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Free surface equatorial two-layer flows in the β-plane approximation with discontinuous stratification

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Abstract. An exact solution of the nonlinear governing equations in the β-plane approximation is established and analysed in this paper. Such solution describes a purely azimuthal equatorial two-layer flows with free surface and discontinuous stratification. We present the explicit solution for the velocity field and the pressure. Moreover, We derive the implicit formulas for the shape of the free surface and the interface. Finally, the monotonicity properties between the free surface and the pressure on the free surface, as well as the regularity result for the interface are proved, respectively.

Contents

1. Introduction

The density fluctuations is a frequent phenomenon in the ocean which is substantially influenced by the changes in temperature [**4, 8, 11, 19, 22**]. This vertical layering greatly affects the velocity field and plays a essential role in geophysical fluid dynamics. The flow stratification represents an intrinsic property of geophysical water flows that especially applicable to large-scale ocean movements [**13, 20, 27**]. There has been a plenty of researches about the density stratification

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with the full governing equations without recourse to approximations, we refer the readers to [**5, 7, 10**]. In the equatorial region, density stratification is greater than anywhere else in the ocean, the water flows are divided into a shallow near-surface layer of relatively warm water and a deep layer of colder and denser water by a rather sharp interface [**22**].

The excellent works about the rigourous mathematical analyses for geophysical ocean flows were initiated by Constantin and Johnson, we refer the readers to [**2, 3, 6, 8, 9**]. It is worth mentioning that in [**6**], an exact solution was presented to describe a steady flow that is moving only in the azimuthal direction. Based on this work, there has been a plenty of researches about the exact azimuthal equatorial flows with continuous stratification, such as [**15, 18**] in the cylindrical coordinates, [**16, 17**] in the spherical coordinates, and [**26**] in the β-plane approximation. Moreover, we introduce some excellent studies related to the exact solutions to the geophysical fluid dynamics equations in the β-plane, see [**14, 21, 24, 25, 28**]. For exact azimuthal solutions for equatorial flows with discontinuous stratification, see [**22**] for the case of a discontinuous density that varies with depth, and [**23**] for a discontinuous density that varies with both depth and latitude.

Motivated by [**23**] and based on [**26**], this paper is devoted to the establishment of an exact solution to the nonlinear governing equations in the β-plane approximation for the inviscid, incompressible fluids with discontinuous stratification related to both depth and latitude, which represents a purely azimuthal equatorial two-layer flows. Although invoking approximations through simplifying the geometry to investigate the governing equations, we still derive the exact solution and prove some properties by a more brief way in this setting.

The rest of the paper is organized as follows. In section 2, we introduce the physical problem and present the governing equations for the geophysical fluid in the β-plane approximation. Section 3 aims to find the explicit solution for the velocity field and the pressure, as well as the implicit formulas for the shape of the free surface and the interface. Finally, we prove the monotonicity properties between the the pressure imposed on the free surface and the resulting surface geometry, and the regularity result for the interface.

2. Preliminaries

In this section, we will briefly introduce some physical problem and present the governing equations for geophysical fluid dynamics in the β-plane approximation. Supposing that the earth is a perfect sphere of radius $R = 6378km$ with the rotating speed $\omega = 7.29 \times 10^{-5} rad/s$. We investigate the geophysical flows in the Equatorial region. The coordinate system we will use is a rotating framework, with the x-axis pointing horizontally due east, the y-axis horizontally due north and the z-axis vertically upward, the corresponding unit vectors of the (x, y, z) system is $(\vec{e}_1, \vec{e}_2, \vec{e}_3)$.

Assuming that the flow consists of the following layers, the upper layer is

$$D_1 := \{(x, y, z) : \eta_2(x, y) \le z \le h_1 + \eta_1(x, y)\}, \tag{2.1}$$

with the density $\rho_1(y, z)$ and the lower layer is

$$D_2 := \{(x, y, z) : -h_2 + d(x, y) \le z \le \eta_2(x, y)\}, \tag{2.2}$$

with the density $\rho_2(y, z)$, where h_1 and h_2 are two positive constants, $d(x, y)$ is a given elevation function, $\eta_1(x, y)$ and $\eta_2(x, y)$ denote the unknown free surface function and the unknown interface functions, respectively. For convenience, we will use the subscript notation 1 and 2 to denote the values in D_1 and D_2, respectively. When we refer to the overall physical variable, we will use $i = 1, 2$ for values. Moreover, as mentioned in [**23**], we suppose that

$$\rho_2 = \rho_1(1+\delta), \tag{2.3}$$

where $\tau \longmapsto \delta(\tau)$ is a positive function with $\delta = \mathcal{O}(10^{-3})$.

The governing equations in the rotating frame are Euler's equations

$$\begin{cases} u_{i,t} + u_i u_{i,x} + v_i u_{i,y} + w_i u_{i,z} + 2\Omega(w_i - \frac{y}{R})v_i = -\frac{1}{\rho_i}p_{i,x}, \\ v_{i,t} + u_i v_{i,x} + v_i v_{i,y} + w_i v_{i,z} + 2\Omega\frac{y}{R}u_i + \Omega^2 y = -\frac{1}{\rho_i}p_{i,y}, \\ w_{i,t} + u_i w_{i,x} + v_i w_{i,y} + w_i w_{i,z} - 2\Omega u_i - \Omega^2 R = -\frac{1}{\rho_i}p_{i,z} - g, \end{cases} \tag{2.4}$$

where $(u_i(x,y,z), v_i(x,y,z), w_i(x,y,z))$ denotes the the velocity field, $p(x,y,z)$ is the pressure in the fluid, Ω denotes the rotational speed of the Earth and g is the gravitational acceleration, and the equation of mass conservation

$$\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w) = 0. \tag{2.5}$$

The equation (2.4) is described in the equatorial β-plane setting, we note here that these are the equations governing the β-plane approximation, which represent a consistent approximation to the governing equations only near the Equator, as emphasized by Dellar [**12**].

The related boundary conditions of the governing equations are given as follows to complete the water wave problem. At the free surface $z = h_1 + \eta_1(x,y)$, there is the kinematic boundary condition

$$w_1 = u_1\eta_{1,x} + v_1\eta_{1,y}, \tag{2.6}$$

and the dynamic boundary condition

$$p_1 = P_1(x,y). \tag{2.7}$$

At the interface $z = \eta_2(x,y)$, we require

$$\begin{aligned} &(u_1\vec{e}_3 + v_1\vec{e}_2 + w_1\vec{e}_1)(\vec{e}_3 - \eta_{2,y}\vec{e}_2 - \eta_{2,x}\vec{e}_1) \\ =& (u_2\vec{e}_3 + v_2\vec{e}_2 + w_2\vec{e}_1)(\vec{e}_3 - \eta_{2,y}\vec{e}_2 - \eta_{2,x}\vec{e}_1) \end{aligned} \tag{2.8}$$

to ensure the normal components of the velocity fields for the two layers are the same, and we also need

$$p_1(x,y,\eta_2(x,y)) = p_2(x,y,\eta_2(x,y)) \tag{2.9}$$

to confirm that the pressure is continuous across the interface.

At the sea-bed of the ocean, we denote the impermeable solid boundary by $z = -h_2 + d(x,y)$ and the related kinematic boundary condition is

$$w_2 = u_2 d_x + v_2 d_y. \tag{2.10}$$

3. Explicit solutions

In order to describe the purely azimuthal steady flows with no variation in this direction, we will seek the solutions of the Eqs. (2.4), (2.5), (2.6), (2.7), (2.8), (2.9) and (2.10) such that the velocity field satisfies $v_1 = v_2 = w_1 = w_2 = 0$ and $u_1 = u_1(y,z), u_2 = u_2(y,z)$. Moreover, the known variables are described by $p_1 = p_1(y,z), p_2 = p_2(y,z), \eta_1 = \eta_1(y), \eta_2 = \eta_2(y), d = d(y)$. The range for the variable y will be limited in the interval $[0, R\varepsilon]$, where ε describes a strip with $100km$ separate from the Equator.

First, we will construct the explicit solutions of the velocity and the pressure for such water wave problem. Noting that in the previous setting, the equation of the mass conservation (2.5) and the boundary conditions (2.6), (2.7), (2.8), (2.9) and (2.10) are automatically satisfied. The Euler equations in the domain D_i are now become

$$\begin{cases} 0 = p_{i,x}, \\ 2\Omega \frac{y}{R} u_i + \Omega^2 y = -\frac{1}{\rho_i} p_{i,y}, \\ -2\Omega u_i - \Omega^2 R = -\frac{1}{\rho_i} p_{i,z} - g. \end{cases} \tag{3.1}$$

We note here that the method was set forth in the paper [**6**] by Constantin and Johnson in the case of homogeneous flows and extended to the case of variable density by Henry and Martin, cf. [**15, 16, 17, 18**]. Employing the method in [**26**], we can get the azimuthal velocities $u_i (i = 1,2)$ are given by

$$u_i(y,z) = -\frac{\Omega R}{2} + \frac{1}{2\Omega\rho_i}\left[\mathcal{F}_i\left(z - \frac{y^2}{2R}\right) + g\int_0^y \rho_{i,y}\Big(\bar{y}(s), \bar{z}(s)\Big)\right],$$

with $x \to \mathcal{F}_i(x), i = 1,2$ are some real-valued functions and $\bar{y}(s) = \frac{s^2 - y^2}{2R} + z, \bar{z}(s) = \frac{s^2}{2R}$. Moreover, using (3.1) we can deduce that the pressure gradient is described by

$$p_{i,y} = -\frac{y}{R}\left[\mathcal{F}_i\left(z - \frac{y^2}{2R}\right) + g\int_0^y \rho_{i,y}\Big(\bar{y}(s), \bar{z}(s)\Big)\right], \tag{3.2}$$

and

$$p_{i,z} = -g\rho_i + \mathcal{F}_i\left(z - \frac{y^2}{2R}\right) + g\int_0^y \rho_{i,y}\Big(\bar{y}(s), \bar{z}(s)\Big). \tag{3.3}$$

Integrating (3.3) with respect to z, we get for all $z \in [-h_2 + d(y), \eta_2(y)]$,

$$p_2(y,z) = \int_{-h_2+d(y)}^{z} [-g\rho_2(y,\psi) + G_2(y,\psi)]d\psi + \int_{-h_2+d(y)-\frac{y^2}{2R}}^{z-\frac{y^2}{2R}} \mathcal{F}_2(\tau)d\tau + M_2(y),$$

with

$$G_2(y,\psi) = g\int_0^y \rho_{2,y}(\bar{y}(s), \frac{\psi s^2}{2Rz})ds,$$

and $y \to M_2(y)$ is a function satisfies that

$$\begin{aligned} M_2(y) = \mathcal{F}_2\left(-h_2 + d(y) - \frac{y^2}{2R}\right) \cdot \frac{d}{dy}\left(d(y) - \frac{y^2}{2R}\right) \\ + G(y, -h_2 + d(y))d'(y) - g\rho_2(y, -h_2 + d(y))d'(y) \end{aligned}$$

Then, we will solve the pressure in the upper layer D_1. Since the interface is featured by $r = \eta_2(y)$, we integrate the equation (3.2) with respect to y and then we conclude that for all z and all $z \in [\eta_2(y), h_1 + \eta_1(y)]$,

$$p_1(y,z) = \int_{\eta_2(y)}^{z} [-g\rho_1(y,\psi) + G_1(y,\psi)]d\psi + \int_{\eta_2(y)-\frac{y^2}{2R}}^{z-\frac{y^2}{2R}} \mathcal{F}_1(\tau)d\tau + M_1(y,\eta_2),$$

with

$$G_1(y,\psi) = g\int_0^y \rho_{1,y}(\bar{y}(s), \frac{\psi s^2}{2Rz})ds,$$

and

$$\begin{aligned} M_1(y,\eta_2) &= \int_0^y [-g\rho_1(\hat{y},\eta_2(\hat{y})) + G_1(\hat{y},\eta_2(\hat{y}))]\eta_2'(\hat{y})d\hat{y} \\ &+ \int_0^y \mathcal{F}_1\left(\eta_2(\hat{y}) - \frac{\hat{y}^2}{2R}\right)[\eta_2'(\hat{y}) - \frac{\hat{y}}{R}]d\hat{y}. \end{aligned} \tag{3.4}$$

Next, we will establish the implicit expressions of the interface and the free surface. The balance of forces at the interface is now becomes

$$p_1(y,\eta_2(y)) = p_2(y,\eta_2(y)), \tag{3.5}$$

where

$$M_1(y,\eta_2) = \int_{-h_2+d(y)}^{\eta_2(y)} [-g\rho_2(y,\psi) + G_2(y,\psi)]d\psi + \int_{-h_2+d(y)-\frac{y^2}{2R}}^{\eta_2(y)-\frac{y^2}{2R}} \mathcal{F}_2(\tau)d\tau + M_2(y).$$

Applying a nondimensionalisation procedure, we can use a functional analytic setting. Namely, we define

$$\mathfrak{H}_2(y) := \frac{\eta_2(y)}{R}$$

and then the equation (3.5) can be written as

$$Q_2(\mathfrak{H}_2) = 0,$$

with Q_2 is an operator which acts from the Banach space $C^1([0, R\varepsilon])$ into itself, featured by

$$\begin{aligned} Q_2(\mathfrak{H}_2) = \frac{1}{P_{atm}}&\left(\int_{-h_2+d(y)}^{R\mathfrak{H}_2(y)} [-g\rho_2(y,\psi) + G_2(y,\psi)]d\psi + \int_{-h_2+d(y)-\frac{y^2}{2R}}^{R\mathfrak{H}_2(y)-\frac{y^2}{2R}} \mathcal{F}_2(\tau)d\tau\right) \\ &+ \frac{M_2(y) - M_1(y, R\mathfrak{H})}{P_{atm}}. \end{aligned}$$

Utilizing the dynamic boundary condition (2.7), we obtain that the pressure $p_1(y)$ at the free surface $z = h_1 + \eta_1(y)$ should be given by

$$\begin{aligned} P_1(y) = &\int_{\eta_2(y)}^{h_1+\eta_1(y)} [-g\rho_1(y,\psi) + G_1(y,\psi)]d\psi + \int_{\eta_2(y)-\frac{y^2}{2R}}^{h_1+\eta_1(y)-\frac{y^2}{2R}} \mathcal{F}_1(\tau)d\tau \\ &+ M_1(y,\eta_2), \end{aligned}$$

which is named the Bernoulli relation between the imposed pressure at the free surface, the resulting distortion of such surface and the shape of the interface. Setting

$$\mathfrak{H}_1(y) := \frac{\eta_1(y)}{R},\ \mathfrak{P}_1(y) := \frac{P_1(y)}{P_{atm}},$$

the previous equation can be rewritten as the following moderatorial equation

$$Q_1(\mathfrak{H}_1, \mathfrak{H}_2, \mathfrak{P}_1) = 0, \tag{3.6}$$

where the operator Q_1 acts from $C([0, R\varepsilon]) \times C^1([0, R\varepsilon]) \times C([0, R\varepsilon]$ into itself and is given as

$$\begin{aligned} Q_1(\mathfrak{H}_1, \mathfrak{H}_2, \mathfrak{P}_1) = \frac{1}{P_{atm}} \Bigg(& \int_{R\mathfrak{H}_2(y)}^{h_1 + R\mathfrak{H}_1(y)} [-g\rho_1(y, \psi) + G_1(y, \psi)] d\psi \\ & + \int_{R\mathfrak{H}_2(y) - \frac{y^2}{2R}}^{h_1 + R\mathfrak{H}_1(y) - \frac{y^2}{2R}} \mathcal{F}_2(\tau) d\tau + M_1(y, \eta_2) \Bigg) - \mathfrak{P}_1(y). \end{aligned} \tag{3.7}$$

Then, we aim to find $(\mathfrak{H}_1, \mathfrak{H}_2)$ such that

$$(Q_1(\mathfrak{H}_1, \mathfrak{H}_2, \mathfrak{P}_1), Q_2(\mathfrak{H}_2)) = 0. \tag{3.8}$$

The natural method to solve the previous problem is the implicitly function theorem [**1**]. First, we need to find a trivial solution $(\mathfrak{H}_{1,0}, \mathfrak{H}_{2,0})$ of (3.8). To this end, we consider the simpler situation when the undisturbed surface and the undisturbed interface along the curvature of the earth. Setting $\mathfrak{H}_1 = \mathfrak{H}_2 = 0$, we obtain that $Q_1(0, 0, \mathfrak{P}_1) = Q_2(0) = 0$ if and only if

$$\begin{aligned} Q_0 = \frac{1}{P_{atm}} \Bigg(& \int_0^{h_1(y)} [-g\rho_1(y, \psi) + G_1(y, \psi)] d\psi + \int_{-\frac{y^2}{2R}}^{h_1(y) - \frac{y^2}{2R}} \mathfrak{F}_1(\tau) d\tau \\ & + \int_0^y \mathfrak{F}_1(-\frac{y^2}{2R})(-\frac{y}{R}) \Bigg) \end{aligned}$$

and

$$\begin{aligned} \frac{1}{P_{atm}} \Bigg(& \int_{-h_2 + d(y)}^0 [-g\rho_1(y, \psi) + G_1(y, \psi)] d\psi + \int_{-\frac{y^2}{2R}}^{h_1(y) - \frac{y^2}{2R}} \mathfrak{F}_1(\tau) d\tau \\ & + \int_0^y \mathfrak{F}_1(-\frac{y^2}{2R})(-\frac{y}{R}) \Bigg) = 0. \end{aligned}$$

The next goal is to compute the value of the limit

$$\lim_{s \to 0} \frac{M_1(y, s\eta_2) - M_1(y, 0)}{s}.$$

Since the equation (3.3) implies

$$\begin{aligned}M_1(y,s\eta_2)-M_1(y,0) &= -g\int_0^y \rho_1(\hat y, s\eta_2(\hat y))s\eta_2'(\hat y)d\hat y\\&+\int_0^y \mathfrak{F}_1(s\eta_2(\hat y)-\frac{\hat y^2}{2R})s\eta_2'(\hat y)d\hat y\\&+\int_0^y\Big(\mathfrak{F}_1(s\eta_2(\hat y)-\frac{\hat y^2}{2R})-\mathfrak{F}_1(-\frac{\hat y^2}{2R})\Big)(-\frac{\hat y}{R})d\hat y\\&+\int_0^y G_1(\hat y, s\eta_2(\hat y))s\eta_2'(\hat y)d\hat y,\end{aligned}\tag{3.9}$$

for the last term of (3.9), we have

$$\begin{aligned}\lim_{s\to 0}\frac{1}{s}\int_0^y G_1(\hat y, s\eta_2(\hat y))s\eta_2'(\hat y)d\hat y &= \lim_{s\to 0}\int_0^y G_1(\hat y, s\eta_2(\hat y))\eta_2'(\hat y)d\hat y\\&=\int_0^y G_1(\hat y,0)\eta_2'(\hat y)d\hat y\\&=G_1(y,0)\eta_2(y)-\int_0^y \eta_2(\hat y)G_{1,y}(\hat y)(\hat y,0)d\hat y\\&=G_1(y,0)\eta_2(y)-\int_0^y \eta_2(\hat y)g\rho_{1,y}(\hat y,0)d\hat y.\end{aligned}\tag{3.10}$$

For the reminder terms of (3.9), we find that

$$\begin{aligned}&\lim_{s\to 0}\frac{1}{s}\int_0^y \mathfrak{F}_1(s\eta_2(\hat y)-\frac{\hat y^2}{2R})s\eta_2'(\hat y)d\hat y\\&+\lim_{s\to 0}\int_0^y \frac{\mathfrak{F}_1(s\eta_2(\hat y)-\frac{\hat y^2}{2R})-\mathfrak{F}_1(-\frac{\hat y^2}{2R})}{s}(-\frac{\hat y}{R})d\hat y\\&=\mathfrak{F}_1(-\frac{y^2}{2R})\eta_2(y)-\int_0^y \mathfrak{F}_1'(-\frac{\hat y^2}{2R})(-\frac{\hat y}{R})\eta_2(\hat y)d\hat y\\&+\lim_{s\to 0}\int_0^y\frac{s\eta_2(\hat y)}{s}\mathfrak{F}_1'(-\frac{\hat y^2}{2R})(-\frac{\hat y}{R})d\hat y\\&=\mathfrak{F}_1(-\frac{y^2}{2R})\eta_2(y).\end{aligned}\tag{3.11}$$

and

$$\begin{aligned}\lim_{s\to 0}\int_0^y\frac{\rho_1(\hat y, s\eta_2(\hat y))s\eta_2'(\hat y)}{s}d\hat y &= \int_0^y \rho_1(\hat y, s\eta_2(\hat y))\eta_2'(\hat y)d\hat y\\&=\rho_1(y,s\eta_2(y))\eta_2(y)-\int_0^y \rho_{1,y}(\hat y,0)\eta_2(\hat y)d\hat y\end{aligned}\tag{3.12}$$

Combing (3.10), (3.11) and (3.12), we get

$$\lim_{s\to 0}\frac{M_1(y,s\eta_2)-M_1(y,0)}{s}=G_1(y,0)\eta_2(y)+\mathfrak{F}_1(-\frac{y^2}{2R})\eta_2(y)-g\rho_1(y,s\eta_2(y))\eta_2(y).$$

Noting that

$$Q_1(0,s\mathfrak{H}_2,\mathfrak{P}_1^0)-Q_1(0,0,\mathfrak{P}_1^0)=\frac{1}{P_{atm}}\bigg(\int_{Rs\mathfrak{H}_2}^{0}[-g\rho_1(y,\psi)+G_1(y,\psi)]d\psi + \int_{Rs\mathfrak{H}_2-\frac{y^2}{2R}}^{-\frac{y^2}{2R}}\mathfrak{F}_1(\tau)d\tau\bigg).$$

Applying the mean value theorem for integrals we arrive

$$\begin{aligned}\lim_{s\to 0}\frac{1}{s}\int_{Rs\mathfrak{H}_2}^{0}[-g\rho_1(y,\psi)+G_1(y,\psi)]d\psi &= [g\rho_1(y,0)-G_1(y,0)]R\mathfrak{H}_2(y)\\ &= [g\rho_1(y,0)-G_1(y,0)]\eta_2(y)\end{aligned} \tag{3.13}$$

and

$$\lim_{s\to 0}\frac{1}{s}\int_{Rs\mathfrak{H}_2-\frac{y^2}{2R}}^{-\frac{y^2}{2R}}\mathfrak{F}_1(\tau)d\tau=-\mathfrak{F}_1(-\frac{y^2}{2R})R\mathfrak{H}_2(y)=-\mathfrak{F}_1(-\frac{y^2}{2R})\eta_2(y). \tag{3.14}$$

Collecting (3.13), (3.14) and the definition of the operator Q_1 in (3.7), we infer that

$$Q_{1,\mathfrak{H}_2}(0,0,\mathfrak{P}_1^0)\mathfrak{H}_2=\lim_{s\to 0}\frac{Q_1(0,s\mathfrak{H}_2,\mathfrak{P}_1^0)-Q_1(0,0,\mathfrak{P}_1^0)}{s}=0.$$

Similarly, due to

$$Q_1(s\mathfrak{H}_1,0,\mathfrak{P}_1^0)-Q_1(0,0,\mathfrak{P}_1^0)=\frac{1}{P_{atm}}\bigg(\int_{h_1}^{h_1+Rs\mathfrak{H}_1}\bigg)[-g\rho_1(y,\psi)+G_1(y,\psi)]d\psi + \int_{h_1-\frac{y^2}{2R}}^{h_1+Rs\mathfrak{H}_2-\frac{y^2}{2R}}\mathfrak{F}_1(\tau)d\tau\bigg),$$

we obtain

$$\begin{aligned}(Q_{1,\mathfrak{H}_1}(0,0,\mathfrak{P}_1^0)\mathfrak{H}_1)(y)&=\lim_{s\to 0}\frac{Q_1(s\mathfrak{H}_1,0,\mathfrak{P}_1^0)-Q_1(0,0,\mathfrak{P}_1^0)}{s}\\ &=\frac{[-g\rho_1(y,\eta_1)+G(y,\eta_1)+\mathfrak{F}_1(\eta_1-\frac{y^2}{2R})]\eta_1(y)}{P_{atm}}\\ &=\frac{\rho_1(y,\eta_1)}{P_{atm}}[2\Omega u_1(y,\eta_1)+\Omega^2R]\eta_1(y).\end{aligned}$$

Moreover, we deduce that

$$\begin{aligned}&(Q_{2,\mathfrak{H}_2}(0)\mathfrak{H}_2)(y)\\ &=\lim_{s\to 0}\frac{Q_2(s\mathfrak{H}_2)(y)-Q_2(0)(y)}{s}\\ &=\frac{[-g\rho_2(y,\eta_2)+G(y,\eta_2)+\mathfrak{F}_2(\eta_2-\frac{y^2}{2R})]\eta_2(y)}{P_{atm}}\\ &\quad-\frac{[-g\rho_1(y,\eta_2)+G(y,\eta_2)+\mathfrak{F}_1(\eta_2-\frac{y^2}{2R})]\eta_2(y)}{P_{atm}}\\ &=\frac{\rho_2(y,\eta_2)[2\Omega u_2(y,\eta_1)+\Omega^2R]-\rho_1(y,\eta_2)[2\Omega u_1(y,\eta_1)+\Omega^2R]}{P_{atm}}\eta_2(y)\end{aligned} \tag{3.15}$$

Furthermore, if we regard Q_2 as an operator of $\mathfrak{H}_1$, $\mathfrak{H}_2$ and $\mathfrak{P}$, then we get $Q_{2,\mathfrak{H}_2}(0,0,\mathfrak{P}_1^0)\mathfrak{H}_2 = 0$ for every $\mathfrak{H}_2$. Hence,

$$(Q_1,Q_2)_{\mathfrak{H}_1,\mathfrak{H}_2}(0,0,\mathfrak{P}_1^0) = \begin{pmatrix} Q_{1,\mathfrak{H}_1}(0,0,\mathfrak{P}_1^0) & Q_{1,\mathfrak{H}_2}(0,0,\mathfrak{P}_1^0) \\ Q_{2,\mathfrak{H}_1}(0,0,\mathfrak{P}_1^0) & Q_{2,\mathfrak{H}_2}(0,0,\mathfrak{P}_1^0) \end{pmatrix}$$

$$= \begin{pmatrix} Q_{1,\mathfrak{H}_1}(0,0,\mathfrak{P}_1^0) & 0 \\ 0 & Q_{2,\mathfrak{H}_2}(0,0,\mathfrak{P}_1^0) \end{pmatrix} \tag{3.16}$$

is a linear operator from $C([0,R\varepsilon]) \times C^1([0,R\varepsilon])$ into itself.

Arguing along the lines of [**23**], we present the following theorem.

THEOREM 3.1. *Suppose that $\mathfrak{P}_1^0$ is disturbed by a small perturbation $\mathfrak{P}$, then there exits a unique tuple $(\mathfrak{H}_1,\mathfrak{H}_2) \in C([0,R\varepsilon]) \times C^1([0,R\varepsilon])$ such that $(Q_1(\mathfrak{H}_1,\mathfrak{H}_2,\mathfrak{P}_1), Q_2(\mathfrak{H}_2)) = 0$.*

PROOF. It can be concluded from (2.3) that

$$\begin{aligned} &\rho_2(y,\eta_2)[2\Omega u_2(y,\eta_1)+\Omega^2 R] - \rho_1(y,\eta_2)[2\Omega u_1(y,\eta_1)+\Omega^2 R] \\ &= \rho_1(y,\eta_2)[2\Omega u_2(u_2-u_1)] + \rho_1\delta[2\Omega u_2(y,\eta_2)+\Omega^2 R] \\ &< \rho_1(y,\eta_2)\Big(2\Omega_2(y,\eta_2) + \delta[2\Omega u_2(y,\eta_2)+\Omega^2 R]\Big) \\ &< Rkg\cdot m^{-2}\cdot s^{-2}. \end{aligned} \tag{3.17}$$

Therefore, taking into account (3.15) and (3.17) we find that one can choose a constant C_1 such that $(Q_{2,\mathfrak{H}_2}(0,0,\mathfrak{P}_1^0))(y) \le C_1$ for all y. Then, we get

$$\begin{aligned} &C^1([0,R\varepsilon]) \to C^1([0,R\varepsilon]), \\ &\mathfrak{H}_2 \longmapsto Q_{2,\mathfrak{H}_2}(0,0,\mathfrak{P}_1^0)\mathfrak{H}_2 \end{aligned} \tag{3.18}$$

is a linear homeomorphism. Similarly, we can deduce that there is a constant C_2 such that $(Q_{1,\mathfrak{H}_1}(0,0,\mathfrak{P}_1^0))(y) \le C_2$ for all y. Moreover, we have

$$\begin{aligned} &C([0,R\varepsilon]) \to ([0,R\varepsilon]), \\ &\mathfrak{H}_1 \longmapsto Q_{1,\mathfrak{H}_2}(0,0,\mathfrak{P}_1^0)\mathfrak{H}_1 \end{aligned} \tag{3.19}$$

is a linear homeomorphism. Using the equations (3.16), (3.18) and (3.19), we find that $(Q_1,Q_2)_{\mathfrak{H}_1,\mathfrak{H}_2}(0,0,\mathfrak{P}_1^0)$ is homeomorphism from $C([0,R\varepsilon]) \times C^1([0,R\varepsilon])$ into itself. By the implicit function theorem in [**1**], we finish the proof. □

4. Properties of exact solutions

In this section, we will present some properties of the exact solutions about the interface and the free surface. Based on [**23**] and [**26**], we have the monotonicity properties between the pressure $\mathfrak{P}_1$ and the function $\mathfrak{H}_1$: the increase of the height of the free surface away from the Equator will generate the decrease of the pressure on the surface, that is,

THEOREM 4.1. *Defining $\mathfrak{P}_1$ as the pressure on the free surface, $\mathfrak{H}_1$ as the height function of such surface, we get*

$$\mathfrak{P}_1'(y) < 0, \text{ if } \mathfrak{H}_1'(y) \ge 0,$$

and

$$\mathfrak{H}_1'(y) < 0, \text{ if } \mathfrak{P}_1'(y) \geq 0,$$

for all $y \in (0, R\varepsilon)$.

PROOF. Differentiating the equation (3.6) with respect y we get

$$\begin{aligned} P_{atm}\mathfrak{P}_1'(y) &= [-g\rho_1(y, h_1 + R\mathfrak{H}_1(y)) + G_1(y, h_1 + R\mathfrak{H}_1)(y)]R\mathfrak{H}_1)'(y) \\ &\quad + \mathfrak{F}_1(h_1 + R\mathfrak{H}_1 - \frac{y^2}{2R})(R\mathfrak{H}_1)'(y) - \frac{y}{R}) \\ &= [2\Omega u_1(y, h_1 + R\mathfrak{H}_1(y)) + \Omega^2 R - g]\rho_1(y, h_1 + R\mathfrak{H}_1(y))R\mathfrak{H}_1'(y) \\ &\quad + \mathfrak{F}_1(h_1 + R\mathfrak{H}_1 - \frac{y^2}{2R})(-\frac{y}{R}). \end{aligned}$$

The proof of the theorem can be finished since

$$2\Omega u_1(y, h_1 + R\mathfrak{H}_1(y)) + \Omega^2 R - g < 0.$$

□

As shown in [**23**], we also have the following theorem.

THEOREM 4.2. *The function* $y \longmapsto \mathfrak{H}_2(y)$ *defining the interface has one degree higher than the smoothness of the velocity field. Namely, if* $\mathcal{F}_1$ *and* $\mathcal{F}_2$ *are* l*-times differentiable, then* $\mathfrak{H}_2(y) \in C^{l+1}[0, R\varepsilon]$. *In addition, if* $\mathcal{F}_1$ *and* $\mathcal{F}_2$ *are Infinitely dimensional differentiable functions, then* $\mathfrak{H}_2(y) \in C^{\infty}[0, R\varepsilon]$.

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