Dynamics of PDE, Vol.20, No.4, 299-309, 2023

Energy conservation and Onsager's conjecture for a surface growth model

Wei Wei, Yulin Ye, and Xue Mei

Communicated by James Robinson, received July 25, 2021.

Abstract. In this paper, it is shown that the energy equality of weak solution v to a surface growth model is valid if $v_x \in L^p(0,T;L^q(\mathbb{T}))$ with $\frac{3}{p}+\frac{1}{q}=1$ and $1 \le q \le 4$, or $v \in L^\infty(0,T;L^\infty(\mathbb{T}))$, or $v_{xx} \in L^p(0,T;L^q(\mathbb{T}))$ with $\frac{2}{p}+\frac{2}{5q}=1$ and $q \ge 1$, which gives an affirmative answer to a question proposed by Yang in [**28**, J. Differential Equations 283: 71–84, 2021]. Furthermore, Onsager's conjecture for this model is also considered.

Contents

1. Introduction

This work is concerned with the scalar surface growth model (SGM)

$$\begin{cases} v_t + v_{xxxx} + \partial_{xx}(v_x)^2 = 0 \text{ in } (0,\,T)\times\mathbb{T}, \\ v|_{t=0} = v_0 \text{ on } \mathbb{T}, \end{cases} \tag{1.1}$$

on the one dimensional torus $\mathbb{T}=[0,1]=\mathbb{R}/\mathbb{Z}$, which is a model of epitaxial growth of monocrystals, with v being the height of a crystalline layer. For more applicational motivations and certain analytical results, we refer the reader to [**8, 9, 23, 24, 26**].

2020 *Mathematics Subject Classification.* Primary 35K25, 35K55, 76D03; Secondary 35Q35, 35Q30.

Key words and phrases. Surface growth model; Energy conservation; Onsager's conjecture.

The well-posedness of this model has been studied by many authors due to its physical background and many striking similarities with the 3D Navier-Stokes equations. More precisely, the following cancellation is valid

$$\int_{\mathbb{T}} v\,\partial_{xx}(v_x)^2 dx = 0, \tag{1.2}$$

which is similar to the convective term in the Navier-Stokes equations on $\mathbb{T}^3$

$$\int_{\mathbb{T}^3} \Big(u \cdot \nabla u\Big) u dx = 0. \tag{1.3}$$

Here u stands for the velocity of flow, and the Navier-Stokes equations can be written as

$$\begin{cases} u_t - \Delta u + u \cdot \nabla u + \nabla \Pi = 0, \\ \operatorname{div} u = 0, \\ u|_{t=0} = u_0. \end{cases} \tag{1.4}$$

According to cancellation (1.2), the natural energy law for surface growth model is formally given by

$$\frac{1}{2}\int_{\mathbb{T}} v^2(t) + \int_0^t \int_{\mathbb{T}} v_{xx}^2 \leq \frac{1}{2}\int_{\mathbb{T}} v_0^2, \text{ a.e. } t \in [0,T].$$

Based on this, similarly as the Leray-Hopf weak solutions of Navier-Stokes equations, the global existence of finite energy weak solutions to (1.1) with initial datum $v_0 \in L^2$ is obtained by Stein and Winkler in [**26**] (see also [**6, 7**]). In 2019, Ożański [**23**] proved an analogue of Serrin type regularity criteria of weak solutions for the surface growth model. The result in [**23**] reads: if v_x satisfies

$$v_x \in L^p(0,T; L^q(\mathbb{T})) \text{ with } \frac{4}{p} + \frac{1}{q} = 1,$$

then $v \in C^\infty((0,T) \times \mathbb{T})$. Later on, the partial regularity of suitable weak solutions to (1.1) was established in [**24**] and improved by Burczak, Ożański and Seregin in [**10**]. Very recently, Yang considered the energy conservation of weak solutions to the surface growth model in [**28**], where it was shown that the energy equality of the SGM (1.1)

$$\frac{1}{2}\int_{\mathbb{T}} v^2(t) + \int_0^t \int_{\mathbb{T}} v_{xx}^2 = \frac{1}{2}\int_{\mathbb{T}} v_0^2, \text{ for } t \in [0,T] \tag{1.5}$$

holds provided

$$\begin{aligned} & v \in L^\infty(0,T; L^2(\mathbb{T})),\ v_{xx} \in L^2(0,T; L^2(\mathbb{T})), \\ & v \in L^\infty(0,T; L^\infty(\mathbb{T})), \\ & v_x \in L^p(0,T; L^q(\mathbb{T})) \text{ with } \begin{cases} p = q = 4, \\ \dfrac{3}{p} + \dfrac{1}{q} = 1,\ 3 \leq q < p. \end{cases} \end{aligned} \tag{1.6}$$

Before going further, we recall the well-known Lions-Shinbrot conditions shown in [**4, 20, 25**] for energy equality of weak solutions to the Navier-Stokes equations

(1.4),

$$u \in L^\infty(0,T;L^2(\mathbb{T}^3)),\ \nabla u \in L^2(0,T;L^2(\mathbb{T}^3)),$$

$$u \in L^p(0,T;L^q(\mathbb{T}^3)) \text{ with } \begin{cases} \dfrac{2}{p}+\dfrac{2}{q}=1,\ q\geq 4, \\ \dfrac{1}{p}+\dfrac{3}{q}=1,\ 3<q<4. \end{cases} \tag{1.7}$$

Compared with (1.7), it is worth noting that (1.6) requires an additional condition $v \in L^\infty(0,T;L^\infty(\mathbb{T}))$. In [**28**, Remark 1.4, P.76], the author mentioned that it remains an open problem how to remove this unnatural condition. The first objective of the present article is to give a positive answer to this question.

Before stating our main results, we introduce the definition of weak solutions to a surface growth model.

Definition 1.1. v is called a weak solution to (1.1) on $Q=(0,T)\times\mathbb{T}$, if v satisfies

(i) $v \in L^\infty(0,T;L^2(\mathbb{T}))\cap L^2(0,T;W^{2,2}(\mathbb{T}))$;

(ii) v satisfies (1.1) in the sense of distributions on Q, i.e.,

$$-\int_{\mathbb{T}} v_0\phi(0)-\int_0^T\int_{\mathbb{T}} v\phi_t+\int_0^T\int_{\mathbb{T}} v_{xx}\phi_{xx}=-\int_0^T\int_{\mathbb{T}} v_x^2\phi_{xx}$$

for every $\phi\in C_0^\infty([0,T)\times\mathbb{T})$;

(iii) (energy inequality) for almost every $t\in[0,T]$,

$$\frac{1}{2}\int_{\mathbb{T}} v^2(t)+\int_0^t\int_{\mathbb{T}} v_{xx}^2\leq\frac{1}{2}\int_{\mathbb{T}} v_0^2\,;$$

(iv) $\|v(t)-v_0\|_{L^2(\mathbb{T})}\to 0$ as $t\to 0^+$.

We formulate our first main result as follows.

Theorem 1.2. *The energy equality* (1.5) *of weak solutions v to the surface growth model* (1.1) *is valid if one of the following three conditions is satisfied*

$$(1)\quad v_x\in L^p(0,T;L^q(\mathbb{T})) \textit{ with } \frac{3}{p}+\frac{1}{q}=1,\ 1\leq q\leq 4; \tag{1.8}$$

$$(2)\quad v\in L^\infty(0,T;L^\infty(\mathbb{T})); \tag{1.9}$$

$$(3)\quad v_{xx}\in L^p(0,T;L^q(\mathbb{T})) \textit{ with } \frac{2}{p}+\frac{2}{5q}=1,\ q\geq 1. \tag{1.10}$$

The proof of Theorem 1.2 is completely different from that in [**28**]. Roughly speaking, in the spirit of [**27**], we shall first prove a new energy conservation criterion in terms of

$$v_x\in L^{\frac{2p}{p-1}}(0,T;L^{\frac{2q}{q-1}}(\mathbb{T})) \text{ and } v_{xx}\in L^p(0,T;L^q(\mathbb{T})).$$

To this end, taking advantage of the cancellation (1.2), we apply commutator estimates involving Friedrich's mollifier developed in [**21, 27**] to achieve this. Then a combination of this new criterion and the natural energy enables us to prove this theorem.

Remark 1.3. Due to (1.8), the Lions's classical energy conservation condition $u\in L_t^4L_x^4$ for Navier-Stokes equations (1.4) is extended to the scalar surface growth model. Moreover, the restrictions on q for the case $q<4$ in (1.8) are weaker than the corresponding case in [**28**].

REMARK 1.4. In [**28**], it is shown that the boundedness of regular solution v guarantees the energy conservation for equations (1.1). Here, our criterion (1.9) ensures that the same result still holds for more general weak solutions.

REMARK 1.5. The energy conservation criterion (1.10) is partially motivated by the recent criteria via the gradient of velocity for energy balance of Navier-Stokes equations [**5, 29**].

Our next goal is to study the Onsager's conjecture for the surface growth model (1.1) without dissipation. The original Onsager's conjecture for the 3D Euler equations is that the critical regularity of weak solutions guarantees energy balance of the Euler equations. In this aspect, the milestone work is due to Constantin-E-Titi [**16**], in which it was proved that the energy of 3D incompressible Euler equations is conserved for every weak solution in $L^3(0,T;B^{\alpha}_{3,\infty})$ with $\alpha > 1/3$. On the other hand, Isett resolved the "negative"part of Onsager's conjecture for 3D incompressible Euler equations in [**19**], where he proved that for any $\alpha < \frac{1}{3}$ there is a nonzero weak solution to the incompressible Euler equations in the class $v \in C^{\alpha}_{t,x}$ and $p \in C^{2\alpha}_{t,x}$ such that v is identically 0 outside a finite time interval. In particular, the solution v fails to conserve the energy. We refer the reader to [**1, 2, 11, 12, 14, 15, 18**] for recent progress in this direction. As for the surface growth model, to the best of authors' knowledge, there are not any corresponding results. The second purpose of this paper is to extend Onsager's conjecture to the following SGM without dissipation

$$v_t + \partial_{xx}(v_x)^2 = 0. \tag{1.11}$$

We state our second main result below.

THEOREM 1.6. *If v is a weak solution to* (1.11)*, then there holds*

$$\int_{\mathbb{T}} |v(t,x)|^2 dx = \int_{\mathbb{T}} |v_0(x)|^2 dx$$

for all $t \in [0,T]$, provided

$$v_x \in L^3(0,T;\dot{B}^{\alpha}_{3,\infty}(\mathbb{T})), \ \alpha > \frac{1}{3}. \tag{1.12}$$

REMARK 1.7. We remark that for any $\alpha > \frac{1}{3}$ our condition (1.12) with the homogeneous Besov space $\dot{B}^{\alpha}_{3,\infty}(\mathbb{T})$ is weaker than the nonhomogeneous condition $v_x \in L^3(0,T;B^{\alpha}_{3,\infty}(\mathbb{T}))$, since $B^{\alpha}_{3,\infty}(\mathbb{T}) = L^3(\mathbb{T}) \bigcap \dot{B}^{\alpha}_{3,\infty}(\mathbb{T})$. For the incompressible Euler equations on the whole space, a similar criterion with the homogeneous Besov space can be referred to [**13**].

REMARK 1.8. With $\alpha > \frac{1}{3}$, one has the following inclusions:

$$C^{\alpha} \subseteq B^{\alpha}_{3,\infty} \subseteq B^{\frac{1}{3}}_{3,c(\mathbb{N})} \subseteq \underline{B}^{\frac{1}{3}}_{3,VMO} \subseteq B^{\frac{1}{3}}_{3,\infty},$$

where $B^{\frac{1}{3}}_{3,c(\mathbb{N})}$ is the class of all tempered distributions u defined by using the Littlewood-Paley decomposition for which $\lim_{q\to\infty} \lambda_q^{1/3} \|\triangle_q u\|_{L^3} = 0$ introduced in [**15**] and $\underline{B}^{\frac{1}{3}}_{3,VMO}$ is a refined Besov-VMO type space introduced in [**17**]. For the 3D incompressible Euler equations, the energy conservation of weak solutions belonging to $B^{\alpha}_{3,\infty}$ was proved by Constantin-E-Titi in [**16**] with $\alpha > \frac{1}{3}$. Later, this result was extended to the solutions belonging to $B^{\frac{1}{3}}_{3,c(\mathbb{N})}$ by Cheskidov *et al.* in [**15**], where it was also shown that this result is almost optimal because one can construct divergence-free vector fields $u \in B^{1/3}_{3,\infty}$ with a non-zero energy flux. Recently, the result was further improved to $\underline{B}^{\frac{1}{3}}_{3,VMO}$ by Fjordholm and Wiedemann in [**17**]. Hence, $\underline{B}^{\frac{1}{3}}_{3,VMO}$ seems to be an almost optimal regularity class for the conservation of energy to 3D incompressible Euler equations. By a slight modification of the proof of Theorem 1.6 and the argument in [**3, 17**], one can consider refining the Besov norm in (1.12) to $\underline{B}^{\frac{1}{3}}_{3,VMO}$. We leave this to the interested readers.

The remainder of this paper is organized as follows. Section 2 is devoted to some notations and auxiliary lemmas involving mollifier estimates. In Section 3, we give the proof of Theorem 1.2 and Theorem 1.6.

2. Notations and some auxiliary lemmas

In this section, we introduce some notations used in the present article. For $p \in [1, \infty]$, the notation $L^p(0, T; X)$ stands for the set of measurable functions on the interval $(0, T)$ with values in X and $\|f(t, \cdot)\|_X$ belonging to $L^p(0, T)$. Given a domain $\Omega \subset \mathbb{R}^d$, the classical Sobolev space $W^{k,p}(\Omega)$ is equipped with the norm $\|f\|_{W^{k,p}(\Omega)} = \sum\limits_{|\alpha|=0}^{k} \|D^\alpha f\|_{L^p(\Omega)}$. For $1 \le q \le \infty$ and $0 < \alpha < 1$, the homogeneous Besov space $\dot{B}^\alpha_{q,\infty}(\mathbb{T}^d)$ is the space of functions f on the d-dimensional torus $\mathbb{T}^d = [0, 1]^d$ for which the seminorm

$$\|f\|_{\dot{B}^\alpha_{q,\infty}(\mathbb{T}^d)} = \Big\| |y|^{-\alpha} \Big\| f(x-y) - f(x) \Big\|_{L^q_x(\mathbb{T}^d)} \Big\|_{L^\infty_y(\mathbb{R}^d)} < \infty,$$

and the nonhomogeneous Besov space $B^\alpha_{q,\infty}(\mathbb{T}^d)$ is the set of functions $f \in L^q(\mathbb{T}^d)$ for which the norm

$$\|f\|_{B^\alpha_{q,\infty}(\mathbb{T}^d)} = \|f\|_{L^q(\mathbb{T}^d)} + \|f\|_{\dot{B}^\alpha_{q,\infty}(\mathbb{T}^d)} < \infty.$$

A similar definition of Besov norms on the whole space $\mathbb{R}^d$ can be referred to [**13**]. For simplicity, we write

$$\int_s^t \int_{\mathbb{T}} f(x, \tau) dx d\tau = \int_s^t \int f \text{ and } \|f\|_{L^p(0,T;X)} = \|f\|_{L^p(X)}.$$

Let $\eta : \mathbb{R}^d \to \mathbb{R}$ be a standard mollifier, i.e. $\eta(x) = C_0 e^{-\frac{1}{1-|x|^2}}$ for $|x| < 1$ and $\eta(x) = 0$ for $|x| \ge 1$, where C_0 is a constant such that $\int_{\mathbb{R}^d} \eta(x) dx = 1$. For $\varepsilon > 0$, we define the rescaled mollifier $\eta_\varepsilon(x) = \frac{1}{\varepsilon^d} \eta(\frac{x}{\varepsilon})$. For any function $f \in L^1_{\text{loc}}(\Omega)$, its mollified version is defined as

$$f^\varepsilon(x) = (f * \eta_\varepsilon)(x) = \int_{\mathbb{R}^d} f(x-y) \eta_\varepsilon(y) dy, \quad x \in \Omega_\varepsilon,$$

where $\Omega_\varepsilon = \{x \in \Omega : \text{dist}(x, \partial\Omega) > \varepsilon\}$.

We now recall some results involving the mollifier established in [**21**].

LEMMA 2.1. *([**21**]) Suppose that $f \in L^p(0, T; L^q(\mathbb{T}^d))$ with $1 \le p, q \le \infty$. Then for any $\varepsilon > 0$, there holds*

$$\|\nabla f^\varepsilon\|_{L^p(0,T;L^q(\mathbb{T}^d))} \le C\varepsilon^{-1} \|f\|_{L^p(0,T;L^q(\mathbb{T}^d))}, \tag{2.1}$$

and if $p, q < \infty$,

$$\limsup_{\varepsilon\to 0} \varepsilon \|\nabla f^\varepsilon\|_{L^p(0,T;L^q(\mathbb{T}^d))} = 0. \tag{2.2}$$

Moreover, if $0 < c_1 \le g \le c_2 < \infty$, *then there holds for any* $\varepsilon > 0$,

$$\Big\|\nabla \frac{f^\varepsilon}{g^\varepsilon}\Big\|_{L^p(0,T;L^q(\mathbb{T}^d))} \le C\varepsilon^{-1}\|f\|_{L^p(0,T;L^q(\mathbb{T}^d))}, \tag{2.3}$$

and if $p, q < \infty$,

$$\limsup_{\varepsilon\to 0} \varepsilon \Big\|\nabla \frac{f^\varepsilon}{g^\varepsilon}\Big\|_{L^p(0,T;L^q(\mathbb{T}^d))} = 0. \tag{2.4}$$

Remark 2.2. We remark that the assertion (2.2) can be deduced by a different argument from that in [**21**]. Indeed, note that $\int_{\mathbb{R}^d} \nabla\eta(x)dx = 0$ and the translation operator is strongly continuous on $L^q(\mathbb{T}^d)$ with $q < \infty$, it follows from the Minkowski inequality that

$$\begin{aligned}\limsup_{\varepsilon\to 0} \varepsilon\|\nabla f^\varepsilon\|_{L^q(\mathbb{T}^d)} =& \limsup_{\varepsilon\to 0} \Big\| \int_{|y|<1} \Big(f(\cdot - \varepsilon y) - f(\cdot)\Big) \nabla\eta(y)dy\Big\|_{L^q(\mathbb{T}^d)} \\ \le& C \limsup_{\varepsilon\to 0} \int_{|y|<1} \Big\|f(\cdot - \varepsilon y) - f(\cdot)\Big\|_{L^q(\mathbb{T}^d)} dy \\ \le& C \limsup_{\varepsilon\to 0} \Big(\sup_{|y|<\varepsilon} \Big\|f(\cdot - y) - f(\cdot)\Big\|_{L^q(\mathbb{T}^d)}\Big) = 0.\end{aligned}$$

Combining this and periodicity of the function f, we may apply the Lebesgue dominated convergence theorem to derive that

$$\limsup_{\varepsilon\to 0} \varepsilon\|\nabla f^\varepsilon\|_{L^p(0,T;L^q(\mathbb{T}^d))} \le C \limsup_{\varepsilon\to 0} \Big\| \sup_{|y|<\varepsilon} \Big\|f(\cdot - y) - f(\cdot)\Big\|_{L^q(\mathbb{T}^d)}\Big\|_{L^p(0,T)} = 0.$$

The next lemma with $p = q, p_1 = q_1, p_2 = q_2$ was also proved in [**21**]. A recent work [**27**] extends it to the integral norms with different exponents in space and time directions as follows.

Lemma 2.3. *(*[**27**]*) Let* $1 \le p, q, p_1, p_2, q_1, q_2 \le \infty$ *with* $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$ *and* $\frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}$. *Assume that* $f \in L^{p_1}(0,T;W^{1,q_1}(\mathbb{T}^d))$ *and* $g \in L^{p_2}(0,T;L^{q_2}(\mathbb{T}^d))$. *Then for any* $\varepsilon > 0$, *there holds*

$$\|(fg)^\varepsilon - f^\varepsilon g^\varepsilon\|_{L^p(0,T;L^q(\mathbb{T}^d))} \le C\varepsilon\|f\|_{L^{p_1}(0,T;W^{1,q_1}(\mathbb{T}^d))}\|g\|_{L^{p_2}(0,T;L^{q_2}(\mathbb{T}^d))}. \tag{2.5}$$

Moreover, if $p_2, q_2 < \infty$, *then*

$$\limsup_{\varepsilon\to 0} \varepsilon^{-1}\|(fg)^\varepsilon - f^\varepsilon g^\varepsilon\|_{L^p(0,T;L^q(\mathbb{T}^d))} = 0. \tag{2.6}$$

We end this section with a new commutator estimate, which will be employed in the proof of Theorem 1.6.

Lemma 2.4. *Assume that* $0 < \alpha, \beta < 1$ *and* $1 \le p, q, p_1, q_1, p_2, q_2 \le \infty$ *with* $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$, $\frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}$. *Then, for any* $f \in L^{p_1}(0,T;\dot{B}^\alpha_{q_1,\infty}(\mathbb{T}^d))$, $g \in L^{p_2}(0,T;\dot{B}^\beta_{q_2,\infty}(\mathbb{T}^d))$ *and* $\varepsilon > 0$, *there holds*

$$\|(fg)^\varepsilon - f^\varepsilon g^\varepsilon\|_{L^p(0,T;L^q(\mathbb{T}^d))} \le C\varepsilon^{\alpha+\beta}\|f\|_{L^{p_1}(0,T;\dot{B}^\alpha_{q_1,\infty}(\mathbb{T}^d))}\|g\|_{L^{p_2}(0,T;\dot{B}^\beta_{q_2,\infty}(\mathbb{T}^d))}. \tag{2.7}$$

Proof. We recall the following identity observed by Constantin-E-Titi in [**16**]

$$(fg)^\varepsilon(x)-f^\varepsilon g^\varepsilon(x)=\int_{\mathbb{R}^d}\eta_\varepsilon(y)[f(x-y)-f(x)][g(x-y)-g(x)]dy-(f-f^\varepsilon)(g-g^\varepsilon)(x).$$

The Hölder's inequality and Minkowski inequality yield that

$$\begin{aligned}
&\|(fg)^\varepsilon-f^\varepsilon g^\varepsilon\|_{L^p(0,T;L^q(\mathbb{T}^d))}\\
\leq&\int_{|y|\leq\varepsilon}\eta_\varepsilon(y)\|f(\cdot-y)-f(\cdot)\|_{L^{p_1}(0,T;L^{q_1}(\mathbb{T}^d))}\|g(\cdot-y)-g(\cdot)\|_{L^{p_2}(0,T;L^{q_2}(\mathbb{T}^d))}dy\\
&+\|f-f^\varepsilon\|_{L^{p_1}(0,T;L^{q_1}(\mathbb{T}^d))}\|g-g^\varepsilon\|_{L^{p_2}(0,T;L^{q_2}(\mathbb{T}^d))}\\
\leq&C\varepsilon^{\alpha+\beta}\|f\|_{L^{p_1}(0,T;\dot{B}^{\alpha}_{q_1,\infty}(\mathbb{T}^d))}\|g\|_{L^{p_2}(0,T;\dot{B}^{\beta}_{q_2,\infty}(\mathbb{T}^d))}.
\end{aligned}$$

Here we have used the following fact that for any $u\in\dot{B}^\tau_{q,\infty}(\mathbb{T}^d)$ with $0<\tau<1$, $1\leq q\leq\infty$ and a.e. $y\in\mathbb{R}^d$,

$$\begin{aligned}
&\|u(\cdot-y)-u(\cdot)\|_{L^q(\mathbb{T}^d)}\leq C|y|^\tau\|u\|_{\dot{B}^\tau_{q,\infty}(\mathbb{T}^d)},\\
&\|u^\varepsilon-u\|_{L^q(\mathbb{T}^d)}\leq C\varepsilon^\tau\|u\|_{\dot{B}^\tau_{q,\infty}(\mathbb{T}^d)},
\end{aligned}$$

which can be deduced from periodicity of the function u in essentially the same manner as derivation of [**13**, Lemma 2.1]. The proof of this lemma is completed. □

3. Proof of the main theorems

In this section, we shall focus our attention on the proof of Theorem 1.2 and Theorem 1.6.

Proof of Theorem 1.2. We start by asserting that the energy law can be derived from the following conditions

$$v_x\in L^{\frac{2p}{p-1}}(0,T;L^{\frac{2q}{q-1}}(\mathbb{T}))\ \text{ and }\ v_{xx}\in L^p(0,T;L^q(\mathbb{T})),\ 1<p,q\leq\infty. \tag{3.1}$$

Indeed, multiplying $(1.1)_1$ by $(v^\varepsilon)^\varepsilon$, then integrating over $(s,t)\times\mathbb{T}$ with $0<s<t<T$, we have

$$\int_s^t\int\Big[v^\varepsilon\partial_\tau(v^\varepsilon)+v^\varepsilon_{xx}v^\varepsilon_{xx}+v^\varepsilon\partial_{xx}[(v_x)^2]^\varepsilon\Big]=0. \tag{3.2}$$

A direct calculation shows

$$\int_s^t\int v^\varepsilon\partial_\tau(v^\varepsilon)=\frac{1}{2}\int_{\mathbb{T}}(v^\varepsilon(t))^2-\frac{1}{2}\int_{\mathbb{T}}(v^\varepsilon(s))^2. \tag{3.3}$$

It follows from the triangle inequality and the Hölder's inequality that

$$\begin{aligned}
&\Big|\int_s^t\int v^\varepsilon_{xx}v^\varepsilon_{xx}-v_{xx}v_{xx}\Big|\leq\int_s^t\int|v^\varepsilon_{xx}v^\varepsilon_{xx}-v_{xx}v^\varepsilon_{xx}|+\int_s^t\int|v_{xx}v^\varepsilon_{xx}-v_{xx}v_{xx}|\\
\leq&\|v^\varepsilon_{xx}\|_{L^2(L^2(\mathbb{T}))}\|v^\varepsilon_{xx}-v_{xx}\|_{L^2(L^2(\mathbb{T}))}+\|v_{xx}\|_{L^2(L^2(\mathbb{T}))}\|v^\varepsilon_{xx}-v_{xx}\|_{L^2(L^2(\mathbb{T}))},
\end{aligned}$$

which together with $v_{xx}\in L^2(0,T;L^2(\mathbb{T}))$ implies that

$$\limsup_{\varepsilon\to0}\Big|\int_s^t\int v^\varepsilon_{xx}v^\varepsilon_{xx}-v_{xx}v_{xx}\Big|=0. \tag{3.4}$$

Thanks to the cancellation (1.2) and integration by parts, we reformulate the nonlinear term as

$$\begin{aligned}\Big|\int_s^t\int v^\varepsilon\partial_{xx}[(v_x)^2]^\varepsilon\Big| &= \Big|\int_s^t\int v^\varepsilon\partial_{xx}\{[(v_x)^2]^\varepsilon - v_x^\varepsilon v_x^\varepsilon\} + v^\varepsilon\partial_{xx}(v_x^\varepsilon v_x^\varepsilon)\Big|\\ &= \Big|\int_s^t\int v^\varepsilon\partial_{xx}\{[(v_x)^2]^\varepsilon - v_x^\varepsilon v_x^\varepsilon\}\Big|\\ &= \Big|\int_s^t\int \{[(v_x)^2]^\varepsilon - v_x^\varepsilon v_x^\varepsilon\}\partial_{xx}v^\varepsilon\Big|.\end{aligned}$$

Invoking Lemma 2.1 and Lemma 2.3, we see that

$$\begin{aligned}&\Big\|((v_x)^2)^\varepsilon - (v_x^\varepsilon)^2\Big\|_{L^{\frac{2p}{p+1}}(L^{\frac{2q}{q+1}}(\mathbb{T}))} \le C\varepsilon\|v_x\|_{L^p(W^{1,q}(\mathbb{T}))}\|v_x\|_{L^{\frac{2p}{p-1}}(L^{\frac{2q}{q-1}}(\mathbb{T}))},\\ &\Big\|\partial_{xx}v^\varepsilon\Big\|_{L^{\frac{2p}{p-1}}(L^{\frac{2q}{q-1}}(\mathbb{T}))} \le C\varepsilon^{-1}\|v_x\|_{L^{\frac{2p}{p-1}}(L^{\frac{2q}{q-1}}(\mathbb{T}))},\\ &\lim_{\varepsilon\to0}\varepsilon\Big\|\partial_{xx}v^\varepsilon\Big\|_{L^{\frac{2p}{p-1}}(L^{\frac{2q}{q-1}}(\mathbb{T}))} = 0.\end{aligned}\tag{3.5}$$

The Gagliardo-Nirenberg inequality [**22**, Theorem 1] and the Young inequality enable us to deduce that

$$\|v_x\|_{L^q(\mathbb{T})} \le C\|v_{xx}\|_{L^q(\mathbb{T})}^{\frac{2q-1}{3q-1}}\|v\|_{L^1(\mathbb{T})}^{\frac{q}{3q-1}} + C\|v\|_{L^1(\mathbb{T})} \le C\Big(\|v_{xx}\|_{L^q(\mathbb{T})} + \|v\|_{L^1(\mathbb{T})}\Big).$$

Combining this, the Hölder's inequality and Lemma 2.1, we find that

$$\begin{aligned}&\Big|\int_s^t\int v^\varepsilon\partial_{xx}[(v_x)^2]^\varepsilon\Big|\\ \le&\Big\|\partial_{xx}v^\varepsilon\Big\|_{L^{\frac{2p}{p-1}}(L^{\frac{2q}{q-1}}(\mathbb{T}))}\Big\|((v_x)^2)^\varepsilon - (v_x^\varepsilon)^2\Big\|_{L^{\frac{2p}{p+1}}(L^{\frac{2q}{q+1}}(\mathbb{T}))}\\ \le&C\varepsilon\Big\|\partial_{xx}v^\varepsilon\Big\|_{L^{\frac{2p}{p-1}}(L^{\frac{2q}{q-1}}(\mathbb{T}))}\|v_x\|_{L^p(W^{1,q}(\mathbb{T}))}\|v_x\|_{L^{\frac{2p}{p-1}}(L^{\frac{2q}{q-1}}(\mathbb{T}))}\\ \le&C\varepsilon\Big\|\partial_{xx}v^\varepsilon\Big\|_{L^{\frac{2p}{p-1}}(L^{\frac{2q}{q-1}}(\mathbb{T}))}\Big(\|v_{xx}\|_{L^p(L^q(\mathbb{T}))} + \|v\|_{L^p(L^1(\mathbb{T}))}\Big)\|v_x\|_{L^{\frac{2p}{p-1}}(L^{\frac{2q}{q-1}}(\mathbb{T}))}\\ \le&C\varepsilon\Big\|\partial_{xx}v^\varepsilon\Big\|_{L^{\frac{2p}{p-1}}(L^{\frac{2q}{q-1}}(\mathbb{T}))}\Big(\|v_{xx}\|_{L^p(L^q(\mathbb{T}))} + \|v\|_{L^\infty(L^2(\mathbb{T}))}\Big)\|v_x\|_{L^{\frac{2p}{p-1}}(L^{\frac{2q}{q-1}}(\mathbb{T}))},\end{aligned}$$

which yields that

$$\lim_{\varepsilon\to0}\Big|\int_s^t\int v^\varepsilon\partial_{xx}[(v_x)^2]^\varepsilon\Big| = 0.\tag{3.6}$$

Then, using the continuity of $\|v(t,\cdot)\|_{L^2(\mathbb{T})}$ at $t=0$, (3.3), (3.4) and (3.6), we derive the energy conservation for weak solutions to the surface growth model (1.1) as $\varepsilon\to0$ and $s\to0$. The assertion (3.1) follows.

Next, due to (3.1), we know that $v_x\in L^4(0,T;L^4(\mathbb{T}))$ can guarantee the energy equality for surface growth model (1.1), since the energy inequality of weak solutions to (1.1) gives $v_{xx}\in L^2(0,T;L^2(\mathbb{T}))$. Hence, in order to complete the proof of Theorem 1.2, it suffices to show $v_x\in L^4(0,T;L^4(\mathbb{T}))$.

Case 1: Suppose that $v_x\in L^p(0,T;L^q(\mathbb{T}))$ with $\frac{3}{p}+\frac{1}{q}=1$, $1\le q\le 4$.

The Gagliardo-Nirenberg inequality [**22**, Theorem 1] ensures that

$$\|v_x\|_{L^4(\mathbb{T})} \le C\|v_{xx}\|_{L^2(\mathbb{T})}^{\frac{4-q}{2q+4}}\|v_x\|_{L^q(\mathbb{T})}^{\frac{3q}{2q+4}} + C\|v_x\|_{L^q(\mathbb{T})}.$$

As a consequence, we further derive that

$$\|v_x\|_{L^4(L^4(\mathbb{T}))} \leq C\|v_{xx}\|^{\frac{4-q}{2q+4}}_{L^2(L^2(\mathbb{T}))}\|v_x\|^{\frac{3q}{2q+4}}_{L^p(L^q(\mathbb{T}))} + C\|v_x\|_{L^p(L^q(\mathbb{T}))} < \infty,$$

where we have used Hölder's inequality and the fact that $1 \leq q \leq 4$.

Case 2: Suppose that $v \in L^\infty(0,T;L^\infty(\mathbb{T}))$.

Using the Gagliardo-Nirenberg inequality once again, we see that

$$\|v_x\|_{L^4(\mathbb{T})} \leq C\|v_{xx}\|^{\frac{1}{2}}_{L^2(\mathbb{T})}\|v\|^{\frac{1}{2}}_{L^\infty(\mathbb{T})} + C\|v\|_{L^\infty(\mathbb{T})},$$

which leads to

$$\|v_x\|_{L^4(L^4(\mathbb{T}))} \leq C\|v_{xx}\|^{\frac{1}{2}}_{L^2(L^2(\mathbb{T}))}\|v\|^{\frac{1}{2}}_{L^\infty(L^\infty(\mathbb{T}))} + C\|v\|_{L^\infty(L^\infty(\mathbb{T}))} < \infty.$$

Case 3: Suppose that $v_{xx} \in L^p(0,T;L^q(\mathbb{T}))$ with $\frac{2}{p} + \frac{2}{5q} = 1$, $q \geq 1$.

Due to the Gagliardo-Nirenberg inequality on $\mathbb{T}$, we have

$$\|v_x\|_{L^4(\mathbb{T})} \leq C\|v\|^{\frac{5q-4}{2(5q-2)}}_{L^2(\mathbb{T})}\|v_{xx}\|^{\frac{5q}{2(5q-2)}}_{L^q(\mathbb{T})} + C\|v\|_{L^2(\mathbb{T})},$$

which implies that

$$\|v_x\|_{L^4(L^4(\mathbb{T}))} \leq C\|v\|^{\frac{5q-4}{2(5q-2)}}_{L^\infty(L^2(\mathbb{T}))}\|v_{xx}\|^{\frac{5q}{2(5q-2)}}_{L^p(L^q(\mathbb{T}))} + C\|v\|_{L^\infty(L^2(\mathbb{T}))} < \infty.$$

Then we have completed the proof of Theorem 1.2. □

Proof of Theorem 1.6. As the above, there holds that for $0 < s < t < T$,

$$\int_s^t \int \Big[v^\varepsilon \partial_t(v^\varepsilon) + v^\varepsilon \partial_{xx}[(v_x)^2]^\varepsilon\Big] = 0. \tag{3.7}$$

We derive from Hölder's inequality that

$$\begin{aligned}\Big|\int_s^t \int v^\varepsilon \partial_{xx}[(v_x)^2]^\varepsilon\Big| =& \Big|\int_s^t \int \{[(v_x)^2]^\varepsilon - v_x^\varepsilon v_x^\varepsilon\}\partial_{xx}v^\varepsilon\Big| \\ \leq& \|[(v_x)^2]^\varepsilon - v_x^\varepsilon v_x^\varepsilon\|_{L^p(0,T;L^q(\mathbb{T}))}\|\partial_{xx}v^\varepsilon\|_{L^{\frac{p}{p-1}}(0,T;L^{\frac{q}{q-1}}(\mathbb{T}))}.\end{aligned} \tag{3.8}$$

It follows from Lemma 2.4 that, for $1/p = 1/p_1 + 1/p_2$ and $1/q = 1/q_1 + 1/q_2$,

$$\|[(v_x)^2]^\varepsilon - (v_x^\varepsilon)^2\|_{L^p(0,T;L^q(\mathbb{T}))} \leq C\varepsilon^{\alpha+\beta}\|v_x\|_{L^{p_1}(0,T;\dot{B}^\alpha_{q_1,\infty}(\mathbb{T}))}\|v_x\|_{L^{p_2}(0,T;\dot{B}^\beta_{q_2,\infty}(\mathbb{T}))}.$$

Plugging this into (3.8), we know that for $p_3 = \frac{p}{p-1}$ and $q_3 = \frac{q}{q-1}$,

$$\begin{aligned}&\Big|\int_s^t \int \{[(v_x)^2]^\varepsilon - v_x^\varepsilon v_x^\varepsilon\}\partial_{xx}v^\varepsilon\Big| \\ \leq& C\varepsilon^{\alpha+\beta+\gamma-1}\|v_x\|_{L^{p_1}(0,T;\dot{B}^\alpha_{q_1,\infty}(\mathbb{T}))}\|v_x\|_{L^{p_2}(0,T;\dot{B}^\beta_{q_2,\infty}(\mathbb{T}))}\|v_x\|_{L^{p_3}(0,T;\dot{B}^\gamma_{q_3,\infty}(\mathbb{T}))}.\end{aligned} \tag{3.9}$$

Here we have used the following fact that for any $u \in \dot{B}^\gamma_{m,\infty}(\mathbb{T}^d)$ with $0 < \gamma < 1$ and $1 \leq m \leq \infty$,

$$\|\nabla u^\varepsilon\|_{L^m(\mathbb{T}^d)} \leq C\varepsilon^{\gamma-1}\|u\|_{\dot{B}^\gamma_{m,\infty}(\mathbb{T}^d)},$$

which can be deduced from periodicity of the function u in essentially the same manner as derivation of [**13**, Lemma 2.1].

Finally, taking $p_1 = p_2 = p_3 = q_1 = q_2 = q_3 = 3$ and $\alpha = \beta = \gamma \in (1/3, 1)$, we arrive at

$$\Big|\int_s^t \int \{[(v_x)^2]^\varepsilon - v_x^\varepsilon v_x^\varepsilon\}\partial_{xx}v^\varepsilon\Big| \leq C\varepsilon^{3\alpha-1}\|v_x\|^3_{L^3(0,T;\dot{B}^\alpha_{3,\infty}(\mathbb{T}))}. \tag{3.10}$$

Letting $\varepsilon \to 0$, we have

$$\Big| \int_s^t \int v^\varepsilon \partial_{xx}[(v_x)^2]^\varepsilon \Big| \longrightarrow 0. \tag{3.11}$$

Thus, in virtue of the continuity of $\|v(t,\cdot)\|_{L^2(\mathbb{T})}$ at $t = 0$, (3.7) and (3.11) enable us to complete the proof of Theorem 1.6. □

Acknowledgements

The authors would like to express their sincere gratitude to Dr. Yanqing Wang at Zhengzhou University of Light Industry, for a lot of useful discussion on this topic. Wei was partially supported by the National Natural Science Foundation of China under grant (No. 11601423, No. 11771352, No. 11871057). Ye was partially supported by the National Natural Science Foundation of China under grant (No.11701145) and China Postdoctoral Science Foundation (No. 2020M672196).

References

[1] D. Albritton, E. Brué, M. Colombo, C. De Lellis, V. Giri, M. Janisch, H. Kwon, Instability and nonuniqueness for the 2d Euler equations in vorticity form, after M. Vishik, *submitted* (2021). arXiv:2112.04943.

[2] C. Bardos, E.S. Titi, Onsager's conjecture for the incompressible Euler equations in bounded domains, *Arch. Ration. Mech. Anal.* **228** (2018), 197–207.

[3] C. Bardos, P. Gwiazda, A. Świerczewska-Gwiazda, E.S.Titi, E. Wiedemann, Onsager's conjecture in bounded domains for the conservation of entropy and other companion laws, *Proc. A.* **475** (2019), no. 2230, 20190289, 18 pp.

[4] H. Beirao, da Veiga, J. Yang, On the Shinbrot's criteria for energy equality to Newtonian fluids: a simplified proof, and an extension of the range of application, *Nonlinear Anal.* **196** (2020), 111809, 4 pp.

[5] L.C. Berselli, E. Chiodaroli, On the energy equality for the 3D Navier-Stokes equations, *Nonlinear Anal.* **192** (2020), 111704, 24 pp.

[6] D. Blömker, C. Gugg, M. Raible, Thin-film-growth models: roughness and correlation functions, *Eur. J. Appl. Math.* **13** (2002), 385-402.

[7] D. Blömker, F. Flandoli, M. Romito, Markovianity and ergodicity for a surface growth PDE, *Ann. Probab.* **37** (2009), 275-313.

[8] D. Blömker, M. Romito, Regularity and blow up in a surface growth model, *Dyn. Partial Differ. Equ.* **6(3)** (2009), 227-252.

[9] D. Blömker, M. Romito, Local existence and uniqueness in the largest critical space for a surface growth model, *NoDEA Nonlinear Differ. Equ. Appl.* **19(3)** (2012), 365-381.

[10] J. Burczak, W.S. Ożański, G. Seregin, On regularity properties of a surface growth model, *Proceedings of the Royal Society of Edinburgh: Section A* **151, no. 6** (2021), 1869-1892.

[11] T. Buckmaster, C. De Lellis, L. Székelyhidi, Jr., V. Vicol, Onsager's conjecture for admissible weak solutions, *Comm. Pure Appl. Math.* **72, no. 2**, (2019), 229-274.

[12] T. Buckmaster, V. Vicol, Convex integration and phenomenologies in turbulence, *EMS Surv. Math. Sci.* **6, no. 1-2** (2019), 173-263.

[13] D. Chae, On the conserved quantities for the weak solutions of the Euler equations and the quasi-geostrophic equations, *Commun. Math. Phys.* **266** (2006), 197–210.

[14] R. M. Chen, C. Yu, Onsager's energy conservation for inhomogeneous Euler equations, *J. Math. Pures Appl.* **131** (2019), 1–16.

[15] A. Cheskidov, P. Constantin, S. Friedlander, R. Shvydkoy, Energy conservation and Onsager's conjecture for the Euler equations,*Nonlinearity* **21** (2008), 1233–1252.

[16] P. Constantin, W. E, E.S. Titi, Onsager's conjecture on the energy conservation for solutions of Euler's equation, *Commun. Math. Phys.* **165** (1994), 207–209.

[17] U.S. Fjordholm, E. Wiedemann, Statistical solutions and Onsager's conjecture, *Phys. D* **376/377** (2018), 259-265.

[18] V. Giri, H. Kwon, On Non-uniqueness of continuous entropy solutions to the isentropic compressible Euler equations, *submitted* (2021). arXiv:2109.12165.

[19] P. Isett, A proof of Onsager's conjecture, *Ann. of Math.* (2) **188 , no. 3** (2018), 871-963.
[20] J.L. Lions, Sur la régularité et l'unicité des solutions turbulentes des équations de Navier Stokes, *Rend. Semin. Mat. Univ. Padova*, **30** (1960) 16–23.
[21] Q. Nguyen, P. Nguyen, B. Tang, Energy equalities for compressible Navier-Stokes equations, *Nonlinearity* **32** (2019), 4206–4231.
[22] L. Nirenberg, An extended interpolation inequality, *Ann. Sc. Norm. Sup. Pisa, cl. Sc. IV. ser.* **20** (1966), 733-737.
[23] W.S. Ożański, A sufficient integral condition for local regularity of solutions to the surface growth model, *J. Funct. Anal.* **276** (2019), 2990–3013.
[24] W.S. Ożański, J.C. Robinson, Partial regularity for a surface growth model, *SIAM J. Math. Anal.* **51(1)** (2019), 228–255.
[25] M. Shinbrot, The energy equation for the Navier-Stokes system, *SIAM J. Math. Anal.* **5** (1974) 948–954.
[26] O. Stein, M. Winkler, Amorphous molecular beam epitaxy: global solutions and absorbing sets, *European J. Appl. Math.* **16(6)** (2005), 767–798.
[27] Y. Wang, Y. Ye, Energy conservation via a combination of velocity and its gradient in the Navier-Stokes system, *submitted* (2021). arXiv:2106.01233.
[28] J. Yang, Energy conservation for weak solutions of a surface growth model, *J. Differential Equations* **283** (2021), 71–84.
[29] Z. Zhang, Remarks on the energy equality for the non-Newtonian fluids, *J. Math. Anal. Appl.* **480** (2019), no. 2, 123443, 9 pp.

(Wei Wei) School of Mathematics and Center for Nonlinear Studies, Northwest University, Xi'an, Shaanxi 710127, P. R. China
Email address: `ww5998198@126.com`

(Yulin Ye) Corresponding author; School of Mathematics and Statistics, Henan University, Kaifeng, 475004, P. R. China
Email address: `ylye@vip.henu.edu.cn`

(Xue Mei) College of Mathematics and Information Science, Zhengzhou University of Light Industry, Zhengzhou, Henan 450002, P. R. China
Email address: `meixue_mx@outlook.com`