

Periodic and quasi-periodic Euler- α flows close to Rankine vortices

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ABSTRACT. In the present contribution, we first prove the existence of \mathbf{m} -fold simply-connected V-states close to the unit disc for Euler- α equations. These solutions are implicitly obtained as bifurcation curves from the circular patches. We also prove the existence of quasi-periodic in time vortex patches close to the Rankine vortices provided that the scale parameter α belongs to a suitable Cantor-like set of almost full Lebesgue measure. The techniques used to prove this result are borrowed from the Berti-Bolle theory in the context of KAM for PDEs.

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1. Euler- α equations and main results

We consider the Euler- α planar model with parameter $\alpha \geq 0$ given by

$$(1.1) \quad \begin{cases} \partial_t \mathbf{u} + (\mathbf{v} \cdot \nabla) \mathbf{u} + (\nabla \mathbf{v})^\top \mathbf{u} + \nabla \pi = 0, & \text{in } \mathbb{R}_+ \times \mathbb{R}^2, \\ \nabla \cdot \mathbf{v} = \nabla \cdot \mathbf{u} = 0, \\ \mathbf{u} = \mathbf{v} - \alpha^2 \Delta \mathbf{v}, \\ \mathbf{v}(0, \cdot) = \mathbf{v}_0. \end{cases}$$

Key words and phrases. KAM theory, Nash-Moser scheme, Bifurcation theory, Vortex patches.

This model is a regularization of Euler equations describing the flow of an incompressible fluid on spatial scales larger than the length scale parameter α . It has been introduced in the context of averaged fluid models, see [53, 54]. In the literature, \mathbf{v} and \mathbf{u} are called the *filtered* and *unfiltered velocities*, respectively. Actually, \mathbf{u} corresponds to the velocity field of the fluid. Notice that $(\nabla \mathbf{v})^\top$ denotes the transpose of the Jacobi matrix for \mathbf{v} , namely

$$(\nabla \mathbf{v})^\top = \left(\frac{\partial v_j}{\partial x_i} \right)_{1 \leq i, j \leq 2}.$$

The pressure term π is linked to the pressure field p of the fluid through the relation

$$\pi \triangleq p - \frac{1}{2} |\mathbf{v}|^2 - \frac{\alpha^2}{2} |\nabla \mathbf{v}|^2.$$

Remark that we formally recover the classical Euler model, by taking $\alpha = 0$ in the above set of equations. The rigorous justification of this convergence can be found in [62]. In the sequel, we assume that $\alpha > 0$. Let us consider the *unfiltered vorticity* $\boldsymbol{\omega}$ defined by

$$(1.2) \quad \boldsymbol{\omega} \triangleq \nabla^\perp \cdot \mathbf{u} = \nabla^\perp \cdot (\text{Id} - \alpha^2 \Delta) \mathbf{v}, \quad \nabla^\perp \triangleq \begin{pmatrix} -\partial_2 \\ \partial_1 \end{pmatrix}.$$

Applying the operator $\nabla^\perp \cdot$ to the first equation in (1.1), we find, after straightforward computations using the divergence-free conditions, that $\boldsymbol{\omega}$ is a solution to the following active scalar equation

$$(1.3) \quad \partial_t \boldsymbol{\omega} + \mathbf{v} \cdot \nabla \boldsymbol{\omega} = 0.$$

Such kind of nonlinear and nonlocal transport PDE has been widely studied during the past decade in fluid dynamics, especially regarding the periodic motions and more recently the quasi-periodic ones. We shall discuss in this introduction two types of active scalar models in the form (1.3) which will be of interest in the sequel. The first example is given by the classical velocity-vorticity formulation of 2D Euler equations, namely

$$\partial_t \boldsymbol{\omega}^E + \mathbf{v}^E \cdot \nabla \boldsymbol{\omega}^E = 0, \quad \mathbf{v}^E = \nabla^\perp \Psi^E, \quad \Delta \Psi^E = \boldsymbol{\omega}^E.$$

In this case, the potential velocity Ψ^E is obtained as the convolution with the Green function associated to the Laplace problem set in the whole plane, namely

$$(1.4) \quad \Psi^E = \mathbf{G}^E * \boldsymbol{\omega}^E, \quad \mathbf{G}^E \triangleq \frac{1}{2\pi} \log(|\cdot|).$$

The second example is given by quasi-geostrophic shallow-water equations with parameter $\lambda > 0$, shorten in what follows into $(QGSW)_\lambda$, which is a geophysical asymptotic model describing the circulation of the atmosphere at large time and space scales [71, p.220]. This model is given by

$$\partial_t \mathbf{q}^{\text{SW}} + \mathbf{v}_\lambda^{\text{SW}} \cdot \nabla \mathbf{q}^{\text{SW}} = 0, \quad \mathbf{v}_\lambda^{\text{SW}} = \nabla^\perp \Psi_\lambda^{\text{SW}}, \quad (\Delta - \lambda^2) \Psi_\lambda^{\text{SW}} = \mathbf{q}^{\text{SW}}.$$

In this case, the potential velocity Ψ_λ^{SW} is obtained as the convolution with the Green function associated to the Helmholtz problem set in the whole plane, namely

$$(1.5) \quad \Psi_\lambda^{\text{SW}} = \mathbf{G}_\lambda^{\text{SW}} * \mathbf{q}^{\text{SW}}, \quad \mathbf{G}_\lambda^{\text{SW}} \triangleq -\frac{1}{2\pi} K_0(\lambda |\cdot|),$$

where K_0 is the modified Bessel function of second kind. We refer to the Appendix A for a presentation of modified Bessel functions and some of their useful properties. The parameter λ is called *Rossby deformation length* or *Rossby radius* and quantifies the rotation/stratification balance for the fluid. Few results are known regarding this model and we may refer to [27, 69] for the mathematical context of

interest in the sequel and to [28, 29] for interesting numerical simulations in the physical literature.

This paper is based on the following fundamental relation, which can be found for instance in [7, 66], giving a nice expression of the Euler- α velocity field \mathbf{v} .

$$(1.6) \quad \mathbf{v} = \nabla^\perp \Psi, \quad \Psi \triangleq \mathbf{G} * \omega, \quad \mathbf{G} \triangleq \mathbf{G}^E - \mathbf{G}_{\frac{1}{\alpha}}^{\text{SW}}.$$

Due to the importance of this formula for our work, we provide in Section 2 its detailed justification. In our approach, we take advantage of this explicit link with Euler and QGSW equations to prove the existence of periodic and quasi-periodic vortex patch motions for the Euler- α equations. In particular, we shall make appeal to the computations carried out in [14, 17, 27, 43, 50, 52] relatively to these contexts. For the Euler- α model, the global existence and uniqueness of classical solutions has been obtained in [18]. The well-posedness of weak solutions in the space of bounded Radon measures for (1.1) has been proved in [66]. Indeed, the formula (1.6) and (A.4) imply that the vector field \mathbf{v} is less singular than the Euler one, which allows to reach the class of measures for the well-posedness. An other consequence is that the Yudovitch theory also applies in this context and the weak solutions are Lagrangian. This fact is the starting point to be able to look for vortex patches. Before presenting the notion of vortex patch solutions together with the main results of this study, we may end this presentation by briefly mentioning that the 2D Euler- α model has also been intensively studied in the case of subset of \mathbb{R}^2 domains with suitable boundary conditions. We may for instance refer the reader to [19, 20, 21, 63, 60].

Now, we shall present the notion of vortex patches and discuss some historical background about the periodic and quasi-periodic settings. The techniques used to find these two kind of solutions are completely different. The periodic solutions are obtained by using bifurcation theory from the stationary solutions and the quasi-periodic ones appear in the context of KAM/Nash-Moser theories. Considering an open bounded simply-connected domain D_0 in \mathbb{R}^2 then the function $\mathbf{1}_{D_0}$ provides an initial datum in the Yudovitch class. Therefore, we have existence and uniqueness of a global weak solution which, according to the transport structure of (1.3), takes the following form

$$\omega(t, \cdot) = \mathbf{1}_{D_t},$$

where D_t is the image of D_0 by the flow map associated to the velocity field \mathbf{v} , namely

$$D_t = \Phi_t(D_0), \quad \Phi_t(x) = x + \int_0^t \mathbf{v}(s, \Phi_s(x)) ds.$$

The resulting solution $t \mapsto \mathbf{1}_{D_t}$ is called a *vortex patch* with initial patch $\mathbf{1}_{D_0}$. Given any parametrization $z(t, \cdot) : \mathbb{T} \rightarrow \partial D_t$ of the boundary of the patch at time t , it is well-known since the works [50, 51] that for an active scalar equation, as in our case, the particles on the boundary move with the flow and remain at the boundary. Hence, at least in the smooth case, one can write the following equation called *vortex patch equation*.

$$(1.7) \quad \left(\partial_t z(t, \theta) - \mathbf{v}(t, z(t, \theta)) \right) \cdot \mathbf{n}(t, z(t, \theta)) = 0,$$

where $\mathbf{n}(t, z(t, \theta))$ is the outward normal vector to the boundary ∂D_t of D_t at the point $z(t, \theta)$. Since the dynamics is planar, it is more convenient to use the complex notation. In particular, the Euclidean structure of \mathbb{R}^2 is transposed into the complex world in the following way:

$$\forall u = a + ib \in \mathbb{C}, \quad \forall v = a' + ib' \in \mathbb{C}, \quad u \cdot v \triangleq \langle u, v \rangle_{\mathbb{R}^2} = \operatorname{Re}(u\bar{v}) = aa' + bb'.$$

Remarking that $\mathbf{n}(t, z(t, \theta))$ is real-proportional to $i\partial_\theta z(t, \theta)$, then we get the complex form of the contour dynamics motion (1.7),

$$(1.8) \quad \operatorname{Im}\left(\partial_t z(t, \theta) \overline{\partial_\theta z(t, \theta)}\right) = \operatorname{Im}\left(\mathbf{v}(t, z(t, \theta)) \overline{\partial_\theta z(t, \theta)}\right).$$

► **Uniformly rotating solutions.**

Uniformly rotating vortex patches, also called *V-states*, form a particular subclass of vortex patch solutions taking the form

$$(1.9) \quad D_t = e^{i\Omega t} D_0.$$

Any solution of this form is rotating with a time independent angular velocity $\Omega \in \mathbb{R}$ around its center of mass fixed at the origin. If $\Omega = 0$, then it is stationary, otherwise it is time periodic with period $\frac{2\pi}{\Omega}$. Historically, the first example of V-states was provided by Kirchhoff [56] who showed that for Euler equations any ellipse with semi-axis a and b uniformly rotates for an angular velocity $\Omega = \frac{ab}{(a+b)^2}$. Observe from (1.6), (1.4) and (1.5) that the stream function Ψ admits a radial Green kernel. Therefore, any radial initial profile would generate a stationary solution. In particular, the Rankine vortices given by the discs provide such examples. The purpose of this study is to prove the existence of periodic (and quasi-periodic) vortex patch structures close to these equilibrium states. Due to the invariance by dilation of the model, it is sufficient to search for these type of solutions close to the unit disc. The analytical study of existence of V-states close to the unit disc for Euler equations goes back to the work of Burbea [17]. We also refer to [50] for a more recent and rigorous point of view. Indeed, combining Crandall-Rabinowitz's Theorem B.1 and complex analysis, one can find the existence of branches of \mathbf{m} -fold V-states bifurcating from the unit disc at the angular velocities

$$(1.10) \quad \Omega_{\mathbf{m}}^E \triangleq \frac{\mathbf{m} - 1}{2\mathbf{m}}.$$

Later on, a lot of attention has been paid to such type of solutions for different nonlinear transport fluid models like Euler equations in the plane or in the unit disc, generalised surface quasi-geostrophic equations $(SQG)_\alpha$ and $(QGSW)_\lambda$ equations. Also, several topological and regularity settings were explored and we may refer to [22, 27, 33, 34, 35, 36, 37, 38, 39, 40, 42, 44, 46, 47, 48, 49, 50, 51, 68, 69]. Nevertheless, it seems that the Euler- α model has been set aside from these studies. Hence, we propose here to study the emergence of simply-connected V-states for this model. For this task, due to the decomposition (1.6), we may emphasize the result in [27], where they found the following angular velocities related to the modified Bessel functions for which the bifurcation of simply-connected V-states from the unit disc occurs

$$(1.11) \quad \Omega_{\mathbf{m}}^{\text{SW}}(\lambda) \triangleq I_1(\lambda)K_1(\lambda) - I_{\mathbf{m}}(\lambda)K_{\mathbf{m}}(\lambda).$$

Now, we shall present the first theorem proved in this study and dealing with periodic simply-connected patches bifurcating from the Rankine vortices.

THEOREM 1.1. *Let $\alpha > 0$ and $\beta \in (0, 1)$. For any $\mathbf{m} \in \mathbb{N}^*$, there exists a branch of \mathbf{m} -fold V -states of regularity $C^{1+\beta}$ bifurcating from the unit disc at the angular velocity*

$$(1.12) \quad \Omega_{\mathbf{m}}^E(\alpha) \triangleq \frac{\mathbf{m} - 1}{2\mathbf{m}} - \left(I_1 \left(\frac{1}{\alpha} \right) K_1 \left(\frac{1}{\alpha} \right) - I_{\mathbf{m}} \left(\frac{1}{\alpha} \right) K_{\mathbf{m}} \left(\frac{1}{\alpha} \right) \right) = \Omega_{\mathbf{m}}^E - \Omega_{\mathbf{m}}^{\text{SW}} \left(\frac{1}{\alpha} \right)$$

for Euler- α equations (1.1).

REMARK 1.1. *Let us make the following remarks.*

- (i) *Observe that the bifurcation frequencies (1.12) are exactly the superposition of those of Euler (1.10) and (QGSW) $_{\frac{1}{\alpha}}$ (1.11) models which is in accordance with the remark in Section 2 and the structure of the vortex patch equation (1.8).*
- (ii) *The following convergences hold true.*

$$\Omega_{\mathbf{m}}^E(\alpha) \xrightarrow{\mathbf{m} \rightarrow \infty} \frac{1}{2} - I_1 \left(\frac{1}{\alpha} \right) K_1 \left(\frac{1}{\alpha} \right) \triangleq \Omega_{\infty}^E(\alpha), \quad \Omega_{\mathbf{m}}^E(\alpha) \xrightarrow{\alpha \rightarrow 0} \frac{\mathbf{m} - 1}{2\mathbf{m}} = \Omega_{\mathbf{m}}^E.$$

- (iii) *The case $\mathbf{m} = 1$ corresponds to a translation of the trivial solution, see Remark 3.1.*

The Theorem 1.1 is proved by using Crandall-Rabinowitz’s Theorem B.1 in the spirit of the previous works mentioned above. For that purpose, we reformulate the vortex patch equation (1.8) in the uniformly rotating framework with conformal mappings. We chose to use the conformal functional setting in order to be able to take advantage of the computations already done in [27, 50]. Nevertheless, one could also use the polar parametrization similarly to the next result on quasi-periodic solutions. Introducing, for an initial domain D_0 close to the unit disc \mathbb{D} , the conformal parametrization $\Phi : \mathbb{C} \setminus \overline{\mathbb{D}} \rightarrow \mathbb{C} \setminus \overline{D_0}$ in the form

$$\Phi(z) = z + f(z), \quad f(z) = \sum_{n=0}^{\infty} \frac{a_n}{z^n}, \quad a_n \in \mathbb{R},$$

we can reformulate the vortex patch equation (1.8) in the uniformly rotating context as the following equation

$$\forall w \in \mathbb{T}, \quad F_{\alpha}(\Omega, f)(w) = 0,$$

with

$$F_{\alpha}(\Omega, f)(w) \triangleq \text{Im} \left\{ \left(\Omega \Phi(w) + I^E(f)(w) + I^{\text{SW}}(f)(\alpha, w) \right) \overline{w \Phi'(w)} \right\},$$

$$I^E(f)(w) \triangleq \int_{\mathbb{T}} \Phi'(\tau) \log(|\Phi(w) - \Phi(\tau)|) d\tau,$$

$$I^{\text{SW}}(f)(\alpha, w) \triangleq \int_{\mathbb{T}} \Phi'(\tau) K_0 \left(\frac{1}{\alpha} |\Phi(\tau) - \Phi(w)| \right) d\tau.$$

Remarking that the disc provides a line of solutions $F_{\alpha}(\Omega, 0) = 0$ for $\Omega \in \mathbb{R}$, we shall apply the local bifurcation theory to find non-trivial solutions. In Proposition 3.1, we prove that the functional F_{α} is of class C^1 with respect to some Hölder regularity spaces and its linearized operator at the equilibrium has the Fredholm property and expresses as the following Fourier multiplier

$$\forall w \in \mathbb{T}, \quad d_f F_{\alpha}(\Omega, 0)(w) = \sum_{n=0}^{\infty} a_n (n + 1) \left(\Omega_{n+1}^E(\alpha) - \Omega \right) \text{Im}(w^{n+1}).$$

The one dimensional kernel condition for applying bifurcation theory is ensured by the strict monotonicity of the sequence $(\Omega_n^E(\alpha))_{n \in \mathbb{N}^*}$ which is checked using some refined estimates on modified Bessel functions, see Lemma 3.1. Finally, the transversality condition is a direct consequence of the Fourier decomposition of the linearized operator and is checked in Proposition 3.2-(iv).

► **Quasi-periodic in time vortex patches.**

This topic is rather new in vortex patch dynamics and the tools used are borrowed from KAM/Nash-Moser theories [2, 11, 58, 64, 65]. Recall that a function $r : \mathbb{R} \rightarrow \mathbb{R}$ is said to be *quasi-periodic* if there exists $d \in \mathbb{N}^*$, $\widehat{r} = \widehat{r}(\varphi) : \mathbb{T}^d \rightarrow \mathbb{R}$ continuous and $\omega \in \mathbb{R}^d$ such that

$$(1.13) \quad r(t) = \widehat{r}(\omega t), \quad \forall l \in \mathbb{Z}^d \setminus \{0\}, \quad \omega \cdot l \neq 0.$$

The existence of quasi-periodic vortex patches close to the unit disc for the QGSW equations have been proved in [52]. These solutions are obtained by selecting the Rossby deformation length in a massive Cantor set. Similarly, in [41], the authors used the parameter inside the equations to generate quasi-periodic vortex patches near the Rankine vortices for $(SQG)_\alpha$ equations for suitable selected values of α . Here our work follows the same idea relying on the free parameter α to obtain these solutions. We mention two other works recently obtained for Euler equations. The first one concerns the quasi-periodic patches close to the Rankine vortices in the unit disc presented in [43]. The second result, which can be found in [14], is relative to quasi-periodic patches close to the Kirchhoff ellipses. In both cases, this is a geometrical parameter which, when taken among a Cantor-like set of admissible parameters, allows to find quasi-periodic solutions. Our second result reads as follows.

THEOREM 1.2. *Let $0 < \alpha_0 < \alpha_1$, $\Omega > 0$ and $\mathbb{S} \subset \mathbb{N}^*$ finite. There exists $\varepsilon_0 \in (0, 1)$ small enough such that for every choice of amplitudes $\mathbf{a} = (\mathbf{a}_j)_{j \in \mathbb{S}} \in (\mathbb{R}_+^*)^{|\mathbb{S}|}$ enjoying the smallness condition*

$$|\mathbf{a}| \leq \varepsilon_0,$$

there exists a Cantor-like set $\mathcal{C}_\infty \subset (\alpha_0, \alpha_1)$ with asymptotically full Lebesgue measure as $\mathbf{a} \rightarrow 0$, i.e.

$$\lim_{\mathbf{a} \rightarrow 0} |\mathcal{C}_\infty| = \alpha_1 - \alpha_0,$$

such that for any $\alpha \in \mathcal{C}_\infty$, the equation (1.3) admits a time quasi-periodic vortex patch solution with diophantine frequency vector $\omega_{pe}(\alpha, \mathbf{a}) \triangleq (\omega_j(\alpha, \mathbf{a}))_{j \in \mathbb{S}} \in \mathbb{R}^{|\mathbb{S}|}$ and taking the form $\omega(t, \cdot) = \mathbf{1}_{D_t}$, with

$$(1.14) \quad D_t = \left\{ \ell e^{i(\theta - \Omega t)}, \quad \theta \in [0, 2\pi], \quad 0 \leq \ell \leq R(t, \theta) \right\}, \quad R(t, \theta) = \sqrt{1 + 2r(t, \theta)},$$

$$r(t, \theta) = \sum_{j \in \mathbb{S}} \mathbf{a}_j \cos(j\theta + \omega_j(\alpha, \mathbf{a})t) + \mathbf{p}(\omega_{pe}(\alpha, \mathbf{a})t, \theta).$$

The diophantine frequency vector satisfies the following asymptotic

$$\omega_{pe}(\alpha, \mathbf{a}) \xrightarrow{\mathbf{a} \rightarrow 0} (-\Omega_j^E(\alpha))_{j \in \mathbb{S}},$$

where $\Omega_j^E(\alpha)$ are the equilibrium frequencies defined by

$$\Omega_j^E(\alpha) \triangleq j \left(\Omega + \Omega_{|j|}^E(\alpha) \right),$$

with $\Omega_{|j|}^{\text{E}}(\alpha)$ as in (1.12). In addition, the perturbation $\mathbf{p} : \mathbb{T}^{|\mathbb{S}|+1} \rightarrow \mathbb{R}$ is an even function satisfying for some large index of regularity s ,

$$\|\mathbf{p}\|_{H^s(\mathbb{T}^{|\mathbb{S}|+1}, \mathbb{R})} \underset{\mathbf{a} \rightarrow 0}{=} o(|\mathbf{a}|).$$

Now we shall briefly mention the key steps of the proof of Theorem 1.2 which are similar to the ones of [41, 43, 52]. The core of the proof of Theorem 1.2 relies on Berti-Bolle theory [11] and [41, Sec. 6]. We mention here some results which have been obtained using this theory [3, 5, 6, 12, 13, 15, 16, 30, 32]. Plugging the ansatz (1.14) into the vortex patch equation (1.8) provides a nonlinear Hamiltonian transport PDE for the radial deformation r in the form

$$(1.15) \quad \partial_t r = \partial_\theta \nabla \mathcal{H}(r),$$

with Hamiltonian \mathcal{H} related to the kinetic energy and the angular momentum. This equation satisfies the reversibility property, namely if $(t, \theta) \mapsto r(t, \theta)$ is a solution, then so is $(t, \theta) \mapsto r(-t, -\theta)$. After a rescaling $r \mapsto \varepsilon r$ the equation (1.15) can be seen as a quasilinear perturbation of its linearization at the equilibrium ($\varepsilon = 0$), see (4.42). As mentioned in Lemmas 4.5 and 4.1, this latter is an integrable Hamiltonian system, namely for any finite set of Fourier modes $\mathbb{S} \subset \mathbb{N}^*$ of cardinal $d \in \mathbb{N}^*$, the linearized equation at the Rankine patch admits, for almost every α in (α_0, α_1) , reversible quasi-periodic solutions with d -dimensional frequency vector $-\omega_{\text{Eq}}(\alpha) \triangleq (-\Omega_j^{\text{E}}(\alpha))_{j \in \mathbb{S}}$ in the form

$$r(t, \theta) = \sum_{j \in \mathbb{S}} r_j \cos(j\theta - \Omega_j^{\text{E}}(\alpha)t), \quad r_j \in \mathbb{R}^*.$$

The property (1.13) for $\omega_{\text{Eq}}(\alpha)$ holds for almost every α by imposing some Diophantine conditions. The measure of the corresponding set is a consequence of the transversality condition in Lemma 4.9-(i) together with the Rüssmann Lemma 4.6. Notice that the transversality condition is itself a consequence of the non-degeneracy of the function $\alpha \mapsto \omega_{\text{Eq}}(\alpha)$ on $[\alpha_0, \alpha_1]$, see Lemma 4.8. The introduction of the Diophantine conditions implies the invertibility of the linearized operator at the equilibrium state but with a fixed loss of derivatives. Therefore, to find quasi-periodic solutions for the nonlinear model, we may apply a Nash-Moser scheme. For this aim we reformulate the problem in terms of embedded tori. This is done by splitting the phase space $L_0^2(\mathbb{T})$ in (4.12) associated with (1.15) into tangential $L_{\mathbb{S}}^2$ and normal L_{\perp}^2 subspaces, see (4.43). Then we introduce in (4.44) the action-angle variables (I, ϑ) by using a symplectic polar change of coordinates for the Fourier coefficients on the tangential subspace. Thus, any function $r \in L^2(\mathbb{T}_{\varphi}^d, L_0^2(\mathbb{T}_{\theta}))$ can be related to an embedded torus, namely

$$r(\varphi, \cdot) = \mathbf{A}(i(\varphi)), \quad \begin{aligned} i : \mathbb{T}^d &\rightarrow \mathbb{T}^d \times \mathbb{R}^d \times L_{\perp}^2 & \mathbf{A} : \mathbb{T}^d \times \mathbb{R}^d \times L_{\perp}^2 &\rightarrow L_0^2(\mathbb{T}). \\ \varphi &\mapsto (\vartheta(\varphi), I(\varphi), z(\varphi)), \end{aligned}$$

Then, the search of reversible quasi-periodic solutions to (1.15) is equivalent to looking for reversible tori solutions to

$$(1.16) \quad \mathcal{F}(i, \kappa, \alpha, \omega, \varepsilon) = 0,$$

where \mathcal{F} is a nonlinear functional whose complete expression can be found in (4.48). This provides a true solution of the original problem for the particular value

$$(1.17) \quad \kappa = -\omega_{\text{Eq}}(\alpha).$$

To find non-trivial solutions of (1.16), we apply a Nash-Moser scheme. At each step, we shall find an approximate right inverse, with nice tame estimates, of the linearized operator $d_{i,\kappa}\mathcal{F}(i_0)$ at the reversible torus i_0 of the current step. Applying the Berti-Bolle theory [11] and [41, Sec. 6], one can conjugate the linearized operator $d_{i,\kappa}\mathcal{F}(i_0)$ by a suitable diffeomorphism of the toroidal phase space $\mathbb{T}^d \times \mathbb{R}^d \times L^2_\perp$ in order to obtain a triangular system in the action-angle-normal variables up to nice error terms. Then to solve the triangular system, it is sufficient to find an almost approximate right inverse for the linearized operator in the normal directions denoted \mathcal{L}_\perp and related to the linearized operator $\mathcal{L}_{\varepsilon_T}$ of (1.15) through the relation (4.53). To do so, we conjugate \mathcal{L}_\perp to a constant coefficients operator up to error terms. This is the content of the Section 4.5. First, in Proposition 4.3 we use a KAM reduction process with quasi-periodic symplectic change of variables to reduce the transport part of $\mathcal{L}_{\varepsilon_T}$. Then, in Proposition 4.4 we look at the localization effects in the normal directions to recover the reduction of the transport part for \mathcal{L}_\perp . This provides a diagonal operator of order 1 plus a remainder of order $(0, -1)$ in the variables (φ, θ) . Finally, in Proposition 4.5 we use a KAM reduction process in the Toeplitz operators class to reduce the remainder term. Each KAM reduction occurs for suitable restrictions of the parameters (α, ω) to a Cantor-like set. In addition, the inversion of the final diagonal operator also requires an extraction of parameters. The Nash-Moser process constructs a non-trivial solution $(\alpha, \omega) \mapsto (i_\infty(\alpha, \omega), \kappa_\infty(\alpha, \omega))$ with reversible torus i_∞ modulo restriction of the parameters to a Cantor set in (α, ω) obtained gathering all the restrictions of all the steps for the construction of the almost approximate right inverses of the linearized operators. Then, coming back to (1.17), we rigidify the frequency ω into $\omega(\alpha, \varepsilon)$ so that

$$(1.18) \quad \kappa_\infty(\alpha, \omega(\alpha, \varepsilon)) = -\omega_{\text{Eq}}(\alpha).$$

We mention that the introduction of the free-parameter κ was required to apply the Berti-Bolle theory along the scheme, more precisely to invert the triangular system. The condition (1.18) gives a final Cantor set in α only that we must check it is not empty. This latter fact is obtained in Proposition 4.8 by estimating the Lebesgue measure of the set through the Rüssmann Lemma 4.6 together with the perturbed transversality conditions.

2. The stream function associated to the filtered velocity

This short section is devoted to the justification of the formula (1.6) which is the fundamental relation for this work. Recall that the identity (1.6) was already observable in the literature, for instance in [7, 66]. The divergence-free property for \mathbf{v} and \mathbf{u} in (1.1) implies the existence of stream functions Ψ and ψ such that

$$\mathbf{v} = \nabla^\perp \Psi, \quad \mathbf{u} = \nabla^\perp \psi$$

and the goal of this section is to find a nice expression for the velocity potential Ψ . Notice that, according to the third equation in (1.1), the stream functions are linked through the relation

$$\psi = (1 - \alpha^2 \Delta) \Psi.$$

Then, (1.2) implies

$$\omega = \Delta \psi = (1 - \alpha^2 \Delta) \Delta \Psi.$$

We deduce from (1.4) and (1.5) that

$$\Psi = \frac{-1}{\alpha^2} \mathbf{G}^E * \mathbf{G}^{\text{SW}}_{\frac{1}{\alpha}} * \omega \triangleq \mathbf{G} * \omega.$$

Now our goal is to compute \mathbf{G} . One can write

$$\begin{aligned} \mathbf{G}(x) &= \frac{1}{4\pi^2\alpha^2} \int_{\mathbb{R}^2} K_0\left(\frac{|y|}{\alpha}\right) \log(|x-y|) dA(y) \\ &= \frac{1}{8\pi^2\alpha^2} \int_{\mathbb{R}^2} K_0\left(\frac{|y|}{\alpha}\right) \log(|x-y|^2) dA(y) \\ &= \frac{1}{8\pi^2\alpha^2} \int_{\mathbb{R}^2} K_0\left(\frac{|y|}{\alpha}\right) \log(|x|^2 + |y|^2 - 2x \cdot y) dA(y), \end{aligned}$$

where dA denotes the planar Lebesgue measure. Writing $x = (R \cos(\theta), R \sin(\theta))$, then a polar change of variables yields

$$\begin{aligned} \mathbf{G}(x) &= \frac{1}{8\pi^2\alpha^2} \int_0^\infty \int_0^{2\pi} r K_0\left(\frac{r}{\alpha}\right) \log(R^2 + r^2 - 2Rr \cos(\theta - \eta)) dr d\eta \\ &= \frac{\log(R)}{2\pi\alpha} \int_0^\infty \frac{r}{\alpha} K_0\left(\frac{r}{\alpha}\right) dr + \frac{1}{8\pi^2\alpha} \int_0^\infty \frac{r}{\alpha} K_0\left(\frac{r}{\alpha}\right) \mathbf{I}_P\left(\frac{r}{R}\right) dr, \end{aligned}$$

where \mathbf{I}_P denotes the Poisson integral defined by

$$\forall x \in \mathbb{R}, \quad \mathbf{I}_P(x) \triangleq \int_{-\pi}^\pi \log(1 + x^2 - 2x \cos(\eta)) d\eta.$$

It is well-known, see for instance [23], that the Poisson integral admits the following explicit formula

$$\begin{cases} \mathbf{I}_P(x) = 0, & \text{if } |x| \leq 1, \\ \mathbf{I}_P(x) = 4\pi \log(|x|), & \text{if } |x| > 1. \end{cases}$$

Therefore,

$$\begin{aligned} \mathbf{G}(x) &= \frac{\log(R)}{2\pi\alpha} \int_0^\infty \frac{r}{\alpha} K_0\left(\frac{r}{\alpha}\right) dr + \frac{1}{2\pi\alpha} \int_R^\infty \frac{r}{\alpha} K_0\left(\frac{r}{\alpha}\right) \log\left(\frac{r}{R}\right) dr \\ &= \frac{\log(R)}{2\pi\alpha} \int_0^R \frac{r}{\alpha} K_0\left(\frac{r}{\alpha}\right) dr + \frac{1}{2\pi\alpha} \int_R^\infty \frac{r}{\alpha} K_0\left(\frac{r}{\alpha}\right) \log(r) dr. \end{aligned}$$

A change of variables together with (A.3), (A.5) and (A.6) give

$$\frac{\log(R)}{2\pi\alpha} \int_0^R \frac{r}{\alpha} K_0\left(\frac{r}{\alpha}\right) dr = \frac{\log(R)}{2\pi} \int_0^{\frac{R}{\alpha}} u K_0(u) du = \frac{\log(R)}{2\pi} \left(1 - \frac{R}{\alpha} K_1\left(\frac{R}{\alpha}\right)\right)$$

and

$$\begin{aligned} \frac{1}{2\pi\alpha} \int_R^\infty \frac{r}{\alpha} K_0\left(\frac{r}{\alpha}\right) \log(r) dr &= \frac{\log(\alpha)}{2\pi} \int_{\frac{R}{\alpha}}^\infty u K_0(u) du + \frac{1}{2\pi} \int_{\frac{R}{\alpha}}^\infty u K_0(u) \log(u) dr \\ &= \frac{R \log(\alpha)}{2\pi\alpha} K_1\left(\frac{R}{\alpha}\right) + \frac{1}{2\pi} \int_{\frac{R}{\alpha}}^\infty u K_0(u) \log(u) du. \end{aligned}$$

An integration by parts in the last integral together with (A.6) lead to

$$\begin{aligned} \frac{1}{2\pi} \int_{\frac{R}{\alpha}}^\infty u K_0(u) \log(u) du &= \frac{R}{2\pi\alpha} K_1\left(\frac{R}{\alpha}\right) \log\left(\frac{R}{\alpha}\right) + \frac{1}{2\pi} \int_{\frac{R}{\alpha}}^\infty K_1(u) du \\ &= \frac{R}{2\pi\alpha} K_1\left(\frac{R}{\alpha}\right) \log(R) - \frac{R \log(\alpha)}{2\pi\alpha} K_1\left(\frac{R}{\alpha}\right) + \frac{1}{2\pi} K_0\left(\frac{R}{\alpha}\right). \end{aligned}$$

Gathering the foregoing computations and reminding that $R = |x|$ we obtain

$$(2.1) \quad \mathbf{G}(x) = \frac{1}{2\pi} \log(|x|) + \frac{1}{2\pi} K_0\left(\frac{|x|}{\alpha}\right) = \mathbf{G}^E(x) - \mathbf{G}^{\text{SW}}_{\frac{1}{\alpha}}(x).$$

Notice that we recover a radial function as a convolution of two radial functions. In particular, any radial profile gives a stationary solution for the active scalar equation (1.3) satisfied by ω .

3. Periodic rigid motion

This section is devoted to the proof of the Theorem 1.1 which is an application of the Crandall-Rabinowitz's Theorem B.1. This latter is applied to a reformulation of the vortex patch equation (1.8) in the uniformly rotating framework.

3.1. Function spaces and reformulation of the vortex patch equation.

The goal of this subsection is to set up the contour dynamics equation for the Euler- α V-states near the Rankine vortices. Consider an ansatz (1.9) with simply-connected domain D_0 and rotating uniformly with some angular velocity Ω . At time $t \geq 0$, the boundary ∂D_t can be parametrized by

$$(3.1) \quad z(t, \theta) = e^{i\Omega t} z(0, \theta), \quad \theta \in [0, 2\pi], \quad z(0, 0) = z(0, 2\pi).$$

With this parametrization, the left hand side of (1.8) becomes

$$(3.2) \quad \text{Im} \left(\partial_t z(t, \theta) \overline{\partial_\theta z(t, \theta)} \right) = \Omega \text{Re} \left(z(0, \theta) \overline{\partial_\theta z(0, \theta)} \right).$$

As for the computation of the right hand-side of (1.8), it is obtained from the Biot-Savart law. Note that the velocity potential Ψ given by (1.6) writes in this context

$$\Psi(t, z) = \frac{1}{2\pi} \int_{D_t} \log(|z - \xi|) dA(\xi) + \frac{1}{2\pi} \int_{D_t} K_0 \left(\frac{1}{\alpha} |z - \xi| \right) dA(\xi).$$

Notice that the real notation $x \in \mathbb{R}^2$ has been replaced by the complex notation $z \in \mathbb{C}$. To get the Biot-Savart law we shall use Stokes' Theorem in complex notation

$$2i \int_D \partial_{\bar{\xi}} f(\xi, \bar{\xi}) dA(\xi) = \int_{\partial D} f(\xi, \bar{\xi}) d\xi.$$

Hence, by making the identification $2i\partial_{\bar{z}} = \nabla^\perp$ leading to $\mathbf{v}(t, z) = 2i\partial_{\bar{z}}\Psi(t, z)$ one deduces, after a regularization procedure similar to the one explained in the proof of [43, Lem. 2.1], that

$$(3.3) \quad \mathbf{v}(t, z) = -\frac{1}{2\pi} \int_{\partial D_t} \log(|z - \xi|) d\xi - \frac{1}{2\pi} \int_{\partial D_t} K_0 \left(\frac{1}{\alpha} |z - \xi| \right) d\xi.$$

One easily obtains from (3.1), (3.3) and change of variables,

$$(3.4) \quad \mathbf{v}(t, z(t, \theta)) = e^{i\Omega t} \mathbf{v}(0, z(0, \theta)).$$

Thus, combining (3.4) and (3.1), we obtain

$$(3.5) \quad \text{Im} \left(\mathbf{v}(t, z(t, \theta)) \overline{\partial_\theta z(t, \theta)} \right) = \text{Im} \left(\mathbf{v}(0, z(0, \theta)) \overline{\partial_\theta z(0, \theta)} \right).$$

Gathering (3.2) and (3.5), the equation (1.8) becomes

$$(3.6) \quad \Omega \text{Re} \left(z(0, \theta) \overline{\partial_\theta z(0, \theta)} \right) = \text{Im} \left(\mathbf{v}(0, z(0, \theta)) \overline{\partial_\theta z(0, \theta)} \right).$$

Therefore, denoting z' a tangent vector to the boundary ∂D_0 at the point z , one gets from (3.3) and (3.6) that for any $z \in \partial D_0$,

$$(3.7) \quad \Omega \text{Re} (zz') + \text{Im} \left(\left[\frac{1}{2\pi} \int_{\partial D_0} \log(|z - \xi|) d\xi + \frac{1}{2\pi} \int_{\partial D_0} K_0 \left(\frac{1}{\alpha} |z - \zeta| \right) d\zeta \right] \overline{z'} \right) = 0.$$

In accordance with the previous works in the field beginning with the one of Burbea [17], we should rewrite the equation (3.7) by using conformal mappings. For that purpose we shall now present the function spaces used throughout this first part on periodic solutions. Notice that we shall identify 2π -periodic functions $g : \mathbb{R} \rightarrow \mathbb{C}$ with functions $f : \mathbb{T} \rightarrow \mathbb{C}$ defined on the torus $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z} \cong \mathbb{U}$ through the relation

$$f(w) = g(\theta), \quad w = e^{i\theta}.$$

In this part, we shall consider the following notation for the mean value line integral of any continuous function f defined on the torus \mathbb{T}

$$\oint_{\mathbb{T}} f(\tau) d\tau \triangleq \frac{1}{2i\pi} \int_{\mathbb{T}} f(\tau) d\tau \triangleq \frac{1}{2\pi} \int_0^{2\pi} f(e^{i\theta}) e^{i\theta} d\theta.$$

Now, we introduce the Hölder spaces on the unit circle. Given $\beta \in (0, 1)$, we denote by $C^\beta(\mathbb{T})$ the space of continuous functions f such that

$$\|f\|_{C^\beta(\mathbb{T})} \triangleq \|f\|_{L^\infty(\mathbb{T})} + \sup_{\substack{(\tau, w) \in \mathbb{T}^2 \\ \tau \neq w}} \frac{|f(\tau) - f(w)|}{|\tau - w|^\beta} < \infty$$

and we denote by $C^{1+\beta}(\mathbb{T})$ the space of C^1 functions with β -Hölder continuous derivative such that

$$\|f\|_{C^{1+\beta}(\mathbb{T})} \triangleq \|f\|_{L^\infty(\mathbb{T})} + \left\| \frac{df}{dw} \right\|_{C^\beta(\mathbb{T})} < \infty.$$

For $\beta \in (0, 1)$, we set

$$\begin{aligned} X^{1+\beta} &\triangleq \left\{ f \in C^{1+\beta}(\mathbb{T}) \quad \text{s.t.} \quad \forall w \in \mathbb{T}, f(w) = \sum_{n=0}^{\infty} f_n \bar{w}^n, f_n \in \mathbb{R} \right\}, \\ Y^\beta &\triangleq \left\{ g \in C^\beta(\mathbb{T}) \quad \text{s.t.} \quad \forall w \in \mathbb{T}, g(w) = \sum_{n=1}^{\infty} g_n e_n(w), g_n \in \mathbb{R} \right\}, \\ e_n(w) &\triangleq \text{Im}(w^n). \end{aligned}$$

Remark that the \mathbf{m} -fold symmetry property can be translated in the functional spaces as follows

$$\begin{aligned} X_{\mathbf{m}}^{1+\beta} &\triangleq \left\{ f \in X^{1+\beta} \quad \text{s.t.} \quad \forall w \in \mathbb{T}, f(w) = \sum_{n=1}^{\infty} f_{n\mathbf{m}-1} \bar{w}^{n\mathbf{m}-1} \right\}, \\ Y_{\mathbf{m}}^\beta &\triangleq \left\{ g \in Y^\beta \quad \text{s.t.} \quad \forall w \in \mathbb{T}, g(w) = \sum_{n=1}^{\infty} g_{n\mathbf{m}} e_{n\mathbf{m}}(w) \right\}. \end{aligned}$$

We shall also consider the following balls of radius $r > 0$ in $X^{1+\beta}$ and $X_{\mathbf{m}}^{1+\beta}$, respectively

$$B_r^{1+\beta} \triangleq \left\{ f \in X^{1+\beta} \quad \text{s.t.} \quad \|f\|_{C^{1+\beta}(\mathbb{T})} < r \right\}, \quad B_{r, \mathbf{m}}^{1+\beta} \triangleq B_r^{1+\beta} \cap X_{\mathbf{m}}^{1+\beta}.$$

Based on the Riemann mapping Theorem, we may parametrize the boundary of D_0 by considering the conformal mapping $\Phi : \mathbb{C} \setminus \mathbb{D} \rightarrow \mathbb{C} \setminus \overline{D_0}$ given by

$$\Phi(z) \triangleq z + f(z), \quad f(z) = \sum_{n=0}^{\infty} \frac{a_n}{z^n}, \quad a_n \in \mathbb{R}.$$

Here, f is in $B_r^{1+\beta}$ for small r . We have $\Phi(\mathbb{T}) = \partial D_0$ and we mention that the link between the regularity of the mapping and of the boundary is given by Kellogg-Warschawski's Theorem [72, 67]. For $w \in \mathbb{T}$, a tangent vector to the boundary ∂D_0 at the point $z = \Phi(w)$ is given by

$$\overline{z'} = -i\overline{w}\overline{\Phi'(w)}.$$

Plugging this into (3.7) and using the change of variables $\xi = \Phi(\tau)$ give

$$\forall w \in \mathbb{T}, \quad F_\alpha(\Omega, f)(w) = 0,$$

with

$$(3.8) \quad \begin{aligned} F_\alpha(\Omega, f)(w) &\triangleq \operatorname{Im} \left\{ \left(\Omega\Phi(w) + I^E(f)(w) + I^{SW}(f)(\alpha, w) \right) \overline{w\Phi'(w)} \right\}, \\ I^E(f)(w) &\triangleq \int_{\mathbb{T}} \Phi'(\tau) \log(|\Phi(w) - \Phi(\tau)|) d\tau, \\ I^{SW}(f)(\alpha, w) &\triangleq \int_{\mathbb{T}} \Phi'(\tau) K_0\left(\frac{1}{\alpha}|\Phi(w) - \Phi(\tau)|\right) d\tau. \end{aligned}$$

Observe that the functional F_α makes appear a term I^E associated with the Euler dynamics and a term I^{SW} corresponding to the QGSW equations.

3.2. Regularity aspects and structure of the linearized operator. Our next task is to study some regularity properties for the functional F_α defined in (3.8), look for the structure of its linearized operator at the Rankine vortex and check some monotonicity property for its spectrum. We first remark that the linearized operator at the equilibrium state acts as a Fourier multiplier according to the functions spaces introduced in Section 3.1. More precisely, we have the following result.

PROPOSITION 3.1. *Let $\alpha > 0$. Then the following properties hold true.*

- (1) *There exists $r > 0$ such that for any $\beta \in (0, 1)$, the following hold true.*
 - (i) $F_\alpha : \mathbb{R} \times B_r^{1+\beta} \rightarrow Y^\beta$ *is well-defined and of class C^1 .*
 - (ii) *For any $\mathbf{m} \in \mathbb{N}^*$, the restriction $F_\alpha : \mathbb{R} \times B_{r, \mathbf{m}}^{1+\beta} \rightarrow Y_{\mathbf{m}}^\beta$ is well-defined.*
 - (iii) *The partial derivative $\partial_\Omega d_f F_\alpha : \mathbb{R} \times B_r^{1+\beta} \rightarrow \mathcal{L}(X^{1+\beta}, Y^\beta)$ exists and is continuous.*
 - (iv) *For any $\Omega \in \mathbb{R}$, one has $F_\alpha(\Omega, 0) = 0$.*
- (2) *Let $\Omega \in \mathbb{R} \setminus \left\{ \frac{1}{2} - I_1\left(\frac{1}{\alpha}\right) K_1\left(\frac{1}{\alpha}\right) \right\}$. Then the operator $d_f F_\alpha(\Omega, 0) : X^{1+\beta} \rightarrow Y^\beta$ is Fredholm with index 0.*

In addition, for $h \in X^{1+\beta}$ writing

$$\forall w \in \mathbb{T}, \quad h(w) = \sum_{n=0}^{\infty} a_n \overline{w}^n, \quad a_n \in \mathbb{R},$$

we have for any $w \in \mathbb{T}$,

$$(3.9) \quad d_f F_\alpha(\Omega, 0)(w) = \sum_{n=0}^{\infty} a_n (n+1) \left(\frac{n}{2(n+1)} - \Omega_{n+1}^{SW}\left(\frac{1}{\alpha}\right) - \Omega \right) e_{n+1}(w),$$

with Ω_n^{SW} as in (1.11).

PROOF. **1. (i)** The proof of the regularity is now classical. We refer the reader to [50] for the computations associated with the Euler part and to [27] for the

computations associated with the QGSW part.

(ii) For $f \in B_{r,m}^{1+\beta}$, the following identities hold

$$\forall w \in \mathbb{T}, \quad \Phi\left(e^{\frac{2i\pi}{m}} w\right) = e^{\frac{2i\pi}{m}} \Phi(w), \quad \Phi'\left(e^{\frac{2i\pi}{m}} w\right) = \Phi'(w).$$

Therefore, the change of variables $\tau \mapsto e^{\frac{2i\pi}{m}} \tau$ implies

$$I^E(f)\left(e^{\frac{2i\pi}{m}} w\right) = e^{\frac{2i\pi}{m}} I^E(f)(w), \quad I^{SW}(f)\left(\alpha, e^{\frac{2i\pi}{m}} w\right) = e^{\frac{2i\pi}{m}} I^{SW}(f)(\alpha, w).$$

Consequently,

$$\forall w \in \mathbb{T}, \quad F_\alpha(\Omega, f)\left(e^{\frac{2i\pi}{m}} w\right) = F_\alpha(\Omega, f)(w).$$

This symmetry property proves the desired result.

(iii) One readily has

$$\partial_\Omega d_f F_\alpha(\Omega, f)[h](w) = \text{Im} \left\{ \overline{h'(w)} \overline{w} \Phi(w) + h(w) \overline{w} \overline{\Phi'(w)} \right\}.$$

Hence, for $(f, g) \in (B_r^{1+\alpha})^2$ and $h \in C^{1+\alpha}(\mathbb{T})$, we get

$$\left\| \partial_\Omega d_f F_\alpha(\Omega, f)[h] - \partial_\Omega d_f F_\alpha(\Omega, g)[h] \right\|_{C^\alpha(\mathbb{T})} \lesssim \|f - g\|_{C^{1+\alpha}(\mathbb{T})} \|h\|_{C^{1+\alpha}(\mathbb{T})}.$$

This proves the continuity of $\partial_\Omega d_f F_\alpha : \mathbb{R} \times B_r^{1+\alpha} \rightarrow \mathcal{L}(X^{1+\alpha}, Y^\alpha)$.

(iv) For $w \in \mathbb{T}$, using the change of variable $\tau \mapsto w\tau$, the fact that $|w| = 1$ and the definition of the line integral, we get

$$\begin{aligned} I^E(0)(w) &= \oint_{\mathbb{T}} \log(|w - \tau|) d\tau \\ &= w \oint_{\mathbb{T}} \log(|1 - \tau|) d\tau \\ &= \frac{w}{2\pi} \int_0^{2\pi} \log(|1 - e^{i\theta}|) e^{i\theta} d\theta. \end{aligned}$$

Now remark that

$$|1 - e^{i\theta}|^2 = 2(1 - \cos(\theta)) = 4 \sin^2\left(\frac{\theta}{2}\right).$$

In particular, this latter quantity is even in θ . Hence, we infer

$$(3.10) \quad I^E(0)(w) = \frac{w}{4\pi} \int_0^{2\pi} \log\left(\sin^2\left(\frac{\theta}{2}\right)\right) \cos(\theta) d\theta = -\frac{w}{2}.$$

The last identity is a consequence of the following formula which can be found for instance in [22, Lem. A.3].

$$(3.11) \quad \frac{1}{2\pi} \int_0^{2\pi} \log\left(\sin^2\left(\frac{\theta}{2}\right)\right) \cos(n\theta) d\theta = -\frac{1}{n}.$$

Similarly,

$$(3.12) \quad I^{SW}(0)(w) = \frac{w}{2\pi} \int_0^{2\pi} K_0\left(\frac{2}{\alpha} \left|\sin\left(\frac{\theta}{2}\right)\right|\right) \cos(\theta) d\theta = w I_1\left(\frac{1}{\alpha}\right) K_1\left(\frac{1}{\alpha}\right).$$

The last identity follows from the following identity which can be found in the proof of [52, Lem. 3.2].

$$(3.13) \quad \frac{1}{2\pi} \int_0^{2\pi} K_0\left(\frac{2}{\alpha} \sin\left(\frac{\theta}{2}\right)\right) \cos(n\theta) d\theta = I_n\left(\frac{1}{\alpha}\right) K_n\left(\frac{1}{\alpha}\right).$$

Inserting (3.10) and (3.12) into (3.8) and using $w\bar{w} = |w|^2 = 1$ gives the desired result.

2. We can write

$$d_f F_\alpha(\Omega, 0)[h](w) = \mathcal{I}[h](w) + \mathcal{K}[h](w),$$

with

$$\begin{aligned} \mathcal{I}[h](w) &\triangleq \left(\Omega - \frac{1}{2} + I_1\left(\frac{1}{\alpha}\right) K_1\left(\frac{1}{\alpha}\right)\right) \operatorname{Im} \left\{ \overline{h'(w)} \right\}, \\ \mathcal{K}[h](w) &\triangleq \operatorname{Im} \left\{ \left(\Omega h(w) + d_f I^E(0)[h](w) + d_f I^{SW}(0)[h](w)\right) \bar{w} \right\}. \end{aligned}$$

One obviously has that $\mathcal{I} : X^{1+\beta} \rightarrow Y^\beta$ is an isomorphism provided that $\Omega \neq \frac{1}{2} - I_1\left(\frac{1}{\alpha}\right) K_1\left(\frac{1}{\alpha}\right)$. One readily has

$$d_f I^E(0)[h](w) = \int_{\mathbb{T}} h'(\tau) \log(|w - \tau|) d\tau + \int_{\mathbb{T}} \frac{h(w) - h(\tau)}{2(w - \tau)} d\tau + \int_{\mathbb{T}} \frac{\overline{h(w)} - \overline{h(\tau)}}{2(\bar{w} - \bar{\tau})} d\tau$$

and

$$\begin{aligned} d_f I^{SW}(0)[h](w) &= \int_{\mathbb{T}} h'(\tau) K_0\left(\frac{1}{\alpha}|w - \tau|\right) d\tau \\ &\quad + \int_{\mathbb{T}} K'_0\left(\frac{1}{\alpha}|w - \tau|\right) \frac{(h(w) - h(\tau))(\bar{w} - \bar{\tau})}{2\alpha|w - \tau|} d\tau \\ &\quad + \int_{\mathbb{T}} K'_0\left(\frac{1}{\alpha}|w - \tau|\right) \frac{(\overline{h(w)} - \overline{h(\tau)})(w - \tau)}{2\alpha|w - \tau|} d\tau. \end{aligned}$$

Now, combining (A.2) and (A.4), we can write

$$K_0(z) = -\log(z) + \mathbf{F}_1(z), \quad K'_0(z) = -K_1(z) = -\frac{1}{z} + \mathbf{F}_2(z),$$

with $\mathbf{F}_1, \mathbf{F}'_1, \mathbf{F}_2, \mathbf{F}'_2$ bounded at 0. Gathering the foregoing computations leads to

$$\begin{aligned} &d_f I^E(0)[h](w) + d_f I^{SW}(0)[h](w) \\ &= \int_{\mathbb{T}} h'(\tau) \mathbf{F}_1\left(\frac{1}{\alpha}|w - \tau|\right) d\tau + \int_{\mathbb{T}} \mathbf{F}_2\left(\frac{1}{\alpha}|w - \tau|\right) \frac{(h(w) - h(\tau))(\bar{w} - \bar{\tau})}{2\alpha|w - \tau|} d\tau \\ &\quad + \int_{\mathbb{T}} \mathbf{F}_2\left(\frac{1}{\alpha}|w - \tau|\right) \frac{(\overline{h(w)} - \overline{h(\tau)})(w - \tau)}{2\alpha|w - \tau|} d\tau. \end{aligned}$$

This can be written in the form

$$d_f I^E(0)[h](w) + d_f I^{SW}(0)[h](w) = \mathcal{T}_{\mathbf{K}_1}(h')(w) + \mathcal{T}_{\mathbf{K}_2}(1)(w) + \overline{\mathcal{T}_{\mathbf{K}_2}}(1)(w),$$

with

$$\begin{aligned} \mathcal{T}_{\mathbf{K}}(u)(w) &\triangleq \int_{\mathbb{T}} u(\tau) \mathbf{K}(w, \tau) d\tau, \\ \mathbf{K}_1(w, \tau) &\triangleq \mathbf{F}_1\left(\frac{1}{\alpha}|w - \tau|\right), \\ \mathbf{K}_2(w, \tau) &\triangleq \mathbf{F}_2\left(\frac{1}{\alpha}|w - \tau|\right) \frac{(h(w) - h(\tau))(\bar{w} - \bar{\tau})}{2\alpha|w - \tau|} d\tau. \end{aligned}$$

Using the boundedness of $\mathbf{F}_1, \mathbf{F}'_1, \mathbf{F}_2$ and \mathbf{F}'_2 close to zero, we get

$$\begin{aligned} |\mathbf{K}_1(w, \tau)| &\leq C, & |\partial_w \mathbf{K}_1(w, \tau)| &\leq \frac{C}{|w - \tau|}, \\ |\mathbf{K}_2(w, \tau)| &\leq C \|h\|_{C^{1+\beta}(\mathbb{T})}, & |\partial_w \mathbf{K}_2(w, \tau)| &\leq \frac{C \|h\|_{C^{1+\beta}(\mathbb{T})}}{|w - \tau|}. \end{aligned}$$

Therefore, applying Lemma B.1, one obtains the continuity of $d_f I^E(0) + d_f I^{SW}(0) : C^{1+\beta}(\mathbb{T}) \rightarrow C^\delta(\mathbb{T})$ for any $\beta \leq \delta < 1$. Coming back to the definition of \mathcal{K} , we deduce that for any $\beta \leq \delta < 1$, the operator $\mathcal{K} : X^{1+\beta} \rightarrow Y^\delta$ is continuous (the symmetry property being easily obtained by straightforward calculations and changes of variables in the integrals). Hence, using the compact embedding of $C^\delta(\mathbb{T})$ into $C^\beta(\mathbb{T})$ for $\beta < \delta < 1$, one deduces that the operator $\mathcal{K} : X^{1+\beta} \rightarrow Y^\beta$ is compact. Consequently, applying [24, Cor. 5.9], $d_f F_\alpha(\Omega, 0) : X^{1+\beta} \rightarrow Y^\beta$ is a Fredholm operator with index 0. Now, we shall compute the Fourier representation of this operator. Fix

$$h(w) = \sum_{n=0}^{\infty} a_n \bar{w}^n \in X^{1+\beta}.$$

First observe that

$$(3.14) \quad \mathcal{I}[h](w) = -\left(\Omega - \frac{1}{2} + I_1\left(\frac{1}{\alpha}\right) K_1\left(\frac{1}{\alpha}\right)\right) \sum_{n=0}^{\infty} n a_n e_{n+1}(w)$$

and

$$(3.15) \quad \text{Im} \{ \Omega h(w) \bar{w} \} = -\Omega \sum_{n=0}^{\infty} a_n e_{n+1}(w).$$

The next task is to compute $d_f I^E(0)$. One has

$$\begin{aligned} \int_{\mathbb{T}} h'(\tau) \log(|w - \tau|) d\tau &= w \int_{\mathbb{T}} h'(w\tau) \log(|1 - \tau|) d\tau \\ &= -\sum_{n=1}^{\infty} n a_n \bar{w}^n \int_{\mathbb{T}} \bar{\tau}^{n+1} \log(|1 - \tau|) d\tau. \end{aligned}$$

The last integral can be computed as follows by using (3.11)

$$\begin{aligned} \int_{\mathbb{T}} \bar{\tau}^{n+1} \log(|1 - \tau|) d\tau &= \frac{1}{2\pi} \int_0^{2\pi} e^{-in\theta} \log(|1 - e^{i\theta}|) d\theta \\ &= \frac{1}{4\pi} \int_0^{2\pi} \log\left(\sin^2\left(\frac{\theta}{2}\right)\right) \cos(n\theta) d\theta \\ &= -\frac{1}{2n}. \end{aligned}$$

Hence,

$$\int_{\mathbb{T}} h'(\tau) \log(|w - \tau|) d\tau = \sum_{n=1}^{\infty} \frac{a_n}{2} \bar{w}^n.$$

Similarly, we obtain

$$\begin{aligned} \int_{\mathbb{T}} \frac{h(w) - h(\tau)}{2(w - \tau)} d\tau &= \sum_{n=0}^{\infty} \frac{a_n}{2} \int_{\mathbb{T}} \frac{\bar{w}^n - \bar{\tau}^n}{w - \tau} d\tau = -\sum_{n=1}^{\infty} \sum_{k=0}^{n-1} \frac{a_n \bar{w}^{k+1}}{2} \int_{\mathbb{T}} \bar{\tau}^{n-k} d\tau \\ &= -\sum_{n=1}^{\infty} \frac{a_n}{2} \bar{w}^n, \\ \int_{\mathbb{T}} \frac{\overline{h(w)} - \overline{h(\tau)}}{2(\bar{w} - \bar{\tau})} d\tau &= \sum_{n=0}^{\infty} \frac{a_n}{2} \int_{\mathbb{T}} \frac{w^n - \tau^n}{\bar{w} - \bar{\tau}} d\tau = -\sum_{n=1}^{\infty} \sum_{k=0}^{n-1} \frac{a_n w^{k+1}}{2} \int_{\mathbb{T}} \tau^{n-k} d\tau = 0. \end{aligned}$$

Consequently,

$$(3.16) \quad d_f I^E(0) = 0.$$

Now, let us turn to the calculation of $\text{Im} \{d_f I^{\text{SW}}(0)[h](w)\bar{w}\}$. The formula (3.13) gives

$$\begin{aligned} \int_{\mathbb{T}} h'(\tau) K_0\left(\frac{1}{\alpha}|w - \tau|\right) d\tau &= - \sum_{n=1}^{\infty} n a_n \bar{w}^n \int_{\mathbb{T}} \bar{\tau}^{n+1} K_0\left(\frac{1}{\alpha}|1 - \tau|\right) d\tau \\ &= - \sum_{n=1}^{\infty} \frac{n a_n \bar{w}^n}{2\pi} \int_0^{2\pi} K_0\left(\frac{2}{\alpha} \sin\left(\frac{\theta}{2}\right)\right) \cos(n\theta) d\theta \\ &= - \sum_{n=1}^{\infty} n I_n\left(\frac{1}{\alpha}\right) K_n\left(\frac{1}{\alpha}\right) a_n \bar{w}^n. \end{aligned}$$

Thus

$$\text{Im} \left\{ \bar{w} \int_{\mathbb{T}} h'(\tau) K_0\left(\frac{1}{\alpha}|w - \tau|\right) d\tau \right\} = \sum_{n=1}^{\infty} n I_n\left(\frac{1}{\alpha}\right) K_n\left(\frac{1}{\alpha}\right) a_n e_{n+1}(w).$$

On the other hand

$$\begin{aligned} &\int_{\mathbb{T}} K'_0\left(\frac{1}{\alpha}|w - \tau|\right) \frac{(h(w) - h(\tau))(\bar{w} - \bar{\tau}) + (\overline{h(w) - h(\tau)})(w - \tau)}{2\alpha|w - \tau|} d\tau \\ &= \sum_{n=0}^{\infty} a_n \left(\int_{\mathbb{T}} \bar{w}^n K'_0\left(\frac{1}{\alpha}|1 - \tau|\right) \frac{(\bar{\tau}^n - 1)(\bar{\tau} - 1)}{2\alpha|1 - \tau|} d\tau \right. \\ &\quad \left. + w^{n+2} \int_{\mathbb{T}} K'_0\left(\frac{1}{\alpha}|1 - \tau|\right) \frac{(\tau^{n+1} - 1)(\tau - 1)}{2\alpha|1 - \tau|} d\tau \right). \end{aligned}$$

One can easily check that the above integrals are real and therefore

$$\begin{aligned} &\text{Im} \left\{ \bar{w} \int_{\mathbb{T}} K'_0\left(\frac{1}{\alpha}|w - \tau|\right) \frac{(h(w) - h(\tau))(\bar{w} - \bar{\tau}) + (\overline{h(w) - h(\tau)})(w - \tau)}{2\alpha|w - \tau|} d\tau \right\} \\ &= \sum_{n=0}^{\infty} a_n \left(\int_{\mathbb{T}} K'_0\left(\frac{1}{\alpha}|1 - \tau|\right) \frac{(\tau^{n+1} - 1)(\tau - 1) - (\bar{\tau}^{n+1} - 1)(\bar{\tau} - 1)}{2\alpha|1 - \tau|} d\tau \right) e_{n+1}(w) \\ &= \sum_{n=0}^{\infty} a_n \left(\int_{\mathbb{T}} K'_0\left(\frac{1}{\alpha}|1 - \tau|\right) \frac{(\tau^{n+1} - \bar{\tau}^{n+1}) - (\tau^n - \bar{\tau}^n) - (\tau - \bar{\tau})}{2\alpha|1 - \tau|} d\tau \right) e_{n+1}(w). \end{aligned}$$

Now, symmetry arguments together with an integration by parts and (3.13) imply for any $k \in \mathbb{N}^*$

$$\begin{aligned} \int_{\mathbb{T}} K'_0\left(\frac{1}{\alpha}|1 - \tau|\right) \frac{\tau^k - \bar{\tau}^k}{2\alpha|1 - \tau|} d\tau &= \frac{-1}{4\pi\alpha} \int_0^{2\pi} K'_0\left(\frac{2}{\alpha} \left| \sin\left(\frac{\theta}{2}\right) \right| \right) \frac{\sin(k\theta) \sin(\theta)}{\left| \sin\left(\frac{\theta}{2}\right) \right|} d\theta \\ &= \frac{-1}{2\pi\alpha} \int_0^{2\pi} K'_0\left(\frac{2}{\alpha} \sin\left(\frac{\theta}{2}\right)\right) \sin(k\theta) \cos\left(\frac{\theta}{2}\right) d\theta \\ &= \frac{k}{2\pi} \int_0^{2\pi} K_0\left(\frac{2}{\alpha} \sin\left(\frac{\theta}{2}\right)\right) \cos(k\theta) d\theta \\ &= k I_k\left(\frac{1}{\alpha}\right) K_k\left(\frac{1}{\alpha}\right). \end{aligned}$$

Combining the foregoing computations leads to

$$(3.17) \quad \begin{aligned} & \operatorname{Im} \left\{ d_f I^{\text{SW}}(0)[h](w)\overline{w} \right\} \\ &= \sum_{n=0}^{\infty} a_n \left((n+1)I_{n+1} \left(\frac{1}{\alpha} \right) K_{n+1} \left(\frac{1}{\alpha} \right) - I_1 \left(\frac{1}{\alpha} \right) K_1 \left(\frac{1}{\alpha} \right) \right) e_{n+1}(w). \end{aligned}$$

Putting together (3.14), (3.15), (3.16) and (3.17) gives the desired result. \square

According to Proposition 3.1 the possible values for Ω from which we can hope to bifurcate are

$$(3.18) \quad \Omega_n^E(\alpha) \triangleq \frac{n-1}{2n} - \left[I_1 \left(\frac{1}{\alpha} \right) K_1 \left(\frac{1}{\alpha} \right) - I_n \left(\frac{1}{\alpha} \right) K_n \left(\frac{1}{\alpha} \right) \right] = \Omega_n^E - \Omega_n^{\text{SW}} \left(\frac{1}{\alpha} \right),$$

for $n \in \mathbb{N}^*$. We shall now prove the monotonicity of the sequence $(\Omega_n^E(\alpha))_{n \in \mathbb{N}^*}$. This is given by the following result.

LEMMA 3.1. *For any $\alpha > 0$, the sequence $(\Omega_n^E(\alpha))_{n \in \mathbb{N}^*}$ is strictly increasing and tends to $\Omega_\infty^E(\alpha)$, with*

$$\Omega_\infty^E(\alpha) \triangleq \frac{1}{2} - I_1 \left(\frac{1}{\alpha} \right) K_1 \left(\frac{1}{\alpha} \right).$$

Moreover,

$$\forall n \in \mathbb{N}^*, \quad \Omega_n^E(\alpha) \xrightarrow{\alpha \rightarrow 0} \frac{n-1}{2n}.$$

PROOF. The convergences are immediate consequences of (A.7) and (A.6). We shall now study the monotonicity. For this aim, we consider for any fixed $\alpha > 0$, the smooth function f_α defined on $(0, \infty)$ by

$$\forall \nu > 0, \quad f_\alpha(\nu) \triangleq \Omega_\nu^E(\alpha) \triangleq \frac{1}{2} - \frac{1}{2\nu} - I_1 \left(\frac{1}{\alpha} \right) K_1 \left(\frac{1}{\alpha} \right) + I_\nu \left(\frac{1}{\alpha} \right) K_\nu \left(\frac{1}{\alpha} \right).$$

According to (A.8), we can write

$$I_\nu \left(\frac{1}{\alpha} \right) K_\nu \left(\frac{1}{\alpha} \right) = \frac{1}{2} \int_0^\infty J_0 \left(\frac{2}{\alpha} \sinh \left(\frac{t}{2} \right) \right) e^{-\nu t} dt.$$

Differentiating with respect to ν , we get

$$\partial_\nu \left(I_\nu \left(\frac{1}{\alpha} \right) K_\nu \left(\frac{1}{\alpha} \right) \right) = -\frac{1}{2} \int_0^\infty t J_0 \left(\frac{2}{\alpha} \sinh \left(\frac{t}{2} \right) \right) e^{-\nu t} dt.$$

The integral representation (A.9) gives

$$\forall x > 0, \quad |J_0(x)| = \frac{1}{\pi} \left| \int_0^\pi \cos(x \sin(\theta)) d\theta \right| \leq 1.$$

Since the function J_0 is continuous and non-constant on $(0, \infty)$, we deduce that

$$\left| \partial_\nu \left(I_\nu \left(\frac{1}{\alpha} \right) K_\nu \left(\frac{1}{\alpha} \right) \right) \right| < \frac{1}{2} \int_0^\infty t e^{-\nu t} dt = \frac{1}{2\nu^2}.$$

Therefore

$$\partial_\nu f_\alpha(\nu) = \frac{1}{2\nu^2} + \partial_\nu \left(I_\nu \left(\frac{1}{\alpha} \right) K_\nu \left(\frac{1}{\alpha} \right) \right) > 0.$$

This achieves the proof of Lemma 3.1. \square

3.3. Construction of local bifurcation branches. We check here the hypothesis of Crandall-Rabinowitz’s Theorem B.1 which immediately imply Theorem 1.1. Notice that the line of trivial solutions has already been obtained in Lemma 3.1-1-(iv).

PROPOSITION 3.2. *Let $\alpha > 0$, $\beta \in (0, 1)$ and $\mathbf{m} \in \mathbb{N}^*$. The following assertions hold true.*

- (i) $F_\alpha : \mathbb{R} \times X_{\mathbf{m}}^{1+\beta} \rightarrow Y_{\mathbf{m}}^\beta$ is well-defined and of class C^1 .
- (ii) The kernel $\ker \left(d_f F_\alpha(\Omega_{\mathbf{m}}^E(\alpha), 0) \right)$ is one dimensional and generated by

$$v_{0,\mathbf{m}} : \begin{array}{l} \mathbb{T} \rightarrow \mathbb{C} \\ w \mapsto \bar{w}^{\mathbf{m}-1}. \end{array}$$

- (iii) The range $\mathcal{R} \left(d_f F_\alpha(\Omega_{\mathbf{m}}^E(\alpha), 0) \right)$ is closed and of codimension one in $Y_{\mathbf{m}}^\beta$.
- (iv) Transversality condition:

$$\partial_\Omega d_f F_\alpha(\Omega_{\mathbf{m}}^E(\alpha), 0)[v_{0,\mathbf{m}}] \notin \mathcal{R} \left(d_f F_\alpha(\Omega_{\mathbf{m}}^E(\alpha), 0) \right).$$

PROOF. (i) Follows immediately from Proposition 3.1-1.
(ii) It is a direct consequence of Proposition 3.1-2 and Lemma 3.1 since for

$$h(w) = \sum_{n=0}^\infty a_n \bar{w}^{n\mathbf{m}-1} \in X_{\mathbf{m}}^{1+\beta},$$

we have by (3.9)

$$(3.19) \quad d_f F_\alpha(\Omega_{\mathbf{m}}^E(\alpha), 0)[h] = \sum_{n=1}^\infty n\mathbf{m}a_n \left(\Omega_{n\mathbf{m}}^E(\alpha) - \Omega_{\mathbf{m}}^E(\alpha) \right) e_{n\mathbf{m}}.$$

(iii) From Proposition 3.1-2, we know that $d_f F_\alpha(\Omega_{\mathbf{m}}^E(\alpha), 0)$ is a Fredholm operator with index zero. Together with the point (ii), we conclude that the range of $d_f F_\alpha(\Omega_{\mathbf{m}}^E(\alpha), 0)$ is closed with codimension one. We endow $Y_{\mathbf{m}}^\beta$ with the scalar product

$$\langle f, g \rangle \triangleq \int_{\mathbb{T}} f(w) \overline{g(w)} dw = \sum_{n=1}^\infty f_{n\mathbf{m}} g_{n\mathbf{m}}, \quad f = \sum_{n=1}^\infty f_{n\mathbf{m}} e_{n\mathbf{m}}, \quad g = \sum_{n=1}^\infty g_{n\mathbf{m}} e_{n\mathbf{m}}.$$

The continuity of $\langle \cdot, \cdot \rangle$ on $Y_{\mathbf{m}}^\beta \times Y_{\mathbf{m}}^\beta$ is a direct consequence of Cauchy-Schwarz inequality and the continuous embedding $C^\beta(\mathbb{T}) \hookrightarrow L^2(\mathbb{T})$. We claim that

$$(3.20) \quad \mathcal{R} \left(d_f F_\alpha(\Omega_{\mathbf{m}}^E(\alpha), 0) \right) = \langle e_{\mathbf{m}} \rangle^\perp,$$

where the orthogonal is understood in the sense of the scalar product $\langle \cdot, \cdot \rangle$. Observe that (3.19) and the point (i) immediately imply the first inclusion in (3.20). The inverse inclusion is deduced from the fact that the range is of codimension one.

(iv) For any $h \in X_{\mathbf{m}}^{1+\beta}$,

$$\forall w \in \mathbb{T}, \quad \partial_\Omega d_f F_\alpha(\Omega_{\mathbf{m}}^E(\alpha), 0)[h](w) = \text{Im} \left\{ h(w)\bar{w} + \overline{h'(w)} \right\}.$$

Consequently, (3.20) implies

$$\partial_\Omega d_f F_\alpha(\Omega_{\mathbf{m}}^E(\alpha), 0)[v_{0,\mathbf{m}}] = -\mathbf{m}e_{\mathbf{m}} \notin \mathcal{R} \left(d_f F_\alpha(\Omega_{\mathbf{m}}^E(\alpha), 0) \right).$$

This ends the proof of Proposition 3.2 and proves Theorem 1.1. □

REMARK 3.1. Observe that for $\mathbf{m} = 1$, a similar calculation to Proposition 3.1-1-(iv), shows that for $\Phi(z) = z + a_0$, with $a_0 \in \mathbb{R}$, one has $F_\alpha(0, \Phi) = 0$. By uniqueness of the constructed branch of bifurcation, this latter corresponds to a translation of the Rankine vortex.

4. Quasi-periodic Euler- α patches

This section is devoted to the proof of Theorem 1.2. It is based on KAM and Nash-Moser techniques in a similar way to the recent works [14, 41, 43, 52]. First let us introduce the notations and topologies used along this section.

4.1. Topologies for functions and operators. This subsection presents the notations and topologies used along this part on the construction of quasi-periodic vortex patches. First we fix

$$(4.1) \quad 0 < \alpha_0 < \alpha_1.$$

The parameter α will live in the interval (α_0, α_1) . More precisely, at the end it will belong to a Cantor set contained in this interval. We shall denote

$$(4.2) \quad d \in \mathbb{N}^*$$

the number of excited frequencies generating the quasi-periodic solutions. Consequently, the frequency vector ω belongs to \mathbb{R}^d . More precisely, at the end ω should be close to the equilibrium frequency vector obtained at the linear level at the Rankine vortex. Our solutions will be searched in the Sobolev class constructed on L^2 in the variables $\varphi \in \mathbb{T}^d$ and $\theta \in \mathbb{T}$. Hence, we decompose any $\rho \in L^2(\mathbb{T}^{d+1}, \mathbb{C})$, in Fourier series as follows

$$\rho = \sum_{(l,j) \in \mathbb{Z}^{d+1}} \rho_{l,j} \mathbf{e}_{l,j}, \quad \rho_{l,j} \triangleq \langle \rho, \mathbf{e}_{l,j} \rangle_{L^2(\mathbb{T}^{d+1}, \mathbb{C})},$$

where $(\mathbf{e}_{l,j})_{(l,j) \in \mathbb{Z}^d \times \mathbb{Z}}$ denotes the classical Hilbert basis of the complex Hilbert space $L^2(\mathbb{T}^{d+1}, \mathbb{C})$. Explicitly, we have

$$\mathbf{e}_{l,j}(\varphi, \theta) \triangleq e^{i(l \cdot \varphi + j\theta)}, \quad \mathbf{e}_j \triangleq \mathbf{e}_{0,j}.$$

The associated Hermitian inner product is

$$\begin{aligned} \langle \rho_1, \rho_2 \rangle_{L^2(\mathbb{T}^{d+1}, \mathbb{C})} &\triangleq \int_{\mathbb{T}^{d+1}} \rho_1(\varphi, \theta) \overline{\rho_2(\varphi, \theta)} d\varphi d\theta, \\ \int_{\mathbb{T}^n} f(x) dx &\triangleq \frac{1}{(2\pi)^n} \int_{[0, 2\pi]^n} f(x) dx. \end{aligned}$$

For $s \in \mathbb{R}$ the complex Sobolev space $H^s(\mathbb{T}^{d+1}, \mathbb{C})$ is given by

$$\begin{aligned} H^s(\mathbb{T}^{d+1}, \mathbb{C}) &\triangleq \left\{ \rho \in L^2(\mathbb{T}^{d+1}, \mathbb{C}) \quad \text{s.t.} \quad \|\rho\|_{H^s}^2 \triangleq \sum_{(l,j) \in \mathbb{Z}^{d+1}} \langle l, j \rangle^{2s} |\rho_{l,j}|^2 < \infty \right\}, \\ \langle l, j \rangle &\triangleq \max(1, |l|, |j|). \end{aligned}$$

The closed sub-vector space of real valued functions is denoted

$$\begin{aligned} H^s &\triangleq H^s(\mathbb{T}^{d+1}, \mathbb{R}) \triangleq \left\{ \rho \in H^s(\mathbb{T}^{d+1}, \mathbb{C}) \quad \text{s.t.} \quad \forall (\varphi, \theta) \in \mathbb{T}^{d+1}, \rho(\varphi, \theta) = \overline{\rho(\varphi, \theta)} \right\} \\ &= \left\{ \rho \in H^s(\mathbb{T}^{d+1}, \mathbb{C}) \quad \text{s.t.} \quad \forall (l, j) \in \mathbb{Z}^{d+1}, \rho_{-l, -j} = \overline{\rho_{l,j}} \right\}. \end{aligned}$$

In order to ensure some suitable embeddings, we shall consider the following restrictions on the Sobolev indices. In particular S is chosen large enough.

$$(4.3) \quad S \geq s \geq s_0 > \frac{d+1}{2} + q + 2.$$

During the scheme, we shall also keep track of the regularity of our functions and operators with respect to the parameters $\mu \triangleq (\alpha, \omega)$. Thus, we introduce the parameters

$$(4.4) \quad \gamma \in (0, 1), \quad q \in \mathbb{N}^*$$

and consider the following weighted spaces

$$\begin{aligned} W^{q,\infty,\gamma}(\mathcal{O}, H^s) &\triangleq \left\{ \rho : \mathcal{O} \rightarrow H^s \quad \text{s.t.} \quad \|\rho\|_{q,s}^{\gamma,\mathcal{O}} < \infty \right\}, \\ \|\rho\|_{q,s}^{\gamma,\mathcal{O}} &\triangleq \sum_{\substack{\alpha \in \mathbb{N}^{d+1} \\ |\alpha| \leq q}} \gamma^{|\alpha|} \sup_{\mu \in \mathcal{O}} \|\partial_\mu^\alpha \rho(\mu, \cdot)\|_{H^{s-|\alpha|}}, \\ W^{q,\infty,\gamma}(\mathcal{O}, \mathbb{C}) &\triangleq \left\{ \rho : \mathcal{O} \rightarrow \mathbb{C} \quad \text{s.t.} \quad \|\rho\|_{q,s}^{\gamma,\mathcal{O}} < \infty \right\}, \\ \|\rho\|_{q,s}^{\gamma,\mathcal{O}} &\triangleq \sum_{\substack{\alpha \in \mathbb{N}^{d+1} \\ |\alpha| \leq q}} \gamma^{|\alpha|} \sup_{\mu \in \mathcal{O}} |\partial_\mu^\alpha \rho(\mu)|. \end{aligned}$$

The set \mathcal{O} is an open bounded subset of \mathbb{R}^{d+1} which will be fixed in (4.50). The parameters γ and q will be fixed in (4.99) and (4.49), respectively. Observe that any $\rho \in W^{q,\infty,\gamma}(\mathcal{O}, H^s)$ decomposes as follows

$$\rho(\mu, \varphi, \theta) = \sum_{(l,j) \in \mathbb{Z}^{d+1}} \rho_{l,j}(\mu) \mathbf{e}_{l,j}(\varphi, \theta).$$

Some classical properties of the weighted Sobolev norm are gathered in the following lemma. We refer for instance to [16, 12, 13] for the ideas of their proofs.

LEMMA 4.1. *Let (d, γ, q, s_0, s) satisfying (4.2), (4.3) and (4.4). Take $\rho_1, \rho_2 \in W^{q,\infty,\gamma}(\mathcal{O}, H^s)$. Then the following assertions hold true.*

(i) *Space translation invariance: for any $\eta \in \mathbb{T}$, we have $\tau_\eta \rho_1 : (\mu, \varphi, \theta) \mapsto \rho_1(\mu, \varphi, \eta + \theta) \in W^{q,\infty,\gamma}(\mathcal{O}, H^s)$, and*

$$\|\tau_\eta \rho_1\|_{q,s}^{\gamma,\mathcal{O}} = \|\rho_1\|_{q,s}^{\gamma,\mathcal{O}}.$$

(ii) *Projectors properties: For all $N \in \mathbb{N}^*$ and for all $t \in \mathbb{R}_+^*$,*

$$\|\Pi_N \rho_1\|_{q,s+t}^{\gamma,\mathcal{O}} \leq N^t \|\rho_1\|_{q,s}^{\gamma,\mathcal{O}} \quad \|\Pi_N^\perp \rho_1\|_{q,s}^{\gamma,\mathcal{O}} \leq N^{-t} \|\rho_1\|_{q,s+t}^{\gamma,\mathcal{O}},$$

where the projectors are defined by

$$\Pi_N \left(\sum_{j \in \mathbb{Z}} (\rho_1)_j \mathbf{e}_j \right) \triangleq \sum_{\substack{j \in \mathbb{Z} \\ |j| \leq N}} (\rho_1)_j \mathbf{e}_j, \quad \Pi_N^\perp \triangleq \text{Id} - \Pi_N.$$

(iii) *Law products: $\rho_1 \rho_2 \in W^{q,\infty,\gamma}(\mathcal{O}, H^s)$ and*

$$\|\rho_1 \rho_2\|_{q,s}^{\gamma,\mathcal{O}} \lesssim \|\rho_1\|_{q,s_0}^{\gamma,\mathcal{O}} \|\rho_2\|_{q,s}^{\gamma,\mathcal{O}} + \|\rho_1\|_{q,s}^{\gamma,\mathcal{O}} \|\rho_2\|_{q,s_0}^{\gamma,\mathcal{O}}.$$

(iv) *Composition law: For $f \in C^\infty(\mathcal{O} \times \mathbb{R}, \mathbb{R})$, if there exists $M > 0$ such that*

$$\|\rho_1\|_{q,s}^{\gamma,\mathcal{O}}, \|\rho_2\|_{q,s}^{\gamma,\mathcal{O}} \leq M,$$

then $f(\rho_1) - f(\rho_2) \in W^{q,\infty,\gamma}(\mathcal{O}, H^s)$ with

$$\|f(\rho_1) - f(\rho_2)\|_{q,s}^{\gamma,\mathcal{O}} \leq C(s, d, q, f, \mathbf{M})\|\rho_1 - \rho_2\|_{q,s}^{\gamma,\mathcal{O}},$$

where we used the notation

$$\forall(\mu, \varphi, \theta) \in \mathcal{O} \times \mathbb{T}^{d+1}, \quad f(\rho)(\mu, \varphi, \theta) \triangleq f(\mu, \rho(\mu, \varphi, \theta)).$$

We shall now present the operator topology used along this section. In particular, we deal with the Toeplitz in time operator class. These notions are based on the one introduced in [3, 12, 13, 16]. We consider parameter dependent operators in the form

$$T : \mu \in \mathcal{O} \mapsto T(\mu) \in \mathcal{L}(H^s(\mathbb{T}^{d+1}, \mathbb{C})),$$

which can be identified with an infinite dimensional matrix $(T_{l_0, j_0}^{l, j}(\mu))_{\substack{(l, l_0) \in (\mathbb{Z}^d)^2 \\ (j, j_0) \in \mathbb{Z}^2}}$

through their action on the Hilbert basis $(\mathbf{e}_{l, j})_{(l, j) \in \mathbb{Z}^d \times \mathbb{Z}}$ as follows

$$T(\mu)\mathbf{e}_{l_0, j_0} = \sum_{(l, j) \in \mathbb{Z}^{d+1}} T_{l_0, j_0}^{l, j}(\mu)\mathbf{e}_{l, j}, \quad T_{l_0, j_0}^{l, j}(\mu) \triangleq \langle T(\mu)\mathbf{e}_{l_0, j_0}, \mathbf{e}_{l, j} \rangle_{L^2(\mathbb{T}^{d+1})}.$$

More precisely, we shall see such operators acting on $W^{q,\infty,\gamma}(\mathcal{O}, H^s(\mathbb{T}^{d+1}, \mathbb{C}))$ in the following sense,

$$\rho \in W^{q,\infty,\gamma}(\mathcal{O}, H^s(\mathbb{T}^{d+1}, \mathbb{C})), \quad (T\rho)(\mu, \varphi, \theta) \triangleq T(\mu)\rho(\mu, \varphi, \theta).$$

We shall now introduce the Toeplitz operators class. An operator $T(\mu)$ is said to be Toeplitz in time iff

$$T_{l_0, j_0}^{l, j}(\mu) = T_{j_0}^j(\mu, l - l_0), \quad T_{j_0}^j(\mu, l) \triangleq T_{0, j_0}^{l, j}(\mu).$$

Its action on a function $\rho = \sum_{(l_0, j_0) \in \mathbb{Z}^{d+1}} \rho_{l_0, j_0} \mathbf{e}_{l_0, j_0}$ writes

$$(4.5) \quad T(\mu)\rho = \sum_{\substack{(l, l_0) \in (\mathbb{Z}^d)^2 \\ (j, j_0) \in \mathbb{Z}^2}} T_{j_0}^j(\mu, l - l_0)\rho_{l_0, j_0} \mathbf{e}_{l, j}.$$

The Toeplitz topology is given by the following off-diagonal norm,

$$\|T\|_{\mathcal{O}\text{-d}, q, s}^{\gamma, \mathcal{O}} \triangleq \sum_{\substack{\alpha \in \mathbb{N}^{d+1} \\ |\alpha| \leq q}} \gamma^{|\alpha|} \sup_{\mu \in \mathcal{O}} \|\partial_\mu^\alpha(T)(\mu)\|_{\mathcal{O}\text{-d}, s-|\alpha|},$$

$$\|T\|_{\mathcal{O}\text{-d}, s}^2 \triangleq \sum_{(l, m) \in \mathbb{Z}^{d+1}} \langle l, m \rangle^{2s} \sup_{j-k=m} |T_j^k(l)|^2.$$

We mention that we will encounter several operators acting only on the variable θ and that can be considered as φ -dependent operators $T(\mu, \varphi)$ taking the form of integral operators

$$T(\mu, \varphi)\rho(\varphi, \theta) = \int_{\mathbb{T}} K(\mu, \varphi, \theta, \eta)\rho(\varphi, \eta)d\eta.$$

One can easily check that those operators are Toeplitz and therefore they satisfy (4.5). We shall now give some classical results on the Toeplitz norm. The proofs are very similar to those in [16] concerning pseudo-differential operators.

LEMMA 4.2. *Let (d, s_0, s, γ, q) satisfying (4.2), (4.3) and (4.4). Let T, T_1 and T_2 be Toeplitz in time operators.*

(i) *Composition law:*

$$\|T_1 T_2\|_{\mathcal{O}\text{-d},q,s}^{\gamma,\mathcal{O}} \lesssim \|T_1\|_{\mathcal{O}\text{-d},q,s}^{\gamma,\mathcal{O}} \|T_2\|_{\mathcal{O}\text{-d},q,s_0}^{\gamma,\mathcal{O}} + \|T_1\|_{\mathcal{O}\text{-d},q,s_0}^{\gamma,\mathcal{O}} \|T_2\|_{\mathcal{O}\text{-d},q,s}^{\gamma,\mathcal{O}}.$$

(ii) *Link between operators and off-diagonal norms:*

$$\|T\rho\|_{q,s}^{\gamma,\mathcal{O}} \lesssim \|T\|_{\mathcal{O}\text{-d},q,s_0}^{\gamma,\mathcal{O}} \|\rho\|_{q,s}^{\gamma,\mathcal{O}} + \|T\|_{\mathcal{O}\text{-d},q,s}^{\gamma,\mathcal{O}} \|\rho\|_{q,s_0}^{\gamma,\mathcal{O}}.$$

In particular

$$\|T\rho\|_{q,s}^{\gamma,\mathcal{O}} \lesssim \|T\|_{\mathcal{O}\text{-d},q,s}^{\gamma,\mathcal{O}} \|\rho\|_{q,s}^{\gamma,\mathcal{O}}.$$

Now, we give the following definition inspired for instance from [4, Def. 2.2].

DEFINITION 4.1. *We defined the following involutions*

$$(\mathcal{S}_2\rho)(\varphi, \theta) = \rho(-\varphi, -\theta), \quad (\mathcal{S}_c\rho)(\varphi, \theta) = \overline{\rho(\varphi, \theta)}.$$

An operator $T = T(\mu)$ is said to be

- *real iff* $\forall \rho \in L^2(\mathbb{T}^{d+1}, \mathbb{C}), \mathcal{S}_c\rho = \rho \Rightarrow \mathcal{S}_c(T\rho) = T\rho.$
- *reversible iff* $T \circ \mathcal{S}_2 = -\mathcal{S}_2 \circ T.$
- *reversibility preserving iff* $T \circ \mathcal{S}_2 = \mathcal{S}_2 \circ T.$

We end this subsection by recalling an important lemma whose proof can be found in [52, Lem. 4.4].

LEMMA 4.3. *Let (d, s_0, s, γ, q) satisfy (4.2), (4.3) and (4.4), then for any integral operator T with a real-valued kernel K , namely*

$$(T\rho)(\mu, \varphi, \theta) = \int_{\mathbb{T}} \rho(\mu, \varphi, \eta) K(\mu, \varphi, \theta, \eta) d\eta, \quad K : (\mu, \varphi, \theta, \eta) \mapsto K(\mu, \varphi, \theta, \eta),$$

the following property holds true.

- *If K is even in (φ, θ, η) , then T is a real and reversibility preserving Toeplitz in time operator.*
- *If K is odd in (φ, θ, η) , then T is a real and reversible Toeplitz in time operator.*

Moreover,

$$\|T\|_{\mathcal{O}\text{-d},q,s}^{\gamma,\mathcal{O}} \lesssim \int_{\mathbb{T}} \|K(*, \cdot, \cdot, \eta + \cdot)\|_{q,s+s_0}^{\gamma,\mathcal{O}} d\eta$$

and

$$\|T\rho\|_{q,s}^{\gamma,\mathcal{O}} \lesssim \|\rho\|_{q,s_0}^{\gamma,\mathcal{O}} \int_{\mathbb{T}} \|K(*, \cdot, \cdot, \eta + \cdot)\|_{q,s}^{\gamma,\mathcal{O}} d\eta + \|\rho\|_{q,s}^{\gamma,\mathcal{O}} \int_{\mathbb{T}} \|K(*, \cdot, \cdot, \eta + \cdot)\|_{q,s_0}^{\gamma,\mathcal{O}} d\eta,$$

where the notation $*, \cdot, \cdot$ denote μ, φ, θ , respectively.

4.2. Hamiltonian contour dynamics equation and its linearization.

Here, we reformulate the contour dynamics equation (1.8) as a Hamiltonian equation on the radial deformation of the patch motion near the Rankine vortex associated with the unit disc. We also compute the linearization of this Hamiltonian equation both at the equilibrium and at a general state close to it. Let us consider the following polar parametrization of the boundary of a patch $t \mapsto \mathbf{1}_{D_t}$ close to the Rankine patch $\mathbf{1}_{\mathbb{D}}$.

$$(4.6) \quad z(t, \theta) \triangleq R(t, \theta)e^{i(\theta - \Omega t)}, \quad R(t, \theta) \triangleq \sqrt{1 + 2r(t, \theta)}.$$

The radial deformation r is assumed to be of small amplitudes and the angular velocity $\Omega > 0$ is introduced to avoid the resonance of the first equilibrium frequency

as in [41, 52]. Therefore, r satisfies an Hamiltonian PDE as explained in the following lemma.

LEMMA 4.4. *The following hold true.*

(i) *At least for short time $T > 0$, the radial deformation r in (4.6) is solution of the following equation: for any $(t, \theta) \in [0, T] \times \mathbb{T}$,*

$$(4.7) \quad \partial_t r(t, \theta) + \Omega \partial_\theta r(t, \theta) - F^E[r](t, \theta) - F^{SW}[r](\alpha, t, \theta) = 0,$$

where

$$(4.8) \quad F^E[r](t, \theta) \triangleq \int_{\mathbb{T}} \log(A_r(t, \theta, \eta)) \partial_{\theta\eta}^2 (R(t, \theta)R(t, \eta) \sin(\eta - \theta)) d\eta,$$

$$(4.9) \quad F^{SW}[r](\alpha, t, \theta) \triangleq \int_{\mathbb{T}} K_0\left(\frac{1}{\alpha} A_r(t, \theta, \eta)\right) \partial_{\theta\eta}^2 (R(t, \theta)R(t, \eta) \sin(\eta - \theta)) d\eta,$$

$$(4.10) \quad A_r(t, \theta, \eta) \triangleq |R(t, \theta)e^{i\theta} - R(t, \eta)e^{i\eta}|.$$

(ii) *The nonlinear and nonlocal transport-type PDE (4.7) can be written in the following Hamiltonian form*

$$(4.11) \quad \partial_t r = \partial_\theta \nabla \mathcal{H}(r),$$

where

$$\mathcal{H}(r) \triangleq \frac{1}{2} (\mathcal{E}(r) - \Omega \mathcal{J}(r)),$$

$$\mathcal{E}(r)(t) \triangleq -\frac{1}{2\pi} \int_{D_t} \Psi(t, z) dA(z), \quad \mathcal{J}(r)(t) \triangleq \frac{1}{2\pi} \int_{D_t} |z|^2 dA(z).$$

The notation ∇ stands for the $L_\theta^2(\mathbb{T})$ -gradient associated with the $L_\theta^2(\mathbb{T})$ normalized inner product

$$\langle \rho_1, \rho_2 \rangle_{L^2(\mathbb{T})} \triangleq \int_{\mathbb{T}} \rho_1(\theta) \rho_2(\theta) d\theta.$$

PROOF. (i) First, from the polar parametrization (4.6), it is easy to check that the left hand-side of (1.8) writes

$$\text{Im} \left(\partial_t z(t, \theta) \overline{\partial_\theta z(t, \theta)} \right) = -\partial_t r(t, \theta) - \Omega \partial_\theta r(t, \theta).$$

Now we study the right hand-side of (1.8). Combining (3.3) and (A.1), we deduce

$$\begin{aligned} \text{Im} \left(\mathbf{v}(t, z(t, \theta)) \overline{\partial_\theta z(t, \theta)} \right) &= - \int_{\mathbb{T}} \log(|z(t, \theta) - z(t, \eta)|) \text{Im} \left(\partial_\eta z(t, \eta) \overline{\partial_\theta z(t, \theta)} \right) d\eta \\ &\quad - \int_{\mathbb{T}} K_0\left(\frac{1}{\alpha} |z(t, \theta) - z(t, \eta)|\right) \text{Im} \left(\partial_\eta z(t, \eta) \overline{\partial_\theta z(t, \theta)} \right) d\eta. \end{aligned}$$

We conclude by remarking that

$$\text{Im} \left(\partial_\eta z(t, \eta) \overline{\partial_\theta z(t, \theta)} \right) = \partial_{\theta\eta}^2 \text{Im} \left(z(t, \eta) \overline{z(t, \theta)} \right) = \partial_{\theta\eta}^2 (R(t, \eta)R(t, \theta) \sin(\eta - \theta)).$$

(ii) Using polar change of coordinates and (4.6), we obtain

$$\mathcal{J}(r)(t) = \int_{\mathbb{T}} \int_0^{R(t, \theta)} \ell^3 d\ell d\theta = \frac{1}{4} \int_{\mathbb{T}} (1 + 2r(t, \theta))^2 d\theta,$$

leading to

$$\nabla \mathcal{J}(r) = 1 + 2r \quad \text{and} \quad \frac{1}{2} \Omega \partial_\theta \nabla \mathcal{J}(r) = \Omega \partial_\theta r.$$

Now that we have treated the corresponding linear term in (4.7), we can assume $\Omega = 0$. From the complex notation, we have

$$\begin{aligned} \partial_\theta \Psi(t, z(t, \theta)) &= \nabla \Psi(t, z(t, \theta)) \cdot \partial_\theta z(t, \theta) \\ &= \text{Im} \left(\mathbf{v}(t, z(t, \theta)) \overline{\partial_\theta z(t, \theta)} \right) \\ &= -F^E[r](t, \theta) - F^{SW}[r](\alpha, t, \theta). \end{aligned}$$

From (1.6), we can write

$$\Psi(t, z) = \int_{D_t} \tilde{\mathbf{G}}(z, \xi) dA(\xi), \quad \tilde{\mathbf{G}}(z, \xi) \triangleq \mathbf{G}(|z - \xi|).$$

The kernel $\tilde{\mathbf{G}}$ satisfies the following symmetry property

$$\tilde{\mathbf{G}}(z, \xi) = \tilde{\mathbf{G}}(\xi, z).$$

Consequently, we can apply the general [43, Prop. 2.1] giving

$$\nabla \mathcal{L}(r)(t, \theta) = -2\Psi(t, z(t, \theta)).$$

This achieves the proof of Lemma 4.4. □

We shall now briefly discuss the symplectic framework behind the Hamiltonian equation (4.11). First notice that one deduces from (4.11) that the space average is preserved along the motion so we may assume it zero and work in the following phase space

$$(4.12) \quad L_0^2(\mathbb{T}) \triangleq \left\{ r = \sum_{j \in \mathbb{Z}^*} r_j \mathbf{e}_j \quad \text{s.t.} \quad r_{-j} = \bar{r}_j, \quad \sum_{j \in \mathbb{Z}^*} |r_j|^2 < \infty \right\}, \quad \mathbf{e}_j(\theta) \triangleq e^{ij\theta}.$$

The symplectic form \mathcal{W} on $L_0^2(\mathbb{T})$ generated by (4.11) writes

$$(4.13) \quad \mathcal{W}(r, \tilde{r}) \triangleq \int_{\mathbb{T}} \partial_\theta^{-1} r(\theta) \tilde{r}(\theta) d\theta, \quad \partial_\theta^{-1} r \triangleq \sum_{j \in \mathbb{Z}^*} \frac{r_j}{ij} \mathbf{e}_j.$$

The associated Hamiltonian vector-field $X_{\mathcal{H}}$ is defined by

$$d\mathcal{H}(r)[\rho] = \mathcal{W}(X_{\mathcal{H}}(r), \rho), \quad X_{\mathcal{H}}(r) \triangleq \partial_\theta \nabla \mathcal{H}(r).$$

We shall also present the reversibility property of the Hamiltonian \mathcal{H} . For that purpose we introduce the following involution on $L_0^2(\mathbb{T})$

$$(4.14) \quad (\mathcal{S}r)(\theta) \triangleq r(-\theta), \quad \mathcal{S}^2 = \text{Id}.$$

Then changes of variables give

$$\mathcal{S} \circ F^E = -F^E \circ \mathcal{S}, \quad \mathcal{S} \circ F^{SW} = -F^{SW} \circ \mathcal{S}$$

implying in turn

$$\mathcal{H} \circ \mathcal{S} = \mathcal{H}, \quad X_{\mathcal{H}} \circ \mathcal{S} = -\mathcal{S} \circ X_{\mathcal{H}}.$$

Now, in view of applying a Nash-Moser scheme, we shall compute the linearized operator both at the equilibrium and at a general state close to it. It is proved in [43, Lem. 3.1 and 3.2] that

$$(4.15) \quad d_r F^E(r)[\rho] = \partial_\theta (V_r^E \rho) - \partial_\theta \mathbf{L}_r^E(\rho),$$

with

$$(4.16) \quad V_r^E(t, \theta) \triangleq \frac{1}{R(t, \theta)} \int_{\mathbb{T}} \log(A_r(t, \theta, \eta)) \partial_\eta (R(t, \eta) \sin(\eta - \theta)) d\eta,$$

$$(4.17) \quad \mathbf{L}_r^E(\rho)(t, \theta) \triangleq \int_{\mathbb{T}} \rho(t, \eta) \log(A_r(t, \theta, \eta)) d\eta$$

and

$$d_r F^E(0)[\rho] = -\frac{1}{2} \partial_\theta \rho - \partial_\theta \mathcal{K} * \rho = -i \sum_{j \in \mathbb{Z}} j \Omega_{|j|}^E \rho_j \mathbf{e}_j, \quad \mathcal{K}(\theta) \triangleq \frac{1}{2} \log(\sin^2(\frac{\theta}{2})).$$

We recall that A_r and R are defined by (4.10) and (4.6), respectively. In addition, it has been proved in [52, Lem. 3.1 and 3.2] that

$$(4.18) \quad d_r F^{SW}(r)[\rho] = \partial_\theta (V_r^{SW} \rho) - \partial_\theta \mathbf{L}_r^{SW}(\rho),$$

with

$$(4.19) \quad V_r^{SW}(\alpha, t, \theta) \triangleq \frac{1}{R(t, \theta)} \int_{\mathbb{T}} K_0(\frac{1}{\alpha} A_r(t, \theta, \eta)) \partial_\eta (R(t, \eta) \sin(\eta - \theta)) d\eta,$$

$$(4.20) \quad \mathbf{L}_r^{SW}(\rho)(\alpha, t, \theta) \triangleq \int_{\mathbb{T}} \rho(t, \eta) K_0(\frac{1}{\alpha} A_r(t, \theta, \eta)) d\eta$$

and

$$d_r F^{SW}(0)[\rho] = I_1(\frac{1}{\alpha}) K_1(\frac{1}{\alpha}) \partial_\theta \rho - \partial_\theta \mathcal{Q}_\alpha * \rho = i \sum_{j \in \mathbb{Z}} j \Omega_{|j|}^{SW}(\frac{1}{\alpha}) \rho_j \mathbf{e}_j,$$

$$\mathcal{Q}_\alpha(\theta) \triangleq K_0(\frac{2}{\alpha} |\sin(\frac{\theta}{2})|).$$

Gathering the previous results leads to the following lemma giving the general expression of the linearized equation and stating that the equilibrium is given by a Fourier multiplier associated with an integrable Hamiltonian system.

LEMMA 4.5. *The following assertions hold true.*

(i) *The linearization of (4.11) at a general state r writes*

$$(4.21) \quad \partial_t \rho = -\partial_\theta (V_r \rho + \mathbf{L}_r(\rho)),$$

where V_r and \mathbf{L}_r are respectively defined by

$$(4.22) \quad V_r(\alpha, t, \theta) \triangleq \Omega - V_r^E(t, \theta) - V_r^{SW}(\alpha, t, \theta), \quad \mathbf{L}_r \triangleq \mathbf{L}_r^E + \mathbf{L}_r^{SW}.$$

In addition, we have the following symmetry property

$$(4.23) \quad r(-t, -\theta) = r(t, \theta) \quad \Rightarrow \quad V_r(\alpha, -t, -\theta) = V_r(\alpha, t, \theta).$$

(ii) (a) *At $r = 0$, the equation (4.21) becomes*

$$(4.24) \quad \partial_t \rho = \partial_\theta L(\alpha) \rho = \partial_\theta \nabla \mathcal{H}_L(\rho),$$

where $L(\alpha)$ is the self-adjoint operator given by

$$(4.25) \quad L(\alpha) \triangleq -V_0(\alpha) - \mathcal{K}_\alpha * \cdot,$$

$$V_0(\alpha) \triangleq \Omega + \frac{1}{2} - I_1(\frac{1}{\alpha}) K_1(\frac{1}{\alpha}), \quad \mathcal{K}_\alpha \triangleq \mathcal{K} + \mathcal{Q}_\alpha.$$

The equation (4.24) is generated by the quadratic Hamiltonian

$$(4.26) \quad \mathcal{H}_L(\rho) \triangleq \frac{1}{2} \langle L(\alpha) \rho, \rho \rangle_{L^2(\mathbb{T})}.$$

(b) In Fourier expansion, the solutions of (4.24) take the form

$$\rho(t, \theta) = \sum_{j \in \mathbb{Z}^*} \rho_j(0) e^{i(j\theta - \Omega_j^E(\alpha)t)},$$

where for all $j \in \mathbb{Z}^*$,

$$\begin{aligned} \Omega_j^E(\alpha) &\triangleq j \left[\Omega + \frac{|j|-1}{2|j|} - \left[I_1 \left(\frac{1}{\alpha} \right) K_1 \left(\frac{1}{\alpha} \right) - I_{|j|} \left(\frac{1}{\alpha} \right) K_{|j|} \left(\frac{1}{\alpha} \right) \right] \right] \\ (4.27) \quad &= j \left[\Omega + \mathbf{\Omega}_{|j|}^E(\alpha) \right], \end{aligned}$$

with $\mathbf{\Omega}_j^E(\alpha)$ are the frequencies obtained in the periodic case and defined in (1.12). The operator $L(\alpha)$ and the Hamiltonian \mathcal{H}_L also write

$$(4.28) \quad L(\alpha)\rho = - \sum_{j \in \mathbb{Z}^*} \frac{\Omega_j^E(\alpha)}{j} \rho_j \mathbf{e}_j \quad \text{and} \quad \mathcal{H}_L \rho = - \sum_{j \in \mathbb{Z}^*} \frac{\Omega_j^E(\alpha)}{2j} |\rho_j|^2.$$

4.3. Transversality and linear quasi-periodic solutions. The aim of this section is to find quasi-periodic solutions for the linearized equation (4.24), which is the basis for expecting to get them at the nonlinear level. The result reads as follows

PROPOSITION 4.1. *Let (α_0, α_1, d) as in (4.1)-(4.2). Take $\mathbb{S} \subset \mathbb{N}^*$ with $|\mathbb{S}| = d$. Then, there exists a Cantor-like set*

$$\mathcal{C}_{E_q} \subset [\alpha_0, \alpha_1], \quad |\mathcal{C}_{E_q}| = \alpha_1 - \alpha_0$$

such that for any $\alpha \in \mathcal{C}_{E_q}$, every function in the form

$$\rho(t, \theta) = \sum_{j \in \mathbb{S}} \rho_j \cos(j\theta - \Omega_j^E(\alpha)t), \quad \rho_j \in \mathbb{R}^*$$

is a time quasi-periodic reversible solution to (4.24) with frequency vector

$$(4.29) \quad \omega_{E_q}(\alpha) = (\Omega_j^E(\alpha))_{j \in \mathbb{S}}.$$

The proof of this proposition is very similar to [52, Prop. 3.1] so we refer the reader to the corresponding paper. We mention that in the proof, to measure the Cantor set \mathcal{C}_{E_q} , we make appeal to the following Rüssmann Lemma which can be found in [70, Thm. 17.1].

LEMMA 4.6. *Let $q_0 \in \mathbb{N}^*$, $a, b \in \mathbb{R}$ with $a < b$ and $\mathbf{m}, \mathbf{M} \in (0, \infty)$. Let $f \in C^{q_0+1}([a, b], \mathbb{R})$ such that*

$$(4.30) \quad \inf_{x \in [a, b]} \max_{q \in \llbracket 0, q_0 \rrbracket} |f^{(q)}(x)| \geq \mathbf{m}.$$

Then, there exists $C = C(a, b, q_0) > 0$ such that

$$\left| \{x \in [a, b] \quad \text{s.t.} \quad |f(x)| \leq \mathbf{M}\} \right| \leq C \frac{\mathbf{M}^{\frac{1}{q_0}}}{\mathbf{m}^{1+\frac{1}{q_0}}} \|f\|_{C^{q_0+1}([a, b], \mathbb{R})}.$$

To apply the previous lemma, we shall check the transversality condition (4.30) for the equilibrium frequency vector ω_{E_q} in (4.29). It is proved in Lemma 4.9-(i). Notice that the measure of the final Cantor set in Section 4.6 generating quasi-periodic solutions for the nonlinear model requires transversality conditions for the perturbed frequency vector. These latter are obtained by perturbative arguments from the one for the equilibrium frequency vector stated in Lemma 4.9 and which are

themselves deduced from the non-degeneracy of the unperturbed frequency vector proved in Lemma 4.8. First we start by giving some properties of the frequencies (4.27).

LEMMA 4.7. *The following properties hold true.*

- (i) $\forall \alpha > 0, \quad \Omega_j^E(\alpha) \underset{j \rightarrow \infty}{\sim} V_0(\alpha)j$, with $V_0(\alpha)$ as in (4.25).
- (ii) For all $\alpha > 0$, the sequence $(\Omega_j^E(\alpha))_{j \in \mathbb{N}^*}$ is strictly increasing.
- (iii) For any $j \in \mathbb{Z}^*$, we have

$$\forall \alpha > 0, \quad \left| \Omega_j^E(\alpha) \right| \geq \Omega|j|.$$

- (iv) For any $j, j' \in \mathbb{Z}^*$, we have

$$\forall \alpha > 0, \quad \left| \Omega_j^E(\alpha) - \Omega_{j'}^E(\alpha) \right| \geq \Omega|j - j'|.$$

- (v) Given $0 < \alpha_0 < \alpha_1$ and $q_0 \in \mathbb{N}$, there exists $C_0 > 0$ such that

$$\forall j, j_0 \in \mathbb{Z}^*, \quad \max_{q \in \llbracket 0, q_0 \rrbracket} \sup_{\alpha \in [\alpha_0, \alpha_1]} \left| \partial_\alpha^q \left(\Omega_j^E(\alpha) - \Omega_{j_0}^E(\alpha) \right) \right| \leq C_0|j - j_0|.$$

PROOF. (i) and (ii) follow immediately from Lemma 3.1 and (4.27).

(iii) Due to the symmetry (4.27), it suffices to study the case $j \in \mathbb{N}^*$. From Lemma 3.1, we get

$$\forall \alpha > 0, \quad \Omega_j^E(\alpha) \geq 0.$$

Consequently,

$$\forall \alpha > 0, \quad \Omega_j^E(\alpha) \geq j\Omega.$$

(iv) Due to the symmetry (4.27), it suffices to prove that

$$\forall \alpha > 0, \quad \forall (j, j') \in (\mathbb{N}^*)^2, \quad \left| \Omega_j^E(\alpha) \pm \Omega_{j'}^E(\alpha) \right| \geq \Omega|j \pm j'|.$$

The point (ii) allows us to restrict the discussion to the case $j \geq j'$. We can write

$$\Omega_j^E(\alpha) \pm \Omega_{j'}^E(\alpha) = \Omega(j \pm j') + j\Omega_j^E(\alpha) \pm j'\Omega_{j'}^E(\alpha).$$

In view of Lemma 3.1, we obtain

$$j\Omega_j^E(\alpha) \pm j'\Omega_{j'}^E(\alpha) \geq 0.$$

Hence

$$\Omega_j^E(\alpha) \pm \Omega_{j'}^E(\alpha) \geq \Omega(j \pm j').$$

(v) As before, by symmetry, it suffices to prove that

$$\forall (j, j_0, \alpha, q) \in (\mathbb{N}^*)^2 \times [\alpha_0, \alpha_1] \times \llbracket 0, q_0 \rrbracket, \quad \left| \partial_\alpha^q \left(\Omega_j^E(\alpha) \pm \Omega_{j_0}^E(\alpha) \right) \right| \leq C_0|j \pm j_0|.$$

Let us start with the difference. We can write from (4.27)

$$\Omega_j^E(\alpha) - \Omega_{j_0}^E(\alpha) = \Omega(j - j_0) + \frac{j - j_0}{2} + \Omega_{j_0}^{\text{SW}}\left(\frac{1}{\alpha}\right) - \Omega_j^{\text{SW}}\left(\frac{1}{\alpha}\right), \quad \Omega_j^{\text{SW}}(\lambda) \triangleq j\Omega_{|j|}^{\text{SW}}(\lambda).$$

Now it has been proved in [52, Lem. 3.3-(vi)] that for some $0 < \lambda_0 < \lambda_1$, there exists $C > 0$ such that

(4.31)

$$\forall (j, j_0, \lambda, q) \in (\mathbb{N}^*)^2 \times [\lambda_0, \lambda_1] \times \llbracket 0, q_0 \rrbracket, \quad \left| \partial_\lambda^q \left(\Omega_j^{\text{SW}}(\lambda) \pm \Omega_{j_0}^{\text{SW}}(\lambda) \right) \right| \leq C|j \pm j_0|.$$

We warn the reader about the difference of definitions of the frequencies Ω_j^{SW} between this article and [52] (with the Ω missing). But this has no impact here. Hence, we conclude by the triangle inequality that for some $C_0 > 0$

$$\forall (j, j_0, \alpha, q) \in (\mathbb{N}^*)^2 \times [\alpha_0, \alpha_1] \times \llbracket 0, q_0 \rrbracket, \quad \left| \partial_\alpha^q \left(\Omega_j^{\text{E}}(\alpha) - \Omega_{j_0}^{\text{E}}(\alpha) \right) \right| \leq C_0 |j - j_0|.$$

We now turn to the additional case. We can write

$$0 < \Omega_j^{\text{E}}(\alpha) + \Omega_{j_0}^{\text{E}}(\alpha) = \Omega(j + j_0) + \left(j \frac{j-1}{2j} + j_0 \frac{j_0-1}{2j_0} \right) - \Omega_{j_0}^{\text{SW}} \left(\frac{1}{\alpha} \right) - \Omega_j^{\text{SW}} \left(\frac{1}{\alpha} \right).$$

Notice that

$$\forall j \in \mathbb{N}^*, \quad \frac{j-1}{2j} \leq \frac{1}{2} \quad \text{and} \quad \Omega_j^{\text{SW}} \left(\frac{1}{\alpha} \right) > 0.$$

Thus,

$$0 < \Omega_j^{\text{E}}(\alpha) + \Omega_{j_0}^{\text{E}}(\alpha) \leq \left(\Omega + \frac{1}{2} \right) (j + j_0).$$

Combined with (4.31), this ends the proof of Lemma 4.7. □

For a fixed finite set of Fourier modes

$$(4.32) \quad \mathbb{S} = \{j_1, \dots, j_d\} \subset \mathbb{N}^*, \quad j_1 < \dots < j_d, \quad d \in \mathbb{N}^*,$$

we define the equilibrium frequency vector

$$(4.33) \quad \omega_{\text{Eq}}(\alpha) = \left(\Omega_j^{\text{E}}(\alpha) \right)_{j \in \mathbb{S}}.$$

Then, in view of the measure of the final Cantor set, we may check the Rüssmann conditions for the unperturbed frequency vector (4.33). They are obtained by using the following non-degeneracy conditions.

LEMMA 4.8. *Let (α_0, α_1) as in (4.1). The equilibrium frequency vector ω_{Eq} and the vector-valued functions $(\omega_{\text{Eq}}, V_0)$ and $(\omega_{\text{Eq}}, V_0, 1)$ are non-degenerate on $[\alpha_0, \alpha_1]$, namely the curves*

$$\begin{aligned} \alpha \in [\alpha_0, \alpha_1] &\mapsto \omega_{\text{Eq}}(\alpha), \\ \alpha \in [\alpha_0, \alpha_1] &\mapsto (\omega_{\text{Eq}}(\alpha), V_0(\alpha)), \\ \alpha \in [\alpha_0, \alpha_1] &\mapsto (\omega_{\text{Eq}}(\alpha), V_0(\alpha), 1) \end{aligned}$$

are not contained in an hyperplane of \mathbb{R}^d , \mathbb{R}^{d+1} and \mathbb{R}^{d+2} , respectively.

PROOF. ► Assume that there exists $(c_1, \dots, c_d) \in \mathbb{R}^d$ such that

$$\forall \alpha \in [\alpha_0, \alpha_1], \quad \sum_{k=1}^d c_k \Omega_{j_k}^{\text{E}}(\alpha) = 0,$$

which is equivalent to

$$(4.34) \quad \forall \alpha \in [\alpha_0, \alpha_1], \quad \sum_{k=1}^d c_k j_k \left(\Omega + \frac{j_k-1}{2j_k} \right) = \sum_{k=1}^d c_k \Omega_{j_k}^{\text{SW}} \left(\frac{1}{\alpha} \right).$$

By analyticity of the product $I_n K_n$ on $\{z \in \mathbb{C} \text{ s.t. } \text{Re}(z) > 0\}$ for any $n \in \mathbb{N}^*$, then by continuation principle, the previous identity is still true for $\alpha > 0$. Taking the limit $\alpha \rightarrow 0$ in the previous relation implies from (A.6)

$$\sum_{k=1}^d c_k j_k \left(\Omega + \frac{j_k-1}{2j_k} \right) = 0.$$

The equation (4.34) is reduced to

$$\sum_{k=1}^d c_k \Omega_{j_k}^{\text{SW}} \left(\frac{1}{\alpha} \right) = 0.$$

Then, proceeding as in [52, Lem. 3.4], the asymptotic expansion of large argument for $I_j K_j$ provides an invertible Vandermonde system leading to $\forall k \in \llbracket 1, d \rrbracket$, $c_k = 0$.

► Assume that there exists $(c_1, \dots, c_d, c_{d+1}) \in \mathbb{R}^{d+1}$ (resp. $(c_1, \dots, c_d, c_{d+1}, c_{d+2}) \in \mathbb{R}^{d+2}$) such that

$$\forall \alpha \in [\alpha_0, \alpha_1], \quad (\text{resp. } c_{d+2}+) \quad c_{d+1} V_0(\alpha) + \sum_{k=1}^d c_k \Omega_{j_k}^{\text{E}}(\alpha) = 0,$$

which is equivalent to the fact that for any $\alpha \in [\alpha_0, \alpha_1]$,

$$(4.35) \quad c_{d+1} \left(\Omega + \frac{1}{2} \right) + \sum_{k=1}^d c_k j_k \left(\Omega + \frac{j_k - 1}{2j_k} \right) = c_{d+1} I_1 \left(\frac{1}{\alpha} \right) K_1 \left(\frac{1}{\alpha} \right) + \sum_{k=1}^d c_k \Omega_{j_k}^{\text{SW}} \left(\frac{1}{\alpha} \right). \\ (\text{resp. } + c_{d+2})$$

As in the previous point, this identity can be extended to $(0, \infty)$ and taking the limit $\alpha \rightarrow 0$, one gets by (A.6)

$$(\text{resp. } c_{d+2}+) \quad c_{d+1} \left(\Omega + \frac{1}{2} \right) + \sum_{k=1}^d c_k j_k \left(\Omega + \frac{j_k - 1}{2j_k} \right) = 0.$$

Inserting this information into (4.35) yields

$$\forall \alpha > 0, \quad c_{d+1} I_1 \left(\frac{1}{\alpha} \right) K_1 \left(\frac{1}{\alpha} \right) + \sum_{k=1}^d c_k \Omega_{j_k}^{\text{SW}} \left(\frac{1}{\alpha} \right) = 0.$$

This equation has also been studied in [52, Lem. 3.4] leading to $c_1 = \dots = c_d = c_{d+1} = 0$ (resp. supplemented by $c_{d+2} = 0$). This achieves the proof of Lemma 4.8. □

Now we shall prove the transversality conditions for the equilibrium frequency vector.

LEMMA 4.9. [Transversality] *Let (α_0, α_1) as in (4.1). Then, there exist $q_0 \in \mathbb{N}$ and $\rho_0 > 0$ such that the following results hold true. Recall that ω_{Eq} and Ω_j^{E} are defined in (4.33) and (4.27) respectively.*

(i) *For any $l \in \mathbb{Z}^d \setminus \{0\}$, we have*

$$\inf_{\alpha \in [\alpha_0, \alpha_1]} \max_{q \in \llbracket 0, q_0 \rrbracket} \left| \partial_\alpha^q \omega_{\text{Eq}}(\alpha) \cdot l \right| \geq \rho_0 \langle l \rangle.$$

(ii) *For any $(l, j) \in \mathbb{Z}^d \times (\mathbb{N}^* \setminus \mathbb{S})$*

$$\inf_{\alpha \in [\alpha_0, \alpha_1]} \max_{q \in \llbracket 0, q_0 \rrbracket} \left| \partial_\alpha^q \left(\omega_{\text{Eq}}(\alpha) \cdot l \pm j V_0(\alpha) \right) \right| \geq \rho_0 \langle l \rangle.$$

(iii) *For any $(l, j) \in \mathbb{Z}^d \times (\mathbb{N}^* \setminus \mathbb{S})$*

$$\inf_{\alpha \in [\alpha_0, \alpha_1]} \max_{q \in \llbracket 0, q_0 \rrbracket} \left| \partial_\alpha^q \left(\omega_{\text{Eq}}(\alpha) \cdot l \pm \Omega_j^{\text{E}}(\alpha) \right) \right| \geq \rho_0 \langle l \rangle.$$

(iv) For any $l \in \mathbb{Z}^d, j, j' \in \mathbb{N}^* \setminus \mathbb{S}$ with $(l, j) \neq (0, j')$, we have

$$\inf_{\alpha \in [\alpha_0, \alpha_1]} \max_{q \in \llbracket 0, q_0 \rrbracket} \left| \partial_\alpha^q \left(\omega_{\text{Eq}}(\alpha) \cdot l + \Omega_j^{\text{E}}(\alpha) \pm \Omega_{j'}^{\text{E}}(\alpha) \right) \right| \geq \rho_0 \langle l \rangle.$$

PROOF. We shall prove the point (iv) which is the most difficult one. The arguments are similar in the other cases using the corresponding non-degeneracy conditions provided by Lemma 4.8. Fix some $l \in \mathbb{Z}^d$ and $j, j' \in \mathbb{N}^* \setminus \mathbb{S}$ with $(l, j) \neq (0, j')$. If for some $c_0 > 0$

$$|j \pm j'| \geq c_0 \langle l \rangle,$$

then applying the triangle inequality together with Lemma 4.7-(iv), we get

$$\left| \omega_{\text{Eq}}(\alpha) \cdot l + \Omega_j^{\text{E}}(\alpha) \pm \Omega_{j'}^{\text{E}}(\alpha) \right| \geq \left| \Omega_j^{\text{E}}(\alpha) \pm \Omega_{j'}^{\text{E}}(\alpha) \right| - |\omega_{\text{Eq}}(\alpha) \cdot l| \geq \Omega |j \pm j'| - C|l| \geq \langle l \rangle.$$

Therefore it remains to check the proof for indices satisfying

$$(4.36) \quad |j \pm j'| < c_0 \langle l \rangle, \quad l \in \mathbb{Z}^d \setminus \{0\}, \quad j, j' \in \mathbb{N}^* \setminus \mathbb{S}.$$

We assume in view of a contradiction that for all $m \in \mathbb{N}$, there exist real numbers $l_m \in \mathbb{Z}^d \setminus \{0\}, j_m, j'_m \in \mathbb{N}^* \setminus \mathbb{S}$ satisfying (4.36) and $\alpha_m \in [\alpha_0, \alpha_1]$ such that

$$\max_{q \in \llbracket 0, m \rrbracket} \left| \partial_\alpha^q \left(\omega_{\text{Eq}}(\alpha) \cdot \frac{l_m}{|l_m|} + \frac{\Omega_{j_m}^{\text{E}}(\alpha) \pm \Omega_{j'_m}^{\text{E}}(\alpha)}{|l_m|} \right) \right|_{\alpha=\alpha_m} < \frac{1}{m+1}.$$

This implies that

$$(4.37) \quad \forall q \in \mathbb{N}, \quad \forall m \geq q, \quad \left| \partial_\alpha^q \left(\omega_{\text{Eq}}(\alpha) \cdot \frac{l_m}{|l_m|} + \frac{\Omega_{j_m}^{\text{E}}(\alpha) \pm \Omega_{j'_m}^{\text{E}}(\alpha)}{|l_m|} \right) \right|_{\alpha=\alpha_m} < \frac{1}{m+1}.$$

By compactness and (4.36), up to considering a subsequence, we can assume that

$$(4.38) \quad \lim_{m \rightarrow \infty} \frac{l_m}{|l_m|} = \bar{c} \neq 0, \quad \lim_{m \rightarrow \infty} \frac{j_m \pm j'_m}{|l_m|} = \bar{d}, \quad \lim_{m \rightarrow \infty} \alpha_m = \bar{\alpha}.$$

Now we shall study separately the cases whether the sequence $(l_m)_m$ is bounded or not.

► Here we assume that the sequence $(l_m)_m$ is bounded. Then, by compactness, we can assume, up to an extraction, that we have the following convergence

$$\lim_{m \rightarrow \infty} l_m = \bar{l} \neq 0.$$

Now according to (4.36) we have two sub-cases to discuss depending whether the sequences $(j_m)_m$ and $(j'_m)_m$ are simultaneously bounded or unbounded.

• We first study the case where the sequences $(j_m)_m$ and $(j'_m)_m$ are bounded. Observe that this is the only case to consider if we work with the sign " + " in (4.36). Since they are sequences of integers, then by compactness we may assume, up to considering an extraction, that they are constant, namely

$$(4.39) \quad \exists \bar{j}, \bar{j}' \in \mathbb{N}^* \setminus \mathbb{S} \quad \text{s.t.} \quad \forall m \in \mathbb{N}, \quad j_m = \bar{j} \quad \text{and} \quad j'_m = \bar{j}'.$$

Hence taking the limit as $m \rightarrow \infty$ in (4.37), we obtain

$$\forall q \in \mathbb{N}, \quad \partial_\alpha^q \left(\omega_{\text{Eq}}(\alpha) \cdot \bar{l} + \Omega_{\bar{j}}^{\text{E}}(\alpha) \pm \Omega_{\bar{j}'}^{\text{E}}(\alpha) \right) \Big|_{\alpha=\bar{\alpha}} = 0.$$

Thus, the analytic function $\alpha \mapsto \omega_{\text{Eq}}(\alpha) \cdot \bar{l} + \Omega_{\bar{j}}^{\text{E}}(\alpha) \pm \Omega_{\bar{j}'}^{\text{E}}(\alpha)$ is identically zero which enters in contradiction with Lemma 4.8 up to replacing ω_{Eq} by $(\omega_{\text{Eq}}, \Omega_{\bar{j}}^{\text{E}})$ or $(\omega_{\text{Eq}}, \Omega_{\bar{j}'}^{\text{E}})$.

• Now we study the case where $(j_m)_m$ and $(j'_m)_m$ are both unbounded and without loss of generality we can assume that

$$\lim_{m \rightarrow \infty} j_m = \lim_{m \rightarrow \infty} j'_m = \infty.$$

Notice that here this case only concerns the situation with a sign " - " in (4.36). Nevertheless, for later purposes, we may treat both sign situations. From the expression (4.27) we can write

$$\begin{aligned} \Omega_{j_m}^E(\alpha) \pm \Omega_{j'_m}^E(\alpha) &= (j_m \pm j'_m)V_0(\alpha) - \left(\frac{1}{2} \pm \frac{1}{2}\right) \\ &\quad + (j_m \pm j'_m)(I_{j_m}K_{j_m})\left(\frac{1}{\alpha}\right) \pm j'_m \left((I_{j'_m}K_{j'_m})\left(\frac{1}{\alpha}\right) - (I_{j_m}K_{j_m})\left(\frac{1}{\alpha}\right) \right), \end{aligned}$$

with $V_0(\alpha)$ as in (4.25). It has been proved in [52, Lem. 3.5] that for any $q \in \mathbb{N}$, and for any $0 < \lambda_0 < \lambda_1$,

$$(4.40) \quad \begin{aligned} \lim_{m \rightarrow \infty} \sup_{\lambda \in [\lambda_0, \lambda_1]} \left| j'_m \partial_\lambda^q (I_{j_m}K_{j_m} - I_{j'_m}K_{j'_m})(\lambda) \right| &= 0, \\ \lim_{m \rightarrow \infty} \sup_{\lambda \in [\lambda_0, \lambda_1]} \left| \partial_\lambda^q (I_{j_m}K_{j_m})(\lambda) \right| &= 0. \end{aligned}$$

Therefore (4.38) and (4.40) imply for any $q \in \mathbb{N}$,

$$\lim_{m \rightarrow \infty} |l_m|^{-1} \partial_\alpha^q \left(\Omega_{j_m}^E(\alpha) \pm \Omega_{j'_m}^E(\alpha) \right) \Big|_{\alpha=\alpha_m} = \partial_\alpha^q \left(\bar{d}V_0(\alpha) + |\bar{l}|^{-1} \left(\frac{1}{2} \pm \frac{1}{2} \right) \right) \Big|_{\alpha=\bar{\alpha}}.$$

By taking the limit as $m \rightarrow \infty$ in (4.37), we find

$$\forall q \in \mathbb{N}, \quad \partial_\alpha^q \left(\omega_{\text{Eq}}(\alpha) \cdot \bar{c} + \bar{d}V_0(\alpha) + |\bar{l}|^{-1} \left(\frac{1}{2} \pm \frac{1}{2} \right) \right) \Big|_{\alpha=\bar{\alpha}} = 0.$$

Thus, the analytic function $\alpha \mapsto \omega_{\text{Eq}}(\alpha) \cdot \bar{c} + \bar{d}V_0(\alpha) + |\bar{l}|^{-1} \left(\frac{1}{2} \pm \frac{1}{2} \right)$ with $(\bar{c}, \bar{d}) \neq 0$ and $(\bar{c}, \bar{d}, |\bar{l}|^{-1}) \neq 0$ is vanishing which contradicts Lemma 4.8.

► Now we treat the case where the sequence $(l_m)_m$ is unbounded. Up to an extraction we can assume that

$$\lim_{m \rightarrow \infty} |l_m| = \infty.$$

We shall distinguish three sub-cases.

• We first assume that the sequences $(j_m)_m$ and $(j'_m)_m$ are bounded. Hence we have the stationary behaviour (4.39) up to extraction. Then taking the limit in (4.37) yields,

$$\forall q \in \mathbb{N}, \quad \partial_\alpha^q \omega_{\text{Eq}}(\bar{\alpha}) \cdot \bar{c} = 0.$$

which leads to a contradiction as before.

• The case where the sequences $(j_m)_m$ and $(j'_m)_m$ are both unbounded is similar to what has been done previously.

• Now we assume that the sequence $(j_m)_m$ is unbounded and $(j'_m)_m$ is bounded (the symmetric case is similar). Without loss of generality we can assume that

$$\lim_{m \rightarrow \infty} j_m = \infty, \quad j'_m = \bar{j}.$$

One obtains from (4.40) and (4.38)

$$\forall q \in \mathbb{N}, \quad \lim_{m \rightarrow \infty} |l_m|^{-1} \partial_\alpha^q \left(\Omega_{j_m}^E(\alpha) \pm \Omega_{j'_m}^E(\alpha) \right) \Big|_{\alpha=\alpha_m} = \bar{d} \partial_\alpha^q V_0(\bar{\alpha}).$$

Consequently, taking the limit $m \rightarrow \infty$ in (4.37) gives

$$\forall q \in \mathbb{N}, \quad \partial_\alpha^q \left(\omega_{\text{Eq}}(\alpha) \cdot \bar{c} + \bar{d}V_0(\alpha) \right) \Big|_{\alpha=\bar{\alpha}} = 0.$$

Thus, the analytic function $\alpha \mapsto \omega_{\text{Eq}}(\alpha) \cdot \bar{c} + \bar{d}V_0(\alpha)$ is identically zero with $(\bar{c}, \bar{d}) \neq 0$ which contradicts Lemma 4.8. This completes the proof of Lemma 4.9. \square

4.4. The functional of interest and associated tame estimates. In this subsection, we shall reformulate the problem in terms of embedded tori through the introduction of action-angle variables. This leads to look for the zeros of a nonlinear functional. Observe that the equation (4.11) can be seen as a quasilinear perturbation of its linearization at the equilibrium state, namely

$$(4.41) \quad \begin{aligned} \partial_t r &= \partial_\theta L(\alpha)(r) + X_{\mathcal{P}}(r), \\ X_{\mathcal{P}}(r) &\triangleq \left[\frac{1}{2} - I_1 \left(\frac{1}{\alpha} \right) K_1 \left(\frac{1}{\alpha} \right) \right] \partial_\theta r + \partial_\theta \mathcal{K}_\alpha * r + F^{\text{E}}[r] + F^{\text{SW}}[r], \end{aligned}$$

with F^{E} , F^{SW} and $L(\alpha)$ as in (4.8), (4.9) and (4.25). The smallness property is encoded by the introduction of a small parameter ε . Then we consider the rescaling $r \mapsto \varepsilon r$ with r bounded. Therefore (4.11) becomes

$$(4.42) \quad \partial_t r = \partial_\theta L(\alpha)(r) + \varepsilon X_{\mathcal{P}_\varepsilon}(r), \quad X_{\mathcal{P}_\varepsilon}(r) \triangleq \varepsilon^{-2} X_{\mathcal{P}}(\varepsilon r).$$

Remark that (4.42) can be written in the Hamiltonian form

$$\partial_t r = \partial_\theta \nabla \mathcal{H}_\varepsilon(r), \quad \mathcal{H}_\varepsilon(r) \triangleq \varepsilon^{-2} \mathcal{H}(\varepsilon r) \triangleq \mathcal{H}_L(r) + \varepsilon \mathcal{P}_\varepsilon(r),$$

with \mathcal{H}_L as in (4.26) and $\varepsilon \mathcal{P}_\varepsilon(r)$ containing the higher order terms more than cubic. We shall now reformulate the problem in terms of embedded tori. For that purpose, we introduce the action-angle variables. This is done in the following way. Introducing the symmetrized tangential sets $\bar{\mathbb{S}}$ and \mathbb{S}_0 associated to \mathbb{S} in (4.32)

$$\bar{\mathbb{S}} \triangleq \mathbb{S} \cup (-\mathbb{S}) = \{\pm j, j \in \mathbb{S}\}, \quad \mathbb{S}_0 \triangleq \bar{\mathbb{S}} \cup \{0\},$$

we can split the phase space $L_0^2(\mathbb{T})$ into tangential and normal subspaces

$$(4.43) \quad L_0^2(\mathbb{T}) = L_{\bar{\mathbb{S}}} \oplus L_{\perp}^2,$$

with

$$L_{\bar{\mathbb{S}}} \triangleq \left\{ v = \sum_{j \in \bar{\mathbb{S}}} v_j \mathbf{e}_j, v_{-j} = \overline{v_j} \right\}, \quad L_{\perp}^2 \triangleq \left\{ z = \sum_{j \in \mathbb{Z} \setminus \mathbb{S}_0} z_j \mathbf{e}_j, z_{-j} = \overline{z_j} \right\}.$$

Therefore, we can decompose any $r \in L_0^2(\mathbb{T})$ as follows

$$r = v + z, \quad v = \Pi_{\mathbb{S}_0} r \triangleq \sum_{j \in \bar{\mathbb{S}}} r_j \mathbf{e}_j \in L_{\bar{\mathbb{S}}}, \quad z = \Pi_{\mathbb{S}_0}^\perp r \triangleq \sum_{j \in \mathbb{Z} \setminus \mathbb{S}_0} r_j \mathbf{e}_j \in L_{\perp}^2.$$

We consider small amplitudes

$$(\mathbf{a}_j)_{j \in \bar{\mathbb{S}}} \in (\mathbb{R}_+^*)^d, \quad \mathbf{a}_{-j} = \mathbf{a}_j$$

and introduce the action-angle variables

$$(I, \vartheta) = \left((I_j)_{j \in \bar{\mathbb{S}}}, (\vartheta_j)_{j \in \bar{\mathbb{S}}} \right), \quad I_{-j} = I_j \in \mathbb{R}, \quad \vartheta_{-j} = -\vartheta_j \in \mathbb{T}$$

such that on the tangential set $L_{\bar{\mathbb{S}}}$ we have

$$(4.44) \quad \forall r \in L_{\bar{\mathbb{S}}}, \quad r = \sum_{j \in \bar{\mathbb{S}}} \sqrt{\mathbf{a}_j^2 + |j| I_j} e^{i\vartheta_j} \mathbf{e}_j \triangleq v(I, \vartheta).$$

This defines an application

$$\begin{aligned} \mathbf{A} : \mathbb{T}^d \times \mathbb{R}^d \times L_{\perp}^2(\mathbb{T}) &\rightarrow L_0^2(\mathbb{T}) \\ (\vartheta, I, z) &\mapsto r = v(\vartheta, I) + z \end{aligned}$$

which is symplectic with respect to the symplectic structure \mathscr{W} defined in (4.13). We refer for instance to [43] for a proof of this result. In these coordinates the new Hamiltonian system is generated by the Hamiltonian

$$(4.45) \quad \mathscr{H}_\varepsilon \triangleq \mathscr{H}_\varepsilon \circ \mathbf{A} = \mathscr{N} + \varepsilon \mathscr{P}_\varepsilon, \quad \mathscr{N} \triangleq -\omega_{\text{Eq}}(\alpha) \cdot I + \frac{1}{2} \langle \mathbf{L}(\alpha) z, z \rangle_{L^2(\mathbb{T})}, \quad \mathscr{P}_\varepsilon \triangleq \mathscr{P}_\varepsilon \circ \mathbf{A}.$$

We look for an embedded invariant torus

$$\begin{aligned} i : \mathbb{T}^d &\rightarrow \mathbb{T}^d \times \mathbb{R}^d \times L^2_\perp \\ \varphi &\mapsto i(\varphi) \triangleq (\vartheta(\varphi), I(\varphi), z(\varphi)) \end{aligned}$$

of the Hamiltonian vector field

$$(4.46) \quad X_{\mathscr{H}_\varepsilon} \triangleq (\partial_I \mathscr{H}_\varepsilon, -\partial_\vartheta \mathscr{H}_\varepsilon, \Pi_{\mathbb{S}^0}^\perp \partial_\theta \nabla_z \mathscr{H}_\varepsilon)$$

filled by quasi-periodic solutions with Diophantine frequency vector ω . Note that for value $\varepsilon = 0$, the Hamiltonian system

$$\omega \cdot \partial_\varphi i(\varphi) = X_{\mathscr{H}_0}(i(\varphi))$$

possesses, for any value of the parameter $\alpha \in (\alpha_0, \alpha_1)$, the flat invariant torus $(\varphi, 0, 0)$. Similarly to [11, 41, 43, 52], we shall introduce a free parameter $\kappa \in \mathbb{R}^d$ to deal with zero φ -average conditions in the construction of an almost approximate right inverse for the linearized operator and therefore consider the following family of modified Hamiltonians,

$$(4.47) \quad \mathscr{H}_\varepsilon^\kappa \triangleq \mathscr{N}_\kappa + \varepsilon \mathscr{P}_\varepsilon, \quad \mathscr{N}_\kappa \triangleq \kappa \cdot I + \frac{1}{2} \langle \mathbf{L}(\alpha) z, z \rangle_{L^2(\mathbb{T})}.$$

We mention that the original problem is recovered by taking $\kappa = -\omega_{\text{Eq}}(\alpha)$. According to (4.46) (replacing \mathscr{H}_ε by $\mathscr{H}_\varepsilon^\kappa$) and (4.47), we are now interested in finding non-trivial zeros of the nonlinear functional

$$(4.48) \quad \begin{aligned} \mathscr{F}(i, \kappa, (\alpha, \omega), \varepsilon) &\triangleq \omega \cdot \partial_\varphi i(\varphi) - X_{\mathscr{H}_\varepsilon^\kappa}(i(\varphi)) \\ &= \begin{pmatrix} \omega \cdot \partial_\varphi \vartheta(\varphi) - \kappa - \varepsilon \partial_I \mathscr{P}_\varepsilon(i(\varphi)) \\ \omega \cdot \partial_\varphi I(\varphi) + \varepsilon \partial_\theta \mathscr{P}_\varepsilon(i(\varphi)) \\ \omega \cdot \partial_\varphi z(\varphi) - \partial_\theta [\mathbf{L}(\alpha) z(\varphi) + \varepsilon \nabla_z \mathscr{P}_\varepsilon(i(\varphi))] \end{pmatrix}. \end{aligned}$$

We point out that we can easily check that the Hamiltonian $\mathscr{H}_\varepsilon^\kappa$ is reversible in the sense of the Definition 4.1, that is,

$$\mathscr{H}_\varepsilon^\kappa \circ \mathbf{S} = \mathscr{H}_\varepsilon^\kappa, \quad \mathbf{S}(\vartheta, I, z) \triangleq (-\vartheta, I, \mathscr{S}z),$$

with \mathscr{S} as in (4.14). Thus, we look for reversible tori solutions of $\mathscr{F}(i, \kappa, (\alpha, \omega), \varepsilon) = 0$, that is satisfying

$$\mathbf{S}i(\varphi) = i(-\varphi), \quad \text{i.e.} \quad \vartheta(-\varphi) = -\vartheta(\varphi), \quad I(-\varphi) = I(\varphi), \quad z(-\varphi) = (\mathscr{S}z)(\varphi).$$

Now we define the periodic component \mathfrak{J} of the torus i together with its Sobolev norm by

$$\begin{aligned} \mathfrak{J}(\varphi) &\triangleq i(\varphi) - (\varphi, 0, 0) = (\Theta(\varphi), I(\varphi), z(\varphi)), \quad \Theta(\varphi) \triangleq \vartheta(\varphi) - \varphi, \\ \|\mathfrak{J}\|_{q,s}^{\gamma,\mathcal{O}} &\triangleq \|\Theta\|_{q,s}^{\gamma,\mathcal{O}} + \|I\|_{q,s}^{\gamma,\mathcal{O}} + \|z\|_{q,s}^{\gamma,\mathcal{O}}. \end{aligned}$$

The norm $\|\cdot\|_{q,s}^{\gamma,\mathcal{O}}$ is defined in Section 4.1. We shall fix q as follows

$$(4.49) \quad q \triangleq q_0 + 2,$$

with the q_0 appearing in Lemma 4.9. This particular choice is relevant in the final subsection when checking the perturbed Rüssmann conditions. We also define the set of parameters \mathcal{O} as follows

$$(4.50) \quad \mathcal{O} \triangleq (\alpha_0, \alpha_1) \times B(0, R_0) \subset \mathbb{R}^{d+1},$$

where the open ball $B(0, R_0) \subset \mathbb{R}^d$ with $R_0 > 0$ is chosen such that

$$\omega_{\text{Eq}}((\alpha_0, \alpha_1)) \subset B(0, \frac{R_0}{2}).$$

This is well-defined by continuity of the application ω_{Eq} in (4.33). This particular structure is chosen so that the perturbed frequency vector in Section 4.6 is included in the corresponding component of \mathcal{O} in order to check trivial inclusions. We shall now prove tame estimates for $X_{\mathcal{O}}$.

LEMMA 4.10. *Let (γ, s_0, s, q) satisfying (4.4), (4.3) and (4.49). There exists $\varepsilon_0 \in (0, 1]$ such that if*

$$\|r\|_{q, s_0+2}^{\gamma, \mathcal{O}} \leq \varepsilon_0,$$

then we have the following estimates for the vector field $X_{\mathcal{O}}$ in (4.41)

- (i) $\|X_{\mathcal{O}}(r)\|_{q, s}^{\gamma, \mathcal{O}} \lesssim \|r\|_{q, s+2}^{\gamma, \mathcal{O}} \|r\|_{q, s_0+2}^{\gamma, \mathcal{O}}$.
- (ii) $\|d_r X_{\mathcal{O}}(r)[\rho]\|_{q, s}^{\gamma, \mathcal{O}} \lesssim \|\rho\|_{q, s+2}^{\gamma, \mathcal{O}} \|r\|_{q, s_0+2}^{\gamma, \mathcal{O}} + \|r\|_{q, s+2}^{\gamma, \mathcal{O}} \|\rho\|_{q, s_0+2}^{\gamma, \mathcal{O}}$.
- (iii) $\|d_r^2 X_{\mathcal{O}}(r)[\rho_1, \rho_2]\|_{q, s}^{\gamma, \mathcal{O}} \lesssim \|\rho_1\|_{q, s_0+2}^{\gamma, \mathcal{O}} \|\rho_2\|_{q, s+2}^{\gamma, \mathcal{O}} + \|\rho_1\|_{q, s+2}^{\gamma, \mathcal{O}} \|\rho_2\|_{q, s_0+2}^{\gamma, \mathcal{O}} + \|r\|_{q, s+2}^{\gamma, \mathcal{O}} \|\rho_1\|_{q, s_0+2}^{\gamma, \mathcal{O}} \|\rho_2\|_{q, s_0+2}^{\gamma, \mathcal{O}}$.

PROOF. It suffices to prove the estimate (iii). Indeed, the estimates (i) and (ii) are consequences of (iii) by a direct application of Taylor formula since $X_{\mathcal{O}}(0) = 0$ and $d_r X_{\mathcal{O}}(0) = 0$. Recall from (4.15) and (4.18) that

$$\begin{aligned} d_r X_{\mathcal{O}}(r)[\rho] &= \left[\frac{1}{2} - I_1 \left(\frac{1}{\alpha} \right) K_1 \left(\frac{1}{\alpha} \right) \right] \partial_\theta \rho + \partial_\theta \mathcal{K}_\alpha * \rho + d_r F^{\text{E}}(r)[\rho] + d_r F^{\text{SW}}(r)[\rho] \\ &= \left[\frac{1}{2} - I_1 \left(\frac{1}{\alpha} \right) K_1 \left(\frac{1}{\alpha} \right) \right] \partial_\theta \rho + \partial_\theta \mathcal{K}_\alpha * \rho \\ &\quad + \partial_\theta (V_r^{\text{E}} \rho) - \partial_\theta \mathbf{L}_r^{\text{E}}(\rho) + \partial_\theta (V_r^{\text{SW}} \rho) - \partial_\theta \mathbf{L}_r^{\text{SW}}(\rho). \end{aligned}$$

Differentiating the last expression with respect to r yields

$$\begin{aligned} d_r^2 X_{\mathcal{O}}(r)[\rho_1, \rho_2] &= \partial_\theta ((d_r V_r^{\text{E}}[\rho_2])\rho_1) - \partial_\theta (d_r \mathbf{L}_r^{\text{E}}[\rho_2]\rho_1) \\ &\quad + \partial_\theta ((d_r V_r^{\text{SW}}[\rho_2])\rho_1) - \partial_\theta (d_r \mathbf{L}_r^{\text{SW}}[\rho_2]\rho_1). \end{aligned}$$

But it has been proved in [43, Lem. 5.2] and [52, Lem. 5.2] that each term in the right hand-side of the previous equality satisfy an estimate as in (iii) in the statement of this lemma which concludes the proof of this latter. \square

Consequently proceeding as for [52, Lem. 5.3], the previous lemma implies the following one stating tame estimates for the perturbed Hamiltonian vector field in the action-angle-normal variables.

LEMMA 4.11. *Let (γ, s_0, s, q) satisfy (4.4), (4.3) and (4.49). There exists $\varepsilon_0 \in (0, 1)$ such that if*

$$\varepsilon \leq \varepsilon_0, \quad \|\mathfrak{J}\|_{q, s_0+2}^{\gamma, \mathcal{O}} \leq 1,$$

then the perturbed Hamiltonian vector field in the new variables

$$X_{\mathcal{P}_\varepsilon} = (\partial_I \mathcal{P}_\varepsilon, -\partial_\vartheta \mathcal{P}_\varepsilon, \Pi_{\mathbb{S}}^\perp \partial_\theta \nabla_z \mathcal{P}_\varepsilon)$$

defined through (4.45) and (4.46) satisfies the following tame estimates

- (i) $\|X_{\mathcal{P}_\varepsilon}(i)\|_{q,s}^{\gamma,\mathcal{O}} \lesssim 1 + \|\mathfrak{J}\|_{q,s+2}^{\gamma,\mathcal{O}}$.
- (ii) $\|d_i X_{\mathcal{P}_\varepsilon}(i)[\widehat{i}]\|_{q,s}^{\gamma,\mathcal{O}} \lesssim \|\widehat{i}\|_{q,s+2}^{\gamma,\mathcal{O}} + \|\mathfrak{J}\|_{q,s+2}^{\gamma,\mathcal{O}} \|\widehat{i}\|_{q,s_0+2}^{\gamma,\mathcal{O}}$.
- (iii) $\|d_i^2 X_{\mathcal{P}_\varepsilon}(i)[\widehat{i}, \widehat{i}]\|_{q,s}^{\gamma,\mathcal{O}} \lesssim \|\widehat{i}\|_{q,s+2}^{\gamma,\mathcal{O}} \|\widehat{i}\|_{q,s_0+2}^{\gamma,\mathcal{O}} + \|\mathfrak{J}\|_{q,s+2}^{\gamma,\mathcal{O}} \left(\|\widehat{i}\|_{q,s_0+2}^{\gamma,\mathcal{O}}\right)^2$.

4.5. Almost approximate right inverse. The aim of this section is to construct for any vector-valued function $\kappa_0 : \mathcal{O} \rightarrow \mathbb{R}^d$ and any reversible torus $i_0 = (\vartheta_0, I_0, z_0)$ close to the flat one an almost approximate right inverse for the linearized operator

$$(4.51) \quad d_{(i,\kappa)} \mathcal{F}(i_0, \kappa_0)[\widehat{i}, \widehat{\kappa}] = \omega \cdot \partial_\varphi \widehat{i} - d_i X_{\mathcal{H}_\varepsilon^{\kappa_0}}(i_0(\varphi))[\widehat{i}] - (\widehat{\kappa}, 0, 0)$$

associated with the functional \mathcal{F} defined in (4.48). For this purpose, we use the Berti-Bolle theory developed in [11] and [41, Sec. 6]. Namely, we can find a linear diffeomorphism of the toroidal phase space $\mathbb{T}^d \times \mathbb{R}^d \times L^2_\perp$ such that the conjugation of (4.51) by this application is a triangular system in the action-angles-normal variables up to error terms either vanishing at an exact solution or small fast decaying. The key point to solve the triangular system is that it is sufficient to almost invert the linearized operator in the normal directions. According to the computations done in [11, 41], this latter admits the following form

$$(4.52) \quad \mathcal{L}_\perp \triangleq \mathcal{L}_\perp(i_0) \triangleq \Pi_{\mathbb{S}_0}^\perp \left(\omega \cdot \partial_\varphi - \partial_\theta \partial_z \nabla_z \mathcal{H}_\varepsilon^{\kappa_0}(i_0(\varphi)) - \varepsilon \partial_\theta \mathcal{R}(\varphi) \right) \Pi_{\mathbb{S}_0}^\perp,$$

where $\mathcal{H}_\varepsilon^{\kappa_0}$ is as in (4.47) and $\mathcal{R}(\varphi)$ is a remainder operator coming from a coupling with the tangential part and given by

$$\mathcal{R}(\varphi) \triangleq L_2^\top(\varphi) \partial_I \nabla_I \mathcal{P}_\varepsilon(i_0(\varphi)) L_2(\varphi) + L_2^\top(\varphi) \partial_z \nabla_I \mathcal{P}_\varepsilon(i_0(\varphi)) + \partial_I \nabla_z \mathcal{P}_\varepsilon(i_0(\varphi)) L_2(\varphi),$$

with \mathcal{P}_ε as in (4.45) and

$$L_2 : \mathbb{R}^d \rightarrow L^2_\perp, \quad L_2(\phi) \triangleq -[(\partial_\vartheta \tilde{z}_0)(\vartheta_0(\phi))]^\top \partial_\theta^{-1}, \quad \tilde{z}_0(\vartheta) \triangleq z_0(\vartheta_0^{-1}(\vartheta)).$$

We used the following definition by duality for the transposed operator $L_2^\top : L^2_\perp \rightarrow \mathbb{R}^d$

$$\forall u \in L^2_\perp, \quad \forall v \in \mathbb{R}^d, \quad \langle L_2^\top(\varphi)u, v \rangle_{\mathbb{R}^d} = \langle u, L_2(\varphi)v \rangle_{L^2(\mathbb{T}^d)}.$$

Furthermore, we can have a more explicit decomposition of the operator \mathcal{L}_\perp . The result is described in the following proposition whose proof is similar to [52, Prop. 6.1] or [43, Prop. 6.1].

PROPOSITION 4.2. *Let (d, γ, s_0, q) satisfy (4.2), (4.4), (4.3) and (4.49). Then, the operator \mathcal{L}_\perp in (4.52) writes*

$$(4.53) \quad \mathcal{L}_\perp = \Pi_{\mathbb{S}_0}^\perp \left(\mathcal{L}_{\varepsilon r} - \varepsilon \partial_\theta \mathcal{R} \right) \Pi_{\mathbb{S}_0}^\perp,$$

where the operator $\mathcal{L}_{\varepsilon r}$ is defined as follows with $V_{\varepsilon r}$ and $\mathbf{L}_{\varepsilon r}$ obtained from (4.21)-(4.22),

$$(4.54) \quad \mathcal{L}_{\varepsilon r} \triangleq \omega \cdot \partial_\varphi + \partial_\theta (V_{\varepsilon r} \cdot) + \partial_\theta \mathbf{L}_{\varepsilon r}.$$

The function r is linked to the reversible torus i_0 in the following way

$$(4.55) \quad r(\varphi, \cdot) \triangleq \mathbf{A}(i_0(\varphi)), \quad r(-\varphi, -\theta) = r(\varphi, \theta)$$

and satisfies the following estimates

$$(4.56) \quad \begin{aligned} \|r\|_{q,s}^{\gamma,\mathcal{O}} &\lesssim 1 + \|\mathfrak{J}_0\|_{q,s}^{\gamma,\mathcal{O}}, \\ \|\Delta_{12} r\|_{q,s}^{\gamma,\mathcal{O}} &\lesssim \|\Delta_{12} i\|_{q,s}^{\gamma,\mathcal{O}} + \|\Delta_{12} i\|_{q,s_0}^{\gamma,\mathcal{O}} \max_{j \in \{1,2\}} \|\mathfrak{J}_j\|_{q,s}^{\gamma,\mathcal{O}}. \end{aligned}$$

Here and in the sequel, we shall use the following notation: given two tori i_1 and i_2 , for any function $f = f(i)$ depending on a generic torus i , we denote

$$\Delta_{12}f(i) \triangleq f(i_2) - f(i_1).$$

Finally, the operator \mathcal{R} is an integral operator with kernel \mathcal{J} satisfying the symmetry property

$$(4.57) \quad \mathcal{J}(-\varphi, -\theta, -\eta) = \mathcal{J}(\varphi, \theta, \eta)$$

and the following estimates for all $\ell \in \mathbb{N}$,

$$(4.58) \quad \sup_{\eta \in \mathbb{T}} \|(\partial_\theta^\ell \mathcal{J})(*, \cdot, \cdot, \eta + \bullet)\|_{q,s}^{\gamma, \mathcal{O}} \lesssim 1 + \|\mathfrak{J}_0\|_{q,s+3+\ell}^{\gamma, \mathcal{O}}$$

and

$$(4.59) \quad \sup_{\eta \in \mathbb{T}} \|\Delta_{12}(\partial_\theta^\ell \mathcal{J})(*, \cdot, \cdot, \eta + \bullet)\|_{q,s}^{\gamma, \mathcal{O}} \lesssim \|\Delta_{12}i\|_{q,s+3+\ell}^{\gamma, \mathcal{O}} + \|\Delta_{12}i\|_{q,s_0+3}^{\gamma, \mathcal{O}} \max_{j \in \{1,2\}} \|\mathfrak{J}_j\|_{q,s+3+\ell}^{\gamma, \mathcal{O}}.$$

where $*, \cdot, \bullet$, denote the variables α, φ, θ and $\mathfrak{J}_\ell(\varphi) \triangleq i_\ell(\varphi) - (\varphi, 0, 0)$.

Now we shall start the reduction procedure of the operator \mathcal{L}_\perp . The first step is the reduction of the transport part of $\mathcal{L}_{\varepsilon r}$ in (4.54), which is done by conjugation with quasi-periodic symplectic change of variables close to the identity similarly to [5, 15, 31, 52]. Actually, we apply here [52, Prop. 6.2]. The result is the following.

PROPOSITION 4.3. *Let $(d, s_0, S, \gamma, q, q_0)$ satisfy (4.2), (4.3), (4.4) and (4.49). We consider the following parameters*

$$(4.60) \quad \begin{aligned} v &\triangleq \frac{1}{q_0+4}, & \tau_1 &\triangleq dq_0 + 1, \\ s_l &\triangleq s_0 + \tau_1 q + \tau_1 + 2, & \bar{\mu}_2 &\triangleq 4\tau_1 q + 6\tau_1 + 3, \\ \bar{s}_l &\triangleq s_l + \tau_2 q + \tau_2, & \bar{s}_h &\triangleq \frac{3}{2}\bar{\mu}_2 + s_l + 1, \\ \sigma_1 &\triangleq s_0 + \tau_1 q + 2\tau_1 + 4, & \sigma_2 &\triangleq s_0 + \sigma_1 + 3. \end{aligned}$$

We also denote

$$N_n \triangleq N_0^{\left(\frac{3}{2}\right)^n}, \quad N_0 \geq 2.$$

For every choice of additional parameters (μ_2, \mathbf{p}, s_h) with the constraints

$$(4.61) \quad \mu_2 \geq \bar{\mu}_2, \quad \mathbf{p} \geq 0, \quad s_h \geq \max\left(\frac{3}{2}\mu_2 + s_l + 1, \bar{s}_h + \mathbf{p}\right),$$

there exists $\varepsilon_0 > 0$ such that if the following smallness condition holds

$$(4.62) \quad \varepsilon \gamma^{-1} N_0^{\mu_2} \leq \varepsilon_0, \quad \|\mathfrak{J}_0\|_{q,s_h+\sigma_2}^{\gamma, \mathcal{O}} \leq 1,$$

then there exist $\mathbf{m}_{i_0}^\infty \in W^{q,\infty,\gamma}(\mathcal{O}, \mathbb{C})$ ε -close to the equilibrium one V_0 in (4.25), namely

$$(4.63) \quad \|\mathbf{m}_{i_0}^\infty - V_0\|_q^{\gamma, \mathcal{O}} \lesssim \varepsilon$$

and an invertible quasi-periodic symplectic change of variables \mathcal{B} in the form

$$\mathcal{B} \triangleq (1 + \partial_\theta \beta)\mathcal{B}, \quad \mathcal{B}\rho(\mu, \varphi, \theta) \triangleq \rho(\mu, \varphi, \theta + \beta(\mu, \varphi, \theta)), \quad \beta \in W^{q,\infty,\gamma}(\mathcal{O}, H^S)$$

with inverse \mathcal{B}^{-1} in the form

$$\mathcal{B}^{-1} = (1 + \partial_y \hat{\beta})\mathcal{B}^{-1}, \quad \mathcal{B}^{-1}\rho(\mu, \varphi, y) = \rho(\mu, \varphi, y + \hat{\beta}(\mu, \varphi, y))$$

with

$$y = \theta + \beta(\mu, \varphi, \theta) \quad \Leftrightarrow \quad \theta = y + \hat{\beta}(\mu, \varphi, y)$$

enjoying the symmetry properties

$$(4.64) \quad \beta(\alpha, \omega, -\varphi, -\theta) = -\beta(\alpha, \omega, \varphi, \theta), \quad \widehat{\beta}(\alpha, \omega, -\varphi, -\theta) = -\widehat{\beta}(\alpha, \omega, \varphi, \theta)$$

and the following estimates for all $s \in [s_0, S]$

$$(4.65) \quad \|\mathcal{B}^{\pm 1} \rho\|_{q,s}^{\gamma, \mathcal{O}} + \|\mathcal{B}^{\pm 1} \rho\|_{q,s}^{\gamma, \mathcal{O}} \lesssim \|\rho\|_{q,s}^{\gamma, \mathcal{O}} + \varepsilon \gamma^{-1} \|\mathcal{J}_0\|_{q,s+\sigma_1}^{\gamma, \mathcal{O}} \|\rho\|_{q,s_0}^{\gamma, \mathcal{O}}$$

and

$$(4.66) \quad \|\widehat{\beta}\|_{q,s}^{\gamma, \mathcal{O}} \lesssim \|\beta\|_{q,s}^{\gamma, \mathcal{O}} \lesssim \varepsilon \gamma^{-1} \left(1 + \|\mathcal{J}_0\|_{q,s+\sigma_1}^{\gamma, \mathcal{O}}\right)$$

such that for any $n \in \mathbb{N}$, then in restriction to the Cantor-like set

$$(4.67) \quad \mathcal{O}_{\text{transport}}^{n, \gamma, \tau_1}(i_0) \triangleq \bigcap_{\substack{(l,j) \in \mathbb{Z}^d \times \mathbb{Z} \setminus \{(0,0)\} \\ |l| \leq N_n}} \left\{ (\alpha, \omega) \in \mathcal{O} \quad \text{s.t.} \quad |\omega \cdot l + j \mathbf{m}_{i_0}^\infty(\alpha, \omega)| > \frac{4\gamma^v \langle j \rangle}{\langle l \rangle^{\tau_1}} \right\},$$

the following reduction holds

$$(4.68) \quad \mathfrak{L}_{\varepsilon r} \triangleq \mathcal{B}^{-1} \mathcal{L}_{\varepsilon r} \mathcal{B} = \omega \cdot \partial_\varphi + \mathbf{m}_{i_0}^\infty \partial_\theta + \partial_\theta \mathcal{K}_\alpha * \cdot + \partial_\theta \mathbf{R}_{\varepsilon r} + \mathbf{E}_n^0,$$

with \mathcal{K}_α as in (4.25) and $\mathbf{E}_n^0 = \mathbf{E}_n^0(\alpha, \omega, i_0)$ a linear operator satisfying

$$(4.69) \quad \|\mathbf{E}_n^0 \rho\|_{q,s_0}^{\gamma, \mathcal{O}} \lesssim \varepsilon N_0^{\mu_2} N_{n+1}^{-\mu_2} \|\rho\|_{q,s_0+2}^{\gamma, \mathcal{O}}.$$

The operator $\mathbf{R}_{\varepsilon r}$ is a real and reversibility preserving integral operator satisfying

$$(4.70) \quad \forall s \in [s_0, S], \quad \max_{k \in \{0,1,2\}} \|\partial_\theta^k \mathbf{R}_{\varepsilon r}\|_{\mathcal{O}\text{-d}, q, s}^{\gamma, \mathcal{O}} \lesssim \varepsilon \gamma^{-1} \left(1 + \|\mathcal{J}_0\|_{q,s+\sigma_2}^{\gamma, \mathcal{O}}\right).$$

In addition, for any tori i_1 and i_2 both satisfying (4.62), we have

$$(4.71) \quad \begin{aligned} \|\Delta_{12} \mathbf{m}_i^\infty\|_{q,s_h+2}^{\gamma, \mathcal{O}} &\lesssim \varepsilon \|\Delta_{12} i\|_{q,s_h+2}^{\gamma, \mathcal{O}}, \\ \|\Delta_{12} \beta\|_{q,s_h+p}^{\gamma, \mathcal{O}} + \|\Delta_{12} \widehat{\beta}\|_{q,s_h+p}^{\gamma, \mathcal{O}} &\lesssim \varepsilon \gamma^{-1} \|\Delta_{12} i\|_{q,s_h+p+\sigma_1}^{\gamma, \mathcal{O}} \end{aligned}$$

and

$$(4.72) \quad \max_{k \in \{0,1\}} \|\Delta_{12} (\partial_\theta^k \mathbf{R}_{\varepsilon r})\|_{\mathcal{O}\text{-d}, q, s_h+p}^{\gamma, \mathcal{O}} \lesssim \varepsilon \gamma^{-1} \|\Delta_{12} i\|_{q,s_h+p+\sigma_2}^{\gamma, \mathcal{O}}.$$

PROOF. Using the computations in [52, Prop. 6.2] and [43, Prop. 6.2] for the $V_{\varepsilon r}^E$ and $V_{\varepsilon r}^{\text{SW}}$ in (4.16)-(4.19), we get the following estimate for $V_{\varepsilon r}$ in (4.22),

$$\|V_{\varepsilon r} - V_0\|_{q,s}^{\gamma, \mathcal{O}} \leq \|V_{\varepsilon r}^E\|_{q,s}^{\gamma, \mathcal{O}} + \frac{1}{2} \|\gamma\|_{q,s}^{\gamma, \mathcal{O}} + \|V_{\varepsilon r}^{\text{SW}} - I_1 K_1\|_{q,s}^{\gamma, \mathcal{O}} \lesssim \varepsilon \left(1 + \|\mathcal{J}_0\|_{q,s+1}^{\gamma, \mathcal{O}}\right).$$

This allows us to start a KAM reduction procedure as detailed in [52, prop 6.2], which provides $\mathcal{O}_{\text{transport}}^{n, \gamma, \tau_1}(i_0)$, $\mathbf{m}_{i_0}^\infty$, β and \mathbf{E}_n^0 as in the statement, with the corresponding estimates (4.65), (4.66), (4.69), (4.71) and symmetry (4.64) such that in restriction to $\mathcal{O}_{\text{transport}}^{n, \gamma, \tau_1}(i_0)$ we have

$$\mathcal{B}^{-1} \left(\omega \cdot \partial_\varphi + \partial_\theta (V_{\varepsilon r} \cdot) \right) \mathcal{B} = \omega \cdot \partial_\varphi + \mathbf{m}_{i_0}^\infty \partial_\theta + \mathbf{E}_n^0.$$

Now, we shall look at the conjugation effect on the nonlocal term. Referring to [52, Lem. 5.1] and [43, Lem. 5.1], we have the following decompositions for the

operators in (4.17)-(4.20),

(4.73)

$$\partial_\theta \mathbf{L}_{\varepsilon r}^E = \partial_\theta \mathcal{K} * \cdot + \partial_\theta \mathbf{L}_{\varepsilon r,1}^E, \quad \mathbf{L}_{\varepsilon r,1}^E(\rho)(\varphi, \theta) \triangleq \int_{\mathbb{T}} \rho(\varphi, \eta) \mathbb{K}_{\varepsilon r}^E(\varphi, \theta, \eta) d\eta,$$

(4.74)

$$\partial_\theta \mathbf{L}_{\varepsilon r}^{\text{SW}} = \partial_\theta \mathcal{Q}_\alpha * \cdot + \partial_\theta \mathbf{L}_{\varepsilon r,1}^{\text{SW}}, \quad \mathbf{L}_{\varepsilon r,1}^{\text{SW}}(\rho)(\alpha, \varphi, \theta) \triangleq \int_{\mathbb{T}} \rho(\varphi, \eta) \mathbb{K}_{\varepsilon r}^{\text{SW}}(\alpha, \varphi, \theta, \eta) d\eta,$$

where the kernels $\mathbb{K}_{\varepsilon r}^E$ and $\mathbb{K}_{\varepsilon r}^{\text{SW}}$ are respectively defined by

$$\begin{aligned} \mathbb{K}_{\varepsilon r}^E(\varphi, \theta, \eta) &\triangleq \log(v_{\varepsilon r}(\varphi, \theta, \eta)), \\ v_{\varepsilon r}(\varphi, \theta, \eta) &\triangleq \left(\left(\frac{R(\varphi, \theta) - R(\varphi, \eta)}{2 \sin(\frac{\theta - \eta}{2})} \right)^2 + R(\varphi, \theta) R(\varphi, \eta) \right)^{\frac{1}{2}}, \\ \mathbb{K}_{\varepsilon r}^{\text{SW}}(\varphi, \theta, \eta) &\triangleq \mathcal{H}(\theta - \eta) \mathcal{H}_{\varepsilon r,1}(\lambda, \varphi, \theta, \eta) + \mathcal{H}_{\varepsilon r,2}(\lambda, \varphi, \theta, \eta), \\ \mathcal{H}(\theta) &\triangleq \sin^2\left(\frac{\theta}{2}\right) \log(|\sin(\frac{\theta}{2})|), \\ \mathcal{H}_{\varepsilon r,1}(\lambda, \varphi, \theta, \eta) &\triangleq \sum_{m=1}^{\infty} \frac{(2\lambda)^{2m}}{(m!)^2} \sin^{2m-2}\left(\frac{\theta - \eta}{2}\right) (1 - v_{\varepsilon r}(\varphi, \theta, \eta)), \quad \lambda \triangleq \frac{1}{\alpha}, \\ \mathcal{H}_{\varepsilon r,2}(\lambda, \varphi, \theta, \eta) &\triangleq \log(\lambda) \sin^2\left(\frac{\eta - \theta}{2}\right) \mathcal{H}_{\varepsilon r,1}(\lambda, \varphi, \theta, \eta) \\ &\quad - \mathbb{K}_{\varepsilon r}^E(\varphi, \theta, \eta) I_0(\lambda A_{\varepsilon r}(\varphi, \theta, \eta)) \\ &\quad + f(\lambda A_{\varepsilon r}(\varphi, \theta, \eta)) - f\left(2\lambda \left| \sin\left(\frac{\eta - \theta}{2}\right) \right|\right), \quad f \text{ analytic.} \end{aligned}$$

One can easily check that (4.55) implies

$$(4.75) \quad \mathbb{K}_{\varepsilon r}^E(-\varphi, -\theta, -\eta) = \mathbb{K}_{\varepsilon r}^E(\varphi, \theta, \eta), \quad \mathbb{K}_{\varepsilon r}^{\text{SW}}(\alpha, -\varphi, -\theta, -\eta) = \mathbb{K}_{\varepsilon r}^{\text{SW}}(\alpha, \varphi, \theta, \eta).$$

Hence, in restriction to $\mathcal{O}_{\text{transport}}^{n, \gamma, \tau_1}(i_0)$ we have by (4.73)-(4.74),

$$\begin{aligned} \mathcal{B}^{-1} \mathcal{L}_{\varepsilon r} \mathcal{B} &= \omega \cdot \partial_\varphi + \mathfrak{m}_{i_0}^\infty \partial_\theta + \partial_\theta \mathcal{B}^{-1} \mathbf{L}_{\varepsilon r}^E \mathcal{B} + \partial_\theta \mathcal{B}^{-1} \mathbf{L}_{\varepsilon r}^{\text{SW}} \mathcal{B} + \mathbf{E}_n^0 \\ &= \omega \cdot \partial_\varphi + \mathfrak{m}_{i_0}^\infty \partial_\theta + \partial_\theta \mathcal{K}_\alpha * \cdot + \partial_\theta \mathbf{R}_{\varepsilon r} + \mathbf{E}_n^0, \end{aligned}$$

with

$$\mathbf{R}_{\varepsilon r} \triangleq \mathcal{B}^{-1} \mathbf{L}_{\varepsilon r,1}^E \mathcal{B} + \mathcal{B}^{-1} \mathbf{L}_{\varepsilon r,1}^{\text{SW}} \mathcal{B} + \left(\mathcal{B}^{-1} (\mathcal{K} * \cdot) \mathcal{B} - \mathcal{K} * \cdot \right) + \left(\mathcal{B}^{-1} (\mathcal{Q}_\alpha * \cdot) \mathcal{B} - \mathcal{Q}_\alpha * \cdot \right).$$

The estimates (4.70) and (4.72) are obtained similarly to [52, Prop. 6.2] and [43, Prop. 6.2] by using Lemma 4.3, Lemma 4.1 and (4.66)-(4.71). In particular, the reversibility property follows from (4.75)-(4.64). This concludes the proof of Proposition 4.3. \square

Then we study the action of the localization in the normal directions. For that, we introduce the operator

$$\mathcal{B}_\perp \triangleq \Pi_{S_0}^\perp \mathcal{B} \Pi_{S_0}^\perp,$$

which satisfies by virtue of (4.65) the following estimate

$$(4.76) \quad \|\mathcal{B}_\perp^{\pm 1} \rho\|_{q,s}^{\gamma, \mathcal{O}} \lesssim \|\rho\|_{q,s}^{\gamma, \mathcal{O}} + \varepsilon \gamma^{-1} \|\mathcal{J}_0\|_{q,s+\sigma_1}^{\gamma, \mathcal{O}} \|\rho\|_{q,s_0}^{\gamma, \mathcal{O}}.$$

Hence, the result reads as follows.

PROPOSITION 4.4. *Let $(d, s_0, S, \gamma, q, q_0, \tau_1, s_h, \bar{s}_h, \sigma_2, \mathbf{p})$ satisfy (4.2), (4.3), (4.4), (4.49), (4.60) and (4.61). There exist $\varepsilon_0 > 0$ and $\sigma_3 = \sigma_3(\tau_1, q, d, s_0) \geq \sigma_2$ such that if the following smallness condition holds*

$$(4.77) \quad \varepsilon\gamma^{-1}N_0^{\mu_2} \leq \varepsilon_0, \quad \|\mathfrak{J}_0\|_{q, s_h + \sigma_3}^{\gamma, \mathcal{O}} \leq 1,$$

then for any $n \in \mathbb{N}^*$, in restriction to the Cantor-like set $\mathcal{O}_{\text{transport}}^{n, \gamma, \tau_1}(i_0)$ defined in (4.67), the following decomposition holds

$$(4.78) \quad \begin{aligned} \mathcal{B}_\perp^{-1} \mathcal{L}_\perp \mathcal{B}_\perp &= (\omega \cdot \partial_\varphi + \mathfrak{m}_{i_0}^\infty \partial_\theta + \partial_\theta \mathcal{K}_\alpha * \cdot) \Pi_{\mathbb{S}_0}^\perp + \mathcal{R}_0 + \mathbf{E}_n^1 \\ &\triangleq \omega \cdot \partial_\varphi \Pi_{\mathbb{S}_0}^\perp + \mathcal{D}_0 + \mathcal{R}_0 + \mathbf{E}_n^1 \\ &\triangleq \mathcal{L}_0 + \mathbf{E}_n^1. \end{aligned}$$

The operator $\mathcal{D}_0 = \Pi_{\mathbb{S}_0}^\perp \mathcal{D}_0 \Pi_{\mathbb{S}_0}^\perp$ is a reversible diagonal operator described by

$$(4.79) \quad \begin{aligned} \mathcal{D}_0 \mathbf{e}_{l,j} &= i d_j^0 \mathbf{e}_{l,j}, \quad d_j^0(\alpha, \omega, i_0) \triangleq \Omega_j^E(\alpha) + j r^0(\alpha, \omega, i_0), \\ r^0(\alpha, \omega, i_0) &\triangleq \mathfrak{m}_{i_0}^\infty(\alpha, \omega) - V_0(\alpha), \end{aligned}$$

with

$$(4.80) \quad \|r^0\|_q^{\gamma, \mathcal{O}} \lesssim \varepsilon, \quad \|\Delta_{12} r^0\|_q^{\gamma, \mathcal{O}} \lesssim \varepsilon \|\Delta_{12} i\|_{q, \bar{s}_h + 2}^{\gamma, \mathcal{O}}.$$

The operator $\mathcal{R}_0 = \Pi_{\mathbb{S}_0}^\perp \mathcal{R}_0 \Pi_{\mathbb{S}_0}^\perp$ is a real and reversible Toeplitz in time integral operator satisfying

$$(4.81) \quad \forall s \in [s_0, S], \quad \max_{k \in \{0,1\}} \|\partial_\theta^k \mathcal{R}_0\|_{\mathcal{O}-d, q, s}^{\gamma, \mathcal{O}} \lesssim \varepsilon\gamma^{-1} \left(1 + \|\mathfrak{J}_0\|_{q, s + \sigma_3}^{\gamma, \mathcal{O}}\right)$$

and

$$(4.82) \quad \|\Delta_{12} \mathcal{R}_0\|_{\mathcal{O}-d, q, \bar{s}_h + \mathbf{p}}^{\gamma, \mathcal{O}} \lesssim \varepsilon\gamma^{-1} \|\Delta_{12} i\|_{q, \bar{s}_h + \mathbf{p} + \sigma_3}^{\gamma, \mathcal{O}}.$$

The operator \mathbf{E}_n^1 satisfies the following estimate

$$(4.83) \quad \|\mathbf{E}_n^1 \rho\|_{q, s_0}^{\gamma, \mathcal{O}} \lesssim \varepsilon N_0^{\mu_2} N_{n+1}^{-\mu_2} \|\rho\|_{q, s_0 + 2}^{\gamma, \mathcal{O}}.$$

In addition, the operator \mathcal{L}_0 satisfies

$$(4.84) \quad \forall s \in [s_0, S], \quad \|\mathcal{L}_0 \rho\|_{q, s}^{\gamma, \mathcal{O}} \lesssim \|\rho\|_{q, s+1}^{\gamma, \mathcal{O}} + \varepsilon\gamma^{-1} \|\mathfrak{J}_0\|_{q, s + \sigma_3}^{\gamma, \mathcal{O}} \|\rho\|_{q, s_0}^{\gamma, \mathcal{O}}.$$

PROOF. \blacktriangleright The identities (4.53) and $\text{Id} = \Pi_{\mathbb{S}_0} + \Pi_{\mathbb{S}_0}^\perp$ imply

$$\begin{aligned} \mathcal{B}_\perp^{-1} \mathcal{L}_\perp \mathcal{B}_\perp &= \mathcal{B}_\perp^{-1} \Pi_{\mathbb{S}_0}^\perp (\mathcal{L}_{\varepsilon r} - \varepsilon \partial_\theta \mathcal{R}) \mathcal{B}_\perp \\ &= \mathcal{B}_\perp^{-1} \Pi_{\mathbb{S}_0}^\perp \mathcal{L}_{\varepsilon r} \mathcal{B} \Pi_{\mathbb{S}_0}^\perp - \mathcal{B}_\perp^{-1} \Pi_{\mathbb{S}_0}^\perp \mathcal{L}_{\varepsilon r} \Pi_{\mathbb{S}_0} \mathcal{B} \Pi_{\mathbb{S}_0}^\perp - \varepsilon \mathcal{B}_\perp^{-1} \Pi_{\mathbb{S}_0}^\perp \partial_\theta \mathcal{R} \mathcal{B}_\perp. \end{aligned}$$

Now using (4.68) and the fact that

$$\mathcal{B}_\perp^{-1} \Pi_{\mathbb{S}_0}^\perp = \mathcal{B}_\perp^{-1}, \quad [\Pi_{\mathbb{S}_0}^\perp, F] = 0 = [\Pi_{\mathbb{S}_0}, F], \quad F \text{ Fourier multiplier,}$$

we obtain in restriction to the Cantor set $\mathcal{O}_{\text{transport}}^{n, \gamma, \tau_1}(i_0)$ the following decomposition

$$\begin{aligned} \mathcal{B}_\perp^{-1} \Pi_{\mathbb{S}_0}^\perp \mathcal{L}_{\varepsilon r} \mathcal{B} \Pi_{\mathbb{S}_0}^\perp &= \mathcal{B}_\perp^{-1} \Pi_{\mathbb{S}_0}^\perp \mathcal{B} \mathcal{L}_{\varepsilon r} \Pi_{\mathbb{S}_0}^\perp \\ &= (\omega \cdot \partial_\varphi + \mathfrak{m}_{i_0}^\infty \partial_\theta + \partial_\theta \mathcal{K}_\alpha * \cdot) \Pi_{\mathbb{S}_0}^\perp + \Pi_{\mathbb{S}_0}^\perp \partial_\theta \mathbf{R}_{\varepsilon r} \Pi_{\mathbb{S}_0}^\perp \\ &\quad + \mathcal{B}_\perp^{-1} \mathcal{B} \Pi_{\mathbb{S}_0} \partial_\theta \mathbf{R}_{\varepsilon r} \Pi_{\mathbb{S}_0}^\perp + \mathcal{B}_\perp^{-1} \Pi_{\mathbb{S}_0}^\perp \mathcal{B} \mathbf{E}_n^0 \Pi_{\mathbb{S}_0}^\perp. \end{aligned}$$

Hence, using also (4.73) and (4.74), one gets that in the Cantor set $\mathcal{O}_{\text{transport}}^{n,\gamma,\tau_1}(i_0)$, the following identity holds

$$\begin{aligned} \mathcal{B}_\perp^{-1} \mathcal{L}_\perp \mathcal{B}_\perp &= (\omega \cdot \partial_\varphi + \mathbf{m}_{i_0}^\infty \partial_\theta + \partial_\theta \mathcal{K}_\alpha * \cdot) \Pi_{\mathbb{S}_0}^\perp + \Pi_{\mathbb{S}_0}^\perp \partial_\theta \mathbf{R}_{\varepsilon r} \Pi_{\mathbb{S}_0}^\perp \\ &\quad - \mathcal{B}_\perp^{-1} \Pi_{\mathbb{S}_0}^\perp (\partial_\theta (V_{\varepsilon r} \cdot) + \partial_\theta \mathbf{L}_{\varepsilon r,1}^E + \partial_\theta \mathbf{L}_{\varepsilon r,1}^{\text{SW}}) \Pi_{\mathbb{S}_0} \mathcal{B} \Pi_{\mathbb{S}_0}^\perp \\ &\quad + \mathcal{B}_\perp^{-1} \mathcal{B} \Pi_{\mathbb{S}_0} \partial_\theta \mathbf{R}_{\varepsilon r} \Pi_{\mathbb{S}_0}^\perp - \varepsilon \mathcal{B}_\perp^{-1} \partial_\theta \mathcal{R} \mathcal{B}_\perp + \mathcal{B}_\perp^{-1} \Pi_{\mathbb{S}_0}^\perp \mathcal{B} \mathbf{E}_n^0 \Pi_{\mathbb{S}_0}^\perp \\ &\triangleq \omega \cdot \partial_\varphi \Pi_{\mathbb{S}_0}^\perp + \mathcal{D}_0 + \mathcal{R}_0 + \mathbf{E}_n^1, \end{aligned}$$

with

$$(4.85) \quad \mathcal{D}_0 \triangleq (\mathbf{m}_{i_0}^\infty \partial_\theta + \partial_\theta \mathcal{K}_\alpha * \cdot) \Pi_{\mathbb{S}_0}^\perp, \quad \mathbf{E}_n^1 \triangleq \mathcal{B}_\perp^{-1} \Pi_{\mathbb{S}_0}^\perp \mathcal{B} \mathbf{E}_n^0 \Pi_{\mathbb{S}_0}^\perp.$$

► The estimate (4.83) is obtained gathering (4.85), (4.65), (4.76), (4.69) and Lemma 4.1-(iii).

► Recalling the notations in (4.25), we can write

$$\begin{aligned} \mathcal{D}_0 &\triangleq (\mathbf{m}_{i_0}^\infty - V_0(\alpha)) \partial_\theta \Pi_{\mathbb{S}_0}^\perp + (V_0(\alpha) \partial_\theta + \partial_\theta \mathcal{K}_\alpha * \cdot) \Pi_{\mathbb{S}_0}^\perp \\ &= (\mathbf{m}_{i_0}^\infty - V_0(\alpha)) \partial_\theta \Pi_{\mathbb{S}_0}^\perp - \partial_\theta \mathbf{L}(\alpha) \Pi_{\mathbb{S}_0}^\perp. \end{aligned}$$

Therefore, the expression (4.79) follows from the Fourier representation (4.28).

► The estimates in (4.80) are just a reformulations of (4.63)-(4.71).

► The estimates (4.81) and (4.82) are obtained by the same method as in [52, Prop 6.3] by using some duality representations of $\mathcal{B}_\perp^{\pm 1}$ (see [52, Lem. 6.3]). The computations are very long but straightforward so we prefer to refer the reader to the reference for the details. We just mention that along the calculations, in order to get (4.81) and (4.82), we shall use in particular (4.58), (4.59), (4.70), (4.71) and (4.72). Similarly, during these computations, we can check that the reversibility property of \mathcal{R}_0 is a consequence of (4.23), (4.57), (4.64), (4.75) and the reversibility property of $\mathbf{R}_{\varepsilon r}$.

► Now, we shall prove (4.84). Using (4.78), (4.25) and the product law 4.1-(iii), we have for any $s \in [s_0, S]$,

$$\begin{aligned} \|\mathcal{L}_0 \rho\|_{q,s}^{\gamma,\mathcal{O}} &\leq \|\omega \cdot \partial_\varphi \rho\|_{q,s}^{\gamma,\mathcal{O}} + \|\mathbf{m}_{i_0}^\infty \partial_\theta \rho\|_{q,s}^{\gamma,\mathcal{O}} + \|\partial_\theta \mathcal{K}_\alpha * \rho\|_{q,s}^{\gamma,\mathcal{O}} + \|\mathcal{R}_0\|_{q,s}^{\gamma,\mathcal{O}} \\ &\leq (1 + C \|\mathbf{m}_{i_0}^\infty\|_{q,s}^{\gamma,\mathcal{O}}) \|\rho\|_{q,s+1}^{\gamma,\mathcal{O}} + \|\partial_\theta \mathcal{K} * \rho\|_{q,s}^{\gamma,\mathcal{O}} + \|\partial_\theta \mathcal{Q}_\alpha * \rho\|_{q,s}^{\gamma,\mathcal{O}} + \|\mathcal{R}_0\|_{q,s}^{\gamma,\mathcal{O}}. \end{aligned}$$

The estimate (4.63) implies in particular $\|\mathbf{m}_{i_0}^\infty\|_{q,s}^{\gamma,\mathcal{O}} \lesssim 1$. Besides, from [52, Lem. 5.1], we know that

$$\|\partial_\theta \mathcal{Q}_\alpha * \rho\|_{q,s}^{\gamma,\mathcal{O}} \lesssim \|\rho\|_{q,s}^{\gamma,\mathcal{O}}.$$

Similarly,

$$\|\partial_\theta \mathcal{K} * \rho\|_{q,s}^{\gamma,\mathcal{O}} \lesssim \|\rho\|_{q,s}^{\gamma,\mathcal{O}}.$$

Applying Lemma 4.2-(ii) together with (4.81) and (4.77), we infer

$$\begin{aligned} \|\mathcal{R}_0\|_{q,s}^{\gamma,\mathcal{O}} &\lesssim \|\mathcal{R}_0\|_{\mathcal{O}\text{-d},q,s_0}^{\gamma,\mathcal{O}} \|\rho\|_{q,s}^{\gamma,\mathcal{O}} + \|\mathcal{R}_0\|_{\mathcal{O}\text{-d},q,s}^{\gamma,\mathcal{O}} \|\rho\|_{q,s_0}^{\gamma,\mathcal{O}} \\ &\lesssim \|\rho\|_{q,s}^{\gamma,\mathcal{O}} + \varepsilon \gamma^{-1} \|\mathfrak{J}_0\|_{\mathcal{O}\text{-d},q,s+\sigma_3}^{\gamma,\mathcal{O}} \|\rho\|_{q,s_0}^{\gamma,\mathcal{O}}. \end{aligned}$$

Combining the foregoing calculations leads to (4.84). This ends the proof of Proposition 4.4. \square

The next step is to get rid of the remainder term \mathcal{R}_0 . The properties (4.81), (4.82), (4.84) and Lemma 4.7-(v) allows to start a KAM iterative procedure similar to [52, Prop. 6.5]. We also refer to [43, Prop. 6.4] for a brief explanation of this method. The result reads as follows.

PROPOSITION 4.5. *Let $(d, s_0, S, \gamma, q, q_0, \tau_1, s_l, \bar{s}_l, \bar{s}_h, \bar{\mu}_2, \sigma_3)$ as in (4.2), (4.3), (4.4), (4.49), (4.60) and Proposition 4.4. Set*

$$(4.86) \quad \tau_2 \triangleq (\tau_1 + d)q_0 + 1.$$

For every choice of additional parameters (μ_2, s_h) enjoying the following constraints compatibles with (4.61) for the choice $\mathbf{p} = 4\tau_2q + 4\tau_2$,

$$(4.87) \quad \mu_2 \geq \bar{\mu}_2 + 2\tau_2q + 2\tau_2, \quad s_h \geq \frac{3}{2}\mu_2 + \bar{s}_l + 1,$$

there exist $\varepsilon_0 \in (0, 1)$ and $\sigma_4 = \sigma_4(\tau_1, \tau_2, q, d) \geq \sigma_3$ such that if the following smallness condition holds

$$(4.88) \quad \varepsilon\gamma^{-2-q}N_0^{\mu_2} \leq \varepsilon_0, \quad \|\mathfrak{J}_0\|_{q, s_h + \sigma_4}^{\gamma, \mathcal{O}} \leq 1,$$

then there exists an operator $\Phi_\infty : \mathcal{O} \rightarrow \mathcal{L}(H^s \cap L^2_\perp)$ satisfying

$$(4.89) \quad \forall s \in [s_0, S], \quad \|\Phi_\infty^{\pm 1}\rho\|_{q, s}^{\gamma, \mathcal{O}} \lesssim \|\rho\|_{q, s}^{\gamma, \mathcal{O}} + \varepsilon\gamma^{-2}\|\mathfrak{J}_0\|_{q, s + \sigma_4}^{\gamma, \mathcal{O}}\|\rho\|_{q, s_0}^{\gamma, \mathcal{O}}$$

and a diagonal operator $\mathcal{L}_\infty = \mathcal{L}_\infty(\alpha, \omega, i_0)$ in the form

$$(4.90) \quad \mathcal{L}_\infty = \omega \cdot \partial_\varphi \Pi_{\mathbb{S}_0^\perp}^\perp + \mathcal{D}_\infty,$$

with $\mathcal{D}_\infty = \Pi_{\mathbb{S}_0^\perp}^\perp \mathcal{D}_\infty \Pi_{\mathbb{S}_0^\perp}^\perp$ a reversible Fourier multiplier operator given by,

$$(4.91) \quad \begin{aligned} \mathcal{D}_\infty \mathbf{e}_{l, j} &= i d_j^\infty \mathbf{e}_{l, j}, \quad d_j^\infty(\alpha, \omega, i_0) \triangleq d_j^0(\alpha, \omega, i_0) + r_j^\infty(\alpha, \omega, i_0), \\ \sup_{j \in \mathbb{S}_0^c} |j| \|r_j^\infty\|_q^{\gamma, \mathcal{O}} &\lesssim \varepsilon\gamma^{-1} \end{aligned}$$

such that in restriction to the Cantor-like set

$$(4.92) \quad \mathcal{O}_{\text{remainder}}^{n, \gamma, \tau_1, \tau_2}(i_0) \triangleq \bigcap_{\substack{(l, j, j_0) \in \mathbb{Z}^d \times (\mathbb{S}_0^c)^2 \\ |l| \leq N_n \\ (l, j) \neq (0, j_0)}} \mathcal{O}_{l, j, j_0, i_0}^{n, \gamma, \tau_1, \tau_2},$$

with

$$\mathcal{O}_{l, j, j_0, i_0}^{n, \gamma, \tau_1, \tau_2} \triangleq \left\{ (\alpha, \omega) \in \mathcal{O}_{\text{transport}}^{n, \gamma, \tau_1}(i_0), |\omega \cdot l + d_j^\infty(\alpha, \omega, i_0) - d_{j_0}^\infty(\alpha, \omega, i_0)| > \frac{2\gamma \langle j - j_0 \rangle}{\langle l \rangle^{\tau_2}} \right\},$$

we have

$$(4.93) \quad \Phi_\infty^{-1} \mathcal{L}_0 \Phi_\infty = \mathcal{L}_\infty + \mathbf{E}_n^2, \quad \|\mathbf{E}_n^2 \rho\|_{q, s_0}^{\gamma, \mathcal{O}} \lesssim \varepsilon\gamma^{-2} N_0^{\mu_2} N_{n+1}^{-\mu_2} \|\rho\|_{q, s_0+1}^{\gamma, \mathcal{O}}.$$

Recall that $\mathcal{O}_{\text{transport}}^{n, \gamma, \tau_1}(i_0)$, \mathcal{L}_0 and the frequencies $(d_j^0(\alpha, \omega, i_0))_{j \in \mathbb{S}_0^c}$ are respectively given by (4.67), (4.78) and (4.79). Moreover, for two tori i_1 and i_2 both satisfying (4.88), we have for any $j \in \mathbb{S}_0^c$,

$$(4.94) \quad \begin{aligned} \|\Delta_{12} r_j^\infty\|_q^{\gamma, \mathcal{O}} &\lesssim \varepsilon\gamma^{-1} \|\Delta_{12} i\|_{q, \bar{s}_h + \sigma_4}^{\gamma, \mathcal{O}}, \\ \|\Delta_{12} d_j^\infty\|_q^{\gamma, \mathcal{O}} &\lesssim \varepsilon\gamma^{-1} |j| \|\Delta_{12} i\|_{q, \bar{s}_h + \sigma_4}^{\gamma, \mathcal{O}}. \end{aligned}$$

Now, that we have completely diagonalized the operator \mathcal{L}_\perp in (4.53) up to error terms, we can find an almost approximate right inverse for it. This is done by almost inverting the operator \mathcal{L}_∞ in (4.90). The result is stated below and its proof follows word by word [52, Prop. 6.6] or [43, Prop. 6.5]. Consequently, we omit the proof and only mention that the estimates mainly follows from (4.76)-(4.89)-(4.69)-(4.83)-(4.93) and Lemma 4.1-(ii).

PROPOSITION 4.6. *Let $(d, s_0, S, \gamma, q, \tau_1, \tau_2, s_h, \mu_2, \sigma_4)$ as in (4.2), (4.3), (4.4), (4.49), (4.60), (4.86), (4.87) and Proposition 4.5. There exists $\sigma \triangleq \sigma(\tau_1, \tau_2, q, d) \geq \sigma_4$ such that if the following smallness condition holds*

$$(4.95) \quad \varepsilon \gamma^{-2-q} N_0^{\mu_2} \leq \varepsilon_0, \quad \|\mathfrak{J}_0\|_{q, s_h + \sigma}^{\gamma, \mathcal{O}} \leq 1,$$

(i) *There exists a family of operators $(\mathbf{T}_n)_{n \in \mathbb{N}}$ defined on \mathcal{O} satisfying*

$$\forall s \in [s_0, S], \quad \sup_{n \in \mathbb{N}} \|\mathbf{T}_n \rho\|_{q, s}^{\gamma, \mathcal{O}} \lesssim \gamma^{-1} \|\rho\|_{q, s + \tau_1 q + \tau_1}^{\gamma, \mathcal{O}}$$

and such that for any $n \in \mathbb{N}$, in restriction to the Cantor set

$$(4.96) \quad \mathcal{O}_{\text{inversion}}^{n, \gamma, \tau_1}(i_0) \triangleq \bigcap_{\substack{(l, j) \in \mathbb{Z}^d \times \mathbb{S}_0^c \\ |l| \leq N_n}} \mathcal{O}_{l, j, i_0}^{n, \gamma, \tau_1}$$

$$\mathcal{O}_{l, j, i_0}^{n, \gamma, \tau_1} \triangleq \left\{ (\alpha, \omega) \in \mathcal{O} \quad \text{s.t.} \quad |\omega \cdot l + d_j^\infty(\alpha, \omega, i_0)| > \frac{\gamma \langle j \rangle}{\langle l \rangle^{\tau_1}} \right\},$$

the following identity holds

$$\mathcal{L}_\infty \mathbf{T}_n = \text{Id} + \mathbf{E}_n^3,$$

with

$$\forall s_0 \leq s \leq \bar{s} \leq S, \quad \|\mathbf{E}_n^3 \rho\|_{q, s}^{\gamma, \mathcal{O}} \lesssim N_n^{s - \bar{s}} \gamma^{-1} \|\rho\|_{q, \bar{s} + 1 + \tau_1 q + \tau_1}^{\gamma, \mathcal{O}}.$$

This means that we have an almost approximate right inverse for the operator \mathcal{L}_∞ in (4.90).

(ii) *There exists a family of operators $(\mathbf{T}_{\perp, n})_{n \in \mathbb{N}}$ defined on \mathcal{O} satisfying for any $s \in [s_0, S]$,*

$$\sup_{n \in \mathbb{N}} \|\mathbf{T}_{\perp, n} \rho\|_{q, s}^{\gamma, \mathcal{O}} \lesssim \gamma^{-1} \left(\|\rho\|_{q, s + \sigma}^{\gamma, \mathcal{O}} + \|\mathfrak{J}_0\|_{q, s + \sigma}^{\gamma, \mathcal{O}} \|\rho\|_{q, s_0 + \sigma}^{\gamma, \mathcal{O}} \right)$$

and such that in restriction to the Cantor set

$$(4.97) \quad \mathcal{G}_n(\gamma, \tau_1, \tau_2, i_0) \triangleq \mathcal{O}_{\text{transport}}^{n, \gamma, \tau_1}(i_0) \cap \mathcal{O}_{\text{remainder}}^{n, \gamma, \tau_1, \tau_2}(i_0) \cap \mathcal{O}_{\text{inversion}}^{n, \gamma, \tau_1}(i_0),$$

the following identity holds

$$\mathcal{L}_\perp \mathbf{T}_{\perp, n} = \text{Id} + \mathbf{E}_n,$$

with \mathbf{E}_n satisfying for any $s \in [s_0, S]$,

$$\|\mathbf{E}_n \rho\|_{q, s_0}^{\gamma, \mathcal{O}} \lesssim N_n^{s_0 - s} \gamma^{-1} \left(\|\rho\|_{q, s + \sigma}^{\gamma, \mathcal{O}} + \varepsilon \gamma^{-2} \|\mathfrak{J}_0\|_{q, s + \sigma}^{\gamma, \mathcal{O}} \|\rho\|_{q, s_0}^{\gamma, \mathcal{O}} \right) + \varepsilon \gamma^{-3} N_0^{\mu_2} N_{n+1}^{-\mu_2} \|\rho\|_{q, s_0 + \sigma}^{\gamma, \mathcal{O}}.$$

Recall that \mathcal{L}_\perp , $\mathcal{O}_{\text{transport}}^{n, \gamma, \tau_1}(i_0)$ and $\mathcal{O}_{\text{remainder}}^{n, \gamma, \tau_1, \tau_2}(i_0)$ are respectively defined in (4.53), (4.67) and (4.92).

(iii) *When restricted to the Cantor set $\mathcal{G}_n(\gamma, \tau_1, \tau_2, i_0)$, we have also have the following splitting required to apply the Berti-Bolle theory*

$$\mathcal{L}_\perp = \mathbf{L}_{\perp, n} + \mathbf{R}_{\perp, n}, \quad \mathbf{L}_{\perp, n} \mathbf{T}_{\perp, n} = \text{Id}, \quad \mathbf{R}_{\perp, n} = \mathbf{E}_n \mathbf{L}_{\perp, n},$$

with the operators $L_{\perp,n}$ and $R_{\perp,n}$ defined in \mathcal{O} and satisfying for any $s \in [s_0, S]$,

$$\begin{aligned} \sup_{n \in \mathbb{N}} \|L_{\perp,n} \rho\|_{q,s}^{\gamma,\mathcal{O}} &\lesssim \|\rho\|_{q,s+1}^{\gamma,\mathcal{O}} + \varepsilon \gamma^{-2} \|\mathfrak{J}_0\|_{q,s+\sigma}^{\gamma,\mathcal{O}} \|\rho\|_{q,s_0+1}^{\gamma,\mathcal{O}}, \\ \|R_{\perp,n} \rho\|_{q,s_0}^{\gamma,\mathcal{O}} &\lesssim N_n^{s_0-s} \gamma^{-1} \left(\|\rho\|_{q,s+\sigma}^{\gamma,\mathcal{O}} + \varepsilon \gamma^{-2} \|\mathfrak{J}_0\|_{q,s+\sigma}^{\gamma,\mathcal{O}} \|\rho\|_{q,s_0+\sigma}^{\gamma,\mathcal{O}} \right) \\ &\quad + \varepsilon \gamma^{-3} N_0^{\mu_2} N_{n+1}^{-\mu_2} \|\rho\|_{q,s_0+\sigma}^{\gamma,\mathcal{O}}. \end{aligned}$$

Finally, as mentioned at the beginning of this subsection, we can apply the Berti-Bolle theory [11] and [41, Sec. 6] to construct an approximate right inverse for the full linearized operator $d_{i,\kappa} \mathcal{F}(i_0, \kappa_0)$. For a complete proof of the result, the reader is referred to [41, Thm. 6.1].

THEOREM 4.1. (Almost approximate right inverse)

Let $(d, s_0, S, \gamma, q, \tau_1, \tau_2, s_h, \mu_2)$ satisfy (4.2), (4.3), (4.4), (4.49), (4.60), (4.86) and (4.87). Then there exists $\bar{\sigma} = \bar{\sigma}(\tau_1, \tau_2, d, q) > 0$ and a family of reversible operators $(T_{0,n})_{n \in \mathbb{N}}$ such that if the smallness condition (4.95) holds, then for all $g = (g_1, g_2, g_3)$, satisfying

$$g_1(-\varphi) = g_1(\varphi), \quad g_2(-\varphi) = -g_2(\varphi) \quad g_3(-\varphi) = (\mathcal{S}g_3)(\varphi),$$

the function $T_{0,n}g$ satisfies

$$\forall s \in [s_0, S], \quad \|T_{0,n}g\|_{q,s}^{\gamma,\mathcal{O}} \lesssim \gamma^{-1} \left(\|g\|_{q,s+\bar{\sigma}}^{\gamma,\mathcal{O}} + \|\mathfrak{J}_0\|_{q,s+\bar{\sigma}}^{\gamma,\mathcal{O}} \|g\|_{q,s_0+\bar{\sigma}}^{\gamma,\mathcal{O}} \right).$$

Moreover $T_{0,n}$ is an almost-approximate right inverse of $d_{i,\kappa} \mathcal{F}(i_0, \kappa_0)$ in the Cantor set $\mathcal{G}_n(\gamma, \tau_1, \tau_2, i_0)$. More precisely, for all $(\alpha, \omega) \in \mathcal{G}_n(\gamma, \tau_1, \tau_2, i_0)$ we can write

$$d_{i,\kappa} \mathcal{F}(i_0, \kappa_0) \circ T_{0,n} - \text{Id} = \mathcal{E}_1^{(n)} + \mathcal{E}_2^{(n)} + \mathcal{E}_3^{(n)},$$

where the operators $\mathcal{E}_1^{(n)}$, $\mathcal{E}_2^{(n)}$ and $\mathcal{E}_3^{(n)}$ are defined in the whole set \mathcal{O} with the estimates

$$\|\mathcal{E}_1^{(n)}g\|_{q,s_0}^{\gamma,\mathcal{O}} \lesssim \gamma^{-1} \|\mathcal{F}(i_0, \kappa_0)\|_{q,s_0+\bar{\sigma}}^{\gamma,\mathcal{O}} \|g\|_{q,s_0+\bar{\sigma}}^{\gamma,\mathcal{O}},$$

for any $\mathbf{b} \geq 0$,

$$\|\mathcal{E}_2^{(n)}g\|_{q,s_0}^{\gamma,\mathcal{O}} \lesssim \gamma^{-1} N_n^{-\mathbf{b}} \left(\|g\|_{q,s_0+\mathbf{b}+\bar{\sigma}}^{\gamma,\mathcal{O}} + \varepsilon \|\mathfrak{J}_0\|_{q,s_0+\mathbf{b}+\bar{\sigma}}^{\gamma,\mathcal{O}} \|g\|_{q,s_0+\bar{\sigma}}^{\gamma,\mathcal{O}} \right),$$

and for any $\mathbf{b} \in [0, S - s_0]$,

$$\begin{aligned} \|\mathcal{E}_3^{(n)}g\|_{q,s_0}^{\gamma,\mathcal{O}} &\lesssim N_n^{-\mathbf{b}} \gamma^{-2} \left(\|g\|_{q,s_0+\mathbf{b}+\bar{\sigma}}^{\gamma,\mathcal{O}} + \varepsilon \gamma^{-2} \|\mathfrak{J}_0\|_{q,s_0+\mathbf{b}+\bar{\sigma}}^{\gamma,\mathcal{O}} \|g\|_{q,s_0+\bar{\sigma}}^{\gamma,\mathcal{O}} \right) \\ &\quad + \varepsilon \gamma^{-4} N_0^{\mu_2} N_n^{-\mu_2} \|g\|_{q,s_0+\bar{\sigma}}^{\gamma,\mathcal{O}}. \end{aligned}$$

4.6. Construction of a non-trivial quasi-periodic solution. In this final section, we construct a non-trivial zero for the functional \mathcal{F} defined by (4.48). It is obtained by a Nash-Moser iteration procedure. At each step, we can find an approximate right inverse of the linearized operator following the construction explained in the previous subsection. The proof is technical and already detailed in the previous works [16, 41, 52]. Here, we may use the version exposed in [52, Prop. 7.1 and Cor. 7.1].

PROPOSITION 4.7. (**Nash-Moser**)

(i) Let $(d, s_0, S, q, \tau_1, \tau_2, \bar{\sigma})$ as in (4.2), (4.3), (4.49), (4.60), (4.86) and Theorem 4.1. Consider the parameters fixed by

$$(4.98) \quad \begin{aligned} \mu_1 &\triangleq 3q(\tau_2 + 2) + 6\bar{\sigma} + 6, & a_1 &\triangleq 6q(\tau_2 + 2) + 12\bar{\sigma} + 15, \\ \mu_2 &\triangleq 2q(\tau_2 + 2) + 5\bar{\sigma} + 7, & a_2 &\triangleq 3q(\tau_2 + 2) + 6\bar{\sigma} + 9, \\ s_h &\triangleq s_0 + 4q(\tau_2 + 2) + 9\bar{\sigma} + 11, & b_1 &\triangleq 2s_h - s_0, & \bar{a} &\triangleq \tau_2 + 2 \end{aligned}$$

and

$$(4.99) \quad 0 < a < \frac{1}{\mu_2 + q + 2}, \quad \gamma \triangleq \varepsilon^a, \quad N_0 \triangleq \gamma^{-1}.$$

We consider the finite dimensional subspaces

$$\mathbf{E}_n \triangleq \left\{ \mathfrak{J} = (\Theta, I, z) \quad \text{s.t.} \quad \Theta = P_{N_n} \Theta, \quad I = P_{N_n} I, \quad z = P_{N_n} z \right\},$$

where P_N is the projector defined by

$$P_N \left(\sum_{(l,j) \in \mathbb{Z}^d \times \mathbb{Z}} f_{l,j} e^{i(l \cdot \varphi + j\theta)} \right) = \sum_{\substack{(l,j) \in \mathbb{Z}^d \times \mathbb{Z} \\ (l,j) \leq N}} f_{l,j} e^{i(l \cdot \varphi + j\theta)}.$$

There exist $C_* > 0$ and $\varepsilon_0 > 0$ such that for any $\varepsilon \in [0, \varepsilon_0]$ we get for all $n \in \mathbb{N}$ the following properties,

(a) There exists a q -times differentiable function

$$\begin{aligned} \mathbf{W}_n : \quad \mathcal{O} &\rightarrow \mathbf{E}_{n-1} \times \mathbb{R}^d \times \mathbb{R}^{d+1} \\ (\alpha, \omega) &\mapsto (\mathfrak{J}_n(\alpha, \omega), \kappa_n(\alpha, \omega) - \omega, 0) \end{aligned}$$

satisfying $\mathbf{W}_0 = 0$ and if $n \in \mathbb{N}^*$,

$$\begin{aligned} \|\mathbf{W}_n\|_{q, s_0 + \bar{\sigma}}^{\gamma, \mathcal{O}} &\leq C_* \varepsilon \gamma^{-1} N_0^{q\bar{a}}, \\ \|\mathbf{W}_n\|_{q, b_1 + \bar{\sigma}}^{\gamma, \mathcal{O}} &\leq C_* \varepsilon \gamma^{-1} N_{n-1}^{\mu_1}. \end{aligned}$$

We set

$$\mathbf{U}_0 \triangleq ((\varphi, 0, 0), \omega, (\alpha, \omega))$$

and if $n \in \mathbb{N}^*$,

$$\mathbf{U}_n \triangleq \mathbf{U}_0 + \mathbf{W}_n, \quad \mathbf{H}_n \triangleq \mathbf{U}_n - \mathbf{U}_{n-1}.$$

Then

$$\begin{aligned} \forall s \in [s_0, S], \|\mathbf{H}_1\|_{q, s}^{\gamma, \mathcal{O}} &\leq \frac{1}{2} C_* \varepsilon \gamma^{-1} N_0^{q\bar{a}}, \\ \forall 2 \leq k \leq n, \|\mathbf{H}_k\|_{q, s_0 + \bar{\sigma}}^{\gamma, \mathcal{O}} &\leq C_* \varepsilon \gamma^{-1} N_{k-1}^{-a_2}. \end{aligned}$$

We also have for $n \geq 2$,

$$(4.100) \quad \|\mathbf{H}_n\|_{q, \bar{s}_h + \sigma_4}^{\gamma, \mathcal{O}} \leq C_* \varepsilon \gamma^{-1} N_{n-1}^{-a_2}.$$

(b) Define

$$(4.101) \quad i_n \triangleq (\varphi, 0, 0) + \mathfrak{J}_n, \quad \gamma_n \triangleq \gamma(1 + 2^{-n}) \in [\gamma, 2\gamma].$$

The torus i_n is reversible. Define also

$$(4.102) \quad \mathcal{A}_0^\gamma \triangleq \mathcal{O}, \quad \mathcal{A}_{n+1}^\gamma \triangleq \mathcal{A}_n^\gamma \cap \mathcal{G}_n(\gamma_{n+1}, \tau_1, \tau_2, i_n),$$

with $\mathcal{G}_n(\gamma_{n+1}, \tau_1, \tau_2, i_n)$ as in (4.97). Consider the open sets

$$\begin{aligned} \mathcal{O}_n^\gamma &\triangleq \left\{ (\alpha, \omega) \in \mathcal{O} \quad \text{s.t.} \quad \text{dist}((\alpha, \omega), \mathcal{A}_n^\gamma) < \gamma N_n^{-\bar{a}} \right\}, \\ \text{dist}(x, A) &\triangleq \inf_{y \in A} \|x - y\|. \end{aligned}$$

Then we have

$$\|\mathcal{F}(\mathbf{U}_n)\|_{q, s_0}^{\gamma, \mathcal{O}_n^\gamma} \leq C_* \varepsilon N_{n-1}^{-a_1}.$$

(ii) There exists $\varepsilon_0 > 0$ such that, for any $\varepsilon \in (0, \varepsilon_0)$, the following hold true. We consider the Cantor set $\mathcal{G}_\infty^\gamma$ (related to ε through γ) and given by

$$(4.103) \quad \mathcal{G}_\infty^\gamma \triangleq \bigcap_{n \in \mathbb{N}} \mathcal{A}_n^\gamma.$$

There exists a q -times differentiable function

$$\begin{aligned} \mathbf{U}_\infty : \quad \mathcal{O} &\rightarrow (\mathbb{T}^d \times \mathbb{R}^d \times H^{s_0} \cap L^2_\perp) \times \mathbb{R}^d \times \mathbb{R}^{d+1} \\ (\alpha, \omega) &\mapsto (i_\infty(\alpha, \omega), \kappa_\infty(\alpha, \omega), (\alpha, \omega)) \end{aligned}$$

such that

$$\forall (\alpha, \omega) \in \mathcal{G}_\infty^\gamma, \quad \mathcal{F}(\mathbf{U}_\infty(\alpha, \omega)) = 0.$$

The torus i_∞ is reversible and $\kappa_\infty \in W^{q, \infty, \gamma}(\mathcal{O}, \mathbb{R}^d)$. Furthermore, there exists a q -times differentiable function $\alpha \in (\alpha_0, \alpha_1) \mapsto \omega(\alpha, \varepsilon)$ such that

$$(4.104) \quad \begin{aligned} \kappa_\infty(\alpha, \omega(\alpha, \varepsilon)) &= -\omega_{\text{Eq}}(\alpha), \quad \omega(\alpha, \varepsilon) = -\omega_{\text{Eq}}(\alpha) + \bar{r}_\varepsilon(\alpha), \\ \|\bar{r}_\varepsilon\|_q^{\gamma, \mathcal{O}} &\lesssim \varepsilon \gamma^{-1} N_0^{q\bar{a}} \end{aligned}$$

and

$$\forall \alpha \in \mathcal{C}_\infty^\varepsilon, \quad \mathcal{F}(\mathbf{U}_\infty(\alpha, \omega(\alpha, \varepsilon))) = 0,$$

where the Cantor set $\mathcal{C}_\infty^\varepsilon$ is defined by

$$(4.105) \quad \mathcal{C}_\infty^\varepsilon \triangleq \left\{ \alpha \in (\alpha_0, \alpha_1) \quad \text{s.t.} \quad (\alpha, \omega(\alpha, \varepsilon)) \in \mathcal{G}_\infty^\gamma \right\}.$$

Now to conclude the proof of Theorem 1.2, it remains to prove that the Cantor set $\mathcal{C}_\infty^\varepsilon$ in (4.105) is not empty. Actually, we can prove a lower bound measure for $\mathcal{C}_\infty^\varepsilon$ which shows that when the magnitude of the perturbation tends to zero, then the set of admissible parameters tends to be of full Lebesgue measure in (α_0, α_1) .

PROPOSITION 4.8. Let (q_0, v, a) as in Lemma 4.9, (4.60) and (4.99). Then there exists $C > 0$ such that

$$|\mathcal{C}_\infty^\varepsilon| \geq (\alpha_1 - \alpha_0) - C\varepsilon^{\frac{av}{q_0}}, \quad \text{implying in turn} \quad \lim_{\varepsilon \rightarrow 0} |\mathcal{C}_\infty^\varepsilon| = \alpha_1 - \alpha_0.$$

PROOF. In view of (4.105) and (4.103), we can write

$$(4.106) \quad \mathcal{C}_\infty^\varepsilon = \bigcap_{n \in \mathbb{N}} \mathcal{C}_n^\varepsilon, \quad \mathcal{C}_n^\varepsilon \triangleq \left\{ \alpha \in (\alpha_0, \alpha_1) \quad \text{s.t.} \quad (\alpha, \omega(\alpha, \varepsilon)) \in \mathcal{A}_n^\gamma \right\}.$$

Hence,

$$(4.107) \quad (\alpha_0, \alpha_1) \setminus \mathcal{C}_\infty^\varepsilon = ((\alpha_0, \alpha_1) \setminus \mathcal{C}_0^\varepsilon) \sqcup \bigsqcup_{n=0}^{\infty} (\mathcal{C}_n^\varepsilon \setminus \mathcal{C}_{n+1}^\varepsilon).$$

Observe that (4.104), (4.98) and (4.99) imply

$$\sup_{\alpha \in (\alpha_0, \alpha_1)} |\omega(\alpha, \varepsilon) + \omega_{\text{Eq}}(\alpha)| \leq \|\bar{r}_\varepsilon\|_q^{\gamma, \mathcal{O}} \leq C\varepsilon \gamma^{-1} N_0^{q\bar{a}} = C\varepsilon^{1-a(1+q\bar{a})}$$

and

$$0 < a < \frac{1}{1 + q\bar{a}}.$$

Now by construction (4.50), the previous estimate implies for ε small enough

$$\forall \alpha \in (\alpha_0, \alpha_1), \quad \omega(\alpha, \varepsilon) \in B(0, R_0).$$

We immediately deduce that

$$(4.108) \quad \mathcal{C}_0^\varepsilon = (\alpha_0, \alpha_1), \quad \left| (\alpha_0, \alpha_1) \setminus \mathcal{C}_\infty^\varepsilon \right| \leq \sum_{n=0}^\infty \left| \mathcal{C}_n^\varepsilon \setminus \mathcal{C}_{n+1}^\varepsilon \right|.$$

According to the notations (4.91) and (4.79), we denote the perturbed frequencies associated with the reduced linearized operator at state i_n in the following way

$$(4.109) \quad d_j^{\infty,n}(\alpha, \varepsilon) \triangleq d_j^\infty(\alpha, \omega(\alpha, \varepsilon), i_n) = \Omega_j^E(\alpha) + jr^{0,n}(\alpha, \varepsilon) + r_j^{\infty,n}(\alpha, \varepsilon),$$

with

$$(4.110) \quad r^{0,n}(\alpha, \varepsilon) \triangleq m_n^\infty(\alpha, \varepsilon) - V_0(\alpha), \quad m_n^\infty(\alpha, \varepsilon) \triangleq m_{i_n}^\infty(\alpha, \omega(\alpha, \varepsilon)), \\ r_j^{\infty,n}(\alpha, \varepsilon) \triangleq r_j^\infty(\alpha, \omega(\alpha, \varepsilon), i_n).$$

Now, by construction, one can write in view of (4.106), (4.102), (4.97), (4.67), (4.92) and (4.96) for any $n \in \mathbb{N}$,

$$(4.111) \quad \mathcal{C}_n^\varepsilon \setminus \mathcal{C}_{n+1}^\varepsilon = \bigcup_{\substack{(l,j) \in \mathbb{Z}^d \times \mathbb{Z} \setminus \{(0,0)\} \\ |l| \leq N_n}} \mathcal{R}_{l,j}^{(0)}(i_n) \quad \bigcup_{\substack{(l,j,j_0) \in \mathbb{Z}^d \times (\mathbb{S}_0^c)^2 \\ |l| \leq N_n}} \mathcal{R}_{l,j,j_0}^{(2)}(i_n) \quad \bigcup_{\substack{(l,j) \in \mathbb{Z}^d \times \mathbb{S}_0^c \\ |l| \leq N_n}} \mathcal{R}_{l,j}^{(1)}(i_n),$$

where

$$\mathcal{R}_{l,j}^{(0)}(i_n) \triangleq \left\{ \alpha \in \mathcal{C}_n^\varepsilon \quad \text{s.t.} \quad \left| \omega(\alpha, \varepsilon) \cdot l + j m_n^\infty(\alpha, \varepsilon) \right| \leq \frac{4\gamma_{n+1}^v \langle j \rangle}{\langle l \rangle^{\tau_1}} \right\}, \\ \mathcal{R}_{l,j,j_0}^{(2)}(i_n) \triangleq \left\{ \alpha \in \mathcal{C}_n^\varepsilon \quad \text{s.t.} \quad \left| \omega(\alpha, \varepsilon) \cdot l + d_j^{\infty,n}(\alpha, \varepsilon) - d_{j_0}^{\infty,n}(\alpha, \varepsilon) \right| \leq \frac{2\gamma_{n+1} \langle j-j_0 \rangle}{\langle l \rangle^{\tau_2}} \right\}, \\ \mathcal{R}_{l,j}^{(1)}(i_n) \triangleq \left\{ \alpha \in \mathcal{C}_n^\varepsilon \quad \text{s.t.} \quad \left| \omega(\alpha, \varepsilon) \cdot l + d_j^{\infty,n}(\alpha, \varepsilon) \right| \leq \frac{\gamma_{n+1} \langle j \rangle}{\langle l \rangle^{\tau_1}} \right\}.$$

Copying the proof of [52, Lem. 7.1] we can show, using in particular (4.100)-(4.80)-(4.94), that for any $n \in \mathbb{N} \setminus \{0, 1\}$ and any $l \in \mathbb{Z}^d$ such that $|l| \leq N_{n-1}$, the following properties hold.

- For $j \in \mathbb{Z}$ with $(l, j) \neq (0, 0)$, we get $\mathcal{R}_{l,j}^{(0)}(i_n) = \emptyset$.
- For $(j, j_0) \in (\mathbb{S}_0^c)^2$ with $(l, j) \neq (0, j_0)$, we get $\mathcal{R}_{l,j,j_0}^{(2)}(i_n) = \emptyset$.
- For $j \in \mathbb{S}_0^c$, we get $\mathcal{R}_{l,j}^{(1)}(i_n) = \emptyset$.

This leads to write for any $n \in \mathbb{N} \setminus \{0, 1\}$,

$$(4.112) \quad \mathcal{C}_n^\varepsilon \setminus \mathcal{C}_{n+1}^\varepsilon = \bigcup_{\substack{(l,j) \in \mathbb{Z}^d \times \mathbb{Z} \setminus \{(0,0)\} \\ N_{n-1} < |l| \leq N_n}} \mathcal{R}_{l,j}^{(0)}(i_n) \cup \bigcup_{\substack{(l,j,j_0) \in \mathbb{Z}^d \times (\mathbb{S}_0^c)^2 \\ N_{n-1} < |l| \leq N_n}} \mathcal{R}_{l,j,j_0}^{(2)}(i_n) \\ \cup \bigcup_{\substack{(l,j) \in \mathbb{Z}^d \times \mathbb{S}_0^c \\ N_{n-1} < |l| \leq N_n}} \mathcal{R}_{l,j}^{(1)}(i_n).$$

To estimate, we shall make use of Rüssmann’s Lemma 4.6. First notice that

$$W^{q,\infty,\gamma}(\mathcal{O}, \mathbb{C}) \hookrightarrow C^{q-1}(\mathcal{O}, \mathbb{C}), \quad q = q_0 + 2,$$

imply that for any $n \in \mathbb{N}$, the C^{q_0+1} regularity for the curves

$$\begin{aligned} \alpha &\mapsto \omega(\alpha, \varepsilon) \cdot l + j m_n^\infty(\alpha, \varepsilon), & (l, j) &\in \mathbb{Z}^d \times \mathbb{Z} \setminus \{(0, 0)\}, \\ \alpha &\mapsto \omega(\alpha, \varepsilon) \cdot l + d_j^{\infty, n}(\alpha, \varepsilon) - d_{j_0}^{\infty, n}(\alpha, \varepsilon), & (l, j, j_0) &\in \mathbb{Z}^d \times (\mathbb{S}_0^c)^2, \\ \alpha &\mapsto \omega(\alpha, \varepsilon) \cdot l + d_j^{\infty, n}(\alpha, \varepsilon), & (l, j) &\in \mathbb{Z}^d \times \mathbb{S}_0^c. \end{aligned}$$

In addition, the following perturbed Rüssmann conditions hold. They are obtained by a perturbative argument from the equilibrium transversality conditions in Lemma 4.9 similarly to [52, Lem. 7.3]: for q_0 , C_0 and ρ_0 as in Lemma 4.9, there exist $\varepsilon_0 > 0$ small enough such that for any $\varepsilon \in [0, \varepsilon_0]$ we have

- For all $(l, j) \in \mathbb{Z}^{d+1} \setminus \{(0, 0)\}$ such that $|j| \leq C_0 \langle l \rangle$, we have

$$\forall n \in \mathbb{N}, \quad \inf_{\alpha \in [\alpha_0, \alpha_1]} \max_{k \in \llbracket 0, q_0 \rrbracket} |\partial_\alpha^k (\omega(\alpha, \varepsilon) \cdot l + j m_n^\infty(\alpha, \varepsilon))| \geq \frac{\rho_0 \langle l \rangle}{2}.$$

- For all $(l, j) \in \mathbb{Z}^d \times \mathbb{S}_0^c$ such that $|j| \leq C_0 \langle l \rangle$, we have

$$\forall n \in \mathbb{N}, \quad \inf_{\alpha \in [\alpha_0, \alpha_1]} \max_{k \in \llbracket 0, q_0 \rrbracket} |\partial_\alpha^k (\omega(\alpha, \varepsilon) \cdot l + d_j^{\infty, n}(\alpha, \varepsilon))| \geq \frac{\rho_0 \langle l \rangle}{2}.$$

- For all $(l, j, j_0) \in \mathbb{Z}^d \times (\mathbb{S}_0^c)^2$ such that $|j - j_0| \leq C_0 \langle l \rangle$, we have

$$\forall n \in \mathbb{N}, \quad \inf_{\alpha \in [\alpha_0, \alpha_1]} \max_{k \in \llbracket 0, q_0 \rrbracket} |\partial_\alpha^k (\omega(\alpha, \varepsilon) \cdot l + d_j^{\infty, n}(\alpha, \varepsilon) - d_{j_0}^{\infty, n}(\alpha, \varepsilon))| \geq \frac{\rho_0 \langle l \rangle}{2}.$$

Notice that the proof of the previous transversality conditions requires in particular the estimates (4.118) and (4.119). Thus, applying Lemma 4.6, we get for all $n \in \mathbb{N}$,

$$(4.113) \quad \begin{aligned} \left| \mathcal{R}_{l,j}^{(0)}(i_n) \right| &\lesssim \gamma^{\frac{v}{q_0}} \langle j \rangle^{\frac{1}{q_0}} \langle l \rangle^{-1 - \frac{\tau_1}{q_0}} \langle l, j \rangle, \\ \left| \mathcal{R}_{l,j}^{(1)}(i_n) \right| &\lesssim \gamma^{\frac{1}{q_0}} \langle j \rangle^{\frac{1}{q_0}} \langle l \rangle^{-1 - \frac{\tau_1 + 1}{q_0}} \langle l, j \rangle, \\ \left| \mathcal{R}_{l,j,j_0}^{(2)}(i_n) \right| &\lesssim \gamma^{\frac{1}{q_0}} \langle j - j_0 \rangle^{\frac{1}{q_0}} \langle l \rangle^{-1 - \frac{\tau_2 + 1}{q_0}} \langle l, j - j_0 \rangle. \end{aligned}$$

Since (4.112) is valid for $n \in \mathbb{N} \setminus \{0, 1\}$, we first need to estimate the first two terms in the right hand-side of (4.108). Fix $k \in \{0, 1\}$, then we have by Lemma 4.12

$$(4.114) \quad \begin{aligned} \left| \mathcal{E}_k \setminus \mathcal{E}_{k+1} \right| &\lesssim \sum_{\substack{(l,j) \in \mathbb{Z}^d \times \mathbb{Z} \setminus \{(0,0)\} \\ |j| \leq C_0 \langle l \rangle, |l| \leq N_k}} \left| \mathcal{R}_{l,j}^{(0)}(i_k) \right| + \sum_{\substack{(l,j,j_0) \in \mathbb{Z}^d \times (\mathbb{S}_0^c)^2 \\ |j-j_0| \leq C_0 \langle l \rangle, |l| \leq N_k \\ \min(|j|, |j_0|) \leq c_2 \gamma_{k+1}^{-v} \langle l \rangle^{\tau_1}}} \left| \mathcal{R}_{l,j,j_0}^{(2)}(i_k) \right| \\ &+ \sum_{\substack{(l,j) \in \mathbb{Z}^d \times \mathbb{S}_0^c \\ |j| \leq C_0 \langle l \rangle, |l| \leq N_k}} \left| \mathcal{R}_{l,j}^{(1)}(i_k) \right|. \end{aligned}$$

Observe that for $|j - j_0| \leq C_0 \langle l \rangle$ and $\min(|j|, |j_0|) \leq c_2 \gamma_{k+1}^{-v} \langle l \rangle^{\tau_1}$, then

$$(4.115) \quad \max(|j|, |j_0|) \leq \min(|j|, |j_0|) + |j - j_0| \leq c_2 \gamma_{k+1}^{-v} \langle l \rangle^{\tau_1} + C_0 \langle l \rangle \lesssim \gamma^{-v} \langle l \rangle^{\tau_1}.$$

Combining (4.113), (4.114) and (4.115), we get for $k \in \{0, 1\}$,

$$\begin{aligned} \left| \mathcal{E}_k \setminus \mathcal{E}_{k+1} \right| &\lesssim \gamma^{\frac{1}{q_0}} \sum_{\substack{(l,j) \in \mathbb{Z}^d \times (\mathbb{S}_0)^c \\ |j| \leq C_0(l)}} \langle j \rangle^{\frac{1}{q_0}} \langle l \rangle^{-1 - \frac{\tau_1 + 1}{q_0}} \langle l, j \rangle \\ &\quad + \gamma^{\frac{1}{q_0}} \sum_{\substack{(l,j,j_0) \in \mathbb{Z}^d \times (\mathbb{S}_0^c)^2 \\ |j - j_0| \leq C_0(l), |l| \leq N_k \\ \min(|j|, |j_0|) \leq c_2 \gamma_{k+1}^{-\frac{v}{q_0}}(l)^{\tau_1}}} \langle j - j_0 \rangle^{\frac{1}{q_0}} \langle l \rangle^{-1 - \frac{\tau_2 + 1}{q_0}} \langle l, j - j_0 \rangle \\ &\quad + \gamma^{\frac{v}{q_0}} \sum_{\substack{(l,j) \in \mathbb{Z}^d \times \mathbb{Z} \setminus \{(0,0)\} \\ |j| \leq C_0(l)}} \langle j \rangle^{\frac{1}{q_0}} \langle l \rangle^{-1 - \frac{\tau_1 + 1}{q_0}} \langle l, j \rangle. \end{aligned}$$

Thus, by using (4.60) and (4.86), we deduce

(4.116)

$$\begin{aligned} \max_{k \in \{0,1\}} \left| \mathcal{E}_k \setminus \mathcal{E}_{k+1} \right| &\lesssim \gamma^{\frac{1}{q_0}} \left(\sum_{l \in \mathbb{Z}^d} \langle l \rangle^{-\frac{\tau_1}{q_0}} + \gamma^{-v} \sum_{l \in \mathbb{Z}^d} \langle l \rangle^{\tau_1 - \frac{\tau_2}{q_0}} \right) + \gamma^{\frac{v}{q_0}} \sum_{l \in \mathbb{Z}^d} \langle l \rangle^{-\frac{\tau_1}{q_0}} \\ &\lesssim \gamma^{\min(\frac{v}{q_0}, \frac{1}{q_0} - v)} = \gamma^{\frac{v}{q_0}}. \end{aligned}$$

Now fix some $n \geq 2$. Using (4.112) and Lemma 4.12, we get

$$\begin{aligned} \left| \mathcal{E}_n \setminus \mathcal{E}_{n+1} \right| &\leq \sum_{\substack{(l,j) \in \mathbb{Z}^d \times \mathbb{Z} \setminus \{(0,0)\} \\ |j| \leq C_0(l), N_{n-1} < |l| \leq N_n}} \left| \mathcal{R}_{l,j}^{(0)}(i_n) \right| + \sum_{\substack{(l,j,j_0) \in \mathbb{Z}^d \times (\mathbb{S}_0^c)^2 \\ |j - j_0| \leq C_0(l), N_{n-1} < |l| \leq N_n \\ \min(|j|, |j_0|) \leq c_2 \gamma_{n+1}^{-\frac{v}{q_0}}(l)^{\tau_1}}} \left| \mathcal{R}_{l,j,j_0}^{(2)}(i_n) \right| \\ &\quad + \sum_{\substack{(l,j) \in \mathbb{Z}^d \times \mathbb{S}_0^c \\ |j| \leq C_0(l), N_{n-1} < |l| \leq N_n}} \left| \mathcal{R}_{l,j}^{(1)}(i_n) \right|. \end{aligned}$$

Therefore, we obtain from (4.113) and (4.115),

$$\begin{aligned} \left| \mathcal{E}_n \setminus \mathcal{E}_{n+1} \right| &\lesssim \gamma^{\frac{1}{q_0}} \left(\sum_{\substack{l \in \mathbb{Z}^d \\ |l| > N_{n-1}}} \langle l \rangle^{-\frac{\tau_1}{q_0}} + \gamma^{-v} \sum_{\substack{l \in \mathbb{Z}^d \\ |l| > N_{n-1}}} \langle l \rangle^{\tau_1 - \frac{\tau_2}{q_0}} \right) \\ &\quad + \gamma^{\frac{v}{q_0}} \sum_{\substack{l \in \mathbb{Z}^d \\ |l| > N_{n-1}}} \langle l \rangle^{-\frac{\tau_1}{q_0}}. \end{aligned}$$

Hence, we infer

$$(4.117) \quad \sum_{n=2}^{\infty} \left| \mathcal{E}_n \setminus \mathcal{E}_{n+1} \right| \lesssim \gamma^{\frac{v}{q_0}}.$$

Plugging (4.117) and (4.116) into (4.108) and using (4.99) yields

$$\left| (\alpha_0, \alpha_1) \setminus \mathcal{E}_{\infty}^{\varepsilon} \right| \lesssim \gamma^{\frac{v}{q_0}} = \varepsilon^{\frac{av}{q_0}}.$$

This achieves the proof of Proposition 4.8. □

To conclude, it remains to prove the following lemma providing necessary constraints between time and space Fourier modes so that the sets in (4.111) are not void.

LEMMA 4.12. *There exists ε_0 such that for any $\varepsilon \in [0, \varepsilon_0]$ and $n \in \mathbb{N}$ the following assertions hold true.*

- (i) *Let $(l, j) \in \mathbb{Z}^d \times \mathbb{Z} \setminus \{(0, 0)\}$. If $\mathcal{R}_{l,j}^{(0)}(i_n) \neq \emptyset$, then $|j| \leq C_0 \langle l \rangle$.*
- (ii) *Let $(l, j, j_0) \in \mathbb{Z}^d \times (\mathbb{S}_0^c)^2$. If $\mathcal{R}_{l,j,j_0}^{(2)}(i_n) \neq \emptyset$, then $|j - j_0| \leq C_0 \langle l \rangle$.*
- (iii) *Let $(l, j) \in \mathbb{Z}^d \times \mathbb{S}_0^c$. If $\mathcal{R}_{l,j}^{(1)}(i_n) \neq \emptyset$, then $|j| \leq C_0 \langle l \rangle$.*
- (iv) *Let $(l, j, j_0) \in \mathbb{Z}^d \times (\mathbb{S}_0^c)^2$. There exists $c_2 > 0$ such that if $\min(|j|, |j_0|) \geq c_2 \gamma_{n+1}^{-v} \langle l \rangle^{\tau_1}$, then*

$$\mathcal{R}_{l,j,j_0}^{(2)}(i_n) \subset \mathcal{R}_{l,j-j_0}^{(0)}(i_n).$$

PROOF. (i) Assume that $\mathcal{R}_{l,j}^{(0)}(i_n) \neq \emptyset$. Then, we can find $\alpha \in (\alpha_0, \alpha_1)$ such that

$$|\omega(\alpha, \varepsilon) \cdot l + j \mathfrak{m}_n^\infty(\alpha, \varepsilon)| \leq \frac{4\gamma_{n+1}^v \langle j \rangle}{\langle l \rangle^{\tau_1}}.$$

By using the triangle and Cauchy-Schwarz inequalities together with (4.101), (4.99) and the fact that $(\alpha, \varepsilon) \mapsto \omega(\alpha, \varepsilon)$ is bounded, we infer

$$\begin{aligned} |\mathfrak{m}_n^\infty(\alpha, \varepsilon)| |j| &\leq 4|j| \gamma_{n+1}^v \langle l \rangle^{-\tau_1} + |\omega(\alpha, \varepsilon) \cdot l| \\ &\leq 4|j| \gamma_{n+1}^v + C \langle l \rangle \\ &\leq 8\varepsilon^{av} |j| + C \langle l \rangle. \end{aligned}$$

Now, by (4.110), we can write

$$\mathfrak{m}_n^\infty(\alpha, \varepsilon) = V_0(\alpha) + r^{0,n}(\alpha, \varepsilon).$$

Hence, applying (4.63), (4.99) and Proposition 4.7-(i)-(a) we find

$$\begin{aligned} \forall k \in \llbracket 0, q \rrbracket, \sup_{n \in \mathbb{N}} \sup_{\alpha \in (\alpha_0, \alpha_1)} |\partial_\alpha^k r^{0,n}(\alpha, \varepsilon)| &\leq \gamma^{-k} \sup_{n \in \mathbb{N}} \|r^{0,n}\|_q^{\gamma, \mathcal{O}} \\ &\lesssim \varepsilon \gamma^{-k} \\ &\lesssim \varepsilon^{1-ak}. \end{aligned} \tag{4.118}$$

Thus, for ε small enough, we obtain by (4.25) and the decay property of $I_1 K_1$ on $(0, \infty)$,

$$\inf_{n \in \mathbb{N}} \inf_{\alpha \in (\alpha_0, \alpha_1)} |\mathfrak{m}_n^\infty(\alpha, \varepsilon)| \geq \frac{1}{2} V_0 \left(\frac{1}{\alpha_0} \right).$$

Hence, a suitable choice of ε small enough provides the constraint $|j| \leq C_0 \langle l \rangle$ for some $C_0 > 0$.

(ii) Observe that $\mathcal{R}_{l,j_0,j_0}^{(2)}(i_n) = \mathcal{R}_{l,0}^{(0)}(i_n)$, so the case $j = j_0$ can be included in the previous point. Now we assume $j \neq j_0$ and $\mathcal{R}_{l,j,j_0}^{(2)}(i_n) \neq \emptyset$. There exists $\alpha \in (\alpha_0, \alpha_1)$ such that

$$|\omega(\alpha, \varepsilon) \cdot l + d_j^{\infty,n}(\alpha, \varepsilon) - d_{j_0}^{\infty,n}(\alpha, \varepsilon)| \leq \frac{2\gamma_{n+1} |j - j_0|}{\langle l \rangle^{\tau_2}}.$$

Applying the triangle and Cauchy-Schwarz inequalities, (4.101) and (4.99), we infer

$$\begin{aligned} |d_j^{\infty,n}(\alpha, \varepsilon) - d_{j_0}^{\infty,n}(\alpha, \varepsilon)| &\leq 2\gamma_{n+1} |j - j_0| \langle l \rangle^{-\tau_2} + |\omega(\alpha, \varepsilon) \cdot l| \\ &\leq 2\gamma_{n+1} |j - j_0| + C \langle l \rangle \\ &\leq 4\varepsilon^a |j - j_0| + C \langle l \rangle. \end{aligned}$$

Now, similarly to (4.118), we may obtain from (4.91)-(4.99),

$$\begin{aligned}
 \forall k \in \llbracket 0, q \rrbracket, \quad \sup_{n \in \mathbb{N}} \sup_{j \in \mathbb{S}_0^c} \sup_{\alpha \in (\alpha_0, \alpha_1)} |j| |\partial_\alpha^k r_j^{\infty, n}(\alpha, \varepsilon)| &\leq \gamma^{-k} \sup_{n \in \mathbb{N}} \sup_{j \in \mathbb{S}_0^c} |j| \|r_j^{\infty, n}\|_{q, \mathcal{O}} \\
 &\lesssim \varepsilon \gamma^{-1-k} \\
 (4.119) \qquad \qquad \qquad &\lesssim \varepsilon^{1-a(1+k)}.
 \end{aligned}$$

One obtains from the triangle inequality, Lemma 4.7-(iv), (4.118) and (4.119), up to taking ε sufficiently small

$$\begin{aligned}
 |d_j^{\infty, n}(\alpha, \varepsilon) - d_{j_0}^{\infty, n}(\alpha, \varepsilon)| &\geq |\Omega_j^E(\alpha) - \Omega_{j_0}^E(\alpha)| - |r^{0, n}(\alpha, \varepsilon)| |j - j_0| \\
 &\quad - |r_j^{\infty, n}(\alpha, \varepsilon)| - |r_{j_0}^{\infty, n}(\alpha, \varepsilon)| \\
 &\geq (\Omega - C\varepsilon^{1-a}) |j - j_0| \\
 &\geq \frac{\Omega}{2} |j - j_0|.
 \end{aligned}$$

The foregoing inequalities together give for ε small enough, the constraint $|j - j_0| \leq C_0 \langle l \rangle$, for some $C_0 > 0$.

(iii) As previously, we can forget the case $j = 0$. Assume that $j \neq 0$ and $\mathcal{R}_{l, j}^{(1)}(i_n) \neq \emptyset$. There exists $\alpha \in (\alpha_0, \alpha_1)$ such that

$$|\omega(\alpha, \varepsilon) \cdot l + d_j^{\infty, n}(\alpha, \varepsilon)| \leq \frac{\gamma_{n+1} |j|}{\langle l \rangle^{\tau_1}}.$$

By the same techniques as in the other cases, we find

$$\begin{aligned}
 |d_j^{\infty, n}(\alpha, \varepsilon)| &\leq \gamma_{n+1} |j| \langle l \rangle^{-\tau_1} + |\omega(\alpha, \varepsilon) \cdot l| \\
 &\leq 2\varepsilon^a |j| + C \langle l \rangle.
 \end{aligned}$$

Now (4.109), the triangle inequality, Lemma 4.7-(iii), (4.118) and (4.119) imply

$$\begin{aligned}
 |d_j^{\infty, n}(\alpha, \varepsilon)| &\geq \Omega |j| - |j| |r^{0, n}(\alpha, \varepsilon)| - |r_j^{\infty, n}(\alpha, \varepsilon)| \\
 &\geq \Omega |j| - C\varepsilon^{1-a} |j|.
 \end{aligned}$$

Gathering the foregoing inequalities gives

$$(\Omega - C\varepsilon^{1-a} - 2\varepsilon^a) |j| \leq C \langle l \rangle.$$

Thus, for ε small enough we deduce the constraint $|j| \leq C_0 \langle l \rangle$, for some $C_0 > 0$.

(iv) We can forget the case $j = j_0$. Assume $j \neq j_0$. The symmetry property $d_{-j}^{\infty, n} = -d_j^{\infty, n}$ implies that without loss of generality we can assume that $0 \leq j < j_0$. Take $\alpha \in \mathcal{R}_{l, j, j_0}^{(2)}(i_n)$, i.e.

$$|\omega(\alpha, \varepsilon) \cdot l + d_j^{\infty, n}(\alpha, \varepsilon) \pm d_{j_0}^{\infty, n}(\alpha, \varepsilon)| \leq \frac{2\gamma_{n+1} \langle j \pm j_0 \rangle}{\langle l \rangle^{\tau_2}}.$$

Putting together (4.109), (4.27) and the triangle inequality, we find

$$\begin{aligned}
 |\omega(\alpha, \varepsilon) \cdot l + (j \pm j_0) \mathfrak{m}_n^\infty(\alpha, \varepsilon)| &\leq |\omega(\alpha, \varepsilon) \cdot l + d_j^{\infty, n}(\alpha, \varepsilon) \pm d_{j_0}^{\infty, n}(\alpha, \varepsilon)| + \frac{1}{2} (1 \pm 1) \\
 &\quad + |j I_j \left(\frac{1}{\alpha}\right) K_j \left(\frac{1}{\alpha}\right) \pm j_0 I_{j_0} \left(\frac{1}{\alpha}\right) K_{j_0} \left(\frac{1}{\alpha}\right)| \\
 &\quad + |r_j^{\infty, n}(\alpha, \varepsilon) \pm r_{j_0}^{\infty, n}(\alpha, \varepsilon)|.
 \end{aligned}$$

Hence, we deduce

$$(4.120) \quad \begin{aligned} |\omega(\alpha, \varepsilon) \cdot l + (j \pm j_0)\mathfrak{m}_n^\infty(\alpha, \varepsilon)| &\leq \frac{2\gamma_{n+1}\langle j \pm j_0 \rangle}{\langle l \rangle^{\tau_2}} + \frac{1}{2}(1 \pm 1) \\ &\quad + |jI_j\left(\frac{1}{\alpha}\right)K_j\left(\frac{1}{\alpha}\right) \pm j_0I_{j_0}\left(\frac{1}{\alpha}\right)K_{j_0}\left(\frac{1}{\alpha}\right)| \\ &\quad + |r_j^{\infty, n}(\alpha, \varepsilon) - r_{j_0}^{\infty, n}(\alpha, \varepsilon)|. \end{aligned}$$

It has been proved in [52, Lem. 7.2-(iv)] that

$$\forall x > 0, \quad |jI_j(x)K_j(x) \pm j_0I_{j_0}(x)K_{j_0}(x)| \leq \frac{\langle j \pm j_0 \rangle}{\min(|j|, |j_0|)}.$$

Additionally, (4.91) gives

$$\begin{aligned} |r_j^{\infty, n}(\alpha, \varepsilon) \pm r_{j_0}^{\infty, n}(\alpha, \varepsilon)| &\leq C\varepsilon^{1-a}(|j|^{-1} + |j_0|^{-1}) \\ &\leq C\varepsilon^{1-a} \frac{\langle j \pm j_0 \rangle}{\min(|j|, |j_0|)}. \end{aligned}$$

Also one has the trivial bound

$$\frac{1}{2}(1 \pm 1) \leq \frac{\langle j \pm j_0 \rangle}{\min(|j|, |j_0|)}.$$

Inserting the foregoing estimates into (4.120) yields

$$|\omega(\alpha, \varepsilon) \cdot l + (j \pm j_0)\mathfrak{m}_n^\infty(\alpha, \varepsilon)| \leq \frac{2\gamma_{n+1}\langle j \pm j_0 \rangle}{\langle l \rangle^{\tau_2}} + C \frac{\langle j \pm j_0 \rangle}{\min(|j|, |j_0|)}.$$

Thus, for $\min(|j|, |j_0|) \geq \frac{1}{2}C\gamma_{n+1}^{-\nu}\langle l \rangle^{\tau_1}$, then using (4.86), we get

$$|\omega(\alpha, \varepsilon) \cdot l + (j \pm j_0)\mathfrak{m}_n^\infty(\alpha, \varepsilon)| \leq \frac{4\gamma_{n+1}^\nu \langle j \pm j_0 \rangle}{\langle l \rangle^{\tau_1}}.$$

This proves Lemma 4.12, taking $c_2 \triangleq \frac{C}{2}$. □

Appendix A. Properties of modified Bessel functions

We collect here the definitions and useful properties of modified Bessel functions. The literature is huge as regards these special functions and we may refer the reader to [1, 73] for a nice introduction. The modified Bessel functions of first and second kind are defined as follows, for $|\arg(z)| < \pi$,

$$I_\nu(z) \triangleq \sum_{m=0}^{\infty} \frac{\left(\frac{z}{2}\right)^{\nu+2m}}{m!\Gamma(\nu+m+1)},$$

and

$$\forall \nu \in \mathbb{C} \setminus \mathbb{Z}, \quad K_\nu(z) \triangleq \frac{\pi}{2} \frac{I_{-\nu}(z) - I_\nu(z)}{\sin(\nu\pi)}, \quad \forall n \in \mathbb{Z}, \quad K_n(z) \triangleq \lim_{\nu \rightarrow n} K_\nu(z).$$

Symmetry and positivity properties (see [1, p. 375]):

$$(A.1) \quad \forall n \in \mathbb{N}, \quad \forall x > 0, \quad I_{-n}(x) = I_n(x) \in \mathbb{R}_+^* \quad \text{and} \quad K_{-n}(x) = K_n(x) \in \mathbb{R}_+^*.$$

Derivatives and Anti-derivatives:

If we set $\mathcal{Z}_\nu(z) \triangleq I_\nu(z)$ or $e^{i\nu\pi}K_\nu(z)$, then for all $\nu \in \mathbb{R}$, we have

$$(A.2) \quad \mathcal{Z}_\nu'(z) = \mathcal{Z}_{\nu-1}(z) - \frac{\nu}{z}\mathcal{Z}_\nu(z) = \mathcal{Z}_{\nu+1}(z) + \frac{\nu}{z}\mathcal{Z}_\nu(z)$$

and

$$(A.3) \quad \int z^{\nu+1}\mathcal{Z}_\nu(z)dz = z^{\nu+1}\mathcal{Z}_{\nu+1}(z).$$

Power series extension for K_n (see [1, p. 375]):

$$K_n(z) = \frac{1}{2} \left(\frac{z}{2}\right)^{-n} \sum_{k=0}^{n-1} \frac{(n-k-1)!}{k!} \left(\frac{-z}{4}\right)^k + (-1)^{n+1} \ln\left(\frac{z}{2}\right) I_n(z) + \frac{1}{2} \left(\frac{-z}{2}\right)^n \sum_{k=0}^{\infty} (\psi(k+1) + \psi(n+k+1)) \frac{\left(\frac{z^2}{4}\right)^k}{k!(n+k)!},$$

where

$$\psi(1) \triangleq -\gamma \text{ (Euler's constant)} \quad \forall m \in \mathbb{N}^*, \quad \psi(m+1) \triangleq \sum_{k=1}^m \frac{1}{k} - \gamma.$$

In particular

$$(A.4) \quad K_0(z) = -\log\left(\frac{z}{2}\right) I_0(z) + \sum_{m=0}^{\infty} \frac{\left(\frac{z}{2}\right)^{2m}}{(m!)^2} \psi(m+1),$$

so K_0 behaves like a logarithm at 0.

Decay property for the product $I_\nu K_\nu$ (see [8] and [27]):

The application $(x, \nu) \mapsto I_\nu(x)K_\nu(x)$ is strictly decreasing in each variable $x, \nu > 0$.

Asymptotic expansion of small argument (see [1, p. 375]):

$$(A.5) \quad \forall n \in \mathbb{N}^*, \quad I_n(x) \underset{x \rightarrow 0}{\sim} \frac{\left(\frac{1}{2}x\right)^n}{\Gamma(n+1)} \quad \text{and} \quad K_n(x) \underset{x \rightarrow 0}{\sim} \frac{\Gamma(n)}{2\left(\frac{1}{2}x\right)^n}.$$

Asymptotic expansion of large argument (see [1, p. 375]):

$$(A.6) \quad \forall n \in \mathbb{N}^*, \quad I_n(x) \underset{x \rightarrow \infty}{\sim} \frac{e^x}{\sqrt{2\pi x}} \quad \text{and} \quad K_n(x) \underset{x \rightarrow \infty}{\sim} \sqrt{\frac{\pi}{2x}} e^{-x}.$$

Asymptotic of high order (see [1, p. 377]):

$$(A.7) \quad \forall x > 0, \quad I_\nu(x) \underset{\nu \rightarrow \infty}{\sim} \frac{1}{\sqrt{2\pi\nu}} \left(\frac{ex}{2\nu}\right)^\nu \quad \text{and} \quad K_\nu(x) \underset{\nu \rightarrow \infty}{\sim} \sqrt{\frac{\pi}{2\nu}} \left(\frac{ex}{2\nu}\right)^{-\nu}.$$

Integral representations (see [59, p. 140]):

$$(A.8) \quad \forall x > 0, \quad \forall \nu > 0, \quad I_\nu(x)K_\nu(x) = \frac{1}{2} \int_0^\infty J_0\left(2x \sinh\left(\frac{t}{2}\right)\right) e^{-\nu t} dt,$$

where J_0 is the classical Bessel function which is defined by

$$\forall z \in \mathbb{C}, \quad J_0(z) \triangleq \sum_{m=0}^{\infty} \frac{(-1)^m}{(m!)^2} \left(\frac{z}{2}\right)^{2m}$$

and which admits the following integral representation (see [59, p. 115])

$$(A.9) \quad \forall x > 0, \quad J_0(x) = \frac{1}{\pi} \int_0^\pi \cos(x \sin(\theta)) d\theta.$$

Appendix B. Local bifurcation theorem and singular integrals

We recall here the Crandall-Rabinowitz’s Theorem of local bifurcation theory which was used to find the periodic solutions in Section 3. Its proof can be found in [25] and [55, p.15].

THEOREM B.1 (Crandall-Rabinowitz). *Let X and Y be two Banach spaces. Let V be a neighborhood of 0 in X and let*

$$\begin{aligned} F : \mathbb{R} \times V &\rightarrow Y \\ (\Omega, x) &\mapsto F(\Omega, x) \end{aligned}$$

be a function of classe C^1 with the following properties

- (i) (Trivial solution) $\forall \Omega \in \mathbb{R}, F(\Omega, 0) = 0$.
- (ii) (Regularity) $\partial_\Omega F, d_x F$ and $\partial_\Omega d_x F$ exist and are continuous.
- (iii) (Fredholm property) $d_x F(0, 0)$ is a Fredholm operator with index 0 and $\ker(d_x F(0, 0)) = \langle x_0 \rangle$.
- (iv) (Transversality assumption) $\partial_\Omega d_x F(0, 0)[x_0] \notin R(d_x F(0, 0))$.

If χ denotes any complement of $\ker(d_x F(0, 0))$ in X , then there exist

- U a neighborhood of $(0, 0)$ in $\mathbb{R} \times V$,
- an interval $(-a, a)$ for some $a > 0$,
- continuous functions $\psi : (-a, a) \rightarrow \mathbb{R}$ and $\phi : (-a, a) \rightarrow \chi$ satisfying $\psi(0) = 0$ and $\phi(0) = 0$

such that the set of the zeros of F in U can be described as the following two curves intersecting at $(0, 0)$

$$\begin{aligned} \left\{ (\Omega, x) \in U \text{ s.t. } F(\Omega, x) = 0 \right\} &= \left\{ (\psi(s), sx_0 + s\phi(s)) \text{ s.t. } |s| < a \right\} \\ &\cup \left\{ (\Omega, 0) \in U \right\}. \end{aligned}$$

Now, we also state some continuity properties of singular integral operators. We may refer to [37, 45, 57, 61] for a proof of the following result.

LEMMA B.1. *Consider a function $K : \mathbb{T} \times \mathbb{T} \rightarrow \mathbb{C}$ such that for some $C_0 > 0$ we have*

- K is measurable on $\mathbb{T} \times \mathbb{T} \setminus \{(w, w), w \in \mathbb{T}\}$ and
- $$\forall (w, \tau) \in \mathbb{T}^2, \quad w \neq \tau \quad \Rightarrow \quad |K(w, \tau)| \leq C_0.$$
- For any $\tau \in \mathbb{T}$, the application $w \mapsto K(w, \tau)$ is differentiable in $\mathbb{T} \setminus \{\tau\}$ and

$$\forall (w, \tau) \in \mathbb{T}^2, \quad w \neq \tau \quad \Rightarrow \quad |\partial_w K(w, \tau)| \leq \frac{C_0}{|w - \tau|}.$$

Consider the operator \mathcal{T}_K defined by

$$\mathcal{T}_K(f)(w) = \int_{\mathbb{T}} K(w, \tau) f(\tau) d\tau.$$

Then, the operator \mathcal{T}_K is continuous from $L^\infty(\mathbb{T})$ to $C^\delta(\mathbb{T})$ for any $0 < \delta < 1$ and there exists $C_\delta > 0$ such that

$$\|\mathcal{T}_K(f)\|_{C^\delta(\mathbb{T})} \leq C_\delta C_0 \|f\|_{L^\infty(\mathbb{T})}.$$

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