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Global regularity of multi-dimensional Burgers equation with critical dissipation only in one direction

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ABSTRACT. In this paper, we are concerned with the multi-dimensional Burgers equation with the critical dissipation only in one direction. We make use of the elegant method introduced by Constantin and Vicol to show the unique global existence of smooth solution.

CONTENTS

1. Introduction

The n-dimensional vector Burgers equation with the critical dissipation in j-th direction takes the form

$$\begin{cases} \partial_t u + (u \cdot \nabla) u + \Lambda_{x_j} u = 0, \\ u(x,0) = u_0(x), \end{cases} \tag{1.1}$$

where $u = (u_1(t,x), u_2(t,x), \cdots, u_n(t,x))$ is a vector valued unknown function of $t > 0$ and $x = (x_1, x_2, \cdots, x_n)$, $j \in \{1, 2, \cdots, n\}$. (1.1) is called critical due to the invariance with respect to the scaling transformation given by $u_\lambda(x,t) = u(\lambda x, \lambda t)$. The fractional operator $\Lambda_{x_j} \triangleq (-\partial_{x_j}^2)^{\frac{1}{2}}$ is defined through the Fourier transform, namely

$$\widehat{\Lambda_{x_j} f}(\xi) = |\xi_j| \hat{f}(\xi),$$

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where $\xi = (\xi_1, \xi_2, \cdots, \xi_n)$. The Burgers equation has been widely used to model different areas of physics, such as the surface perturbations, acoustical waves, electromagnetic waves, density waves, population growth, magnetohydrodynamic waves and so on (see the review [**3**]). In the case when $n = 1$, (1.1) becomes well-known one-dimensional Burgers equation [**4**], which is the simplest nonlinear model in fluid dynamics. A detailed picture of the global regularity theory of (1.1) in one-dimensional case was shown in [**10**]. More precisely, based on the method of "moduli of continuity", they prove that smooth initial data gives global smooth solutions. This method was introduced in [**11**] in the context of the critical SQG equation. Here it is worthwhile to point out that "nonlinear maximum principle" is also powerful to prove the global regularity of critical Burgers equation [**7, 6**]. The multi-dimensional ($n \geq 2$) vector Burgers equation with the classical critical dissipation reads

$$\begin{cases} \partial_t u + (u \cdot \nabla)u + \Lambda u = 0, \\ u(x, 0) = u_0(x), \end{cases} \tag{1.2}$$

where the fractional Laplacian operator Λ is defined through the Fourier transform,

$$\widehat{\Lambda f}(\xi) = |\xi| \hat{f}(\xi).$$

For the global wellposedness of (1.2), we refer the reader to [**7**]. It is well-known that the solutions blow up in finite time when the dissipation term Λu is replaced by $\Lambda^\gamma u$ with $\gamma < 1$ (see [**1, 10, 9**]). Consequently, it is natural to ask if the global regularity of (1.2) remains valid when the critical dissipation occurs only in some direction(s). As a matter of fact, based on "nonlinear maximum principle" and some new observations, we are still able to show the global regularity of the multi-dimensional Burgers equation with the critical dissipation in one direction. Without loss of generality, we just focus on the following multi-dimensional vector Burgers equation with the critical dissipation in the first direction

$$\begin{cases} \partial_t u + (u \cdot \nabla)u + \Lambda_{x_1} u = 0, \\ u(x, 0) = u_0(x). \end{cases} \tag{1.3}$$

The main goal of this paper is to establish the following theorem.

THEOREM 1.1. *Let $u_0 \in H^s(\mathbb{R}^n)$ for $s > 1 + \frac{n}{2}$, then (1.3) admits a unique global solution such that, for any given $T > 0$*

$$u \in C([0, T]; H^s(\mathbb{R}^n)).$$

REMARK 1.1. Our arguments are also valid for proving the global regularity of the multi-dimensional scalar Burgers equation with the critical dissipation in one direction [**5**]

$$\begin{cases} \partial_t \theta + \sum_{i=1}^n \theta \partial_{x_i} \theta + \Lambda_{x_j} \theta = 0, \\ \theta(x, 0) = \theta_0(x), \end{cases}$$

where $\theta : [0, \infty) \times \mathbb{R}^n \to \mathbb{R}$ and $j \in \{1, 2, \cdots, n\}$.

REMARK 1.2. Theorem 1.1 indicates that the critical dissipation only in one direction can ensure the global regularity of the multi-dimensional Burgers equation. However, at present we are not able to show this interesting phenomenon is valid for the corresponding SQG equation due to the nonlocal relationship $u = \mathcal{R}^\perp \theta$.

2. The proof of Theorem 1.1

The proof is divided into two steps, namely the Hölder estimate of u and the differentiability of u.

2.1. Hölder estimate of u. We first show the maximum principle of u, namely,

$$\|u(t)\|_{L^\infty} \le \|u_0\|_{L^\infty}. \tag{2.1}$$

Dotting (1.3) with $2u$ gives

$$\partial_t |u|^2 + (u\cdot\nabla)|u|^2 + 2\Lambda_{x_1} u\cdot u = 0. \tag{2.2}$$

Recalling [**8**, Proposition 2.3], we have

$$2\Lambda_{x_1} u\cdot u \ge \Lambda_{x_1}|u|^2,$$

which along with (2.2) implies

$$\partial_t |u|^2 + (u\cdot\nabla)|u|^2 + \Lambda_{x_1}|u|^2 \le 0. \tag{2.3}$$

Since u decays at infinity, there exists a point $x_t \in \mathbb{R}^2$ where $|u|$ attains its maximum value, we let

$$|u|(x_t,t) = \|u(t)\|_{L^\infty}.$$

It should be pointed out that $\nabla_x u(x_t,t) = 0$ and $\Lambda_{x_1}|u|(x_t,t) \ge 0$ (based on the definition of Λ_{x_1}). This together with (2.3) allows us to get

$$\frac{d}{dt}\|u(t)\|_{L^\infty}^2 \le \partial_t|u|^2(x_t,t) \le 0, \quad \text{for all } t>0.$$

Integrating in time yields the desired estimate (2.1).

Now let us focus on the Hölder estimate of u. For simplicity, we denote

$$x = (x_1,\widetilde{x}), \quad \widetilde{x} = (x_2,\cdots,x_n).$$

We consider the difference

$$\delta_h u(x,t) \triangleq u(x+h,t) - u(x,t).$$

We will use the pointwise equality (see, e.g., [**8, 7**])

$$2\delta_h u(x)\cdot \Lambda_{x_1}\delta_h u(x) = \Lambda_{x_1}\big(\delta_h u(x)\big)^2 + \widetilde{D}^{(1)}[\delta_h u](x), \tag{2.4}$$

where

$$\widetilde{D}^{(1)}[\delta_h u](x) = C_0\,\text{P.V.}\int_{\mathbb{R}} \frac{\big(\delta_h u(x_1,\widetilde{x}) - \delta_h u(x_1+y_1,\widetilde{x})\big)^2}{|y_1|^2}\,dy_1$$

with an absolute constant $C_0>0$. Here P.V. stands for the principal value of the integral. Although in this paper we may sometimes omit the P.V. in front of the integral, the integral is always understood in the principal value sense. We also remark that the original statement (2.4) is for a scalar function. Actually, it can be extended for a vector valued function by viewing u^2 as u dot product u, namely $u^2 = u\cdot u$. It follows from (1.3) that

$$\Big(\partial_t + u\cdot\nabla_x + (\delta_h u)\cdot\nabla_h + \Lambda_{x_1}\Big)(\delta_h u) = 0. \tag{2.5}$$

We denote

$$v(x,t;h) \triangleq \frac{|\delta_h u(x,t)|}{|h|^\alpha}.$$

It is worthwhile to state that $\|v\|_{L^\infty_{x,h}}$ is equivalent to the Hölder seminorm $[u(t)]_{C^\alpha}$. Now multiplying (2.5) by $\frac{\delta_h u}{|h|^{2\alpha}}$ with $\alpha > 0$ gives (see also [**6**])

$$\begin{aligned}
\Big(\partial_t + u\cdot\nabla_x + (\delta_h u)\cdot\nabla_h + \Lambda_{x_1}\Big)v^2 + C_0\frac{\widetilde{D}^{(1)}[\delta_h u]}{|h|^{2\alpha}} &= 4\alpha(\delta_h u)\cdot\frac{hv^2}{|h|^2}\\
&\le 4\alpha|\delta_h u|\frac{v^2}{|h|}\\
&\le \frac{4\alpha v^3}{|h|^{1-\alpha}}.
\end{aligned}\tag{2.6}$$

Now let us show the lower bound of $\widetilde{D}^{(1)}[\delta_h u]$. To this end, we appeal to the following proposition.

Proposition 2.1. *Let $n \ge 2$, then it holds*

$$\int_{\mathbb{R}^n}\frac{\big(\delta_h u(x_1,\widetilde{x}) - \delta_h u(x_1+y_1,\widetilde{x})\big)^2}{|y|^{n+1}}\,dy_1 d\widetilde{y} \approx \widetilde{D}^{(1)}[\delta_h u](x).\tag{2.7}$$

Proof. Actually, a calculation shows that

$$\begin{aligned}
&\int_{\mathbb{R}^n}\frac{\big(\delta_h u(x_1,\widetilde{x}) - \delta_h u(x_1+y_1,\widetilde{x})\big)^2}{|y|^{n+1}}\,dy\\
&=\int_{\mathbb{R}}\int_{\mathbb{R}^{n-1}}\frac{\big(\delta_h u(x_1,\widetilde{x}) - \delta_h u(x_1+y_1,\widetilde{x})\big)^2}{|y|^{n+1}}\,d\widetilde{y}dy_1\\
&=\int_{\mathbb{R}}\frac{\big(\delta_h u(x_1,\widetilde{x}) - \delta_h u(x_1+y_1,\widetilde{x})\big)^2}{|y_1|^2}\,dy_1\int_{\mathbb{R}^{n-1}}\frac{|y_1|^2}{|y|^{n+1}}d\widetilde{y}\\
&\approx\int_{\mathbb{R}}\frac{\big(\delta_h u(x_1,\widetilde{x}) - \delta_h u(x_1+y_1,\widetilde{x})\big)^2}{|y_1|^2}\,dy_1\int_{\mathbb{R}^{n-1}}\frac{|y_1|^2}{|y_1|^{n+1}+|\widetilde{y}|^{n+1}}d\widetilde{y}\\
&=C(n)\int_{\mathbb{R}}\frac{\big(\delta_h u(x_1,\widetilde{x}) - \delta_h u(x_1+y_1,\widetilde{x})\big)^2}{|y_1|^2}\,dy_1\int_0^\infty\frac{|y_1|^2r^{n-2}}{|y_1|^{n+1}+r^{n+1}}dr\\
&=C(n)\int_{\mathbb{R}}\frac{\big(\delta_h u(x_1,\widetilde{x}) - \delta_h u(x_1+y_1,\widetilde{x})\big)^2}{|y_1|^2}\,dy_1\int_0^\infty\frac{(\frac{r}{|y_1|})^{n-2}}{1+(\frac{r}{|y_1|})^{n+1}}d(\frac{r}{|y_1|})\\
&=C(n)\int_{\mathbb{R}}\frac{\big(\delta_h u(x_1,\widetilde{x}) - \delta_h u(x_1+y_1,\widetilde{x})\big)^2}{|y_1|^2}\,dy_1\int_0^\infty\frac{\tau^{n-2}}{1+\tau^{n+1}}d\tau\\
&=C\int_{\mathbb{R}}\frac{\big(\delta_h u(x_1,\widetilde{x}) - \delta_h u(x_1+y_1,\widetilde{x})\big)^2}{|y_1|^2}\,dy_1\\
&=C\widetilde{D}^{(1)}[\delta_h u](x),
\end{aligned}$$

which yields (2.7). This finishes the proof of Proposition 2.1. □

With (2.7) in hand, we are able to show the lower bound of $\widetilde{D}^{(1)}[\delta_h u]$. Inspired by [**6**], let χ be a smooth radially cutoff function that vanishes on $|x| \le 1$ and is identically 1 for $|x| \geq 2$ and such that $|\chi'| \leq 2$. For $R \geq 4|h|$, we can conclude

$$\begin{aligned}
\widetilde{D}^{(1)}[\delta_h u](x) \geq & C \int_{\mathbb{R}^n} \frac{\big(\delta_h u(x_1, \widetilde{x}) - \delta_h u(x_1 + y_1, \widetilde{x})^2}{|y|^{n+1}} \chi\Big(\frac{|y|}{R}\Big)\, dy \\
\geq & C[\delta_h u(x_1, \widetilde{x})]^2 \int_{\mathbb{R}^n} \frac{1}{|y|^{n+1}} \chi\Big(\frac{|y|}{R}\Big)\, dy \\
& - 2C|\delta_h u(x_1, \widetilde{x})| \left| \int_{\mathbb{R}^n} \frac{\delta_h u(x_1 + y_1, \widetilde{x})}{|y|^{n+1}} \chi\Big(\frac{|y|}{R}\Big)\, dy \right| \\
\geq & C[\delta_h u(x)]^2 \int_{|y| \geq 2R} \frac{1}{|y|^{n+1}}\, dy \\
& - 2C|\delta_h u(x)| \left| \int_{\mathbb{R}^n} \frac{\delta_h u(x_1 + y_1, \widetilde{x})}{|y|^{n+1}} \chi\Big(\frac{|y|}{R}\Big)\, dy \right| \\
\geq & C\frac{[\delta_h u(x)]^2}{R} - 2C|\delta_h u(x)| \left| \int_{\mathbb{R}^n} u(x_1 + y_1, \widetilde{x}) \delta_{-h}\Big(\frac{\chi(\frac{|y|}{R})}{|y|^{n+1}}\Big)\, dy \right| \\
\geq & C\frac{[\delta_h u(x)]^2}{R} - 2C\|u\|_{L^\infty} |\delta_h u(x)| \int_{\mathbb{R}^n} \left| \delta_{-h}\Big(\frac{\chi(\frac{|y|}{R})}{|y|^{n+1}}\Big) \right| dy \\
\geq & C\frac{[\delta_h u(x)]^2}{R} - 2C|h|\, |\delta_h u(x)|\, \|u_0\|_{L^\infty} \int_{|y| \geq R} \frac{1}{|y|^{n+2}}\, dy \\
\geq & C\frac{[\delta_h u(x)]^2}{R} - 2C|h|\, |\delta_h u(x)|\, \|u_0\|_{L^\infty} \frac{1}{R^2}, \qquad (2.8)
\end{aligned}$$

where we have used (2.1) and the following fact, for $|y| \geq \frac{3}{4}R$,

$$\left| \delta_{-h}\Big(\frac{\chi(\frac{|y|}{R})}{|y|^{n+1}}\Big) \right| \leq C|h| \left(\frac{1}{|y|^{n+2}} + \frac{|\chi'(\frac{|y|}{R})| I_{R \leq |y| \leq 2R}}{R|y|^{n+1}} \right) \leq \frac{C|h|}{|y|^{n+2}}. \qquad (2.9)$$

The above estimate (2.9) follows from the mean value theorem, please refer to (4.18)-(4.19) of [**7**] for more details. Taking R as

$$R \approx \frac{\|u_0\|_{L^\infty} |h|}{|\delta_h u(x)|},$$

we thus get from (2.8) that

$$\widetilde{D}^{(1)}[\delta_h u](x) \geq \frac{|\delta_h u(x)|^3}{C|h|\|u_0\|_{L^\infty}}. \qquad (2.10)$$

Concerning (2.10), it follows that

$$\frac{\widetilde{D}^{(1)}[\delta_h u]}{|h|^{2\alpha}} \geq \frac{|\delta_h u|^3}{C|h|^{1+2\alpha}} \geq \frac{v^3}{C_*|h|^{1-\alpha}}. \qquad (2.11)$$

Next, (2.11) and (2.6) yields

$$\Big(\partial_t + u \cdot \nabla_x + (\delta_h u) \cdot \nabla_h + \Lambda_{x_1}\Big) v^2 + \frac{v^3}{C_*|h|^{1-\alpha}} \leq \frac{4\alpha v^3}{|h|^{1-\alpha}}. \qquad (2.12)$$

Therefore, if we choose α such that

$$\alpha \leq \frac{1}{8C_*},$$

then we have from (2.12) that

$$\Big(\partial_t + u\cdot\nabla_x + (\delta_h u)\cdot\nabla_h + \Lambda_{x_1}\Big)v^2 + \frac{v^3}{2C_*|h|^{1-\alpha}} \le 0. \tag{2.13}$$

By the definition of v,

$$\|v(0)\|_{L^\infty_{x,h}} \le \|u_0\|_{C^\alpha}.$$

Note that the continuity of v with respect to variable x and h and $|v(x,t)| \to 0$ as $|x| \to \infty$, we find that for each τ, there exist $(\widehat{x}, \widehat{h}) = (\widehat{x}(\tau), \widehat{h}(\tau))$ be the point at which v^2 attains its maximum value. We define

$$G(\tau) \triangleq v^2(\widehat{x}(\tau), \tau; \widehat{h}(\tau)).$$

At this point, we have

$$\partial_x v^2 = \partial_h v^2 = 0,\ \ \Lambda_{x_1} v^2 \ge 0.$$

Therefore, (2.13) yields

$$G'(\tau) \le 0,$$

which implies

$$G(\tau) \le G(0) \le \|u_0\|_{C^\alpha}.$$

Consequently, we get

$$[u(t)]_{C^\alpha} \le \|u_0\|_{C^\alpha}.$$

Due to the maximum principle, one has

$$\|u(t)\|_{L^\infty} \le \|u_0\|_{L^\infty},$$

which yields

$$\|u(t)\|_{C^\alpha} \le \|u_0\|_{C^\alpha}.$$

2.2. Differentiability of u. We apply ∇ to (1.3) and pointwise in x take inner product with ∇u to get

$$\Big(\partial_t + u\cdot\nabla + \Lambda_{x_1}\Big)|\nabla u|^2 + \widetilde{D}^{(1)}[\nabla u] \approx 2|\nabla u|^3, \tag{2.14}$$

where

$$\widetilde{D}^{(1)}[\nabla u](x) = P.V.\int_{\mathbb{R}} \frac{\big(\nabla u(x_1,\widetilde{x}) - \nabla u(x_1+y_1,\widetilde{x})\big)^2}{|y_1|^2}\,dy_1.$$

It follows from (2.7) that

$$\int_{\mathbb{R}^n} \frac{\big(\nabla u(x_1,\widetilde{x}) - \nabla u(x_1+y_1,\widetilde{x})\big)^2}{|y|^{n+1}}\,dy \approx \widetilde{D}^{(1)}[\nabla u](x). \tag{2.15}$$

We notice that

$$\begin{aligned}
&\big(\nabla u(x_1,\widetilde{x}) - \nabla u(x_1+y_1,\widetilde{x})\big)^2 \\
&= \sum_{k=1}^{n} \big[(\partial_k u)(x_1,\widetilde{x}) - (\partial_k u)(x_1+y_1,\widetilde{x})\big]^2 \\
&= \sum_{k=1}^{n} \big[(\partial_k u)^2(x_1,\widetilde{x}) - 2(\partial_k u)(x_1,\widetilde{x})(\partial_k u)(x_1+y_1,\widetilde{x}) + (\partial_k u)^2(x_1+y_1,\widetilde{x})\big] \\
&\geq \sum_{k=1}^{n} \big[(\partial_k u)^2(x_1,\widetilde{x}) - 2(\partial_k u)(x_1,\widetilde{x})(\partial_k u)(x_1+y_1,\widetilde{x})\big] \\
&= |\nabla u(x_1,\widetilde{x})|^2 - 2\sum_{k=1}^{n}(\partial_k u)(x_1,\widetilde{x})(\partial_k u)(x_1+y_1,\widetilde{x}),
\end{aligned}$$

which along with (2.8) yields

$$\begin{aligned}
\widetilde{D}^{(1)}[\nabla u](x) \geq & C\int_{\mathbb{R}^n} \frac{\big(\nabla u(x_1,\widetilde{x}) - \nabla u(x_1+y_1,\widetilde{x})\big)^2}{|y|^{n+1}}\chi\Big(\frac{|y|}{R}\Big)\,dy \\
\geq & C[\nabla u(x_1,\widetilde{x})]^2\int_{\mathbb{R}^n} \frac{1}{|y|^{n+1}}\chi\Big(\frac{|y|}{R}\Big)\,dy \\
& - 2C|\nabla u(x_1,\widetilde{x})|\left|\int_{\mathbb{R}^n} \frac{(\partial_1 u)(x_1+y_1,\widetilde{x})}{|y|^{n+1}}\chi\Big(\frac{|y|}{R}\Big)\,dy\right| \\
& - 2C|\nabla u(x_1,\widetilde{x})|\sum_{2\leq k\leq n}\left|\int_{\mathbb{R}^n} \frac{(\partial_k u)(x_1+y_1,\widetilde{x})}{|y|^{n+1}}\chi\Big(\frac{|y|}{R}\Big)\,dy\right| \\
\geq & C[\nabla u(x)]^2\int_{|y|\geq 2R} \frac{1}{|y|^{n+1}}\,dy \\
& - 2C|\nabla u(x)|\left|\int_{\mathbb{R}^n} \partial_{y_1}[u(x_1+y_1,\widetilde{x}) - u(x_1,\widetilde{x})]\frac{\chi(\frac{|y|}{R})}{|y|^{n+1}}\,dy\right| \\
& - 2C|\nabla u(x_1,\widetilde{x})|\sum_{2\leq k\leq n}\left|\int_{\mathbb{R}^n} \frac{(\partial_k u)(x_1+y_1,\widetilde{x})}{|y|^{n+1}}\chi\Big(\frac{|y|}{R}\Big)\,dy\right| \\
\geq & C\frac{[\nabla u(x)]^2}{R} \\
& - 2C|\nabla u(x)|\left|\int_{\mathbb{R}^n} \frac{u(x_1+y_1,\widetilde{x}) - u(x_1,\widetilde{x})}{|y_1|^{\alpha}}|y_1|^{\alpha}\partial_{y_1}\Big(\frac{\chi(\frac{|y|}{R})}{|y|^{n+1}}\Big)\,dy\right| \\
& - 2C|\nabla u(x_1,\widetilde{x})|\sum_{2\leq k\leq n}\left|\int_{\mathbb{R}^n} \frac{(\partial_k u)(x_1+y_1,\widetilde{x})}{|y|^{n+1}}\chi\Big(\frac{|y|}{R}\Big)\,dy\right| \\
\geq & C\frac{[\nabla u(x)]^2}{R} - 2C|\nabla u(x)|\int_{\mathbb{R}^n}\|u\|_{C_1^{\alpha}}|y|^{\alpha}\left|\partial_{y_1}\Big(\frac{\chi(\frac{|y|}{R})}{|y|^{n+1}}\Big)\right|\,dy \\
& - 2C|\nabla u(x_1,\widetilde{x})|\sum_{2\leq k\leq n}\left|\int_{\mathbb{R}^n} \frac{(\partial_k u)(x_1+y_1,\widetilde{x})}{|y|^{n+1}}\chi\Big(\frac{|y|}{R}\Big)\,dy\right|,
\end{aligned} \tag{2.16}$$

where $\|u\|_{C_1^\alpha}$ is defined by

$$\|u\|_{C_1^\alpha} \triangleq \sup_{y_1\neq 0} \frac{|u(x_1+y_1,\widetilde{x})-u(x_1,\widetilde{x})|}{|y_1|^\alpha}.$$

Noticing the following fact

$$\begin{aligned}\|u\|_{C^\alpha} &\triangleq \sup_{y=(y_1,\widetilde{y})\neq 0} \frac{|u(x_1+y_1,\widetilde{x}+\widetilde{y})-u(x_1,\widetilde{x})|}{|y|^\alpha}\\ &\geq \sup_{y_1\neq 0} \frac{|u(x_1+y_1,\widetilde{x})-u(x_1,\widetilde{x})|}{|y_1|^\alpha}\\ &= \|u\|_{C_1^\alpha},\end{aligned}$$

we have

$$\|u\|_{C^\alpha} \geq \|u\|_{C_1^\alpha}.$$

This yields

$$\begin{aligned}\int_{\mathbb{R}^n} \|u\|_{C_1^\alpha}|y|^\alpha \Big|\partial_{y_1}\Big(\frac{\chi(\frac{|y|}{R})}{|y|^{n+1}}\Big)\Big|\,dy \leq & \|u\|_{C^\alpha}\int_{|y|\geq R}|y|^\alpha \frac{1}{|y|^{n+2}}\,dy\\ \leq & C\|u\|_{C^\alpha}\frac{1}{R^{2-\alpha}}.\end{aligned} \tag{2.17}$$

Invoking the duality in Besov space (see [**2**, Proposition 2.29]) and the sharp interpolation inequality (see [**2**, Proposition 2.22]), we have

$$\begin{aligned}&\sum_{2\leq k\leq n}\Big|\int_{\mathbb{R}^n}\frac{(\partial_k u)(x_1+y_1,\widetilde{x})}{|y|^{n+1}}\chi\Big(\frac{|y|}{R}\Big)\,dy\Big|\\ &\leq C\|(\partial_k u)(x_1+y_1,\widetilde{x})\|_{\dot{B}^{\alpha-1}_{\infty,\infty}}\left\|\frac{\chi\Big(\frac{|y|}{R}\Big)}{|y|^{n+1}}\right\|_{\dot{B}^{1-\alpha}_{1,1}}\\ &\leq C\|u\|_{\dot{B}^{\alpha}_{\infty,\infty}}\left\|\frac{\chi\Big(\frac{|y|}{R}\Big)}{|y|^{n+1}}\right\|_{\dot{B}^{1-\alpha}_{1,1}}\\ &\leq C\|u\|_{\dot{B}^{\alpha}_{\infty,\infty}}\left\|\frac{\chi\Big(\frac{|y|}{R}\Big)}{|y|^{n+1}}\right\|^{\alpha}_{\dot{B}^{0}_{1,\infty}}\left\|\frac{\chi\Big(\frac{|y|}{R}\Big)}{|y|^{n+1}}\right\|^{1-\alpha}_{\dot{B}^{1}_{1,\infty}}\\ &\leq C\|u\|_{\dot{B}^{\alpha}_{\infty,\infty}}\left\|\frac{\chi\Big(\frac{|y|}{R}\Big)}{|y|^{n+1}}\right\|^{\alpha}_{\dot{B}^{0}_{1,\infty}}\left\|\nabla\left(\frac{\chi\Big(\frac{|y|}{R}\Big)}{|y|^{n+1}}\right)\right\|^{1-\alpha}_{\dot{B}^{0}_{1,\infty}}\\ &\leq C\|u\|_{\dot{B}^{\alpha}_{\infty,\infty}}\left\|\frac{\chi\Big(\frac{|y|}{R}\Big)}{|y|^{n+1}}\right\|^{\alpha}_{L^1}\left\|\nabla\left(\frac{\chi\Big(\frac{|y|}{R}\Big)}{|y|^{n+1}}\right)\right\|^{1-\alpha}_{L^1}\\ &\leq C\|u\|_{\dot{B}^{\alpha}_{\infty,\infty}}\Big(\frac{1}{R}\Big)^{\alpha}\Big(\frac{1}{R^2}\Big)^{1-\alpha}\\ &\leq C\|u\|_{C^\alpha}\frac{1}{R^{2-\alpha}}.\end{aligned} \tag{2.18}$$

Putting (2.17) and (2.18) into (2.16) yields that

$$\begin{aligned}\widetilde{D}^{(1)}[\nabla u](x) \geq & C\frac{[\nabla u(x)]^2}{R} - C|\nabla u(x)|\, \|u\|_{C^\alpha}\frac{1}{R^{2-\alpha}} \\ \geq & C|\nabla u(x)|^{\frac{3-2\alpha}{1-\alpha}},\end{aligned} \tag{2.19}$$

where we have fixed R as

$$R \approx \left(\frac{\|u\|_{C^\alpha}}{|\nabla u(x)|}\right)^{\frac{1}{1-\alpha}}.$$

Inserting (2.15) and (2.19) into (2.14) implies

$$\Big(\partial_t + u\cdot\nabla + \Lambda_{x_1}\Big)|\nabla u|^2 + C|\nabla u|^{\frac{3-2\alpha}{1-\alpha}} \leq 2|\nabla u|^3. \tag{2.20}$$

We suppose that $(\widehat{x_1}, \widehat{\widehat{x}}) = (\widehat{x_1}(t), \widehat{\widehat{x}}(t))$ be the point at which $|\nabla u(x,t)|^2$ attains its maximum value. We define

$$H(t) \triangleq |\nabla u(\widehat{x_1}(t), \widehat{\widehat{x}}(t))|^2.$$

At this point, we have

$$\partial_x|\nabla u|^2 = 0,\ \ \Lambda_{x_1}|\nabla u|^2 \geq 0.$$

Therefore, (2.20) yields

$$H'(t) + CH^{\frac{3-2\alpha}{2(1-\alpha)}} \leq CH^{\frac{3}{2}},$$

which along with the Young inequality and the fact $\frac{3-2\alpha}{2(1-\alpha)} > \frac{3}{2}$ imply

$$H'(t) + \frac{C}{2}H^{\frac{3-2\alpha}{2(1-\alpha)}} \leq C_\star \tag{2.21}$$

with an absolute constant $C_\star > 0$. Consequently, we get

$$H(t) \leq \max\left\{H(0),\ \left(\frac{2C_\star}{C}\right)^{\frac{2(1-\alpha)}{3-2\alpha}}\right\}. \tag{2.22}$$

We prove (2.22) by contradiction. If not, then by continuity of H, there exists $T > 0$ and $\delta > 0$ such that

$$H(T) = \max\left\{H(0),\ \left(\frac{2C_\star}{C}\right)^{\frac{2(1-\alpha)}{3-2\alpha}}\right\},$$

and for any $t \in (T, T+\delta]$, we have

$$H(t) > \max\left\{H(0),\ \left(\frac{2C_\star}{C}\right)^{\frac{2(1-\alpha)}{3-2\alpha}}\right\}. \tag{2.23}$$

This along with (2.21) yields for any $t \in [T, T+\delta]$

$$\begin{aligned}C_\star &\geq H'(t) + \frac{C}{2}H^{\frac{3-2\alpha}{2(1-\alpha)}} \\ &\geq H'(t) + C_\star,\end{aligned}$$

which gives

$$H'(t) \leq 0, \quad \forall\, t \in [T, T+\delta].$$

We thus obtain that for any $t \in [T, T+\delta]$

$$H(T+\delta) \le H(T) = \max\left\{ H(0),\ \left(\frac{2C_\star}{C}\right)^{\frac{2(1-\alpha)}{3-2\alpha}} \right\},$$

which contradicts with (2.23). In particular, we have from (2.22) that

$$\|\nabla u(t)\|_{L^\infty} \le C. \tag{2.24}$$

Finally, we are ready to prove Theorem 1.1.

Proof of Theorem 1.1. With a priori estimates achieved above, it is a standard procedure to complete the proof of Theorem 1.1. The first step is to construct a local-in-time solution which can be achieved by quite standard arguments. For brevity, we show only the simple *a priori* bounds for the solution. Following the arguments of [**13, 12**], one can modify this to prove it with Fourier truncation or standard mollification. Assume that u is the sufficiently regular solution of (1.3). We apply $\mathcal{J}^s \triangleq (I+\Lambda)^s$ to the equation u and take the L^2 inner product of the resulting equation with $\mathcal{J}^s u$ to obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\mathcal{J}^s u(t)\|_{L^2}^2 + \|\mathcal{J}^s \Lambda_{x_1}^{\frac{1}{2}} u\|_{L^2}^2 \\
&= -\int_{\mathbb{R}^n} [\mathcal{J}^s, u\cdot\nabla]u\cdot \mathcal{J}^s u\,dx - \int_{\mathbb{R}^n} u\cdot\nabla \mathcal{J}^s u\cdot \mathcal{J}^s u\,dx \\
&\le C\|[\mathcal{J}^s, u\cdot\nabla]u\|_{L^2}\|\mathcal{J}^s u\|_{L^2} - \int_{\mathbb{R}^n} u_k \mathcal{J}^s \partial_k u_l \mathcal{J}^s u_l\,dx \\
&\le C\|\nabla u\|_{L^\infty}\|\mathcal{J}^s u\|_{L^2}^2 - \frac{1}{2}\int_{\mathbb{R}^n} u_k \partial_k (\mathcal{J}^s u_l)^2\,dx \\
&\le C\|\nabla u\|_{L^\infty}\|\mathcal{J}^s u\|_{L^2}^2 + \frac{1}{2}\int_{\mathbb{R}^n} \partial_k u_k (\mathcal{J}^s u_l)^2\,dx \\
&\le C\|\nabla u\|_{L^\infty}\|\mathcal{J}^s u\|_{L^2}^2 \\
&\le C\|\mathcal{J}^s u\|_{L^2}\|\mathcal{J}^s u\|_{L^2}^2,
\end{aligned} \tag{2.25}$$

which along with the embedding $H^s(\mathbb{R}^n) \hookrightarrow \mathrm{Lip}(\mathbb{R}^n)$ for $s > 1 + \frac{n}{2}$ gives

$$\frac{d}{dt}\|\mathcal{J}^s u(t)\|_{L^2}^2 \le C\|\mathcal{J}^s u\|_{L^2}^3.$$

An application of ODE theorem yields that there exists a time $\widetilde{T} = \frac{1}{C\|\mathcal{J}^s u_0\|_{L^2}}$ such that

$$u \in C([0, \widetilde{T}); H^s(\mathbb{R}^n)).$$

The second step is to show the global H^s-bound, which is an easy consequence of (2.25) and (2.24). In fact, it follows from (2.25) that

$$\frac{d}{dt}\|\mathcal{J}^s u(t)\|_{L^2}^2 + \|\mathcal{J}^s \Lambda_{x_1}^{\frac{1}{2}} u\|_{L^2}^2 \le C\|\nabla u\|_{L^\infty}\|\mathcal{J}^s u\|_{L^2}^2.$$

It follows from the Gronwall inequality and (2.24) that

$$\|\mathcal{J}^s u(t)\|_{L^2} \le C(t), \tag{2.26}$$

which is the global solution.

Finally, let us show the uniqueness. To this end, let $\widehat{u}$ and $\widetilde{u}$ be two solutions of (1.3) satisfying (2.26). We denote $\delta u = \widehat{u} - \widetilde{u}$, which satisfies

$$\begin{cases} \partial_t \delta u + (\widehat{u} \cdot \nabla)\delta u + \Lambda_{x_1} \delta u = -(\delta u \cdot \nabla)\widetilde{u}, \\ \delta u(x,0) = 0. \end{cases} \tag{2.27}$$

Taking the L^2-inner product of (2.27) with δu gives

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|\delta u(t)\|_{L^2}^2 + \|\Lambda_{x_1}^{\frac{1}{2}} \delta u\|_{L^2}^2 &= -\int_{\mathbb{R}^2} (\delta u \cdot \nabla)\widetilde{u} \cdot \delta u \, dx - \int_{\mathbb{R}^2} (\widehat{u} \cdot \nabla)\delta u \cdot \delta u \, dx \\ &= -\int_{\mathbb{R}^2} (\delta u \cdot \nabla)\widetilde{u} \cdot \delta u \, dx - \frac{1}{2}\int_{\mathbb{R}^2} (\widehat{u} \cdot \nabla)|\delta u|^2 \, dx \\ &= -\int_{\mathbb{R}^2} (\delta u \cdot \nabla)\widetilde{u} \cdot \delta u \, dx + \frac{1}{2}\int_{\mathbb{R}^2} \partial_k \widehat{u}_k |\delta u|^2 \, dx \\ &\leq C(\|\nabla \widetilde{u}\|_{L^\infty} + \|\nabla \widehat{u}\|_{L^\infty})\|\delta u\|_{L^2}^2 \\ &\leq C(\|\mathcal{J}^s \widetilde{u}\|_{L^2} + \|\mathcal{J}^s \widehat{u}\|_{L^2})\|\delta u\|_{L^2}^2, \end{aligned}$$

which implies

$$\frac{d}{dt}\|\delta u(t)\|_{L^2}^2 \leq C(\|\mathcal{J}^s \widetilde{u}\|_{L^2} + \|\mathcal{J}^s \widehat{u}\|_{L^2})\|\delta u\|_{L^2}^2.$$

This together with (2.26) as well as the Gronwall inequality show

$$\delta u = 0,$$

which is the desired uniqueness. Thus, this ends the proof of Theorem 1.1. □

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