

$$\left\|\lambda^2(\nu\lambda+1)A_1^{1/4}V_\lambda^{-1}\partial_n U_\lambda f_2\right\|_{L^2(\Gamma_1)} \leqslant C\left\|(\nu\lambda+1)U_\lambda f_2\right\|_{H^2(\Omega)} \leqslant C\left\|f_2\right\|_{L^2(\Omega)}, \tag{4.12}$$

$$\left\|\lambda A_1^{1/4}V_\lambda^{-1}A_1 g_1\right\|_{L^2(\Gamma_1)} \leqslant C\left\|A_1^{3/4}g_1\right\|_{L^2(\Gamma_1)}, \tag{4.13}$$

$$\left\|\lambda^2 A_1^{1/4}V_\lambda^{-1}g_2\right\|_{L^2(\Gamma_1)} \leqslant C\left\|A_1^{1/4}g_2\right\|_{L^2(\Gamma_1)}. \tag{4.14}$$

Combining (4.10) with (4.11)–(4.14), we deduce that

$$\left\|\lambda\eta_2\right\|_{\mathcal{D}(A_1^{1/4})} \leqslant C\left\|F\right\|_{\mathcal{H}}. \tag{4.15}$$

Moreover, we deduce from (4.10) and from (4.3) that

$$\lambda\eta_1 = -V_\lambda^{-1}\partial_n f_1 - (\nu\lambda+1)V_\lambda^{-1}\partial_n U_\lambda\left(\Delta f_1\right) - \lambda(\nu\lambda+1)V_\lambda^{-1}\partial_n U_\lambda f_2 + g_1 - V_\lambda^{-1}A_1 g_1 + \lambda V_\lambda^{-1} g_2. \tag{4.16}$$

Similarly as above, we apply Theorem 3.1 ((with $(\theta,\beta)=(3/4,-1/4)$ or $(\theta,\beta)=(3/4,1/4)$) and Lemma 2.1, and we deduce

$$\|\lambda\eta_1\|_{\mathcal{D}(A_1^{3/4})} \leqslant C\,\|F\|_{\mathcal{H}}\,. \tag{4.17}$$

Now, coming back to (4.5), we obtain

$$\|\lambda u_2\|_{L^2(\Omega)} \leqslant \|\lambda W_\lambda \eta_2\|_{L^2(\Omega)} + \left\|\lambda^2 U_\lambda f_2\right\|_{L^2(\Omega)} + \|\lambda U_\lambda \Delta f_1\|_{L^2(\Omega)}\,.$$

Combining the above relation with (2.11), Lemma 2.1 and (4.15), we deduce

$$\|\lambda u_2\|_{L^2(\Omega)} \leqslant C\left(\|\lambda\eta_2\|_{H^{-1/2}(\Gamma_1)} + \|f_2\|_{L^2(\Omega)} + \|f_1\|_{H^2(\Omega)}\right) \leqslant C\,\|F\|_{\mathcal{H}}\,.$$

Finally, (4.3) and (4.5) yield

$$\|\lambda u_1\|_{H^2(\Omega)} \leqslant \|W_\lambda \eta_2\|_{H^2(\Omega)} + \|\lambda U_\lambda f_2\|_{H^2(\Omega)} + \|U_\lambda \Delta f_1\|_{H^2(\Omega)} + \|f_1\|_{H^2(\Omega)}\,.$$

Combining the above relation with (2.12), Lemma 2.1 and (4.3), we deduce

$$\begin{aligned}\|\lambda u_1\|_{H^2(\Omega)} &\leqslant C\left(\|\eta_2\|_{H^{3/2}(\Gamma_1)} + \|\lambda\eta_2\|_{H^{-1/2}(\Gamma_1)} + \|f_2\|_{L^2(\Omega)} + \|f_1\|_{H^2(\Omega)}\right)\\ &\leqslant C\left(\|\lambda\eta_1\|_{H^{3/2}(\Gamma_1)} + \|g_1\|_{H^{3/2}(\Gamma_1)} + \|\lambda\eta_2\|_{H^{-1/2}(\Gamma_1)} + \|f_2\|_{L^2(\Omega)} + \|f_1\|_{H^2(\Omega)}\right).\end{aligned}$$

Combining this with (4.15) and (4.17) yields

$$\|\lambda u_1\|_{H^2(\Omega)} \leqslant C\,\|F\|_{\mathcal{H}}\,.$$

We thus deduce (4.9) and therefore (1.9). Combined with (4.4), this allows us to conclude the proof of Theorem 1.1.

Appendix A. Formal derivation of Ventcel boundary conditions

In this section, we present a formal way to derive the system (1.1) and in particular the Ventcel boundary conditions. The approach is the same as in [**6**, Appendix A] and we write this part only for sake of completeness.

We consider for any $\delta > 0$ small enough,

$$\omega_\delta := \{x + sn\ ;\ s \in (0,\delta),\ x \in \Gamma_1\}$$

and we assume that

$$\overline{\Omega} \cap \overline{\omega_\delta} = \Gamma_1.$$

We define the domain Ω_δ by

$$\overline{\Omega_\delta} := \overline{\Omega \cup \omega_\delta}.$$

Then, we consider the following system coupling two wave equations:

$$\left\{\begin{array}{rl} \partial_{tt}u - \nu\Delta\partial_t u - \Delta u = 0 & \text{in } (0,\infty)\times\Omega,\\ \partial_{tt}v - \kappa\Delta v = 0 & \text{in } (0,\infty)\times\omega_\delta,\\ u = v & \text{on } (0,\infty)\times\Gamma_1,\\ \partial_n(\nu\partial_t u + u) = \kappa\partial_n v & \text{on } (0,\infty)\times\Gamma_1,\\ u = 0 & \text{on } (0,\infty)\times\Gamma_0,\\ \partial_n v = 0 & \text{on } (0,\infty)\times\partial\omega_\delta\setminus\Gamma_1. \end{array}\right. \tag{A.1}$$

In the above system, the two wave equations are coupled at the interface Γ_1 through standard transmission conditions. Note that we choose to consider a Neumann boundary condition on the $\partial\omega_\delta\setminus\Gamma_1$. Similar computations for a Dirichlet boundary condition lead to a slightly different model.

First we use a standard formula for the Laplace operator in Γ_1 as follows (see, for instance, [**11**, p.220, formula (5.59)]):

$$\Delta v = \partial_n^2 v + (\operatorname{div} n)\partial_n v + \Delta_b v \quad \text{on } \Gamma_1.$$

Then using Taylor's formula and the Neumann boundary condition on $\partial\omega_\delta \setminus \Gamma_1$, we deduce

$$0 = \partial_n v(x + \delta n) = \partial_n v(x) + \delta \partial_n^2 v(x) + O(\delta^2) \quad \text{on } \Gamma_1.$$

We thus deduce from the wave equation on ω_δ and from the two above relations that

$$\partial_{tt} v - \kappa \Delta_b v = -\frac{\kappa}{\delta} \partial_n v + O(\delta) \quad \text{on } (0, \infty) \times \Gamma_1.$$

Then using the transmission conditions, and denoting by η the trace of v on Γ_1, we deduce

$$\partial_{tt} \eta - \kappa \Delta_b \eta = -\frac{1}{\delta} \partial_n (\nu \partial_t u + u) + O(\delta) \quad \text{on } (0, \infty) \times \Gamma_1.$$

Neglecting the remainder, we recover the Ventcel condition in (1.1). Note that in this system, we take $\delta = 1$ since this constant does not play any role in the proof of our main result.

References

[1] Mehdi Badra and Takéo Takahashi. Gevrey regularity for a system coupling the Navier-Stokes system with a beam equation. *SIAM J. Math. Anal.*, 51(6):4776–4814, 2019.

[2] Mehdi Badra and Takéo Takahashi. Gevrey regularity for a system coupling the Navier-Stokes system with a beam: the non-flat case. *Funkc. Ekvacioj, Ser. Int.*, 65(1):63–109, 2022.

[3] Mehdi Badra and Takéo Takahashi. Maximal regularity for the Stokes system coupled with a wave equation: application to the system of interaction between a viscous incompressible fluid and an elastic wall. *J. Evol. Equ.*, 22(3):55, 2022. Id/No 71.

[4] Alain Bensoussan, Giuseppe Da Prato, Michel C. Delfour, and Sanjoy K. Mitter. *Representation and control of infinite dimensional systems*. Systems & Control: Foundations & Applications. Birkhäuser Boston, Inc., Boston, MA, second edition, 2007.

[5] Haïm Brezis. *Analyse fonctionnelle*. Collection Mathématiques Appliquées pour la Maîtrise. [Collection of Applied Mathematics for the Master's Degree]. Masson, Paris, 1983. Théorie et applications. [Theory and applications].

[6] Rémi Buffe. Stabilization of the wave equation with Ventcel boundary condition. *J. Math. Pures Appl. (9)*, 108(2):207–259, 2017.

[7] Shu Ping Chen and Roberto Triggiani. Proof of extensions of two conjectures on structural damping for elastic systems. *Pacific J. Math.*, 136(1):15–55, 1989.

[8] Nicolas Fourrier and Irena Lasiecka. Regularity and stability of a wave equation with a strong damping and dynamic boundary conditions. *Evol. Equ. Control Theory*, 2(4):631–667, 2013.

[9] Philip Jameson Graber and Irena Lasiecka. Analyticity and Gevrey class regularity for a strongly damped wave equation with hyperbolic dynamic boundary conditions. *Semigroup Forum*, 88(2):333–365, 2014.

[10] Pierre Grisvard. Caractérisation de quelques espaces d'interpolation. *Arch. Rational Mech. Anal.*, 25:40–63, 1967.

[11] Antoine Henrot and Michel Pierre. *Shape variation and optimization*, volume 28 of *EMS Tracts in Mathematics*. European Mathematical Society (EMS), Zürich, 2018.

[12] Günter Leugering. Optimal controllability in viscoelasticity of rate type. *Math. Methods Appl. Sci.*, 8(3):368–386, 1986.

Institut de Mathématiques de Toulouse ; UMR5219; Université de Toulouse, CNRS, UPS IMT, F-31062 Toulouse Cedex 9, France
Email address: `mehdi.badra@math.univ-toulouse.fr`

Université de Lorraine, CNRS, Inria, IECL, F-54000 Nancy, France
Email address: `takeo.takahashi@inria.fr`
URL: `http://iecl.univ-lorraine.fr/membre-iecl/takahashi-takeo/`