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Threshold solutions for the 3D focusing cubic-quintic nonlinear Schrödinger equation at low frequencies

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Abstract. This paper addresses the focusing cubic-quintic nonlinear Schrödinger equation in three space dimensions. Especially, we study the global dynamics of solutions whose energy and mass equal to those of the ground state in the spirits of Duyckaerts and Merle [**14**]. When we try to obtain the corresponding results of [**14**], we meet several difficulties due to the cubic-quintic nonlinearity. We overcome them by using the one-pass theorem (no return theorem) developed by Nakanishi and Schlag [**38**].

Contents

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1. Introduction

In this paper, we consider the following nonlinear Schrödinger equation:

$$i\partial_t\psi + \Delta\psi + |\psi|^2\psi + |\psi|^4\psi = 0 \qquad \text{in } \mathbb{R}\times\mathbb{R}^3, \tag{1.1}$$

where Δ is the Laplace operator on $\mathbb{R}^3$. Several studies have been made on the asymptotic behavior of solutions to double power nonlinear Schrödinger equations (see e.g. [**3, 6, 11, 19, 20, 22, 23, 26, 27, 28, 30, 32, 33, 39, 40, 42**] and references therein). Here, we are concerned with global dynamics of solutions whose mass and energy equal to those of the ground state.

For any $\psi_0 \in H^1(\mathbb{R}^3)$, there exists a unique solution ψ in $C(I_{\max}; H^1(\mathbb{R}^3))$ with $\psi|_{t=0} = \psi_0$ for some interval $I_{\max} = (-T_{\max}^-, T_{\max}^+) \subset \mathbb{R}$, a maximal existence interval including 0. We say that ψ blows up in finite time if $T_{\max}^+ < \infty$ or $T_{\max}^- < \infty$. The solution ψ satisfies the following conservation laws of the mass and the energy in this order:

$$\mathcal{M}(\psi(t)) = \mathcal{M}(\psi_0), \qquad \mathcal{E}(\psi(t)) = \mathcal{E}(\psi_0), \tag{1.2}$$

where

$$\mathcal{M}(u) := \frac{1}{2}\|u\|_{L^2}^2, \qquad \mathcal{E}(u) := \frac{1}{2}\|\nabla u\|_{L^2}^2 - \frac{1}{4}\|u\|_{L^4}^4 - \frac{1}{6}\|u\|_{L^6}^6 \qquad \text{for } u \in H^1(\mathbb{R}^3).$$

If, in addition, $\psi_0 \in L^2(\mathbb{R}^3, |x|^2dx)$, then the corresponding solution ψ also belongs to $C(I_{\max}; L^2(\mathbb{R}^3, |x|^2dx))$ and satisfies the so-called virial identity:

$$\begin{aligned}\int_{\mathbb{R}^3} |x|^2|\psi(t,x)|^2dx &= \int_{\mathbb{R}^3} |x|^2|\psi_0(x)|^2dx + 2t\ \mathrm{Im}\int_{\mathbb{R}^3} x\cdot\nabla\psi_0(x)\overline{\psi_0(x)}dx \\ &\quad + 16\int_0^t\int_0^{t'} \mathcal{K}(\psi(t''))dt''dt' \qquad \text{for any } t \in I_{\max},\end{aligned} \tag{1.3}$$

where

$$\mathcal{K}(u) := \|\nabla u\|_{L^2}^2 - \frac{3}{4}\|u\|_{L^4}^4 - \|u\|_{L^6}^6 \qquad \text{for } u \in H^1(\mathbb{R}^3).$$

See e.g. Cazenave [**10**, Section 6.5] for details.

By a *standing wave*, we mean a solution to (1.1) of the form

$$\psi(t,x) = e^{i\omega t}Q_\omega(x)$$

for some $\omega > 0$ and $Q_\omega \in H^1(\mathbb{R}^3)\setminus\{0\}$. Then, we see that Q_ω should solve the following semilinear elliptic equation:

$$-\Delta Q + \omega Q - |Q|^2Q - |Q|^4Q = 0 \qquad \text{in } \mathbb{R}^3. \tag{1.4}$$

If we define the action functional $\mathcal{S}_\omega$ by

$$\mathcal{S}_\omega(u) := \mathcal{E}(u) + \omega\mathcal{M}(u) \qquad \text{for } u \in H^1(\mathbb{R}^3), \tag{1.5}$$

then $\mathcal{S}_\omega'(Q_\omega) = 0$ in $H^{-1}(\mathbb{R}^3)$ if and only if $Q_\omega \in H^1(\mathbb{R}^3)$ is a weak solution to (1.4). To seek a solution to (1.4), we consider the following minimization problem:

$$m_\omega := \inf\left\{\mathcal{S}_\omega(u)\colon u \in H^1(\mathbb{R}^3)\setminus\{0\},\ \mathcal{K}(u) = 0\right\}. \tag{1.6}$$

It is known that if there is a minimizer for m_ω, it satisfies (1.4). Here we call Q_ω a *ground state* to (1.6) provided Q_ω is a minimizer for m_ω. Concerning the existence of a ground state, the following results hold:

THEOREM 1.1 ([**7, 43**]). *There exists $\omega_c > 0$ such that m_ω has a ground state for $0 < \omega < \omega_c$ and no ground state for $\omega > \omega_c$.*

REMARK 1.2. We do not know whether the ground state exists or not at $\omega = \omega_c$.

There are several results on the global dynamics of solutions to nonlinear Schrödinger equations. See e.g. [**1, 2, 3, 5, 6, 11, 12, 13, 14, 15, 17, 21, 24, 25, 27, 28, 29, 31, 38, 44, 45**] and references therein. Let us recall some of them which are concerned with the following nonlinear Schrödinger equations:

$$i\partial_t\psi + \Delta\psi + |\psi|^{p-1}\psi = 0 \qquad \text{in } \mathbb{R} \times \mathbb{R}^d. \tag{1.7}$$

where $d \in \mathbb{N}, 1 < p < 2^* - 1$ and $2^* = \frac{2d}{d-2}$. The equation (1.7) is scale invariant. More precisely, putting

$$\psi_\lambda(t,x) := \lambda^{\frac{2}{p-1}}\psi(\lambda^2 t, \lambda x) \qquad (\lambda > 0), \tag{1.8}$$

we see that if $\psi(t,x)$ satisfies (1.7), so does ψ_λ. The scaling (1.8) preserves the mass $\mathcal{M}$ and the corresponding energy when $p = 1 + 4/d$ and $p = (d+2)/(d-2)$, respectively. Thus, the exponent $p = 1 + 4/d$ is referred to as "mass critical" and $p = (d+2)/(d-2)$ as "energy critical". [1]

It is known that (1.7) has a stationary solution, which neither scatters [2] nor blows up. More precisely, when the energy critical case, (1.7) has the following explicit static solution,

$$W(x) := \left(1 + \frac{|x|^2}{d(d-2)}\right)^{-\frac{d-2}{2}}.$$

The solution W is called by *Aubin-Talenti function*. Similarly, when $1 < p < (d+2)/(d-2)$, the equation (1.7) also has a standing wave $\psi(t,x) = e^{i\omega t}R_\omega$ $(\omega > 0)$. Then, R_ω satisfies the following semilinear equation:

$$-\Delta R + \omega R - |R|^{p-1}R = 0 \qquad \text{in } \mathbb{R}^d. \tag{1.9}$$

For the energy critical case $p = (d+2)/(d-2)$, Kenig and Merle [**25**] employed the concentration-compactness and showed that the radial solution to (1.7) whose energy is less than that of the Aubin-Talenti function W blows up in finite time or scatters as $t \to \pm\infty$ for $d = 3, 4, 5$. Killip and Visan [**29**] extended the result of [**25**] for $d \geq 5$, removing the radial condition. Dodson [**12**] obtained the corresponding result of [**29**] for $d = 4$.

For the mass supercritical and the energy subcritical case $1 + 4/d < p < (d+2)/(d-2)$, Holmer and Roudenko [**24**] considered the three dimensional cubic nonlinear Schrödinger equation ($d = p = 3$) and proved that the radial solution below the ground state scatters or blows up in finite time. Then, Duyckaerts, Holmer and Roudenko [**13**] extended the result of [**24**] to non-radial H^1 initial data. Then, Akahori and Nawa [**5**] and Fang, Xie and Cazenave [**18**] extended the result to general dimension and power nonlinearity.

Duyckaerts and Merle [**14**] studied the threshold solution to the energy critical nonlinear Schrödinger equations, that is, the solution whose energy equals to the Aubin-Talenti function for $d = 3, 4, 5$. They constructed special solutions $W^{\pm}$, which converge to the Aubin-Talenti function W in the positive time direction while W^+ blows up and W^- scatters in the negative time direction, respectively. They also classified the threshold solutions under the radial assumption. Li and

[1]Note that the quintic power nonlinearity $|\psi|^4\psi$ in three space dimensions which is involved in (1.1) corresponds to the energy critical one.

[2]Here, we say that a solution scatters if the solution converges to the one of the linear Schrödinger equation.

Zhang [**31**] extended the result of [**14**] to the higher dimensions $d \geq 6$. Duyckaerts and Roudenko [**16**] studied the threshold solution for the three dimensional cubic nonlinear Schrödinger equations. They also constructed special solutions and classify all solutions (not necessarily radially symmetric) at the threshold level. Recently, Campos, Farah and Roudenko [**9**] generalized the result of Duyckaerts and Roudenko [**16**] to any dimension and any power of the nonlinearity. They also considered the energy critical case and gave an alternative proof of the result of Li and Zhang [**31**]. See also [**1, 2, 3, 15, 21, 33, 41**] for the threshold solutions to other nonlinear Schrödinger equations.

In this paper, we address the threshold solution to (1.1). To state our results, we put

$$\begin{aligned}
\mathcal{BA}_\omega &:= \left\{u \in H^1_{\text{rad}}(\mathbb{R}^3) \colon \mathcal{S}_\omega(u) = m_\omega,\ \mathcal{M}(u) = \mathcal{M}(Q_\omega)\right\},\\
\mathcal{BA}_{\omega,+} &:= \{u \in \mathcal{BA}_\omega \colon \mathcal{K}(u) > 0\},\\
\mathcal{BA}_{\omega,-} &:= \{u \in \mathcal{BA}_\omega \colon \mathcal{K}(u) < 0\},\\
\mathcal{BA}_{\omega,0} &:= \{u \in \mathcal{BA}_\omega \colon \mathcal{K}(u) = 0\}.
\end{aligned}$$

Clearly, we have $\mathcal{BA}_\omega = \mathcal{BA}_{\omega,-} \cup \mathcal{BA}_{\omega,0} \cup \mathcal{BA}_{\omega,+}$. We see from Proposition 2.1 below that

$$\mathcal{BA}_{\omega,0} = \left\{e^{i\theta} Q_\omega \colon \theta \in \mathbb{R}\right\}. \tag{1.10}$$

In addition, we can easily find that the sets $\mathcal{BA}_{\omega,\pm}$ and $\mathcal{BA}_{\omega,0}$ are invariant under the flow of (1.1) (see e.g. Lemma 4.1 below). Then, by a similar argument to [**14**], we can construct the following special solutions to (1.1):

THEOREM 1.3. *There exists a sufficiently small $\omega_* > 0$ such that for $\omega \in (0, \omega_*)$,* (1.1) *has two radial solutions $Q_\omega^+ \in \mathcal{BA}_{\omega,+}$ and $Q_\omega^- \in \mathcal{BA}_{\omega,-}$ satisfying the following:*

(i) *$Q_\omega^\pm$ exists on $[0,\infty)$, and there exist constants $e_\omega, C_\omega > 0$ such that*
$$dist_{H^1}(Q_\omega^\pm(t), \mathcal{O}(Q_\omega)) \leq C_\omega e^{-e_\omega t} \qquad \text{for all } t \geq 0,$$
where
$$dist_{H^1}\big(u, \mathcal{O}(Q_\omega)\big) := \inf_{\theta\in\mathbb{R}} \|u - e^{i\theta} Q_\omega\|_{H^1}.$$

(ii) *$\mathcal{K}(Q_\omega^-) < 0$ and the negative time of existence of Q_ω^- is finite.*

(iii) *$\mathcal{K}(Q_\omega^+) > 0$, Q_ω^+ exists on $(-\infty, \infty)$ and scatters for negative time, that is, there exists $\phi_- \in H^1(\mathbb{R}^3)$ such that*
$$\lim_{t\to-\infty} \|Q_\omega^+(t) - e^{it\Delta}\phi_-\|_{H^1} = 0.$$

In what follows, we say that $\psi = \phi$ up to the symmetries if there exist $t_0 \in \mathbb{R}$ and θ_0 such that

$$\psi(t,x) = e^{i\theta_0}\phi(t + t_0, x) \qquad \text{or} \qquad \psi(t,x) = e^{i\theta_0}\overline{\phi}(-t + t_0, x).$$

Our main result is as follows:

THEOREM 1.4. *Let $\omega_* > 0$ be the constant given in Theorem 1.3 and ψ be a radial solution to* (1.1) *with $\psi|_{t=0} = \psi_0 \in \mathcal{BA}_\omega$ for $\omega \in (0, \omega_*)$. Then, the following holds:*

(i) *If $\psi_0 \in \mathcal{BA}_{\omega,-}$, then either ψ blows up in finite time or $\psi = Q_\omega^-$ up to the symmetries.*

(ii) *If* $\psi_0 \in \mathcal{BA}_{\omega,0}$*, then* $\psi = e^{i\omega t}Q_\omega$ *up to the symmetries.*
(iii) *If* $\psi_0 \in \mathcal{BA}_{\omega,+}$*, then either* ψ *scatters or* $\psi = Q_\omega^+$ *up to the symmetries.*

REMARK 1.5. (i) Nakanishi and Schlag [**38**] proved that the solutions whose energy is slightly larger than that of the ground state is classified into 9 sets (combination of blows up, scattering and trapped by the ground state generated by the phase for $t > 0$ and $t < 0$). See also [**6, 17, 27, 34, 35**] for the global dynamics above the ground state. In particular, it was studied in [**6**] that the behavior of the solutions ψ to (1.1) with $\psi|_{t=0} = \psi_0$ satisfying $\mathcal{S}_\omega(\psi_0) < m_\omega + \varepsilon$ are also classified into the 9 sets. However, it seems that from the result of [**6**], we could not determine the behavior of solutions by the initial data as in Theorem 1.4. Another difference between the result of [**6**] and ours is that we obtain a kind of uniqueness of solution which converges to the orbit of the ground state (see Proposition 7.1 below).

(ii) We may extend our results to general dimensions and power nonlinearities by using the argument of [**9**]. However, for the simplicity of our presentation, we restrict ourselves to three space dimension and cubic-quintic nonlinearity.

The proof of Theorem 1.4 is based on that of [**14, 16**]. However, it seems that due to the cubic and quintic nonlinearities, some part of the argument in [**14, 16**] does not work for our equation (1.1). For example, in [**16**], a Cauchy-Schwarz type inequality plays an important role (see [**16**, Claim 5.4] in detail). In contrast, it seems difficult to obtain a corresponding inequality for our equation (1.1). To overcome the difficulty, we employ the one-pass theorem (no return theorem) which was introduced by Nakanishi and Schlag [**38**] for the equation (1.7) with $d = p = 3$. Roughly speaking, one-pass theorem states that if a solution moves away from a neighborhood of the ground states, then the solution never return to the neighborhood. We employ the one-pass theorem to prove that if a threshold solution neither blows up nor scatters, the solution converges to the ground state exponentially (see Propositions 4.1 and 5.1 below).

REMARK 1.6. (i) The reason why we need the radially symmetry for solutions is due to the one-pass theorem. Indeed, a kind of Ogawa-Tsutsumi's saturated virial identity was used for the proof of the one-pass theorem. Except for the theorem, we do not require the condition.

(ii) Recently, Ardila and Murphy [**3**] studied the threshold solutions to the following cubic-quintic nonlinear Schrödinger equation:

$$i\partial_t\psi + \Delta\psi + |\psi|^2\psi - |\psi|^4\psi = 0 \qquad \text{in } \mathbb{R}\times\mathbb{R}^3. \tag{1.11}$$

Note that the quintic power nonlinearity is defocusing, which is different from our equation (1.1) and any solutions to (1.11) are global ($T^{\pm}_{\max} = \infty$). They also classified the threshold solutions, which are not necessary radially symmetric, to (1.11). Let ψ be the threshold solution whose sign of the virial functional (the one corresponding to $\mathcal{K}$) is positive. Then, they showed that the solution either ψ scatters in both time directions and coincide with a special solution. To this end, they employed the modulation analysis and the concentration-compactness method. Their method might work for our equation (1.1). However, we would like to stress that we can study the threshold solutions which blow up or scatter in a unified way by using the one-pass theorem.

This paper is organized as follows. In Section 2, we recall several properties of the ground state and its linearized operator. In Section 3, we recall the one-pass theorem, which was proved in [**6**]. In Sections 4 and 5, we study dynamics of solutions which start from the sets $\mathcal{BA}_{\omega,\pm}$, respectively. In Section 6, we give a sketch of the proof of Theorem 1.3. In Section 7, we obtain the uniqueness of the special solution and give the proof of Theorem 1.4. In Appendix, we give the proof of a convergence property which we admit in Section 4.

Notation.

(i) We use $(\cdot,\cdot)_{L^2}$ to denote the inner product in $L^2(\mathbb{R}^3)$:

$$(u,v)_{L^2} := \int_{\mathbb{R}^3} u(x)\overline{v(x)}dx \qquad \text{for all } u,v \in L^2(\mathbb{R}^3).$$

(ii) We also use $L^2_{\mathrm{real}}(\mathbb{R}^3)$ to denote the real Hilbert space of complex-valued functions in $L^2(\mathbb{R}^3)$ which is equipped with the inner product

$$(u,v)_{L^2_{\mathrm{real}}} := \mathrm{Re}\int_{\mathbb{R}^3} u(x)\overline{v(x)}dx \qquad \text{for all } u,v \in L^2(\mathbb{R}^3).$$

(iii) We use $\langle\cdot,\cdot\rangle_{H^{-1},H^1}$ to denote the duality pair of $u \in H^1(\mathbb{R}^3)$ and $v \in H^{-1}(\mathbb{R}^3)$:

$$\langle u,v\rangle_{H^{-1},H^1} := ((1-\Delta)^{\frac{1}{2}}u,(1-\Delta)^{-\frac{1}{2}}v)_{L^2_{\mathrm{real}}} \qquad \text{for all } u \in H^1(\mathbb{R}^3) \text{ and } v \in H^{-1}(\mathbb{R}^3).$$

2. Properties of ground state and its linearized operator

In this section, we recall several properties of the ground state and its linearized operator, which are mainly obtained in [**6**]. First, we recall that the uniqueness of ground state and that the following slope condition holds:

PROPOSITION 2.1 (Proposition 2.0.4 of [**6**]). *The following properties hold:*

(i) *There exists $\omega_1 > 0$ such that for $\omega \in (0,\omega_1)$, the positive radial ground state Q_ω is unique up to phase. Namely, if $u \in H^1(\mathbb{R}^3)$ satisfies $\mathcal{S}_\omega(u) = m_\omega$ and $\mathcal{K}(u) = 0$, then we have $u = e^{i\theta}Q_\omega$ for some $\theta \in \mathbb{R}$.*

(ii) *The mapping $\omega \in (0,\omega_1) \mapsto Q_\omega \in H^1(\mathbb{R}^3)$ is continuously differentiable,*

(iii)

$$\frac{d}{d\omega}\mathcal{M}(Q_\omega) = (Q_\omega,\partial_\omega Q_\omega)_{L^2_{real}} < 0 \qquad \text{for } \omega \in (0,\omega_1).$$

Let ψ be a solution to (1.1). We consider the following decomposition of the form

$$\psi(t,x) = e^{i\theta(t)}\big(Q_\omega(x) + \eta(t,x)\big), \tag{2.1}$$

where $\theta(t)$ is a function of $t \in I_{\max}$ to be chosen later (see (3.8) below) and η is the remainder. Let $\eta_1 = \mathrm{Re}\,\eta$ and $\eta_2 = \mathrm{Im}\,\eta$. We will identify $\mathbb{C}$ and $\mathbb{R}^2$ and consider $\eta = \eta_1 + i\eta_2$ as an element $\begin{pmatrix}\eta_1\\ \eta_2\end{pmatrix}$ of $\mathbb{R}^2$. Then, η satisfies

$$\frac{\partial\eta}{\partial t}(t) = -i\mathcal{L}_\omega\eta(t) - i\Big\{\frac{d\theta}{dt}(t) - \omega\Big\}(Q_\omega + \eta(t)) + N_\omega(\eta(t)), \tag{2.2}$$

where

$$-i\mathcal{L}_\omega = \begin{pmatrix} 0 & L_{\omega,-} \\ -L_{\omega,+} & 0 \end{pmatrix},$$

and the self-adjoint operators $L_{\omega,\pm}$ and the reminder $N_\omega(\eta)$ are defined by

$$L_{\omega,+} := -\Delta + \omega - 3Q_\omega^2 - 5Q_\omega^4, \qquad L_{\omega,-} := -\Delta + \omega - Q_\omega^2 - Q_\omega^4, \tag{2.3}$$

$$N_\omega(\eta) := N_{\omega,1}(\eta) + N_{\omega,2}(\eta) \tag{2.4}$$

Here, $N_{\omega,1}(\eta)$ and $N_{\omega,2}(\eta)$ are defined by

$$N_{\omega,1}(\eta) := i\left(|Q_\omega+\eta|^2(Q_\omega+\eta) - Q_\omega^3 - 2Q_\omega^2\eta - Q_\omega^2\overline{\eta}\right),$$
$$N_{\omega,2}(\eta) := i\left(|Q_\omega+\eta|^4(Q_\omega+\eta) - Q_\omega^5 - 3Q_\omega^4\eta - 2Q_\omega^4\overline{\eta}\right).$$

We note that for any $u, v \in H^1(\mathbb{R}^3)$,

$$\left[\mathcal{S}_\omega''(Q_\omega)u\right]v = \langle \mathcal{L}_\omega u, v\rangle_{H^{-1},H^1}. \tag{2.5}$$

Moreover, since Q_ω is a solution to (1.4), we can verify that

$$\mathcal{L}_\omega Q_\omega = L_{\omega,+}Q_\omega = -2Q_\omega^3 - 4Q_\omega^5, \tag{2.6}$$

$$\mathcal{L}_\omega(iQ_\omega) = \mathcal{L}_\omega \begin{pmatrix} 0 \\ Q_\omega \end{pmatrix} = L_{\omega,-}Q_\omega = 0, \tag{2.7}$$

$$\mathcal{L}_\omega \partial_\omega Q_\omega = L_{\omega,+}\partial_\omega Q_\omega = -Q_\omega. \tag{2.8}$$

Theorem 2.1 (Proposition 4.0.1, Lemmas B.0.1 and B.0.2 of [**6**]). *Let $\sigma(-i\mathcal{L}_\omega)$ be the spectrum of the operator $-i\mathcal{L}_\omega$ and $\sigma_{ess}(-i\mathcal{L}_\omega)$ be its essential spectrum. Then, we have the following:*

(i) $\sigma_{ess}(-i\mathcal{L}_\omega) = \{i\xi \colon \xi \in \mathbb{R}, |\xi| \geq \omega\}$.

(ii) *There exists $\omega_2 \in (0, \omega_1)$ such that for $\omega \in (0, \omega_2)$, $-i\mathcal{L}_\omega$ has positive and negative eigenvalues e_ω and $-e_\omega$ with eigenfunction $\mathcal{Y}_{\omega,+}$ and $\mathcal{Y}_{\omega,-}$, respectively. Furthermore, Ker $(-i\mathcal{L}_\omega) =$ Span $\{iQ_\omega, \partial_{x_i}Q_\omega$ for $i = 1,2,3\}$.*

It is known that $\overline{\mathcal{Y}_{\omega,+}} = \mathcal{Y}_{\omega,-}$. Then, we put

$$\mathcal{Y}_{\omega,1} := \frac{\mathcal{Y}_{\omega,+} + \mathcal{Y}_{\omega,-}}{2} = \mathrm{Re}\,[\mathcal{Y}_{\omega,+}], \qquad \mathcal{Y}_{\omega,2} := \frac{\mathcal{Y}_{\omega,+} - \mathcal{Y}_{\omega,-}}{2i} = \mathrm{Im}\,[\mathcal{Y}_{\omega,+}], \tag{2.9}$$

that is,

$$\mathcal{Y}_{\omega,+} = \begin{pmatrix} \mathcal{Y}_{\omega,1} \\ \mathcal{Y}_{\omega,2} \end{pmatrix}.$$

The equation $-i\mathcal{L}_\omega \mathcal{Y}_{\omega,+} = e_\omega \mathcal{Y}_{\omega,+}$ is equivalent to

$$\begin{cases} L_{\omega,+}\mathcal{Y}_{\omega,1} = -e_\omega \mathcal{Y}_{\omega,2}, \\ L_{\omega,-}\mathcal{Y}_{\omega,2} = e_\omega \mathcal{Y}_{\omega,1}, \end{cases} \tag{2.10}$$

Concerning the eigenfunctions $\mathcal{Y}_{\omega,1}$ and $\mathcal{Y}_{\omega,2}$, we know the following:

Lemma 2.2 (Lemma 4.0.7 of [**6**]). *Let $\omega \in (0, \omega_2)$. We have the following orthogonalities:*

$$\left(Q_\omega, \mathcal{Y}_{\omega,1}\right)_{L^2} = \left(\partial_\omega Q_\omega, \mathcal{Y}_{\omega,2}\right)_{L^2} = 0. \tag{2.11}$$

Furthermore, we have

$$\left(\mathcal{Y}_{\omega,1}, \mathcal{Y}_{\omega,2}\right)_{L^2} > 0 \tag{2.12}$$

and

$$\big(Q_\omega, \mathcal{Y}_{\omega,2}\big)_{L^2} \neq 0. \tag{2.13}$$

The relation (2.13) in Lemma 2.2 allows us to choose $\mathcal{Y}_{\omega,2}$ so that

$$(Q_\omega, \mathcal{Y}_{\omega,2})_{L^2} < 0. \tag{2.14}$$

Note that since Q_ω is positive (especially, real-valued) and $\overline{\mathcal{Y}_{\omega,+}} = \mathcal{Y}_{\omega,-}$, we have, by (2.9) and (2.11), that

$$(Q_\omega, \mathcal{Y}_{\omega,\pm})_{L^2_{\mathrm{real}}} = (Q_\omega, \mathcal{Y}_{\omega,1})_{L^2} = 0. \tag{2.15}$$

In addition, we need the following technical lemma:

Lemma 2.3 (Lemma 4.0.8 of [**6**]). *There exists a frequency $\omega_3 \in (0, \omega_2)$ such that for any $\omega \in (0, \omega_3)$,*

$$\frac{e_\omega}{2}\big|(Q_\omega, \mathcal{Y}_{\omega,2})_{L^2}\big| \geq 4\big|(Q_\omega^5, \mathcal{Y}_{\omega,1})_{L^2}\big|. \tag{2.16}$$

3. One-pass theorem

This section is devoted to the one-pass theorem (no return theorem) which was first obtained by Nakanishi and Schlag [**37, 38**] for the cubic nonlinear Klein-Gordon and Schrödinger equations in three space dimensions. In [**6**], the one-pass theorem for the double power nonlinear Schrödinger equations was proved. Here, we recall the set-up and the one-pass theorem of [**6**].

3.1. Symplectic decomposition and parameter choice. For a positive radial ground state Q_ω to (1.4) and a solution ψ to (1.1), we consider the decomposition (2.1). We will work in the symplectic space $(L^2(\mathbb{R}^3), \Omega)$, where Ω is the symplectic form defined by

$$\Omega(f,g) := \big(f, ig\big)_{L^2_{\mathrm{real}}} = \mathrm{Im}\int_{\mathbb{R}^3} f(x)\overline{g(x)}\,dx.$$

We apply the "symplectic decomposition" corresponding to the discrete modes of $i\mathcal{L}_\omega$ to the remainder η in (2.1) and determine the function $\theta(t)$ in (2.1).

We assume that $\mathcal{Y}_{\omega,+}$ and $\mathcal{Y}_{\omega,-}$ are normalized in the following sense:

$$\Omega\big(\mathcal{Y}_{\omega,+}, \mathcal{Y}_{\omega,-}\big) = 1, \quad \Omega\big(\mathcal{Y}_{\omega,-}, \mathcal{Y}_{\omega,+}\big) = -1. \tag{3.1}$$

We can easily find that

$$\Omega\big(f,f\big) = \mathrm{Im}\int_{\mathbb{R}^3} |f|^2 dx = 0$$

for all $f \in L^2(\mathbb{R}^3)$. Furthermore, it follows from (2.7), (2.8), (2.15) and $\mathcal{L}_\omega \mathcal{Y}_{\omega,\pm} = \pm i e_\omega \mathcal{Y}_{\omega,\pm}$ that

$$\begin{aligned}\Omega\big(iQ_\omega, \mathcal{Y}_{\omega,\pm}\big) &= \big(iQ_\omega, i\mathcal{Y}_{\omega,\pm}\big)_{L^2_{\mathrm{real}}} = \pm\frac{1}{e_\omega}\big(iQ_\omega, \mathcal{L}_\omega \mathcal{Y}_{\omega,\pm}\big)_{L^2_{\mathrm{real}}} \\ &= \pm\frac{1}{e_\omega}\big(\mathcal{L}_\omega(iQ_\omega), \mathcal{Y}_{\omega,\pm}\big)_{L^2_{\mathrm{real}}} = 0,\end{aligned} \tag{3.2}$$

$$\begin{aligned}\Omega\big(\partial_\omega Q_\omega, \mathcal{Y}_{\omega,\pm}\big) &= \pm\frac{1}{e_\omega}(\partial_\omega Q_\omega, \mathcal{L}_\omega \mathcal{Y}_{\omega,\pm})_{L^2_{\mathrm{real}}} = \pm\frac{1}{e_\omega}(\mathcal{L}_\omega \partial_\omega Q_\omega, \mathcal{Y}_{\omega,\pm})_{L^2_{\mathrm{real}}} \\ &= \mp\frac{1}{e_\omega}(Q_\omega, \mathcal{Y}_{\omega,\pm})_{L^2_{\mathrm{real}}} = 0.\end{aligned} \tag{3.3}$$

Now, we expand the remainder $\eta(t)$ in the decomposition (2.1) by $-i\mathcal{L}_\omega$:

$$\eta(t) = \lambda_+(t)\mathcal{Y}_{\omega,+} + \lambda_-(t)\mathcal{Y}_{\omega,-} + a(t)iQ_\omega + b(t)\partial_\omega Q_\omega + \gamma(t), \tag{3.4}$$

where

$$\Omega\big(\gamma(t), \mathcal{Y}_{\omega,\pm}\big) = \Omega\big(\gamma(t), iQ_\omega\big) = \Omega\big(\gamma(t), \partial_\omega Q_\omega\big) = 0. \tag{3.5}$$

We see from Proposition 2.1 (iii) and (3.1)–(3.5) that the coefficients are as follows:

$$\lambda_+(t) = \Omega(\eta(t), \mathcal{Y}_{\omega,-}), \qquad \lambda_-(t) = -\Omega(\eta(t), \mathcal{Y}_{\omega,+}), \tag{3.6}$$

$$a(t) = \frac{\Omega(\eta(t), \partial_\omega Q_\omega)}{(Q_\omega, \partial_\omega Q_\omega)_{L^2_{\mathrm{real}}}}, \qquad b(t) = -\frac{\Omega(\eta(t), iQ_\omega)}{(Q_\omega, \partial_\omega Q_\omega)_{L^2_{\mathrm{real}}}}. \tag{3.7}$$

We require that $a(t) \equiv 0$. To this end, we choose the function $\theta(t)$ in (2.1) so that [3]

$$\Omega(e^{-i\theta(t)}\psi(t), \partial_\omega Q_\omega) \equiv 0. \tag{3.8}$$

Then, it follows from (3.8), (2.1) and $\Omega(Q_\omega, \partial_\omega Q_\omega) = 0$ that

$$0 \equiv \Omega(e^{-i\theta(t)}\psi(t), \partial_\omega Q_\omega) \equiv \Omega(Q_\omega + \eta(t), \partial_\omega Q_\omega) \equiv \Omega(\eta(t), \partial_\omega Q_\omega). \tag{3.9}$$

This together with (3.7) implies that $a(t) \equiv 0$. Furthermore, since

$$\mathcal{M}(\psi) = \mathcal{M}(Q_\omega) + \mathcal{M}(\eta(t)) + \big(Q_\omega, \eta(t)\big)_{L^2_{\mathrm{real}}},$$

the condition $\mathcal{M}(\psi) = \mathcal{M}(Q_\omega)$ implies that for any $t \in I_{\max}$,

$$\big(Q_\omega, \eta(t)\big)_{L^2_{\mathrm{real}}} = -\mathcal{M}(\eta(t)). \tag{3.10}$$

This together with $(Q_\omega, \partial_\omega Q_\omega)_{L^2_{\mathrm{real}}} < 0$ yields that

$$\begin{aligned}(e^{-i\theta(t)}\psi(t), \partial_\omega Q_\omega)_{L^2_{\mathrm{real}}} &= (Q_\omega, \partial_\omega Q_\omega)_{L^2_{\mathrm{real}}} + (\eta(t), \partial_\omega Q_\omega)_{L^2_{\mathrm{real}}} \\ &= (Q_\omega, \partial_\omega Q_\omega)_{L^2_{\mathrm{real}}} - \mathcal{M}(\eta(t)) < 0\end{aligned} \tag{3.11}$$

as long as $\mathcal{M}(\eta)$ is small. In what follows, we assume that ψ satisfies (3.8), (3.11) and $\mathcal{M}(\psi) = \mathcal{M}(Q_\omega)$.

From [**6**, Section 4.0.3], ordinary differential equations for θ, λ_- and λ_+ under the condition $\mathcal{M}(\psi) = \mathcal{M}(Q_\omega)$ are following:

$$\Big\{\frac{d\theta}{dt}(t) - \omega\Big\}\big(Q_\omega + \eta(t), \partial_\omega Q_\omega\big)_{L^2_{\mathrm{real}}} = -\mathcal{M}(\eta(t)) + \big(N_\omega(\eta(t)), \partial_\omega Q_\omega\big)_{L^2_{\mathrm{real}}}. \tag{3.12}$$

$$\frac{d\lambda_+}{dt}(t) = e_\omega\lambda_+(t) - \Big(\Big\{\frac{d\theta}{dt}(t) - \omega\Big\}\eta(t) - N_\omega(\eta(t)),\, \mathcal{Y}_{\omega,-}\Big)_{L^2_{\mathrm{real}}}, \tag{3.13}$$

$$\frac{d\lambda_-}{dt}(t) = -e_\omega\lambda_-(t) + \Big(\Big\{\frac{d\theta}{dt}(t) - \omega\Big\}\eta(t) - N_\omega(\eta(t)),\, \mathcal{Y}_{\omega,+}\Big)_{L^2_{\mathrm{real}}}. \tag{3.14}$$

[3]Let $\psi(t) = |\psi(t)|e^{i\alpha(t)}$ ($\alpha(t) \in \mathbb{R}/2\pi\mathbb{Z}$). It suffices to choose $\theta(t)$ so that $\theta(t) = \alpha(t)$. Then, $\Omega(e^{-i\theta(t)}\psi(t), \partial_\omega Q_\omega) = (|\psi(t)|, i\partial_\omega Q_\omega)_{L^2_{\mathrm{real}}} = 0$.

3.2. Linearized energy norm. In this subsection, we introduce the "linearized energy norm" for the remainder η in the decomposition (2.1). Put

$$\Gamma(t) := b(t)\partial_\omega Q_\omega + \gamma(t). \tag{3.15}$$

Since $a(t) \equiv 0$ (see (3.8)), the decomposition (3.4) is reduced to

$$\eta(t) = \lambda_+(t)\mathcal{Y}_{\omega,+} + \lambda_-(t)\mathcal{Y}_{\omega,-} + \Gamma(t). \tag{3.16}$$

Then, we have the following relationship:

LEMMA 3.1 (Lemma 4.0.2 of [**6**]). *The function* Γ *in the decomposition* (3.16) *satisfies*

$$\langle \mathcal{L}_\omega \Gamma(t), \Gamma(t)\rangle_{H^{-1},H^1} \sim \|\Gamma(t)\|_{H^1}^2 \tag{3.17}$$

for all $t \in I_{\max}$.

We also recall that

$$\mathcal{E}(\psi) - \mathcal{E}(Q_\omega) = -e_\omega \lambda_+(t)\lambda_-(t) + \frac{1}{2}\langle \mathcal{L}_\omega \Gamma(t), \Gamma(t)\rangle_{H^{-1},H^1} + O(\,\|\eta(t)\|_{H^1}^3\,). \tag{3.18}$$

See [**6**, (4.69)]. Defining the linearized energy norm $\|\eta(t)\|_E$ by

$$\|\eta(t)\|_E^2 := \frac{e_\omega}{2}\big(\lambda_+^2(t) + \lambda_-^2(t)\big) + \frac{1}{2}\langle \mathcal{L}_\omega \Gamma(t), \Gamma(t)\rangle_{H^{-1},H^1}, \tag{3.19}$$

we have, by (3.18), that

$$\mathcal{E}(\psi) - \mathcal{E}(Q_\omega) + \frac{e_\omega}{2}\big(\lambda_+(t) + \lambda_-(t)\big)^2 - \|\eta(t)\|_E^2 = O(\,\|\eta(t)\|_{H^1}^3\,). \tag{3.20}$$

Then, we can also have the following lemma:

LEMMA 3.2 (Lemma 4.0.3 of [**6**]). *The function* Γ *in the decomposition* (3.16) *satisfies*

$$\|\Gamma(t)\|_{H^1} \lesssim \|\eta(t)\|_{H^1} \tag{3.21}$$

for all $t \in I_{\max}$. *Moreover, we have*

$$\|\eta(t)\|_{H^1} \sim \|\eta(t)\|_E. \tag{3.22}$$

As a summary, we obtain the following:

PROPOSITION 3.1 (Proposition 4.0.4 of [**6**]). *Let* ψ *be a function in* $H^1(\mathbb{R}^3)$ *satisfying* $\mathcal{M}(\psi) = \mathcal{M}(Q_\omega)$. *We have the following the decomposition:*

$$\begin{aligned}\psi(t,x) &= e^{i\theta(t)}(Q_\omega + \eta(t,x)), \quad \Omega(e^{-i\theta(t)}\psi, \partial_\omega Q_\omega) \equiv 0,\\ &(e^{-i\theta(t)}\psi, \partial_\omega Q_\omega)_{L^2_{real}} < 0,\end{aligned} \tag{3.23}$$

$$\eta(t,x) = \lambda_+(t)\mathcal{Y}_{\omega,+} + \lambda_-(t)\mathcal{Y}_{\omega,-} + \Gamma(t). \tag{3.24}$$

Furthermore, there exists a constant $\delta_E(\omega) > 0$ *such that if* $\|\eta(t)\|_E \le 4\delta_E(\omega)$, *we obtain*

$$\Big|\ \mathcal{E}(\psi) - \mathcal{E}(Q_\omega) + \frac{e_\omega}{2}\big(\lambda_+(t) + \lambda_-(t)\big)^2 - \|\eta(t)\|_E^2\ \Big| \le \frac{\|\eta(t)\|_E^2}{10}.$$

3.3. Distance function from the ground state. In this subsection, we introduce a distance function from the ground state Q_ω by using the linearized energy norm (3.19). For this, we fix a non-increasing smooth function χ on $[0,\infty)$ such that

$$\chi(r) = \begin{cases} 1 & \text{if} \quad 0 \le r \le 1, \\ 0 & \text{if} \quad r \ge 2. \end{cases}$$

Then, we define a function $d_\omega \colon [0, T_{\max}) \to [0,\infty)$ by

$$d_\omega^2(t) := \|\eta(t)\|_E^2 + \chi\left(\frac{\|\eta(t)\|_E}{2\delta_E(\omega)} \right) C_\omega(\psi(t)), \tag{3.25}$$

where $\delta_E(\omega)$ is the constant given by Proposition 3.1, and

$$C_\omega(\psi(t)) := \mathcal{E}(\psi) - \mathcal{E}(Q_\omega) + \frac{e_\omega}{2}\big(\lambda_+(t) + \lambda_-(t)\big)^2 - \|\eta(t)\|_E^2.$$

Now, we introduce new parameters $\lambda_1(t)$ and $\lambda_2(t)$ defined by

$$\lambda_1(t) := \frac{\lambda_+(t) + \lambda_-(t)}{2}, \qquad \lambda_2(t) := \frac{\lambda_+(t) - \lambda_-(t)}{2}. \tag{3.26}$$

It follows from (3.13) and (3.14) that

$$\frac{d\lambda_1}{dt}(t) = e_\omega \lambda_2(t) + (\Big\{ \frac{d\theta}{dt}(t) - \omega \Big\}\eta(t) - N_\omega(\eta(t)),\, i\mathcal{Y}_{\omega,2})_{L^2_{\text{real}}}, \tag{3.27}$$

$$\frac{d\lambda_2}{dt}(t) = e_\omega \lambda_1(t) - (\Big\{ \frac{d\theta}{dt}(t) - \omega \Big\}\eta(t) - N_\omega(\eta(t)),\, \mathcal{Y}_{\omega,1})_{L^2_{\text{real}}}. \tag{3.28}$$

We recall a property of the distance function $d_\omega(t)$:

LEMMA 3.3 (Lemma 4.0.5 of [**6**]). *Assume that there exists an interval I on which*

$$\sup_{t \in I} d_\omega(t) \le \delta_E(\omega), \tag{3.29}$$

$\delta_E(\omega)$ is the constant given by Proposition 3.1. Then, all of the following hold for all $t \in I$:

$$\frac{1}{2}\|\eta(t)\|_E^2 \le d_\omega^2(t) \le \frac{3}{2}\|\eta(t)\|_E^2, \tag{3.30}$$

$$d_\omega^2(t) = \mathcal{E}(\psi) - \mathcal{E}(Q_\omega) + 2e_\omega \lambda_1^2(t), \tag{3.31}$$

$$\begin{aligned} \frac{d}{dt} d_\omega^2(t) = {} & 4e_\omega^2 \lambda_1(t)\lambda_2(t) + 4e_\omega \lambda_1(t)\Big(\Big\{ \frac{d\theta}{dt}(t) - \omega \Big\}\eta(t) \\ & - N_\omega(\eta(t)), i\mathcal{Y}_{\omega,2}\Big)_{L^2_{real}}. \end{aligned} \tag{3.32}$$

Next, we introduce a "modified distance function" $\widetilde{d}_\omega$.

LEMMA 3.4 (Lemma 6.0.1 of [**6**]). *Let ω_2 be the frequency given by Theorem 2.1. Then, for any $\omega \in (0, \omega_2)$, there exists $\gamma_1(\omega) > 0$ with the following property: let ψ be a solution to* (1.1) *satisfying $\mathcal{M}(\psi) = \mathcal{M}(Q_\omega)$. If $\operatorname{dist}_{H^1}\big(\psi(t), \mathcal{O}(Q_\omega)\big) \le \gamma_1(\omega)$, then*

$$\operatorname{dist}_{H^1}\big(\psi(t), \mathcal{O}(Q_\omega)\big) \sim \|\eta(t)\|_E. \tag{3.33}$$

Then, we can obtain the following proposition from Lemma 3.4:

PROPOSITION 3.2 (Proposition 6.0.2 of [**6**]). *Let* ω_2 *be the frequency given by Theorem 2.1. Then, for any* $\omega \in (0, \omega_2)$*, there exist a constant* $\widetilde{\gamma}(\omega) \in (0, \delta_E(\omega))$ [4] *and a continuous function* $\widetilde{d}_\omega \colon [0, T_{\max}) \to [0, \infty)$ *such that:*

$$\widetilde{d}_\omega(t) \sim \operatorname{dist}_{H^1}\big(\psi(t), \mathcal{O}(Q_\omega)\big)$$

and if $\widetilde{d}_\omega(t) \leq \widetilde{\gamma}(\omega)$*, then*

$$\widetilde{d}_\omega(t) = d_\omega(t), \tag{3.34}$$

where d_ω *is the distance function defined by* (3.25)*.*

3.4. One-pass theorem. We are now in a position to state the one-pass theorem:

THEOREM 3.5 (One-pass theorem (Theorem 7.0.1 of [**6**])). *Let* ω_3 *be the frequency given by Lemma 2.3. Then, for any* $\omega \in (0, \omega_3)$*, there exists a positive constant* $R_* > 0$ *such that for any* $R \in (0, R_*)$ *and any radial solution* ψ *to* (1.1) *with* $\psi|_{t=0} = \psi_0$ *satisfying*

$$\mathcal{S}_\omega(\psi_0) = m_\omega, \qquad \mathcal{M}(\psi_0) = \mathcal{M}(Q_\omega), \qquad \widetilde{d}_\omega(\psi_0) < R, \tag{3.35}$$

we have either

(i) $\widetilde{d}_\omega(\psi(t)) < R + R^{\frac{3}{2}}$ *for all* $t \in [0, T^+_{\max}]$*; or*

(ii) *there exists* $t_* > 0$ *such that* $\widetilde{d}_\omega(\psi(t)) \geq R + R^{\frac{3}{2}}$ *for all* $t \in [t_*, T^+_{\max}]$*.*
Here, $T^+_{\max}$ *denotes the maximal existence time of* ψ *in the positive direction.*

REMARK 3.6. Actually, the result of Theorem 7.0.1 of [**6**] can be more general and we can treat the solutions satisfying $\mathcal{S}_\omega(\psi_0) < m_\omega + \varepsilon$ for sufficiently small $\varepsilon > 0$. However, for simplicity of our presentation, we restrict ourselves to the threshold solutions.

4. Analysis on $\mathcal{BA}_{\omega,-}$

In this section, we study dynamics of solutions which start from $\mathcal{BA}_{\omega,-}$.

4.1. Convergence to the orbit of the ground state. First, we shall show that if a solution with (1.1) $\psi|_{t=0} = \psi_0 \in \mathcal{BA}_{\omega,-}$ exists globally in positive time direction, the solution converges to the orbit of the ground state exponentially as t goes to infinity. More precisely, we obtain the following:

PROPOSITION 4.1. *Assume that* $\omega \in (0, \omega_3)$*, where* $\omega_3 > 0$ *is the constant given in Lemma 2.3. Suppose that a solution* ψ *to* (1.1) *with* $\psi|_{t=0} = \psi_0 \in \mathcal{BA}_{\omega,-}$ *exists on* $(-T^-_{\max}, \infty)$*, Then, there exist constants* $C > 0$ *and* $c > 0$ *such that*

$$dist_{H^1}(\psi(t), \mathcal{O}(Q_\omega)) \leq Ce^{-ct} \qquad \text{for all } t > 0. \tag{4.1}$$

We first show that the set $\mathcal{BA}_{\omega,-}$ is invariant under the flow of (1.1).

LEMMA 4.1. *If* $\psi_0 \in \mathcal{BA}_{\omega,-}$*, we have* $\psi(t) \in \mathcal{BA}_{\omega,-}$ *for all* $t \in I_{\max}$*, where* ψ *is the solution to* (1.1) *with* $\psi|_{t=0} = \psi_0$*.*

[4]We recall that $\delta_E(\omega)$ denotes the constant given by Proposition 3.1.

Proof. It follows from the conservation laws (1.2) that $\mathcal{S}_\omega(\psi(t)) = m_\omega$. We shall show $\mathcal{K}(\psi(t)) < 0$ for all $t \in I_{\max}$ by contradiction. Suppose to the contrary that there exist $\omega_* \in (0, \omega_3), \psi_{0,*} \in \mathcal{BA}_{\omega,-}$ and $t_* \in I_{\max}$ such that $\mathcal{K}(\psi(t_*)) \geq 0$. Then, from the continuity of $\mathcal{K}(\psi(t))$ and $\mathcal{K}(\psi_{0,*}) < 0$, there exists $t_1 \in (0, t_*]$ such that $\mathcal{K}(\psi(t_1)) = 0$. Thus, $\psi(t_1)$ satisfies $\mathcal{S}_{\omega_*}(\psi(t_1)) = m_{\omega_*}$ and $\mathcal{K}(\psi(t_1)) = 0$. Since $\psi(t_1)$ is a minimizer of m_{ω_*} (ground state of m_{ω_*}), we have that $\psi(t) = e^{i\omega_* t + \theta} Q_{\omega_*}$ for some $\theta \in \mathbb{R}$, which contradicts $\mathcal{K}(\psi(0)) = \mathcal{K}(\psi_{0,*}) < 0$. □

Next, we see that the following convergence result holds.

Proposition 4.2. *Let $\{u_n\}$ be a sequence in $\mathcal{BA}_\omega$ satisfying* $\lim_{n\to\infty} \mathcal{K}(u_n) = 0$. *Then, we have* $\lim_{n\to\infty}$ *$dist_{H^1}(u_n, \mathcal{O}(Q_\omega)) = 0$.*

We can prove Proposition 4.2 by a standard argument. However, for the sake of the completeness, we shall give the proof in Appendix A below. Next, we recall several estimates, which are needed later.

Lemma 4.2. *Let $\theta(t)$ and η be the functions given in* (2.1), *$N_\omega(\eta)$ be the nonlinear function defined by* (2.4), *$\mathcal{Y}_{\omega,\pm}$ be the eigenfunctions of $-i\mathcal{L}_\omega$, $\mathcal{Y}_{\omega,i}$ $(i = 1, 2)$ be the function defined by* (2.9) *and $\lambda_1(t)$ be the parameter given in* (3.26). *There exists sufficiently small $\delta_X > 0$ and $C_1 > 0$ such that as long as the solution ψ to* (1.1) *satisfies $d_\omega(\psi(t)) < \delta_X$, we have*

$$\left|\Omega\Big(\Big\{\frac{d\theta}{dt}(t) - \omega\Big\}\eta(t) - N_\omega(\eta(t)), \mathcal{Y}_{\omega,i}\Big)\right| \leq C_1 \big|\lambda_1(t)\big|^2 \qquad \textit{for } i = 1, 2, \tag{4.2}$$

$$\left|\Big(\Big\{\frac{d\theta}{dt}(t) - \omega\Big\}\eta(t) - N_\omega(\eta(t)), \mathcal{Y}_{\omega,\pm}\Big)_{L^2_{real}}\right| \leq C_1 \big|\lambda_1(t)\big|^2. \tag{4.3}$$

See [**6**, (5.30) and (5.31)] for the proof of Lemma 4.2. We are now in a position to prove Proposition 4.1.

Proof of Proposition 4.1 . We divide the proof into four steps.

(Step 1). We claim that there exists a sequence $\{t_n\}$ in $(0, \infty)$ with $\lim_{n\to\infty} t_n = \infty$ such that

$$\lim_{n\to\infty} \mathrm{dist}_{H^1}(\psi(t_n), \mathcal{O}(Q_\omega)) = 0. \tag{4.4}$$

Since $\mathcal{BA}_{\omega,-}$ is an invariant set (see Lemma 4.1), we see that $\mathcal{S}_\omega(\psi(t)) = m_\omega$ and $\mathcal{K}(\psi(t)) < 0$ for all $t \in (-T^-_{\max}, \infty)$. Then, we obtain $\limsup_{t\to\infty} \mathcal{K}(\psi(t)) = 0$. Otherwise, the solution $\psi(t)$ blows up in finite time (see the proof of Theorem 1.3 in [**4**]). Thus, there exists a sequence $\{t_n\}$ in $[0, \infty)$ with $\lim_{n\to\infty} t_n = \infty$ such that $\lim_{n\to\infty} \mathcal{K}(\psi(t_n)) = 0$. Then, up to a subsequence, we see from Proposition 4.2 that $\lim_{n\to\infty} \mathrm{dist}_{H^1}(\psi(t_n), \mathcal{O}(Q_\omega)) = 0$. Thus, (4.4) holds.

(Step 2). (4.4) together with one-pass theorem (Theorem 3.5) yields that for any $R > 0$, there exists $N \in \mathbb{N}$ such that $\widetilde{d}_\omega(\psi(t)) < R + R^{\frac{3}{2}}$ for all $t \in [t_N, \infty)$. Since $R > 0$ is arbitrary, we have $\lim_{t\to\infty} \widetilde{d}_\omega(\psi(t)) = 0$. Then, it follows from (3.34) that

$$d_\omega(\psi(t)) < 2R \qquad \text{for all } t \in [t_N, \infty).$$

In addition, by (3.17), (3.19) and (3.30), we have

$$\lambda_+^2(\psi(t)) + \lambda_-^2(\psi(t)) \leq \frac{2}{e_\omega}\|\eta(t)\|_E^2 \lesssim R^2 \qquad \text{for all } t \in [t_N, \infty) \tag{4.5}$$

Therefore, letting

$$v_0 := \max_{t\geq 0}\{|\lambda_1(t)|\} > 0. \tag{4.6}$$

we may assume that $v_0 \in (0, \widetilde{\gamma}_\omega)$ by replacing t by $t+T$ ($T \gg 1$).

(Step 3). Using an idea of Nakanishi and Schlag [**38**], we shall show that $|\lambda_1(t)|$ is a decreasing function of t. Suppose to the contrary that there exists $t_0 > 0$ such that $|\lambda_1(t_0)| = \max_{t\geq 0}|\lambda_1(t)| = v_0$. Observe from $\psi_0 \in \mathcal{BA}_{\omega,-}$ that $\mathcal{E}(\psi) = \mathcal{E}(Q_\omega)$. Then, it follows from (3.31) that

$$d_\omega^2(t) = 2e_\omega \lambda_1^2(t). \tag{4.7}$$

This yields that $d_\omega^2(t_0) = \max_{t\geq 0} d_\omega^2(t)$. Then, one has

$$0 = \frac{d}{dt} d_\omega^2\big(\psi(t)\big)\bigg|_{t=t_0}.$$

This together with (3.32) yields

$$0 = e_\omega^2 \lambda_1(t_0)\lambda_2(t_0) + e_\omega\lambda_1(t_0)\Big(\Big\{\frac{d\theta}{dt}(t_0) - \omega\Big\}\eta(t_0) - N_\omega(\eta(t_0)), i\mathcal{Y}_{\omega,2}\Big)_{L^2_{\text{real}}}. \tag{4.8}$$

Combining (4.8) with (4.2), we obtain that

$$0 \leq e_\omega^2 \operatorname{sgn}[\lambda_1(t_0)]|\lambda_1(t_0)|\lambda_2(t_0) + Ce_\omega|\lambda_1(t_0)|^3 \tag{4.9}$$

for some constant $C > 0$. It follows from (4.9) that

$$-C|\lambda_1(t_0)|^2 \leq e_\omega \operatorname{sgn}[\lambda_1(t_0)]\lambda_2(t_0). \tag{4.10}$$

Suppose that $\operatorname{sgn}[\lambda_1(t_0)] = 1$. Then, since $|\lambda_1(t_0)| \ll 1$, we see from (4.6), (4.10) and (3.26) that

$$0 < \lambda_1(t_0) \leq 2\lambda_1(t_0) - \frac{2C}{e_\omega}\big|\lambda_1(t_0)\big|^2 \leq 2(\lambda_1(t_0) + \lambda_2(t_0)) = 2\lambda_+(t_0). \tag{4.11}$$

Suppose next that $\operatorname{sgn}[\lambda_1(t_0)] = -1$. Then, (4.10) becomes $e_\omega\lambda_2(t_0) \leq C|\lambda_1(t_0)|^2$. Since $|\lambda_1(t_0)| = R \ll 1$, we see from (3.26) that

$$\lambda_+(t_0) = \lambda_1(t_0) + \lambda_2(t_0) \leq \lambda_1(t_0) + \frac{C}{e_\omega}|\lambda_1(t_0)|^2 \leq \frac{\lambda_1(t_0)}{2} < 0. \tag{4.12}$$

Thus, we conclude from (4.11) and (4.12) that

$$|\lambda_1(t_0)| \leq 2|\lambda_+(t_0)|. \tag{4.13}$$

From (3.13), (4.3), (4.13) and $|\lambda_1(t_0)| = \max_{t\geq 0}|\lambda_1(t)|$, we have

$$\begin{aligned}
|\lambda_+(t)| &\geq e^{e_\omega(t-t_0)}|\lambda_+(t_0)| - C_1|\lambda_1(t_0)|^2\int_{t_0}^t e^{e_\omega(t-s)}ds \\
&\geq e^{e_\omega(t-t_0)}|\lambda_+(t_0)| - \frac{C_1}{e_\omega}|\lambda_1(t_0)|^2 e^{e_\omega(t-t_0)} \\
&\geq e^{e_\omega(t-t_0)}\left(1 - \frac{4C_1|\lambda_+(t_0)|}{e_\omega}\right)|\lambda_+(t_0)| \geq \frac{e^{e_\omega(t-t_0)}}{2}|\lambda_+(t_0)| \to \infty \text{ as } t\to\infty,
\end{aligned}$$

which contradicts (4.5). We have used the fact that $|\lambda_+(t_0)| \ll 1$ in the last inequality (see (4.5)). Thus, we find that $|\lambda_1(t)|$ is a decreasing function of t.

(Step 4). We shall show (4.1). We first consider the case that $\lambda_1(t)$ is a decreasing function of $t > 0$. Since $\lim_{t\to\infty}\lambda_1(t) = 0$, we have $\lambda_1(t) > 0$.

It follows from (3.28), (4.2) and $\max_{t\geq 0}|\lambda_1(t)| = R \ll 1$ that

$$\frac{d\lambda_2}{dt}(t) \geq e_\omega \lambda_1(t) - C_1|\lambda_1(t)|^2 \geq \frac{e_\omega}{2}\lambda_1(t) > 0.$$

This together with $\lim_{t\to\infty}|\lambda_2(t)| = 0$ implies that $\lambda_2(t) < 0$ for all $t > 0$. Since $\lambda_1(t) > 0$ and $\lambda_2(t) < 0$ for all $t > 0$, we see from (3.26) that $-\lambda_-(t) < \lambda_+(t) < \lambda_-(t)$ for all $t > 0$. Therefore, we have $\lambda_-(t) > 0$ and

$$|\lambda_+(t)| \leq |\lambda_-(t)|. \tag{4.14}$$

From (3.14), (4.3), (4.14) and $|\lambda_-(t)| \ll 1$, we find that

$$\frac{d\lambda_-}{dt}(t) \leq -e_\omega \lambda_-(t) + 4C_1|\lambda_-(t)|^2 \leq -\frac{e_\omega}{2}\lambda_-(t).$$

It follows that $0 < \lambda_-(t) \leq \lambda_-(0)e^{-\frac{e_\omega}{2}t}$. We see from (4.14) that $|\lambda_+(t)| < \lambda_-(0)e^{-\frac{e_\omega}{2}t}$. It follows from (3.26) that $|\lambda_1(t)| \leq \lambda_-(0)e^{-\frac{e_\omega}{2}t}$. This together with Proposition 3.2, (4.7) implies that (4.1) holds.

We can prove the case where $\lambda_1(t)$ is an increasing function of $t > 0$ similarly. Thus, we omit it. □

4.2. Blowup in negative time direction. Next, we shall show that if a solution starts from $\mathcal{BA}_{\omega,-}$ exists globally in positive time direction, the solution must blow up in finite negative time:

PROPOSITION 4.3. *Assume that $\omega \in (0, \omega_3)$, where $\omega_3 > 0$ is the constant given in Lemma 2.3. Suppose that a solution ψ to* (1.1) *with $\psi|_{t=0} = \psi_0 \in \mathcal{BA}_{\omega,-}$ exists on $(-T^-_{\max}, \infty)$. Then, ψ blows up in finite negative time, that is, $T^-_{\max} < \infty$.*

To prove Proposition 4.3, we first consider a solution which has a finite variance. Namely, we shall show the following:

LEMMA 4.3. *Assume that $\omega \in (0, \omega_3)$, where $\omega_3 > 0$ is the constant given in Lemma 2.3. Suppose that a solution ψ to* (1.1) *with $\psi|_{t=0} = \psi_0 \in \mathcal{BA}_{\omega,-}$ defined on $(-T^-_{\max}, \infty)$ satisfies*

$$|x|\psi \in L^2(\mathbb{R}^3).$$

Then, ψ blows up in finite negative time, that is, $T^-_{\max} < \infty$.

We can prove Lemma 4.3 by a similar argument in the proof of Proposition 5.1 (Page 25) of [**16**]. Thus, we omit the proof.

LEMMA 4.4. *Assume that $\omega \in (0, \omega_3)$, where $\omega_3 > 0$ is the constant given in Lemma 2.3. Suppose that a solution ψ to* (1.1) *with $\psi|_{t=0} = \psi_0 \in \mathcal{BA}_{\omega,-}$ is defined on $(-T^-_{\max}, \infty)$ and $T^-_{\max} = \infty$. Then, $\lim_{t\to-\infty} dist_{H^1}(\psi(t), \mathcal{O}(Q_\omega)) = 0$ and there exists a constant $C_1 > 0$ such that*

$$\mathcal{K}(\psi(t)) < -C_1\|\eta(t)\|_{H^1} \qquad \text{for all } t \in (-\infty, 0). \tag{4.15}$$

PROOF. We divide the proof into four steps.

(Step 1). Suppose that there exists $\delta > 0$ such that $\mathcal{K}(\psi(t)) < -\delta$ for all $t \in (-T^-_{\max}, 0)$. Then, we see that $T^-_{\max} < \infty$. Thus, it suffices to consider the case of $\limsup_{t\to-\infty} \mathcal{K}(\psi(t)) = 0$. Then, there exists a sequence $\{t_n\}$ in $(-\infty, 0)$ with $\lim_{n\to\infty} t_n = -\infty$ such that $\lim_{n\to\infty} \mathcal{K}(\psi(t_n)) = 0$. Then, by a similar way to **(Step 1)** –**(Step 3)** in the proof of Proposition 4.1, we see that for any $R > 0$, there exists t_N in $(-\infty, 0)$ such that

$$d_\omega(\psi(t)) < CR \qquad \text{for all } t \in (-\infty, t_N],$$

$$\lambda_+^2(\psi(t)) + \lambda_-^2(\psi(t)) \leq \|\eta(t)\|_E^2 \lesssim R \qquad \text{for all } t \in (-\infty, t_N]. \tag{4.16}$$

Since $R > 0$ is arbitrary, we have $\lim_{t\to-\infty} \operatorname{dist}_{H^1}(\psi(t), \mathcal{O}(Q_\omega)) = 0$. Replacing t by $t - T$ ($T \gg 1$), we may assume that for any $v_0 \in (0, \widetilde{\gamma}_\omega)$,

$$v_0 = \max_{t\leq 0}\{|\lambda_1(t)|\} > 0. \tag{4.17}$$

In addition, we see that

$$|\lambda_1(t)| \text{ is an increasing function of } t. \tag{4.18}$$

(Step 2). We claim that

$$\lim_{t\to-\infty}\left|\frac{\lambda_-(t)}{\lambda_+(t)}\right| = 0. \tag{4.19}$$

Suppose to the contrary that there exists a constant $\delta > 0$ such that

$$\limsup_{t\to-\infty}\left|\frac{\lambda_-(t)}{\lambda_+(t)}\right| \geq \delta. \tag{4.20}$$

Then there exists a sequence $\{t_n\}$ in $(-\infty, 0)$ with $\lim_{n\to\infty} t_n = -\infty$ such that

$$\left|\frac{\lambda_-(t_n)}{\lambda_+(t_n)}\right| \geq \frac{\delta}{2}.$$

This together with (3.26) yields that

$$|\lambda_1(t_n)| \leq |\lambda_+(t_n)| + |\lambda_-(t_n)| \leq \frac{2+\delta}{\delta}|\lambda_-(t_n)|. \tag{4.21}$$

From (3.14), (4.2), (4.18) and (4.21), we have

$$\begin{aligned}
|\lambda_-(t)| &\geq e^{-e_\omega(t-t_n)}|\lambda_-(t_n)| - C_1|\lambda_1(t_n)|^2 \int_t^{t_n} e^{-e_\omega(t-s)} ds \\
&\geq e^{-e_\omega(t-t_n)}|\lambda_-(t_n)| - \frac{C_1}{e_\omega}|\lambda_1(t_n)|^2 e^{-e_\omega(t-t_n)} \\
&\geq \left(1 - \frac{(2+\delta)C_1}{\delta e_\omega}|\lambda_1(t_n)|\right) e^{-e_\omega(t-t_n)}|\lambda_-(t_n)|.
\end{aligned}$$

Since $\lim_{t\to-\infty}|\lambda_1(t)| = 0$, there exists a sufficiently large $n_0 \in \mathbb{N}$ such that

$$1 - \frac{(2+\delta)C_1}{\delta e_\omega}|\lambda_1(t_{n_0})| \geq \frac{1}{2}.$$

Then, we obtain

$$|\lambda_-(t)| \geq \frac{e^{-e_\omega(t-t_{n_0})}}{2}|\lambda_-(t_{n_0})| \to \infty \qquad \text{as } t \to -\infty,$$

which is a contradiction. Thus, (4.19) holds.

(Step 3). Now, we see from (3.10), (3.24) and (2.15) that

$$(Q_\omega, \Gamma(t))_{L^2_{\text{real}}} = -\mathcal{M}(\eta(t)). \tag{4.22}$$

Moreover, we see from (2.6) and (2.7) that

$$\begin{aligned}
\mathcal{K}'(Q_\omega) + 2\omega Q_\omega &= -2\Delta Q_\omega + 2\omega Q_\omega - 3Q_\omega^3 - 6Q_\omega^5 \\
&= \frac{3}{2}L_{\omega,-}Q_\omega + \frac{1}{2}L_{\omega,+}Q_\omega - 2Q_\omega^5 \\
&= \frac{1}{2}L_{\omega,+}Q_\omega - 2Q_\omega^5.
\end{aligned} \tag{4.23}$$

We note that the decomposition (3.24) of $\eta(t)$ is expressed as follows in terms of the functions $\mathcal{Y}_{\omega,1}$ and $\mathcal{Y}_{\omega,2}$:

$$\eta(t) = 2\lambda_1(t)\mathcal{Y}_{\omega,1} + 2i\lambda_2(t)\mathcal{Y}_{\omega,2} + \Gamma(t). \tag{4.24}$$

This together with (4.22), (4.23), (4.24) and (3.10) shows that

$$\begin{aligned}
&\langle \mathcal{K}'(Q_\omega), \eta(t)\rangle_{H^{-1},H^1} \\
&= \langle \mathcal{K}'(Q_\omega) + 2\omega Q_\omega, \eta(t)\rangle_{H^{-1},H^1} - 2\omega(Q_\omega, \eta(t))_{L^2_{\mathrm{real}}} \\
&= \langle \frac{1}{2}L_{\omega,+}Q_\omega - 2Q_\omega^5, 2\lambda_1(t)\mathcal{Y}_{\omega,1} + 2i\lambda_2(t)\mathcal{Y}_{\omega,2} + \Gamma(t)\rangle_{H^{-1},H^1} - 2\omega(Q_\omega, \eta(t))_{L^2_{\mathrm{real}}} \\
&= \langle \frac{1}{2}L_{\omega,+}Q_\omega - 2Q_\omega^5, 2\lambda_1(t)\mathcal{Y}_{\omega,1} + \Gamma(t)\rangle_{H^{-1},H^1} + 2\omega\mathcal{M}(\eta(t)).
\end{aligned}$$

Here, we have used the fact that $Q_\omega, L_{\omega,+}Q_\omega$ and $\mathcal{Y}_{\omega,2}$ are real-valued functions in the last equality. Observe from (2.6) that $\frac{1}{2}L_{\omega,+}Q_\omega - 2Q_\omega^5 = -Q_\omega^3 - 4Q_\omega^5$. This together with (2.10) yields that

$$\begin{aligned}
\langle \mathcal{K}'(Q_\omega), \eta(t)\rangle_{H^{-1},H^1} &= \lambda_1(t)(L_{\omega,+}Q_\omega, \mathcal{Y}_{\omega,1})_{L^2} - 4\lambda_1(t)(Q_\omega^5, \mathcal{Y}_{\omega,1})_{L^2} \\
&\quad - (Q_\omega^3 + 4Q_\omega^5, \Gamma(t))_{L^2_{\mathrm{real}}} + 2\omega\mathcal{M}(\eta(t)) \\
&= -e_\omega\lambda_1(t)(Q_\omega, \mathcal{Y}_{\omega,2})_{L^2} - 4\lambda_1(t)(Q_\omega^5, \mathcal{Y}_{\omega,1})_{L^2} \\
&\quad - (Q_\omega^3 + 4Q_\omega^5, \Gamma(t))_{L^2_{\mathrm{real}}} + 2\omega\mathcal{M}(\eta(t)).
\end{aligned} \tag{4.25}$$

Taylor's expansion of $\mathcal{K}$ around Q_ω together with $\mathcal{K}(Q_\omega) = 0$ and (4.25) shows that

$$\begin{aligned}
\mathcal{K}(\psi(t)) &= \mathcal{K}(Q_\omega + \eta(t)) = \langle \mathcal{K}'(Q_\omega), \eta(t)\rangle + O(\|\eta(t)\|_{H^1}^2) \\
&= -e_\omega\lambda_1(t)(Q_\omega, \mathcal{Y}_{\omega,2})_{L^2} - 4\lambda_1(t)(Q_\omega^5, \mathcal{Y}_{\omega,1})_{L^2} - (Q_\omega^3, \Gamma(t))_{L^2_{\mathrm{real}}} \\
&\quad - 4(Q_\omega^5, \Gamma(t))_{L^2_{\mathrm{real}}} + O(\|\eta(t)\|_{H^1}^2).
\end{aligned} \tag{4.26}$$

(Step 4). From (4.19), for any $\varepsilon > 0$, there exists $T_\varepsilon > 0$ such that

$$|\lambda_-(t)| < \varepsilon|\lambda_+(t)| \qquad \text{for all } t < -T_\varepsilon. \tag{4.27}$$

It follows from (3.18), the condition $\mathcal{E}(\psi) = \mathcal{E}(Q_\omega)$, (3.30), (3.31) and (3.17) that

$$e_\omega|\lambda_+(t)||\lambda_-(t)| + C|\lambda_1(t)|^3 \geq \|\Gamma(t)\|_{H^1}^2.$$

This together with (4.27) and (3.26) yields that

$$\|\Gamma(t)\|_{H^1}^2 \leq e_\omega\varepsilon|\lambda_+(t)|^2 + C|\lambda_1(t)|^3 \leq C\varepsilon|\lambda_1(t)|^2 \tag{4.28}$$

Suppose that there exists $t_* < -T_\varepsilon$ such that $\lambda_1(t_*) > 0$. Then, it follows from (4.26), (2.14), (2.16) and (4.28) that

(4.29)

$$\begin{aligned}
0 > \mathcal{K}(\psi(t_*)) &\geq \frac{e_\omega}{2}\lambda_1(t_*)|(Q_\omega, \mathcal{Y}_{\omega,2})_{L^2}| - C(\|Q_\omega\|_{L^\infty}^2 + \|Q_\omega\|_{L^\infty}^4)\|Q_\omega\|_{L^2}\|\Gamma(t_*)\|_{L^2} \\
&\geq \frac{e_\omega}{4}\lambda_1(t_*)|(Q_\omega, \mathcal{Y}_{\omega,2})_{L^2}| \geq 0,
\end{aligned}$$

which is a contradiction. Thus, we see from (4.18) and $\lim_{t\to-\infty}\lambda_1(t) = 0$ that $\lambda_1(t) < 0$ for $t < -T_\varepsilon$. Then, from (4.7), (3.30) and (3.22), we have $\lambda_1(t) \leq -C\|\eta(t)\|_{H^1}$. Then, by a similar argument as in (4.29), we obtain

$$\mathcal{K}(\psi(t)) \leq \frac{e_\omega}{4}\lambda_1(t)|(Q_\omega, \mathcal{Y}_{\omega,2})_{L^2}| \leq -\frac{e_\omega}{4}C|(Q_\omega, \mathcal{Y}_{\omega,2})_{L^2}|\|\eta(t)\|_{H^1} \qquad \text{for all } t < -T_\varepsilon.$$

Therefore, from the continuity of $\mathcal{K}(\psi(t))$ and $\eta(t)$, we see that (4.15) holds. □

Let φ be a radial function such that

$$\varphi \in C^\infty(\mathbb{R}^3), \qquad \varphi(r) \geq 0, \quad \varphi''(r) \leq 2 \qquad (r \geq 0), \qquad \varphi(r) = \begin{cases} r^2 & (0 \leq r \leq 1), \\ 0 & (r \geq 2). \end{cases}$$

For $R > 0$, we put

$$y_R(t) := \int_{\mathbb{R}^3} R^2\varphi(\frac{x}{R})|\psi(t,x)|^2 dx. \tag{4.30}$$

Then, we obtain

$$y_R''(t) = 8\mathcal{K}(\psi) + A_R(\psi(t)), \tag{4.31}$$

where

$$\begin{aligned} A_R(\psi(t)) := 4\int_{\mathbb{R}^3}\Big(\varphi''(\frac{x}{R}) - 2\Big)|\nabla\psi(t,x)|^2 dx - \int_{\mathbb{R}^3}\Big(\Delta\varphi(\frac{x}{R}) - 6\Big)|\psi(t,x)|^4 dx \\ - \frac{4}{3}\int_{\mathbb{R}^3}\Big(\Delta\varphi(\frac{x}{R}) - 6\Big)|\psi(t,x)|^6 dx + \frac{1}{R^2}\int_{\mathbb{R}^3}\Delta^2\varphi(\frac{x}{R})|\psi(t,x)|^2 dx. \end{aligned}$$

Then, by a similar argument in [**14**, Claim 4.3] (see also [**16**, Section 5.2]), we can prove the following:

LEMMA 4.5. *For any $\varepsilon > 0$, there exists $R_\varepsilon > 0$ such that for any $R \geq R_\varepsilon$, we have*

$$|A_R(\psi(t))| \leq \varepsilon\|\eta(t)\|_{H^1}. \tag{4.32}$$

We are now in a position to prove Proposition 4.3. Actually, the proof is similar to that of [**17**]. However, for the reader's convenience, we shall give the proof.

PROOF OF PROPOSITION 4.3. Suppose to the contrary that $T_{\max}^- = \infty$. We see from (4.31) and Lemmas 4.4 and 4.5 that $y_R''(t) < 0$ for all $t < (-\infty, 0)$. We claim that

$$y_R'(t) < 0 \qquad \text{for all } t < 0. \tag{4.33}$$

Suppose to the contrary that (4.33) fails. Then, one of the following two cases must occur: there exist $t_0 < 0$ and $\varepsilon_0 > 0$ such that $y_R'(t_0) > \varepsilon_0$, or there exists $\widetilde{t}_0 < 0$ such that $y_R'(\widetilde{t}_0) = 0$. If the latter case occurs, we see from $y_R''(t) < 0$ for all $t < (-\infty, 0)$ that there exists $\widetilde{t}_1 \in (\widetilde{t}_0, 0)$ and $\widetilde{\varepsilon}_1 > 0$ such that $y_R'(\widetilde{t}_1) > \widetilde{\varepsilon}_1$. Thus, it suffices to consider only the former case.

We see from $y_R''(t) < 0$ for all $t < (-\infty, 0)$ and $y_R'(t_0) > \varepsilon_0$ that $y_R'(t) > \varepsilon_0$ for all $t < t_0$. Then, it follows from the fundamental theorem of calculus that

$$y_R(t_0) - y_R(t) = \int_t^{t_0} y_R'(s)ds > \varepsilon_0(t_0 - t).$$

This yields that

$$y_R(t) < y_R(t_0) - \varepsilon_0(t_0 - t) = y_R(t_0) - \varepsilon_0 t_0 + \varepsilon_0 t \to -\infty \qquad (t \to -\infty),$$

which contradicts the positivity of $y_R(t)$. Thus, (4.33) holds.

Then, we see from (4.33) that $y_R(t)$ is a decreasing function. In addition, we know from Lemma 4.4 that $\lim_{t\to-\infty} \mathrm{dist}_{H^1}(\psi(t), \mathcal{O}(Q_\omega)) = 0$. These imply that

$$y_R(0) \le \lim_{t\to-\infty} y_R(t) = \int_{\mathbb{R}^3} R^2\varphi(\frac{x}{R})|Q_\omega|^2 dx.$$

Letting R go to infinity, we obtain

$$\int_{\mathbb{R}^3} |x|^2|\psi_0|^2 dx \le \int_{\mathbb{R}^3} |x|^2|Q_\omega|^2 dx < \infty.$$

Then, from Lemma 4.3, we see that $T_{\max}^- < \infty$, which is absurd. Therefore, we conclude the desired result. □

5. Analysis on $\mathcal{BA}_{\omega,+}$

In this section, we investigate the asymptotic behavior of solutions which start from $\mathcal{BA}_{\omega,+}$.

5.1. Convergence to the orbit of the ground state. We can prove that if a solution with (1.1) $\psi|_{t=0} = \psi_0 \in \mathcal{BA}_{\omega,+}$ does not scatter in positive time direction, the solution converges to the orbit of the ground state exponentially. More precisely, we obtain the following:

PROPOSITION 5.1. *Let $\omega_3 > 0$ be the constant given in Lemma 2.3 and assume that $\omega \in (0, \omega_3)$. Suppose that a solution ψ to (1.1) with $\psi|_{t=0} = \psi_0 \in \mathcal{BA}_{\omega,+}$ does not scatter for positive time. Then, there exist constants $C > 0$ and $c > 0$ such that*

$$\text{(5.1)} \qquad dist_{H^1}(\psi(t), \mathcal{O}(Q_\omega)) \le Ce^{-ct} \qquad \text{for all } t > 0.$$

The proof of Proposition 5.1 is similar to that of Proposition 4.1. Thus, we omit the proof.

5.2. Scattering in negative time direction. Secondly, we shall show that if a solution starts from $\mathcal{BA}_{\omega,+}$ does not scatter in positive time direction, the solution must scatter in negative one:

PROPOSITION 5.2. *Assume that $\omega \in (0, \omega_3)$, where $\omega_3 > 0$ is the constant given in Lemma 2.3. Suppose that a solution ψ to (1.1) with $\psi|_{t=0} = \psi_0 \in \mathcal{BA}_{\omega,+}$ does not scatter in positive time direction. Then, ψ exists on $(-\infty, \infty)$ and scatters for negative one, that is, there exists $\varphi_- \in H^1(\mathbb{R}^3)$ such that*

$$\lim_{t\to-\infty} \|\psi(t) - e^{it\Delta}\varphi_-\|_{H^1} = 0.$$

To prove Proposition 5.2, we need the following lemma:

LEMMA 5.1. *Assume that $\omega \in (0, \omega_3)$, where $\omega_3 > 0$ is the constant given in Lemma 2.3. Suppose that a solution ψ to (1.1) with $\psi|_{t=0} = \psi_0 \in \mathcal{BA}_{\omega,+}$ does not scatter in negative time direction. Then, $\lim_{t\to-\infty} dist_{H^1}(\psi(t), \mathcal{O}(Q_\omega)) = 0$ and there exist a constant $C_2 > 0$ such that*

$$\text{(5.2)} \qquad \mathcal{K}(\psi(t)) \ge C_2\|\eta(t)\|_{H^1} \qquad \text{for all } t \in (-\infty, 0).$$

We can prove Lemma 5.1 by a similar argument in the proof of Lemma 4.4. Thus, we omit the proof. We are now in the position to prove Proposition 5.2.

Proof of Proposition 5.2. We will follow the argument of [**16**, Section 6.4]. Suppose that a solution ψ does not scatter in both positive and negative time direction. Let y_R be the function given by (4.30). From the fundamental theorem of calculus, we obtain

$$\int_\sigma^\tau y_R''(t)dt = y_R'(\tau) - y_R'(\sigma). \tag{5.3}$$

(4.31) together with (4.32) and (5.2) yields that

$$\int_\sigma^\tau y_R''(t)dt \geq \frac{C_2}{2}\int_\sigma^\tau \|\eta(t)\|_{H^1}dt > 0 \qquad \text{for all } \sigma < 0 < \tau. \tag{5.4}$$

Note that

$$y_R'(t) = 2R\mathrm{Im}\int_{\mathbb{R}^3} \overline{\eta(t,x)}\nabla\varphi(\frac{x}{R})\cdot\nabla Q_\omega(x)dx + 2R\mathrm{Im}\int_{\mathbb{R}^3} Q_\omega(x)\nabla\varphi(\frac{x}{R})\cdot\nabla\eta(t,x)dx$$
$$+ 2R\mathrm{Im}\int_{\mathbb{R}^3}\overline{\eta(t,x)}\nabla\varphi(\frac{x}{R})\cdot\nabla\eta(x)dx.$$

From this and (5.1), we obtain

$$|y_R'(t)| \leq CR(\|\eta(t)\|_{H^1} + \|\eta(t)\|_{H^1}^2) \to 0 \qquad \text{as } t\to\pm\infty. \tag{5.5}$$

Letting σ go to $-\infty$ and τ to $+\infty$ in (5.3), we have, by (5.4), (5.5) and Lemmas 5.1 and Lemma 4.5, that

$$\frac{C_2}{2}\int_{-\infty}^\infty \|\eta(t)\|_{H^1}dt \leq \lim_{\sigma\to-\infty}|y'(\sigma)| + \lim_{\tau\to\infty}|y'(\tau)| = 0.$$

This implies that $\eta(t) = 0$ for all $t\in\mathbb{R}$. However, this contradicts the assumption $\mathcal{K}(\psi(0)) = \mathcal{K}(\psi_0) > 0$. This completes the proof. ☐

6. Construction of special solutions

In this section, we introduce Strichartz-type spaces and give the proof of Theorem 1.3. First, we recall the Strichartz estimate:

Definition 6.1. We say that a pair of (q, r) is L^2-*admissible* if

$$\frac{1}{r} = \frac{3}{2}\left(\frac{1}{2} - \frac{1}{q}\right).$$

Lemma 6.2 (Strichartz estimate). (i) *For any L^2-admissible pair (q, r), we have*

$$\|e^{it\Delta}u\|_{L^r(\mathbb{R},L^q)} \lesssim \|u\|_{L^2}. \tag{6.1}$$

(ii) *For any admissible pairs (q_1, r_1) and (q_2, r_2), we have*

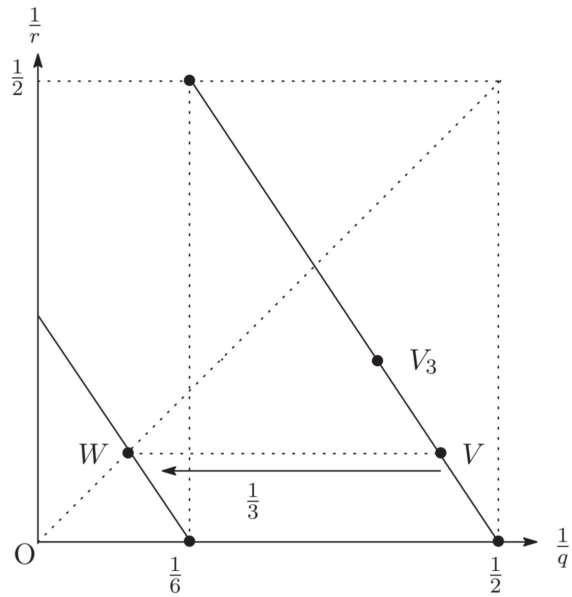
$$\left\|\int_{t_0}^t e^{i(t-t')\Delta}f(t')dt'\right\|_{L^{r_1}(\mathbb{R},L^{q_1})} \lesssim \|f\|_{L^{r_2'}(\mathbb{R},L^{q_2'})} \qquad \text{for any } t_0\in\mathbb{R}, \tag{6.2}$$

where q' and r' denote the Hölder conjugates of q and r respectively.

We shall use the following Strichartz-type spaces:

$$St(I) := L_t^\infty L_x^2(I\times\mathbb{R}^3)\cap L_t^2L_x^6(I\times\mathbb{R}^3), \qquad V(I) := L_t^{10}L_x^{\frac{30}{13}}(I\times\mathbb{R}^3),$$

$$V_3(I) := L_t^5L_x^{\frac{30}{11}}(I\times\mathbb{R}^3), \qquad W(I) := L_t^{10}L_x^{10}(I\times\mathbb{R}^3).$$



We define the norm of $St(I)$ for an interval I by

$$\|u\|_{St(I)} := \sup_{(q,r):\ L^2\text{-admissible}} \|u\|_{L_t^r L_x^q(I\times\mathbb{R}^3)}.$$

From the definition, we see that

$$\|u\|_{V(I)}, \|u\|_{V_3(I)} \leq \|u\|_{St(I)}. \tag{6.3}$$

We also use the following function space:

$$N(I) := L_t^{\frac{5}{3}} L_x^{\frac{30}{23}}(I\times\mathbb{R}^3),$$

which is the dual space of the Strichartz space $L_t^{\frac{5}{2}} L_x^{\frac{30}{7}}(I\times\mathbb{R}^3)$.

By a similar argument to [**14**, Lemma 6,1] (see also [**16**, Proposition 3.4]), we can construct a family of approximate solutions to (1.1). More precisely, we shall show the following:

Proposition 6.1. *Let $\omega_2 > 0$ be the constant given by Theorem 2.1. For any $\omega \in (0,\omega_2)$ and $A \in \mathbb{R}$, there exists a sequence $\{\mathcal{Z}_{j,\omega}^A\}_{j\in\mathbb{N}}$ of functions in $\mathcal{S}(\mathbb{R}^3)\setminus\{0\}$ such that $\mathcal{Z}_{1,\omega}^A = A\mathcal{Y}_{\omega,-}$ and if $k \geq 1$ and $\mathcal{V}_{k,\omega}^A := \sum_{j=1}^k e^{-je_\omega t}\mathcal{Z}_{k,\omega}^A$, then we have*

$$\partial_t \mathcal{V}_{k,\omega}^A + i\mathcal{L}_\omega \mathcal{V}_{k,\omega}^A = N_\omega(\mathcal{V}_{k,\omega}^A) + O(e^{-(k+1)e_\omega t}) \qquad \text{in } \mathcal{S}(\mathbb{R}^3) \text{ as } t\to\infty. \tag{6.4}$$

By using Proposition 6.1 and the contraction argument, we can construct a solution U_ω^A to (1.1) which is close to $e^{i\omega t}(Q_\omega + \mathcal{V}_{k,\omega}^A)$.

Proposition 6.2. *Let $\omega_2 > 0$ be the constant given by Theorem 2.1. For any $\omega \in (0,\omega_2)$ and $A \in \mathbb{R}$, there exists $k_0 > 0$ such that for any $k \geq k_0$, there exist sufficiently large $t_0 = t_0(\omega, A) > 0$ and the radial solution $U_\omega^A \in C^\infty([t_0,\infty), H^\infty(\mathbb{R}^3))$*

to (1.1) *satisfying the following:*

$$\|\langle\nabla\rangle(U_\omega^A - e^{i\omega t}(Q_\omega + \mathcal{V}_{k,\omega}^A))\|_{St(t,\infty)} \le e^{-(k+\frac{1}{2})e_\omega t} \qquad (t \ge t_k). \tag{6.5}$$

Furthermore, U_ω^A *is a unique solution to* (1.1) *satisfying* (6.5) *for large* $t > 0$. *Finally,* U_ω^A *is independent of* $k \in \mathbb{N}$ *and satisfies*

$$\|U_\omega^A - e^{i\omega t}(Q_\omega + Ae^{-e_\omega t}\mathcal{Y}_{\omega,-})\|_{H^1} \le Ce^{-\frac{3}{2}e_\omega t} \qquad \text{for all } t \ge t_0. \tag{6.6}$$

We can prove Proposition 6.2 in a way similar to [**14**, Proposition 6.3]. Thus, we omit the proof. We are now in a position to prove Theorem 1.3.

Proof of Theorem 1.3. We put

$$Q_\omega^\pm(t,x) := e^{-i\omega t_0}U_\omega^{\pm 1}(t+t_0,x). \tag{6.7}$$

Then, we see from $\mathcal{Y}_{\omega,-} = \overline{\mathcal{Y}}_{\omega,+}$, (2.9), (6.6) and (6.7) that

$$\begin{aligned} Q_\omega^\pm(t,x) &= e^{-i\omega t_0}U_\omega^{\pm 1}(t+t_0,x) \\ &= e^{-i\omega t_0}e^{i\omega(t+t_0)}(Q_\omega \pm e^{-e_\omega(t+t_0)}\mathcal{Y}_{\omega,-} + \Gamma^\pm(t,x)) \\ &= e^{i\omega t}(Q_\omega \pm e^{-e_\omega(t+t_0)}\mathcal{Y}_{\omega,1} \mp ie^{-e_\omega(t+t_0)}\mathcal{Y}_{\omega,2} + \Gamma^\pm(t,x)), \end{aligned}$$

where $\Gamma^\pm \in C^\infty([t_0,\infty), H^\infty(\mathbb{R}^3))$ with $\|\Gamma^\pm(t)\|_{H^1} = O(e^{-\frac{3}{2}e_\omega t})$. Then, by a similar argument in (4.26) [5], we obtain

$$\begin{aligned} \mathcal{K}(Q_\omega^\pm) = &\mp e_\omega \frac{e^{-e_\omega(t+t_0)}}{2}(Q_\omega, \mathcal{Y}_{\omega,2})_{L^2_{\text{real}}} \mp 2e^{-e_\omega(t+t_0)}(Q_\omega^5, \mathcal{Y}_{\omega,1})_{L^2_{\text{real}}} \\ &- (Q_\omega^3, \Gamma^\pm(t))_{L^2_{\text{real}}} - 4(Q_\omega^5, \Gamma^\pm(t))_{L^2_{\text{real}}} + O(e^{-2e_\omega t}). \end{aligned}$$

By (2.14), (2.16) and $\|\Gamma^\pm(t)\|_{H^1} = O(e^{-\frac{3}{2}e_\omega t})$, we have

$$\pm\mathcal{K}(Q_\omega^\pm) \ge e_\omega \frac{e^{-e_\omega(t+t_0)}}{8}|(Q_\omega, \mathcal{Y}_{\omega,2})_{L^2_{\text{real}}}| > 0$$

for sufficiently large $t > 0$. Thus, we see that $Q_\omega^\pm \in \mathcal{BA}_{\omega,\pm}$ which satisfy

$$\text{dist}_{H^1}(Q_\omega^\pm(t), \mathcal{O}(Q_\omega)) \le Ce^{-e_\omega t} \qquad \text{for all } t \ge 0.$$

Then, it follows from Propositions 4.3 and 5.2 that Q_ω^+ blows up in finite negative time and Q_ω^- is globally defined and scatters for negative time. This completes the proof. □

7. Uniqueness and proof of Theorem 1.4

In this section, we shall show that a solution which converges to the orbit of the ground state must be the special one obtained in Proposition 6.2. After that, we will give the proof of Theorem 1.4. Our first aim in this section is to prove the following:

Proposition 7.1. *Assume that* $\omega \in (0, \omega_3)$, *where* $\omega_3 > 0$ *is the constant given in Lemma 2.3. Suppose that a solution* ψ *to* (1.1) *with* $\psi|_{t=0} = \psi_0 \in \mathcal{BA}_\omega$ *satisfies*

$$dist_{H^1}(\psi(t), \mathcal{O}(Q_\omega)) \le Ce^{-ct} \qquad (t \ge t_0) \tag{7.1}$$

for some $c, C \in \mathbb{R}$ *and* $t_0 > 0$. *Then, there exists* $A_0 \in \mathbb{R}$ *and* $\theta_0 \in \mathbb{R}$ *such that* $\psi = e^{i\theta_0}U_\omega^{A_0}$, *where* $U_\omega^{A_0}$ *is the solution to* (1.1) *defined in Proposition 6.2.*

[5] $\lambda_1(t), \lambda_2(t)$ and $\Gamma(t)$ in (4.26) correspond to $\pm\frac{e^{-e_\omega(t+t_0)}}{2}, \mp\frac{e^{-e_\omega(t+t_0)}}{2}$ and $\Gamma^\pm(t)$, respectively.

7.1. Exponentially small solution to the linearized equation. In this subsection, in order to prove Proposition 7.1, we consider $\eta \in C^0([t_0,\infty), H^1(\mathbb{R}^3))$ and $g \in C^0([t_0,\infty), L^2(\mathbb{R}^3))$ such that

$$\partial_t \eta + i\mathcal{L}_\omega \eta = g \qquad \text{in } (t,x) \in (t_0,\infty)\times\mathbb{R}^3, \tag{7.2}$$

$$\|\eta(t)\|_{H^1} \le Ce^{-\gamma_1 t} \qquad (t \ge t_0), \tag{7.3}$$

$$\|\langle\nabla\rangle g\|_{N(t,\infty)} + \|g(t)\|_{L^{\frac{6}{5}}(\mathbb{R}^3)} \le Ce^{-\gamma_2 t} \qquad (t\ge t_0), \tag{7.4}$$

where

$$0<\gamma_1<\gamma_2.$$

We shall show the following:

Proposition 7.2. *Assume that η, g satisfy* (7.2)–(7.4)*. Then, the following holds:*

(i) *if $\gamma_2 \le e_\omega$, $\|\eta(t)\|_{H^1} + \|\langle\nabla\rangle\eta\|_{St(t,\infty)} \le Ce^{-\gamma_2^- t}$,*
(ii) *if $\gamma_2 > e_\omega$, there exists $A\in\mathbb{R}$ such that $\eta(t) = Ae^{-e_\omega t}\mathcal{Y}_{\omega,-} + w(t)$ with $\|w(t)\|_{H^1} + \|\langle\nabla\rangle w\|_{St(t,\infty)} \le Ce^{-\gamma_2^- t}$.*

The proof of Proposition 7.2 is similar to that of Proposition 5.9 in [**14**]. However, since we still employ the symplectic decomposition, the detail is a bit different. Therefore, we give the proof for the sake of completeness.

To prove Proposition 7.2, we need several preparations. Note that (7.2) is equivalent to

$$i\partial_t\eta + \Delta\eta - \omega\eta + \mathcal{V}(\eta) = ig, \tag{7.5}$$

where

$$\begin{aligned}\mathcal{V}(\eta) &= Q_\omega^2(2\eta+\overline{\eta}) + Q_\omega^4(3\eta+2\overline{\eta})\\ &= 3Q_\omega^2(\mathrm{Re}\,\eta) + iQ_\omega^2(\mathrm{Im}\,\eta) + 5Q_\omega^4(\mathrm{Re}\,\eta) + iQ_\omega^4(\mathrm{Im}\,\eta).\end{aligned} \tag{7.6}$$

As in [**14**, Lemma 5.5], we obtain the following:

Lemma 7.1 (Linear estimate). (i) *Let $f\in L^6(\mathbb{R}^3)$. Then, there exists a constant $C_1>0$ such that*

$$\|\mathcal{V}(f)\|_{L^{\frac{6}{5}}} \le C_1\|f\|_{L^6}.$$

(ii) *Let I be a finite time interval of length I and $f\in W(I)$ such that $\nabla f \in V(I)$. Then, there exists $C_2>0$ independent of I, f and g such that*

$$\|f\|_{W(I)} \le C_2\|\nabla f\|_{V(I)}, \tag{7.7}$$

$$\|\langle\nabla\rangle\mathcal{V}(f)\|_{N(I)} \le |I|^{\frac{2}{5}}\|\langle\nabla\rangle f\|_{V_3(I)}. \tag{7.8}$$

Proof. We can obtain (i) by the Hölder inequality. (7.7) follows from the Sobolev inequality. We can also obtain (7.8) by the Hölder inequality. ☐

Lemma 7.2. *For any finite time-interval I, of length $|I|$, and any functions g and η such that $g\in L^\infty(I, L^{\frac{6}{5}}(\mathbb{R}^3)), \langle\nabla\rangle g\in N(I), g\in L^\infty(I,L^2(\mathbb{R}^3)), \eta\in L^\infty(I,L^6(\mathbb{R}^3))$ and $\langle\nabla\rangle\eta \in L^{\frac{5}{2}}(I, L^{\frac{30}{7}}(\mathbb{R}^3))$, we have*

$$\begin{aligned}\int_I \left|\langle\mathcal{L}_\omega g(t),\eta(t)\rangle\right| dt \le C\Big[&\|\langle\nabla\rangle g\|_{N(I)}\|\langle\nabla\rangle\eta\|_{L^{\frac{5}{2}}(I,L^{\frac{30}{7}}(\mathbb{R}^3))}\\ &+|I|\|g\|_{L^\infty(I,L^{\frac{6}{5}}(\mathbb{R}^3))}\|\eta\|_{L^\infty(I,L^6(\mathbb{R}^3))}\Big].\end{aligned} \tag{7.9}$$

PROOF. We have

$$\langle \mathcal{L}_\omega g(t), \eta(t)\rangle = a(t) + b(t),$$

where

$$a(t) = \mathrm{Re} \int_{\mathbb{R}^3} \nabla g(t) \cdot \nabla \overline{\eta(t)} dx + \omega \mathrm{Re} \int_{\mathbb{R}^3} g(t)\overline{\eta(t)} dx,$$

$$b(t) = -\int_{\mathbb{R}^3} (3Q_\omega^2 + 5Q_\omega^4) g_1(t)\eta_1(t) dx - \int_{\mathbb{R}^3} (Q_\omega^2 + Q_\omega^4) g_2(t)\eta_2(t) dx.$$

Here, $g_1 = \mathrm{Re}\, g, \eta_1 = \mathrm{Re}\, \eta, g_2 = \mathrm{Im}\, g$ and $\eta_2 = \mathrm{Im}\, \eta$. By the Hölder inequality, we obtain

$$\int_I |a(t)| dt \le C \|\langle \nabla \rangle g\|_{N(I)} \|\langle \nabla \eta \rangle\|_{L_t^{\frac{5}{2}}(I, L_x^{\frac{30}{7}})},$$

$$|b(t)| \le C \|g(t)\|_{L_x^{\frac{6}{5}}(\mathbb{R}^3)} \|\eta(t)\|_{L_x^6(\mathbb{R}^3)} (\|Q_\omega\|_{L^\infty}^2 + \|Q_\omega\|_{L^\infty}^4).$$

Integrating the estimate on $b(t)$ over I, we get the conclusion. □

We recall the following lemma which is obtained by Duyckaerts and Merle [**14**, Claim 5.8].

LEMMA 7.3 (Sums of exponential). *Let $t_0 > 0, p \in [1, \infty), a_0 < 0$, E be a normed vector space, and $f \in L^p_{loc}(t_0, \infty; E)$ such that there exist $\tau_0 > 0, C_0 > 0$ satisfying*

$$\|f\|_{L^p(t, t+\tau_0; E)} \le C_0 e^{a_0 t} \qquad (t \ge t_0).$$

Then, we have

$$\|f\|_{L^p(t, \infty; E)} \le \frac{C_0 e^{a_0 t}}{1 - e^{a_0 \tau_0}} \qquad (t \ge t_0).$$

Using Lemma 7.3, we shall show the following:

LEMMA 7.4. *Let η be a solution to* (7.2). *Assume that η satisfies*

$$\|\eta(t)\|_{H^1} \le C_1 e^{-\gamma t} \qquad (t > 0) \tag{7.10}$$

for some $C_1 > 0$ and $\gamma \in (0, \gamma_2)$. Then, we have

$$\|\langle \nabla \rangle \eta\|_{St(t,\infty)} + \|\eta\|_{W(t,\infty)} \le C_2 e^{-\gamma t} \qquad (t > 0)$$

for some $C_2 > 0$.

PROOF. We shall show Lemma 7.4 following [**14**, Lemma 5.7]. From (7.5), one has

$$i\partial_t \langle \nabla \rangle \eta + \Delta \langle \nabla \rangle \eta - \omega \langle \nabla \rangle \eta + \langle \nabla \rangle (\mathcal{V}(\eta) - ig) = 0. \tag{7.11}$$

Let t and τ such that $t > 0$ and $0 < \tau < 1$. By the Strichartz estimates (6.1), (6.2), (7.10), (7.8) and (7.4), one has

$$\begin{aligned} \|\langle \nabla \rangle \eta\|_{V_3(t,t+\tau)} &\le C(\|\eta(t)\|_{H^1} + \|\langle \nabla \rangle (\mathcal{V}(\eta)\|_{N(t,t+\tau)} + \|\langle \nabla \rangle g\|_{N(t,t+\tau)}) \\ &\le C\left(e^{-\gamma t} + \tau^{\frac{2}{5}} \|\langle \nabla \rangle \eta\|_{V_3(t,t+\tau)} + e^{-\gamma_2 t}\right). \end{aligned} \tag{7.12}$$

Thus, we can take $\tau_0 > 0$ sufficiently small so that

$$\|\langle \nabla \rangle \eta\|_{V_3(t,t+\tau_0)} \le C e^{-\gamma t}. \tag{7.13}$$

for $t > 0$. Note that by the Hölder inequality, the assumption (7.10) and (7.13), we obtain

$$\|\langle \nabla \rangle \eta\|_{V(t,t+\tau_0)} \le \|\eta\|_{L^\infty((t,t+\tau_0),H^1)}^{\frac{1}{2}} \|\langle \nabla \rangle \eta\|_{V_3(t,t+\tau_0)}^{\frac{1}{2}} \le C_1 e^{-\gamma t}. \tag{7.14}$$

By Lemma 7.3 and the inequalities (7.13) and (7.14), we have

$$\|\langle\nabla\rangle\eta\|_{V(t,\infty)} + \|\langle\nabla\rangle\eta\|_{V_3(t,\infty)} \le Ce^{-\gamma t} \qquad (t>0). \tag{7.15}$$

Then, by the Sobolev inequality (7.7), we see that $\|\eta\|_{W(t,\infty)} \le C_2 e^{-\gamma t}$ for some $C_2 > 0$.

Next, we shall estimate $\|\langle\nabla\rangle\eta\|_{St(t,\infty)}$. By the Strichartz estimates (6.1), (6.2), and the estimate similar to (7.12), we obtain

$$\begin{aligned}\|\langle\nabla\rangle\eta\|_{St(t,t+\tau)} &\le C(e^{-\gamma t} + \tau^{\frac{2}{5}}\|\langle\nabla\rangle\eta\|_{V_3(t,t+\tau)} + \|\langle\nabla\rangle g\|_{N(t,t+\tau)}) \\ &\le C(e^{-\gamma t} + \tau^{\frac{2}{5}}\|\langle\nabla\rangle\eta\|_{St(t,t+\tau)} + e^{-\gamma_2 t}).\end{aligned}$$

This implies that $\|\langle\nabla\rangle\eta\|_{St(t,t+\tau)} \le Ce^{-\gamma t}$ for sufficiently small $\tau > 0$. By Lemma 7.3, we get $\|\langle\nabla\rangle\eta\|_{St(t,\infty)} \le C_2 e^{-\gamma t}$ for some $C_2 > 0$. This completes the proof. □

We are now in a position to prove Proposition 7.2.

Proof of Proposition 7.2. **(Step 1).** We claim the following:

$$\frac{d\lambda_+}{dt}(t) - e_\omega\lambda_+(t) = \Omega(g, \mathcal{Y}_{\omega,-}), \tag{7.16}$$

$$\frac{d\lambda_-}{dt}(t) + e_\omega\lambda_-(t) = -\Omega(g, \mathcal{Y}_{\omega,+}), \tag{7.17}$$

$$\frac{d}{dt}\langle\mathcal{L}_\omega\eta(t), \eta(t)\rangle_{H^{-1},H^1} = 2\langle\mathcal{L}_\omega\eta(t), g\rangle_{H^{-1},H^1}, \tag{7.18}$$

where $\lambda_\pm(t)$ are given by (3.6). It follows from (3.6), (7.2) and $\mathcal{L}_\omega\mathcal{Y}_{\omega,-} = -ie_\omega\mathcal{Y}_{\omega,-}$ that

$$\begin{aligned}\frac{d\lambda_+}{dt}(t) &= \Omega(\partial_t\eta(t), \mathcal{Y}_{\omega,-}) = \Omega(-i\mathcal{L}_\omega\eta(t), \mathcal{Y}_{\omega,-}) + \Omega(g, \mathcal{Y}_{\omega,-}) \\ &= e_\omega\lambda_+(t) + \Omega(g, \mathcal{Y}_{\omega,-}).\end{aligned}$$

Similarly, by (3.6), (7.2) and $\mathcal{L}_\omega\mathcal{Y}_{\omega,+} = ie_\omega\mathcal{Y}_{\omega,+}$, we obtain

$$\begin{aligned}\frac{d\lambda_-}{dt}(t) = -\Omega(\partial_t\eta(t), \mathcal{Y}_{\omega,+}) &= -\Omega(-i\mathcal{L}_\omega\eta(t), \mathcal{Y}_{\omega,+}) - \Omega(g, \mathcal{Y}_{\omega,+}) \\ &= -e_\omega\lambda_-(t) - \Omega(g, \mathcal{Y}_{\omega,+}).\end{aligned}$$

Clearly, we have

$$(\mathcal{L}_\omega u, -i\mathcal{L}_\omega u)_{L^2_{\mathrm{real}}} = 0 \qquad \text{for all } u \in H^2(\mathbb{R}^3). \tag{7.19}$$

It follows form (7.2) and (7.19) that

$$\frac{d}{dt}\langle\mathcal{L}_\omega\eta(t), \eta(t)\rangle_{H^{-1},H^1} = 2\langle\mathcal{L}_\omega\eta(t), g\rangle_{H^{-1},H^1}.$$

This yields (7.18).

(Step 2). We now claim the following:

$$|\lambda_+(t)| \le Ce^{-\gamma_2 t}, \tag{7.20}$$

$$|\lambda_-(t)| \le Ce^{-\gamma_2^- t} \qquad \text{if } \gamma_2 \le e_\omega, \tag{7.21}$$

$$\text{there exists } A \in \mathbb{R} \text{ such that } |\lambda_-(t) - Ae^{-e_\omega t}| \le Ce^{-\gamma_2 t} \qquad \text{if } \gamma_2 > e_\omega. \tag{7.22}$$

It follows from (7.16) and (7.4) that

$$\left|\frac{d}{dt}(e^{-e_\omega t}\lambda_+(t))\right| = e^{-e_\omega t}|\Omega(g, \mathcal{Y}_{\omega,-})| \le Ce^{-e_\omega t}\|g\|_{L^{\frac{6}{5}}(\mathbb{R}^3)} \le Ce^{-(e_\omega+\gamma_2)t}.$$

Integrating the above from t to ∞, we obtain (7.20).

Next, we shall show (7.21). From (7.17), we obtain

$$\left|\frac{d}{dt}(e^{e_\omega t}\lambda_-(t))\right| = e^{e_\omega t}|\Omega(g,\mathcal{Y}_{\omega,+})| \le Ce^{e_\omega t}\|g\|_{L^{\frac{6}{5}}(\mathbb{R}^3)} \le Ce^{(e_\omega-\gamma_2)t}. \tag{7.23}$$

Assume that $\gamma_2 \le e_\omega$. By (7.23), one has

$$|e^{e_\omega t}\lambda_-(t)| \le \begin{cases} e^{e_\omega t_0}|\lambda_-(t_0)| + Ce^{(e_\omega-\gamma_2)t} & \text{if } \gamma_2 < e_\omega, \\ e^{e_\omega t_0}|\lambda_+(t_0)| + C(t-t_0) & \text{if } e_\omega = \gamma_2. \end{cases}$$

This yields (7.21). Next, we assume that $\gamma_2 > e_\omega$. Then, we see that

$$\int_{t_0}^{\infty} e^{(e_\omega-\gamma_2)\tau}d\tau < \infty.$$

It follows from (7.23) that

$$|e^{e_\omega s}\lambda_-(s) - e^{e_\omega t}\lambda_-(t)| \le C\int_t^s e^{(e_\omega-\gamma_2)\tau}d\tau \le C\int_t^\infty e^{(e_\omega-\gamma_2)\tau}d\tau \le Ce^{(e_\omega-\gamma_2)t} \tag{7.24}$$

for $s \ge t > t_0$. Thus, there exists $A \in \mathbb{R}$ such that $\lim_{t\to\infty} e^{e_\omega t}\lambda_-(t) = A$. In addition, letting $s \to \infty$ in (7.24), we have

$$|A - e^{e_\omega t}\lambda_-(t)| \le Ce^{(e_\omega-\gamma_2)t}.$$

This implies (7.22).

(Step 3). We next prove

$$\|\Gamma(t)\|_{H^1} \le Ce^{-\frac{\gamma_1+\gamma_2}{2}t}, \tag{7.25}$$

where $\Gamma(t)$ is given by (3.15). We have, by (3.3), (3.5) and (3.15), that

$$\Omega(\Gamma(t),\mathcal{Y}_{\omega,+}) = \Omega(\Gamma(t),\mathcal{Y}_{\omega,-}) = 0.$$

Recall that $\Omega(\mathcal{Y}_{\omega,+},\mathcal{Y}_{\omega,-}) = 1$ and

$$\Omega(\mathcal{Y}_{\omega,+},\mathcal{Y}_{\omega,+}) = \Omega(\mathcal{Y}_{\omega,-},\mathcal{Y}_{\omega,-}) = 0.$$

Using this, we get

$$\begin{aligned} &\langle\mathcal{L}_\omega\eta(t),\eta(t)\rangle_{H^{-1},H^1} \\ &= e_\omega\lambda_+(t)\lambda_-(t)(i\mathcal{Y}_{\omega,+},\mathcal{Y}_{\omega,-})_{L^2_{\text{real}}} - e_\omega\lambda_+(t)\lambda_-(t)(i\mathcal{Y}_{\omega,-},\mathcal{Y}_{\omega,+})_{L^2_{\text{real}}} \\ &\quad + \langle\mathcal{L}_\omega\Gamma(t),\Gamma(t)\rangle_{H^{-1},H^1} \\ &= -2e_\omega\lambda_+(t)\lambda_-(t) + \langle\mathcal{L}_\omega\Gamma(t),\Gamma(t)\rangle_{H^{-1},H^1}. \end{aligned} \tag{7.26}$$

By Lemmas 7.2 and 7.4, (7.18), (7.4) and (7.3), we have

$$\begin{aligned} &\int_t^{t+1}\left|\frac{d}{ds}\langle\mathcal{L}_\omega\eta(s),\eta(s)\rangle_{H^{-1},H^1}\right|ds \\ &= 2\int_t^{t+1}\left|\langle\mathcal{L}_\omega g(s),\eta(s)\rangle_{H^{-1},H^1}\right|ds \\ &\le C\Big[\|\langle\nabla\rangle g\|_{N([t,t+1])}\|\langle\nabla\rangle\eta\|_{L^{\frac{5}{2}}([t,t+1],L^{\frac{30}{7}}(\mathbb{R}^3))} \\ &\quad + \|g\|_{L^\infty([t,t+1],L^{\frac{6}{5}}(\mathbb{R}^3))}\|\eta\|_{L^\infty([t,t+1],L^6(\mathbb{R}^3))}\Big] \\ &\le Ce^{-(\gamma_1+\gamma_2)t}. \end{aligned}$$

Then, from Lemma 7.3, we obtain

$$\int_t^\infty \left| \frac{d}{ds} \langle \mathcal{L}_\omega \eta(s), \eta(s) \rangle_{H^{-1},H^1} \right| ds \leq Ce^{-(\gamma_1+\gamma_2)t}.$$

Since $\lim_{t\to\infty} \langle \mathcal{L}_\omega \eta(t), \eta(t)\rangle_{H^{-1},H^1} = 0$, we see that

$$\left| \langle \mathcal{L}_\omega \eta(t), \eta(t) \rangle_{H^{-1},H^1} \right| \leq \int_t^\infty \left| \frac{d}{ds} \langle \mathcal{L}_\omega \eta(s), \eta(s) \rangle_{H^{-1},H^1} \right| ds \leq Ce^{-(\gamma_1+\gamma_2)t}. \tag{7.27}$$

From Lemma 3.1, (7.20), (7.21), (7.26) and (7.27), we obtain

$$\|\Gamma(t)\|_{H^1}^2 \leq C \langle \mathcal{L}_\omega \Gamma(t), \Gamma(t) \rangle_{H^{-1},H^1} \leq Ce^{-(\gamma_1+\gamma_2)t}.$$

From this, we conclude that (7.25) holds.

(Step 4). We conclude the proof. We first consider the case of $e_\omega \geq \gamma_2$ or the case of $\gamma_2 > e_\omega$ and $A = 0$. By the decomposition (3.4) of η, (7.20), (7.21), (7.22) and (7.25), we have

$$\|\eta(t)\|_{H^1} \leq Ce^{-\frac{\gamma_1+\gamma_2}{2}t}. \tag{7.28}$$

Iterating the above argument, we get the bound

$$\|\eta(t)\|_{H^1} \leq C_1 e^{-\gamma_2^-}.$$

From Lemma 7.4, we have $\|\langle\nabla\rangle\eta\|_{St(t,\infty)} \leq C_2 e^{-\gamma_2^- t}$.

Secondly, we consider the case of $e_\omega < \gamma_2$. Then, by the decomposition (3.16), (7.20), (7.22) and (7.25), we have

$$\|\eta(t) - Ae^{-e_\omega t}\mathcal{Y}_{\omega,-}\|_{H^1} \leq C(e^{-\gamma_2 t} + e^{-\frac{\gamma_1+\gamma_2}{2}t}) \leq Ce^{-\frac{\gamma_1+\gamma_2}{2}t}.$$

Putting $\eta_0(t) := \eta(t) - Ae^{-e_\omega t}\mathcal{Y}_{\omega,-}$, we find that η_0 satisfies (7.2) with γ_1 replaced by $\frac{\gamma_1+\gamma_2}{2} (> \gamma_1)$. By iterating the above argument, we obtain

$$\|\eta_0(t)\|_{H^1} \leq C_1 e^{-\gamma_2^-}.$$

From Lemma 7.4, we have $\|\langle\nabla\rangle\eta_0\|_{St(t,\infty)} \leq C_2 e^{-\gamma_2^- t}$. This completes the proof. □

7.2. Proof of Proposition 7.1. In this subsection, we give the proof of Proposition 7.1. First, we recall the following nonlinear estimates, which are obtained in Duyckaerts and Merle [**14**, Lemma 5.6] (see also Ardila and Murphy [**3**, Lemma 6.2]):

LEMMA 7.5 (Nonlinear estimates).

$$\|N_{\omega,1}(f) - N_{\omega,1}(g)\|_{L_x^{\frac{6}{5}}} \leq C\|f-g\|_{L_x^{\frac{18}{5}}} (\|Q_\omega\|_{L_x^{\frac{18}{5}}} \|f\|_{L_x^{\frac{18}{5}}} + \|Q_\omega\|_{L_x^{\frac{18}{5}}} \|g\|_{L_x^{\frac{18}{5}}} + \|f\|_{L_x^{\frac{18}{5}}}^2 + \|g\|_{L_x^{\frac{18}{5}}}^2),$$

$$\|N_{\omega,1}(f) - N_{\omega,1}(g)\|_{L_x^{\frac{30}{23}}} \leq C\|f-g\|_{L_x^{\frac{30}{11}}} (\|Q_\omega\|_{L_x^5} \|f\|_{L_x^5} + \|Q_\omega\|_{L_x^5} \|g\|_{L_x^5} + \|f\|_{L_x^5}^2 + \|g\|_{L_x^5}^2),$$

$$\|N_{\omega,2}(f) - N_{\omega,2}(g)\|_{L_x^{\frac{6}{5}}} \leq C\|f-g\|_{L_x^6} (\|Q_\omega\|_{L_x^6}^3 \|f\|_{L_x^6} + \|Q_\omega\|_{L_x^6}^3 \|g\|_{L_x^6} + \|f\|_{L_x^6}^4 + \|g\|_{L_x^6}^4),$$

$$\|N_{\omega,2}(f) - N_{\omega,2}(g)\|_{L_x^{\frac{30}{23}}} \leq C\|f-g\|_{L_x^{\frac{30}{11}}} (\|Q_\omega\|_{L_x^{10}}^3 \|f\|_{L_x^{10}} + \|Q_\omega\|_{L_x^{10}}^3 \|g\|_{L_x^{10}} + \|f\|_{L_x^{10}}^4 + \|g\|_{L_x^{10}}^4).$$

Let I be a finite time interval and f, g be functions in $W(I)$ such that $\langle\nabla\rangle f$ and $\langle\nabla\rangle g$ are in $V(I)$. Then, we have

(7.29)
$$\|\langle\nabla\rangle N_{\omega,1}(f) - \langle\nabla\rangle N_{\omega,1}(g)\|_{N(I)} \leq C\|\langle\nabla\rangle(f-g)\|_{V_3(I)} \cdot \Big[|I|^{\alpha_1}(\|\langle\nabla\rangle f\|_{V_3(I)} + \|\langle\nabla\rangle g\|_{V_3(I)}) + \|\langle\nabla\rangle f\|_{V_3(I)}^2 + \|\langle\nabla\rangle g\|_{V_3(I)}^2\Big],$$

(7.30)
$$\|\langle\nabla\rangle N_{\omega,2}(f) - \langle\nabla\rangle N_{\omega,2}(g)\|_{N(I)} \leq C\|\langle\nabla\rangle(f-g)\|_{V_3(I)} \cdot \Big[|I|^{\alpha_2}(\|\langle\nabla\rangle f\|_{V(I)} + \|\langle\nabla\rangle g\|_{V(I)}) + \|\langle\nabla\rangle f\|_{V(I)}^4 + \|\langle\nabla\rangle g\|_{V(I)}^4\Big],$$

where α_1 and α_2 are some positive constants.

We also recall the following estimate of θ.

Lemma 7.6. *There exists $C > 0$ such that for any $t \in [0, T_X]$, one has*

$$\left|\frac{d\theta}{dt} - \omega\right| \leq C\|\eta(t)\|_{H^1}^2, \tag{7.31}$$

See [**6**, (5.28)] for the proof of Lemma 7.6. We are now in a position to prove Proposition 7.1.

Proof of Proposition 7.1. We divide the proof into 4 steps.

(Step 1). Let $\psi = e^{i\theta(t)}(Q_\omega(x) + \eta(t,x))$. It follows from (3.22), (3.33) and the assumption (7.1) that $\|\eta(t)\|_{H^1} \leq Ce^{-ct}$ for $t \geq t_0$. We see from (7.31) that

$$\left|(\theta(t) - \omega t) - (\theta(s) - \omega s)\right| \leq \int_s^t \left|\frac{d\theta}{d\tau} - \omega\right| d\tau \leq \int_s^t C\|\eta(\tau)\|_{H^1}^2 d\tau \leq Ce^{-2cs}.$$

Thus, there exists $\theta_0 \in \mathbb{R}$ such that $\lim_{t\to\infty}(\theta(t) - \omega t) = \theta_0$. From the above inequality, we obtain

$$|\theta(t) - \omega t - \theta_0| \leq Ce^{-2ct}. \tag{7.32}$$

Let $\psi(t,x) = e^{i(\omega t + \theta_0)}(Q_\omega(x) + \eta_0(t,x))$. We claim that η_0 satisfies

$$\|\eta_0(t)\|_{H^1} \leq Ce^{-ct} \qquad (t \geq t_0) \tag{7.33}$$

for some $c, C > 0$. Observe that

$$\eta_0(t,x) = e^{-i(\omega t + \theta_0)}\psi(t,x) - Q_\omega(x) = e^{i(\theta(t) - \omega t - \theta_0)}(Q_\omega(x) + \eta(t,x)) - Q_\omega(x).$$

This yields with (7.32) and $\|\eta(t)\|_{H^1} \leq Ce^{-ct}$ for $t \geq t_0$ that

$$\begin{aligned}\|\eta_0(t)\|_{H^1} &= \left\|e^{i(\theta(t) - \omega t - \theta_0)}(Q_\omega + \eta(t)) - Q_\omega\right\|_{H^1} \\ &\leq \|\eta(t)\|_{H^1} + C\left|\theta(t) - \omega t - \theta_0\right|\|Q_\omega\|_{H^1} \\ &\leq Ce^{-ct} + Ce^{-2ct} \leq Ce^{-ct}.\end{aligned}$$

Thus, (7.33) holds.

(Step 2). We claim that η_0 satisfies

$$\|\eta_0(t)\|_{H^1} \leq Ce^{-e_\omega^- t} \qquad (t \geq t_0). \tag{7.34}$$

By (6.3) and Lemmas 7.4, 7.5 with $f = \eta_0$ and $g = 0$, we obtain

$$\|N_\omega(\eta_0(t))\|_{L_x^{\frac{6}{5}}} + \|\langle\nabla\rangle N_\omega(\eta_0)\|_{N(t,\infty)} \le Ce^{-2c_0 t}.$$

Then, applying Proposition 7.2 with $\gamma_1 = c_0$ and $\gamma_2 = 2c_0$, $g = N_\omega(\eta_0(t))$, we have

$$\|\eta_0(t)\|_{H^1} \le C(e^{-e_\omega t} + e^{-\frac{3}{2}c_0 t}).$$

If $e_\omega \le \frac{3}{2}c_0$, (7.34) holds. If not, we get $\|\eta_0(t)\|_{H^1} \le Ce^{-\frac{3}{2}c_0 t}$. Then, we can verify that (7.34) holds by iterating the above argument.

(Step 3). Applying Proposition 7.2 (ii) again with $\gamma_1 = e_\omega^-, \gamma_2 = 2e_\omega^- (> e_\omega)$ and $g = N_\omega(\eta_0(t))$, we find that there exists $A_0 \in \mathbb{R}$ such that

(7.35) $$\|\eta_0(t) - A_0 e^{-e_\omega t}\mathcal{Y}_{\omega,-}\|_{H^1} + \|\langle\nabla\rangle(\eta_0(t) - A_0 e^{-e_\omega t}\mathcal{Y}_{\omega,-})\|_{St(t,\infty)} \le Ce^{-2e_\omega^- t}.$$

Let U_ω^A be the solution constructed in Proposition 6.2 for each $A \in \mathbb{R}$. We write $U_\omega^A = e^{i\omega t}(Q_\omega + \eta^A(t))$. We claim that for any $\gamma > 0$,

(7.36) $$\|\eta_0(t) - \eta^{A_0}(t)\|_{H^1} + \|\langle\nabla\rangle(\eta_0(t) - \eta^{A_0}(t))\|_{St(t,\infty)} \le Ce^{-\gamma t} \qquad (t \ge t_0)$$

for some $C > 0$. Observe from Proposition 6.2 that η^{A_0} satisfies [6]

(7.37)
$$\|\eta^{A_0} - A_0 e^{-e_\omega t}\mathcal{Y}_{\omega,-}\|_{H^1} + \|\langle\nabla\rangle\left(\eta^{A_0} - A_0 e^{-e_\omega t}\mathcal{Y}_{\omega,-}\right)\|_{St(t,\infty)} \le Ce^{-\frac{3}{2}e_\omega t} \qquad (t \ge t_0).$$

It follows from (7.35) that (7.36) holds with $\gamma = \frac{3}{2}e_\omega$. We can easily verify that $\eta_0 - \eta^{A_0}$ satisfies

$$\partial_t(\eta_0 - \eta^{A_0}) + i\mathcal{L}_\omega(\eta_0 - \eta^{A_0}) = N_\omega(\eta_0) - N_\omega(\eta^{A_0}).$$

Then, by Lemmas 7.4 and 7.5 with $f = \eta_0(t)$ and $g = \eta^{A_0}(t)$, we have

$$\begin{aligned}&\|N_\omega(\eta_0(t)) - N_\omega(\eta^{A_0}(t))\|_{L_x^{\frac{6}{5}}} \\ &+ \|\langle\nabla\rangle N_\omega(\eta_0) - \langle\nabla\rangle N_\omega(\eta^{A_0})\|_{N(t,\infty)} \le Ce^{-\frac{5}{2}e_\omega^- t}.\ (t \ge t_0).\end{aligned}$$

Then, by Proposition 7.2 (ii) with $\gamma_2 = 2e_\omega$, there exists $A_1 \in \mathbb{R}$ such that $\eta_0 - \eta^{A_0} = A_1 e^{-e_\omega t}\mathcal{Y}_{\omega,-} + w_1(t)$ with

(7.38) $$\|w_1\|_{H^1} + \|\langle\nabla\rangle w_1\|_{St(t,\infty)} \le Ce^{-\frac{7}{4}e_\omega t} \qquad (t \ge t_0).$$

It follows from (7.36) with $\gamma = \frac{3}{2}e_\omega$ and (7.38) that

$$|A_1|e^{-e_\omega t}\|\mathcal{Y}_{\omega,-}\|_{H^1} \le \|\eta_0 - \eta^{A_0}\|_{H^1} + \|w_1(t)\|_{H^1} \le Ce^{-\frac{3}{2}e_\omega t},$$

which implies $A_1 = 0$. Thus, we see from (7.38) that (7.36) holds for $\gamma = \frac{7}{4}e_\omega$. Iterating this argument, we see that (7.36) holds.

(Step 4). We derive a conclusion. Using (7.36) with $\gamma = (k_0+1)e_\omega$ and (6.5) with $k = k_0$, we see that

$$\begin{aligned}&\|\langle\nabla\rangle(e^{-i\theta_0}\psi - e^{i\omega t}(Q_\omega + \mathcal{V}_{k_0,\omega}^{A_0}))\|_{St(t,\infty)} \\ &\le \|\langle\nabla\rangle(e^{-i\theta_0}\psi - U_\omega^{A_0})\|_{St(t,\infty)} + \|\langle\nabla\rangle(U_\omega^{A_0} - e^{i\omega t}(Q_\omega + \mathcal{V}_{k_0,\omega}^{A_0}))\|_{St(t,\infty)} \\ &\le \|\langle\nabla\rangle(\eta_0 - \eta^{A_0})\|_{St(t,\infty)} + e^{-(k_0+\frac{1}{2})e_\omega t} \\ &\le Ce^{-(k_0+1)e_\omega t} + e^{-(k_0+\frac{1}{2})e_\omega t} \le e^{-(k_0+\frac{1}{2})e_\omega t}.\end{aligned}$$

Thus, from the uniqueness of the solution satisfying (6.5) (see Proposition 6.2), we find that $\psi = U_\omega^{A_0}$. □

[6] Note that $\mathcal{V}_{1,\omega}^{A_0} = e^{-e_\omega t}\mathcal{Z}_{1,\omega}^{A_0} = A_0 e^{-e_\omega t}\mathcal{Y}_{\omega,-}$.

7.3. Proof of Theorem 1.4. We are now in a position to prove Theorem 1.4

Proof of Theorem 1.4. We divide the proof into two steps.
(Step 1). It follows from (6.6) and (6.7) that

$$\begin{aligned} Q_\omega^+(t) &= e^{-i\omega t_0} U^{+1}(t+t_0, x) \\ &= e^{i\omega t} Q_\omega + e^{-e_\omega t_0} e^{(i\omega - e_\omega)t} \mathcal{Y}_{\omega,-} + O(e^{-\frac{3}{2}e_\omega t}) \qquad \text{in } H^1(\mathbb{R}^3). \end{aligned}$$

Fix $A > 0$. Let $t_1 = -t_0 - \frac{1}{e_\omega}\log A$. This yields that

$$Q_\omega^+(t+t_1) = e^{i\omega t_1} e^{i\omega t} Q_\omega + e^{-e_\omega t_0} e^{(i\omega - e_\omega)t} e^{(i\omega - e_\omega)t_1} \mathcal{Y}_{\omega,-} + O(e^{-\frac{3}{2}e_\omega t}) \qquad \text{in } H^1(\mathbb{R}^3).$$

From (6.6) and $e^{-e_\omega t_1} = Ae^{e_\omega t_0}$, we obtain

$$\begin{aligned} e^{-i\omega t_1} Q_\omega^+(t+t_1) &= e^{i\omega t} Q_\omega + Ae^{(i\omega - e_\omega)t} \mathcal{Y}_{\omega,-} + O(e^{-\frac{3}{2}e_\omega t}) \\ &= e^{i\omega t}(Q_\omega + Ae^{-e_\omega t}\mathcal{Y}_{\omega,-}) + O(e^{-\frac{3}{2}e_\omega t}) \\ &= U_\omega^A + O(e^{-\frac{3}{2}e_\omega t}) \qquad \text{in } H^1(\mathbb{R}^3). \end{aligned} \tag{7.39}$$

From this and (6.6), we see that there exists $C_1 > 0$ such that

$$\|e^{-i\omega t_1} Q_\omega^+(t+t_1) - e^{i\omega t} Q_\omega\|_{H^1} \le C_1 e^{-e_\omega t} \qquad \text{for } t > 0.$$

This together with Proposition 7.1 yields that there exists $\widetilde{A} \in \mathbb{R}$ and $\widetilde{\theta}_0 \in \mathbb{R}$ such that $e^{-i\omega t_1} Q_\omega^+(t+t_1) = e^{i\widetilde{\theta}_0} U_\omega^{\widetilde{A}}$. By (7.39), we have $\widetilde{A} = A$ and $\widetilde{\theta}_0 = 0$, which yields that

$$U_\omega^A = e^{-i\omega t_1} Q_\omega^+(t+t_1). \tag{7.40}$$

(Step 2). Let ψ be a solution to (1.1) with $\psi|_{t=0} = \psi_0 \in \mathcal{BA}_\omega$. If $\mathcal{K}(\psi) = 0$, then ψ is the ground state of m_ω. From the uniqueness of the ground state (see Proposition 2.1 (i)), we see that $\psi(t,x) = e^{i\theta + i\omega t} Q_\omega(x)$ for some $\theta \in \mathbb{R}$.

Assume that $\mathcal{K}(\psi) > 0$ and ψ does not scatter for positive time. By Proposition 5.1, there exist constants $C, c > 0$ such that

$$\operatorname{dist}_{H^1}(\psi(t), \mathcal{O}(Q_\omega)) \le Ce^{-ct} \qquad \text{for } t > 0.$$

Hence, $\psi(t)$ satisfies the assumption of Proposition 7.1 and $\mathcal{K}(\psi) > 0$, which shows that $\psi = U_\omega^A$ for some $A > 0$. Thus, we see from (7.40) that (iii) holds.

Combining Propositions 4.1 and 7.1, we can prove (i) by a similar argument of (iii). □

Appendix A. Proof of Proposition 4.2

This appendix is devoted to the proof of Proposition 4.2. For each $\lambda > 0$ and $u \in H^1(\mathbb{R}^3)$, we define

$$T_\lambda u(\cdot) := \lambda^{\frac{3}{2}} u(\lambda \cdot).$$

We can easily find that for any $u \in H^1(\mathbb{R}^3) \setminus \{0\}$, there exists $\lambda(u) > 0$ such that

$$\mathcal{K}(T_\lambda u) \begin{cases} > 0 & \text{for } 0 < \lambda < \lambda(u), \\ = 0 & \text{for } \lambda = \lambda(u), \\ < 0 & \text{for } \lambda > \lambda(u). \end{cases} \tag{A.1}$$

By a standard argument, one has

$$m_\omega = \inf\left\{\mathcal{J}_\omega(u) \colon u \in H^1(\mathbb{R}^3) \setminus \{0\},\ \mathcal{K}(u) \le 0\right\}, \tag{A.2}$$

where

$$\mathcal{J}_\omega(u) := \mathcal{S}_\omega(u) - \frac{1}{2}\mathcal{K}(u) = \omega\mathcal{M}(u) + \frac{1}{8}\|u\|_{L^4}^4 + \frac{1}{3}\|u\|_{L^6}^6. \tag{A.3}$$

Since the ground state $Q_\omega \in H^1(\mathbb{R}^3) \setminus \{0\}$ exists for $\omega \in (0, \omega_c)$ (see Theorem 1.1), we see that

$$m_\omega = J_\omega(Q_\omega) > \omega\mathcal{M}(Q_\omega) > 0. \tag{A.4}$$

In addition, we have

$$m_\omega < \frac{\sigma^{\frac{3}{2}}}{3} \qquad \text{for } \omega \in (0, \omega_c), \tag{A.5}$$

where

$$\sigma := \inf\left\{\|\nabla u\|_{L^2}^2 \colon u \in \dot{H}^1(\mathbb{R}^3) \text{ with } \|u\|_{L^6} = 1\right\}. \tag{A.6}$$

See [**7**, Theorem 1.4]. To prove Proposition 4.2, we need the following lemmas:

LEMMA A.1 (Brezis and Lieb [**8**]). *Let $\{u_n\}$ be a bounded sequence in $H^1(\mathbb{R}^3)$ such that*

$$\lim_{n\to\infty} u_n(x) = u_\infty(x) \qquad \textit{almost all } x \in \mathbb{R}^3$$

for some function $u_\infty \in H^1(\mathbb{R}^3)$. Then, for any $2 \le r \le 6$,

$$\lim_{n\to\infty} \int_{\mathbb{R}^3} \big||u_n|^r - |u_n - u_\infty|^r - |u_\infty|^r\big|\, dx = 0$$

and

$$\lim_{n\to\infty} \int_{\mathbb{R}^3} \big||\nabla u_n|^2 - |\nabla\{u_n - u_\infty\}|^2 - |\nabla u_\infty|^2\big|\, dx = 0.$$

PROOF OF PROPOSITION 4.2. First, we obtain a H^1-boundedness of the sequence $\{u_n\}$. For sufficiently large $n \in \mathbb{N}$, we have

$$\begin{aligned} 2m_\omega + 1 \ge |\mathcal{S}_\omega(u_n)| + \frac{1}{3}|\mathcal{K}(u_n)| &\ge |\mathcal{S}_\omega(u_n) - \frac{1}{3}\mathcal{K}(u_n)| \\ &= \frac{1}{6}\|\nabla u_n\|_{L^2}^2 + \frac{\omega}{2}\|u_n\|_{L^2}^2 + \frac{1}{6}\|u_n\|_{L^6}^6. \end{aligned}$$

Therefore, we see that the sequence $\{u_n\}$ is bounded in $H^1(\mathbb{R}^3)$.

Then, up to a subsequence, there exists $Q_\infty \in H^1(\mathbb{R}^3)$ such that $\lim_{n\to\infty} u_n = Q_\infty$ weakly in $H^1(\mathbb{R}^3)$. We shall show that $Q_\infty \neq 0$. Suppose to the contrary that $Q_\infty \equiv 0$. From the compactness of the embedding $H^1_{\mathrm{rad}}(\mathbb{R}^3) \subset L^4(\mathbb{R}^3)$, we have

$$0 = \lim_{n\to\infty} \mathcal{K}(u_n) = \lim_{n\to\infty} \left\{\|\nabla u_n\|_{L^2}^2 - \|u_n\|_{L^6}^6\right\}. \tag{A.7}$$

We claim that $\limsup_{n\to\infty} \|\nabla u_n\|_{L^2}^2 > 0$. Suppose to the contrary that

$$\lim_{n\to\infty} \|\nabla u_n\|_{L^2} = 0.$$

Then, it follows from the Sobolev embedding and (A.7) that $\lim_{n\to\infty} \|u_n\|_{L^q} = 0$ for all $2 < q \le 6$. This together with $u_n \in \mathcal{BA}_\omega$ and (A.4) yields that

$$m_\omega = \lim_{n\to\infty} \mathcal{S}_\omega(u_n) = \lim_{n\to\infty} \omega\mathcal{M}(u_n) = \omega\mathcal{M}(Q_\omega) < m_\omega,$$

which is absurd. Thus, by taking a subsequence, we may assume $\lim_{n\to\infty} \|\nabla u_n\|_{L^2} > 0$.

Now, (A.7) with the definition of σ (A.6) gives us

$$\lim_{n\to\infty} \|\nabla u_n\|_{L^2}^2 \geq \sigma \lim_{n\to\infty} \|u_n\|_{L^6}^2 \geq \sigma \lim_{n\to\infty} \|\nabla u_n\|_{L^2}^{\frac{2}{3}}.$$

From this together with $\lim_{n\to\infty} \|\nabla u_n\|_{L^2} > 0$ and (A.7), we have

$$\sigma^{\frac{3}{2}} \leq \lim_{n\to\infty} \|\nabla u_n\|_{L^2}^2 = \lim_{n\to\infty} \|u_n\|_{L^6}^6.$$

Hence, we see that

$$\begin{aligned} m_\omega = \lim_{n\to\infty} \mathcal{S}_\omega(u_n) = \lim_{n\to\infty} \left\{ \mathcal{S}_\omega(u_n) - \frac{1}{2}\mathcal{K}(u_n) \right\} &\geq \lim_{n\to\infty} \left\{ \frac{1}{8} \|u_n\|_{L^4}^4 + \frac{1}{3} \|u_n\|_{L^6}^6 \right\} \\ &\geq \frac{1}{3} \lim_{n\to\infty} \|u_n\|_{L^6}^6 \geq \frac{\sigma^{\frac{2}{3}}}{3}, \end{aligned}$$

which contradicts (A.5). Thus, $Q_\omega \not\equiv 0$.

It follows from $\lim_{n\to\infty} \mathcal{S}_\omega(u_n) = m_\omega$ and $\lim_{n\to\infty} \mathcal{K}(u_n) = 0$ that

$$\lim_{n\to\infty} \mathcal{J}_\omega(u_n) = \lim_{n\to\infty} \left\{ \mathcal{S}_\omega(u_n) - \frac{1}{2}\mathcal{K}(u_n) \right\} = m_\omega \tag{A.8}$$

Using Lemma A.1, we have

$$\mathcal{J}_\omega(u_n) - \mathcal{J}_\omega(u_n - Q_\infty) - \mathcal{J}_\omega(Q_\infty) = o_n(1), \tag{A.9}$$

$$\mathcal{K}(u_n) - \mathcal{K}(u_n - Q_\infty) - \mathcal{K}(Q_\infty) = o_n(1). \tag{A.10}$$

Furthermore, (A.9) together with (A.8) and the positivity of $\mathcal{J}_\omega$ implies that $\mathcal{J}_\omega(Q_\infty) \leq m_\omega$. We claim that $\mathcal{K}(Q_\infty) \leq 0$. Suppose to the contrary that $\mathcal{K}(Q_\infty) > 0$. Then, it follows from $\lim_{n\to\infty} \mathcal{K}(u_n) = 0$ and (A.10) that $\mathcal{K}(u_n - Q_\infty) < 0$ for sufficiently large n. Hence, from (A.1), we can take $\lambda_n \in (0,1)$ such that $\mathcal{K}(T_{\lambda_n}(u_n - Q_\infty)) = 0$. Furthermore, we see from $0 < \lambda_n < 1$, and the definition of $\mathcal{J}_\omega$ that

$$\begin{aligned} m_\omega \leq \mathcal{J}_\omega(T_{\lambda_n}(u_n - Q_\infty)) &= \frac{\omega}{2}\|u_n - Q_\infty\|_{L^2}^2 + \frac{\lambda_n^3}{8}\|u_n - Q_\infty\|_{L^4}^4 + \frac{\lambda_n^6}{3}\|u_n - Q_\infty\|_{L^6}^6 \\ &< \mathcal{J}_\omega(u_n - Q_\infty). \end{aligned}$$

In addition, it follows from (A.8), (A.9) and $Q_\infty \neq 0$ that

$$m_\omega < \mathcal{J}_\omega(u_n - Q_\infty) = \mathcal{J}_\omega(u_n) - \mathcal{J}_\omega(Q_\infty) + o_n(1) = m_\omega - \mathcal{J}_\omega(Q_\infty) + o_n(1) < m_\omega$$

for sufficiently large $n \in \mathbb{N}$, which is a contradiction. Thus, $\mathcal{K}(Q_\infty) \leq 0$.

Since $Q_\infty \not\equiv 0$ and $\mathcal{K}(Q_\infty) \leq 0$, it follows from (A.2) that

$$m_\omega \leq \mathcal{J}_\omega(Q_\infty). \tag{A.11}$$

Moreover, it follows from the weak lower semicontinuity that

$$\mathcal{J}_\omega(Q_\infty) \leq \liminf_{n\to\infty} \mathcal{J}_\omega(u_n) \leq m_\omega. \tag{A.12}$$

Combining (A.11) and (A.12), we obtain $\mathcal{J}_\omega(Q_\infty) = m_\omega$. Thus, we have proved that Q_∞ is a minimizer for m_ω. Then, from the uniqueness of the ground state (see Proposition 2.1 (i)), there exists $\theta \in \mathbb{R}$ such that $Q_\infty = e^{i\theta}Q_\omega$. Thus, we see that $\mathcal{S}_\omega(Q_\infty) = m_\omega$ and $\mathcal{K}(Q_\infty) = 0$. It follows from $\mathcal{J}_\omega(Q_\infty) = m_\omega = \lim_{n\to\infty} \mathcal{J}_\omega(u_n)$ and (A.9) that $\lim_{n\to\infty} \mathcal{J}_\omega(u_n - Q_\infty) = 0$. This together with the Hölder inequality yields that $\lim_{n\to\infty}\|u_n - Q_\infty\|_{L^q} = 0$ for $2 \le q \le 6$. Then, since $\lim_{n\to\infty}\mathcal{S}_\omega(u_n) = m_\omega = S_\omega(Q_\infty)$, we have

$$\begin{aligned}\lim_{n\to\infty}\left\{\frac{1}{2}\|\nabla u_n\|_{L^2}^2 + \frac{\omega}{2}\|u_n\|_{L^2}^2\right\} &= \lim_{n\to\infty}\left\{\mathcal{S}_\omega(u_n) - \frac{1}{4}\|u_n\|_{L^4}^4 - \frac{1}{6}\|u_n\|_{L^6}^6\right\} \\ &= \mathcal{S}_\omega(Q_\infty) - \frac{1}{4}\|Q_\infty\|_{L^4}^4 - \frac{1}{6}\|Q_\infty\|_{L^6}^6 \\ &= \frac{1}{2}\|\nabla Q_\infty\|_{L^2}^2 + \frac{\omega}{2}\|Q_\infty\|_{L^2}^2.\end{aligned}$$

This together with the weak convergence of u_n to Q_∞ in $H^1(\mathbb{R}^3)$ implies that

$$\lim_{n\to\infty} u_n = Q_\infty = e^{i\theta}Q_\omega \qquad \text{strongly in } H^1(\mathbb{R}^3).$$

Thus, we infer that $\lim_{n\to\infty}\mathrm{dist}_{H^1}(u_n, \mathcal{O}(Q_\omega)) = 0$.

□

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