Dynamics of PDE, Vol.21, No.3, 209-233, 2024

Stability of a class of solutions of the barotropic vorticity equation on a sphere

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Communicated by Alexander Kiselev, received April 11, 2021.

Abstract. The linear and nonlinear stability of modons and Wu-Verkley waves, which are weak solutions of the barotropic vorticity equation on a rotating sphere, are analyzed. Necessary conditions for normal mode instability are obtained, the growth rate of unstable modes is estimated, and the orthogonality of unstable modes to the basic flow is shown. The Liapunov instability of dipole modons in the norm associated with enstrophy is proven.

Contents

2020 *Mathematics Subject Classification.* 35B10, 35D30, 76B47, 76E09.

Key words and phrases. barotropic vorticity equation on a sphere, modons and Wu-Verkley waves, linear and Liapunov instability.

The author thanks the National System of Researchers (SNI, CONACYT, Mexico) for grant 14539.

1. Introduction

The barotropic vorticity equation (BVE) governing the motion of a nondivergent ideal fluid on a rotating sphere is considered. This equation also describes the main features of large-scale atmospheric dynamics [**19**]. The Liapunov and exponential stability of infinitely differentiable BVE solutions (Rossby-Haurwitz waves and Legendre polynomial flows) were analyzed in [**22**]. In this work, we study the stability of weak BVE solutions, namely, Verkley's modons [**24, 25, 26**] and antisymmetric Wu-Verkley (WV) waves [**28**].

Since the pioneering work of Larichev and Reznik [**10**], vortex pairs known as modons have attracted a lot of attention due to their potential applications in geophysical fluid dynamics and plasma physics. Modon stability has been an important issue in all modon applications [**6, 8, 17, 21, 25**]. In particular, attempts have been made to use the geometric structure and relative stability of modons to explain the phenomena of atmospheric blocking [**12, 14, 15**]. It was also shown that the trajectories of non-stationary solutions of the forced and dissipative BVE can alternately approach one of the two main atmospheric regimes (zonal circulation or blocking-like circulation) [**27**]. Therefore, the study of the stability of solutions to this equation can be helpful for understanding the mechanisms of low-frequency variability of the atmosphere.

Hilbert spaces of functions on a sphere are introduced in Section 2. The main properties of BVE solutions and the structure of modons are described in Sections 3 and 4. In Section 5, formulas are derived that determine the distance between two dipole modons in the norms associated with energy and enstrophy, and in Section 6 these formulas are refined for the case when two modons move along the same latitudinal circle. Stationary modons and WV waves are considered in Section 7, and the conservation law for infinitesimal perturbations of these solutions and conditions for their normal mode stability are obtained in Sections 8 and 9, respectively. Unlike flows in the form of Legendre polynomials (LP-flows) and Rossby-Haurwitz waves (RH-waves) [**22**], these conditions depend not only on the degree of the spherical polynomial representing the basic solution, but also on the spectral distribution of the energy of the normal mode. The new conditions are used in Section 10 to estimate the maximum growth rate of unstable modes. In addition, we show the orthogonality of unstable modes to the basic flow in the inner product associated with kinetic energy. The results allow testing numerical algorithms and software packages developed for the study of stability. Liapunov instability of dipole modons in the norm associated with enstrophy is proved in Section 11, while the main conclusions are given in Section 12.

2. Function spaces

Let $S = \{x \in \mathbb{R}^3 : |x| = 1\}$ be a unit sphere in the three-dimensional Euclidean space $\mathbb{R}^3$, and let $\mathbb{C}_0^\infty(S)$ be a set of infinitely differentiable functions $\psi(x)$ on S which are orthogonal to any constant:

$$\mathrm{Y}_0(\psi) = \int_S \psi(x)\, dS = 0 \tag{2.1}$$

We define the inner product in $\mathbb{C}_0^\infty(S)$ as

$$\langle \psi, g \rangle = \int_S \psi(x)\, \overline{g(x)}\, dS \tag{2.2}$$

Here $x = (\lambda, \mu)$ is a point of the sphere, $dS = d\lambda d\mu$ is an infinitesimal element of the sphere surface, $\mu = \sin\phi$; $\mu \in [-1, 1]$, ϕ is the latitude, $\lambda \in [0, 2\pi)$ is the longitude, and $\overline{g(x)}$ is the complex conjugate of $g(x)$. The Hilbert space obtained by the closure of $\mathbb{C}_0^\infty(S)$ in the norm

$$\|\psi\| = \langle \psi, \psi \rangle^{1/2} \tag{2.3}$$

is denoted as $\mathbb{H}^0$.

It is well known [**16**] that the spherical harmonics

$$Y_n^m(\lambda, \mu) = \left[\frac{2n+1}{4\pi} \frac{(n-m)!}{(n+m)!} \right]^{1/2} P_n^m(\mu)\, e^{im\lambda} \ , \quad n \geq 1, \quad |m| \leq n \ ,$$

form the orthonormal basis in $\mathbb{H}^0$: $\left\langle Y_n^m, Y_l^k \right\rangle = \delta_{mk}\delta_{nl}$, where δ_{mk} is the Kronecker delta,

$$P_n^m(\mu) = \frac{\left(1-\mu^2\right)^{m/2}}{2^n n!} \frac{d^{n+m}}{d\mu^{n+m}} \left(\mu^2 - 1\right)^n \tag{2.4}$$

is the associated Legendre function of degree n and zonal number m. Thus, for any $\psi \in \mathbb{H}^0$,

$$\psi(x) = \sum_{n=1}^{\infty} \mathrm{Y}_n(\psi)\, , \qquad \mathrm{Y}_n(\psi) = \sum_{m=-n}^{n} \psi_n^m\, Y_n^m(x)$$

where $\mathrm{Y}_n(\psi)$ is a homogeneous spherical polynomial of degree n. Note that $\mathrm{Y}_n(\psi)$ represents the orthogonal projection of ψ onto the $(2n+1)$-dimensional eigensubspace

$$\mathbf{H}_n = \{\psi : -\Delta\psi = \chi_n \psi\}, \quad \chi_n = n\,(n+1) \tag{2.5}$$

of symmetric and positive definite spherical Laplace operator $-\Delta$, corresponding to the eigenvalue χ_n [**22**].

For $k = 1, 2$, we define in $\mathbb{C}_0^\infty(S)$ the inner product

$$\langle \psi, g \rangle_k = \langle (-\Delta)^k \psi, g \rangle = \sum_{n=1}^{\infty} \chi_n^k \ \langle \mathrm{Y}_n(\psi)\, , \mathrm{Y}_n(g) \rangle \tag{2.6}$$

and denote by $\mathbb{H}^k$, the Hilbert space obtained by the closure of $\mathbb{C}_0^\infty(S)$ in the norm

$$\|\psi\|_k = \langle \psi, \psi \rangle_k^{1/2} = \left\{ \sum_{n=1}^{\infty} \chi_n^k \, \|\mathrm{Y}_n(\psi)\|^2 \right\}^{1/2} \tag{2.7}$$

LEMMA 2.1. [**22**] *Let* $\psi \in \mathbb{H}^k$, $k = 1, 2$. *Then* $\|\psi\|_1 = \|\nabla\psi\|$ *and* $\|\psi\|_{k-1} \leq 2^{-1/2}\|\psi\|_k$ *where* $\|\psi\|_0 \equiv \|\psi\|$.

If $\psi \in \mathbb{H}^2$ *then* $\|\psi\|_2 = \|\Delta\psi\|$.

3. Governing equation

Let us consider the nonlinear barotropic vorticity equation (BVE)

$$\frac{\partial}{\partial t}\Delta\psi + J\left(\psi, \Delta\psi + 2\mu\right) = 0 \tag{3.1}$$

describing the dynamics of an inviscid incompressible fluid on a rotating sphere S. Here ψ and $\Delta\psi$ are the streamfunction and potential vorticity of the nondivergent fluid, respectively; the nonlinear term $J(\psi, \Delta\psi) = (\overrightarrow{n} \times \nabla\psi) \cdot \nabla\Delta\psi$ describes the fluid advection, and $\overrightarrow{n}$ is the unit normal to the surface of the sphere. This equation expresses the conservation of the absolute vorticity $\Omega = \Delta\psi + 2\mu$ in each material particle of the fluid. Let $\psi(t, x)$ be a real solution of (3.1). Since this solution is determined up to a constant, it is assumed that all functions in this work are orthogonal to any constant on the sphere, that is, (2.1) is valid.

The total kinetic energy $\mathbf{K}$ and the enstrophy η of the solution ψ of equation (3.1) are invariants of motion [**3, 4**]:

$$\frac{d}{dt}\mathbf{K} = \frac{1}{2}\frac{d}{dt}\left\|\nabla\psi\right\|^2 = 0 \quad \text{and} \quad \frac{d}{dt}\boldsymbol{\eta} = \frac{1}{2}\frac{d}{dt}\left\|\Delta\psi\right\|^2 = 0 \tag{3.2}$$

where

$$\mathbf{K} = \frac{1}{2}\sum_{n=1}^{\infty}\chi_n\left\|\mathrm{Y}_n\left(\psi\right)\right\|^2 = \sum_{n=1}^{\infty}\mathbf{K}_n$$

and

$$\boldsymbol{\eta} = \frac{1}{2}\sum_{n=1}^{\infty}\chi_n^2\left\|\mathrm{Y}_n\left(\psi\right)\right\|^2 = \sum_{n=1}^{\infty}\chi_n\mathbf{K}_n \ .$$

Here $\mathbf{K}_n$ is the part of the energy concentrated in the subspace $\mathbf{H}_n$, and χ_n is defined in (2.5).

Due to (3.2), the mean spectral number

$$\chi_a = \boldsymbol{\eta}/\mathbf{K} = Const \tag{3.3}$$

of the solution ψ also persists throughout the motion. The value $\sqrt{\chi_a}$ is known as the average spectral number by Fjörtoft [**4**]. According to law (3.3), over time, the energy $\mathbf{K}$ can not concentrate only on small scales (that is, in the subspaces $\mathbf{H}_n$ with $\chi_n > \chi_a$) or only on large scales (in the subspaces $\mathbf{H}_n$ with $\chi_n < \chi_a$).

Note some properties of the Jacobian determinant. Obviously,

$$J(\psi, h) = -J(h, \psi) \quad \text{and} \quad J(\psi, \psi) = 0 \ . \tag{3.4}$$

It is also well known [**7**] that $\int_S \nabla \cdot \overrightarrow{X}\, dS = 0$ where $\overrightarrow{X}$ is a vector field with a compact support $K \subset S$. Therefore, the use of a partition of unity $\sum_i f_i(x) = 1$ of the sphere implies

$$\int_S J(\psi, h)\, dS = \sum_i \int_S \nabla \cdot \left[f_i h\, \left(\overrightarrow{n} \times \nabla\psi\right)\right] dS = 0 \ . \tag{3.5}$$

LEMMA 3.1. [**22**] *Let k be a natural number, and let ψ and h be continuously differentiable complex-valued functions on S. Then*

$$Re\,\langle J(\psi,\mu),\psi\rangle = 0 \quad \text{and} \quad Re\,\langle J(\psi,\mu),\Delta\psi\rangle = 0\ .$$

In particular, if ψ is a real function then

$$Re\,\langle J(h,\psi),h\rangle = 0 \quad \text{and} \quad \langle J(\psi,h),\psi\rangle = 0\ . \tag{3.6}$$

4. Modons

The first modons constructed on the sphere by Verkley [**24**] were similar in structure to the localized modons built on a plane [**10**]. They decay quickly in the outer region, and can be stationary only in the eastern background flow. The latter did not allow using these solutions even for a qualitative interpretation of the observed large-scale stationary atmospheric eddies formed in the mid-latitudes of the northern hemisphere in the background flow directed from east to west. The two later modons [**25, 26**] are less localized and have more in common with the observed atmospheric situations. In particular, the modons constructed in [**25**] can be stationary in the solid-state rotation, which is westerly, while the absolute vorticity of the modons built in [**26**] is constant in the inner region, resembling the atmospheric blocking structure characterized by rather low and uniform values of potential vorticity in the blocking region. Moreover, the latter have a better chance of retaining their structure in the inner region under the influence of small perturbations.

In addition to the coordinate system (λ,μ), consider a primed coordinate system (λ',μ'), the pole N' of which with coordinates $\lambda=\lambda_0$ and $\mu=\mu_0$ moves in a circle of latitude at a constant velocity C in accordance with the law $\lambda_0 = Ct$, $\mu_0 = Const.$ Then the Verkley modon can be written as

$$\widetilde{\psi}(\lambda',\mu') = R(\mu')\cdot\cos\lambda' + G(\mu') \tag{4.1}$$

where

$$\begin{aligned} R(\mu') &= A_i P^1_\alpha(\mu') + \omega_i\sqrt{1-\mu_0^2}\,P^1_1(\mu') \\ G(\mu') &= B_i P^0_\alpha(\mu') - \omega_i\mu_0 P^0_1(\mu') + D_i \end{aligned} \tag{4.2}$$

in the inner region $S_i = \{(\lambda',\mu')\in S:\ \mu' > \mu_a\}$ of sphere S, and

$$\begin{aligned} R(\mu') &= A_0 P^1_\sigma(-\mu') + \omega_0\sqrt{1-\mu_0^2}\,P^1_1(\mu') \\ G(\mu') &= B_0 P^0_\sigma(-\mu') - \omega_0\mu_0 P^0_1(\mu') + D_0 \end{aligned} \tag{4.3}$$

in the outer region $S_0 = \{(\lambda',\mu')\in S:\ \mu' < \mu_a\}$. Here D_i and D_0 are constants,

$$\begin{aligned} A_i = (C-\omega_0)a_i\,,&\quad A_0 = (C-\omega_0)a_0 \\ \mu_0 A_i = -\sqrt{1-\mu_0^2}\,B_i\,,&\quad \mu_0 A_0 = \sqrt{1-\mu_0^2}\,B_0 \end{aligned} \tag{4.4}$$

$$\begin{aligned} a_i &= b_0(\sigma)\,[b_i(\alpha)]^{-1}\left[P^1_\alpha(\mu_a)\right]^{-1}\sqrt{1-\mu_a^2}\sqrt{1-\mu_0^2} \\ a_0 &= \left[P^1_\sigma(-\mu_a)\right]^{-1}\sqrt{1-\mu_a^2}\sqrt{1-\mu_0^2} \end{aligned} \tag{4.5}$$

$$b_i(\alpha) = \chi_\alpha - 2, \;\; b_0(\sigma) = \chi_\sigma - 2 \tag{4.6}$$

while $\chi_\alpha = \alpha(\alpha+1)$ and $\chi_\sigma = \sigma(\sigma+1)$. The modon (4.1) is the solution of equation (3.1) provided that

$$C = \omega_0 - 2(\omega_0 + 1)/\chi_\sigma \tag{4.7}$$

Besides, the dispersion relation

$$-\frac{b_0(\sigma)P_\sigma^1(-\mu_a)}{P_\sigma^2(-\mu_a)} = \frac{b_i(\alpha)P_\alpha^1(\mu_a)}{P_\alpha^2(\mu_a)}$$

between α, σ and μ_a is the necessary condition for the modon (4.1) to exist [**24**]. Note that the degree α is real ($\alpha \geq 2$ and $\chi_\alpha > 0$), while σ may be complex. For example, if $\sigma = -\frac{1}{2} + ik$ where $k > 0$, the modon is localized only in a small neighborhood of inner region S_i, besides, $\chi_\sigma = -(k^2 + \frac{1}{4}) < 0$. Since $\chi_\alpha \neq \chi_\sigma$,

$$J(\psi, \Delta\psi) = -r(\lambda', \mu')\frac{\partial \psi}{\partial \lambda}$$

where

$$r(\lambda', \mu') = \begin{cases} C\chi_\alpha + 2, & \text{if} \quad (\lambda', \mu') \in S_i \\ C\chi_\sigma + 2, & \text{if} \quad (\lambda', \mu') \in S_0 \end{cases} .$$

If $|\mu_0| \neq 1$ then (4.1) is called dipole modon. A dipole modon has the purely dipole structure $\psi(\lambda', \mu') = R(\mu') \cdot \cos\lambda'$ if $\mu_0 = 0$. At $\mu = \mu_a$, the functions $R(\mu)$ and $G(\mu)$ have continuous derivatives only up to the second order.

Lemma 4.1. *Let ψ be a dipole modon* (4.1). *Then*

$$G_n = -\chi_n^{-1/2}\mu_0\,(1 - \mu_0^2)^{-1/2}\,R_n \tag{4.8}$$

where

$$R_n = \int_{-1}^{1} R(\mu)Q_n^1(\mu)d\mu \quad , \quad G_n = \int_{-1}^{1} G(\mu)Q_n^0(\mu)d\mu \tag{4.9}$$

are the Fourier coefficients of functions $R(\mu)$ and $G(\mu)$ of the modon.

The proof is given in Appendix B.

5. Distance between two modons

We now calculate the distance between two dipole modons. A perturbation $\psi(t,x)$ of a solution $\widetilde{\psi}(t,x)$ of Eq. (3.1) can be written as $\psi(t,x) = \widetilde{\psi}(t,x) - \widehat{\psi}(t,x)$ where $\widehat{\psi}(t,x)$ is another solution of the same equation. Then

$$K(t) = \frac{1}{2}\left\|\nabla\psi\right\|^2 = \frac{1}{2}\left\|\nabla(\widetilde{\psi} - \widehat{\psi})\right\|^2 = \mathbf{K}_a - \left\langle \widetilde{\psi}, \widehat{\psi} \right\rangle_1 \tag{5.1}$$

and

$$\eta(t) = \frac{1}{2}\left\|\Delta\psi\right\|^2 = \frac{1}{2}\left\|\Delta(\widetilde{\psi} - \widehat{\psi})\right\|^2 = \boldsymbol{\eta}_a - \left\langle \widetilde{\psi}, \widehat{\psi} \right\rangle_2 \tag{5.2}$$

are the kinetic energy and enstrophy of the perturbation $\psi(t,x)$, while $\mathbf{K}_a = \frac{1}{2}(\left\|\widetilde{\psi}\right\|_1^2 + \left\|\widehat{\psi}\right\|_1^2)$ and $\boldsymbol{\eta}_a = \frac{1}{2}(\left\|\widetilde{\psi}\right\|_2^2 + \left\|\widehat{\psi}\right\|_2^2)$ are constant values due to (3.2) and Lemma 2.1. Thus, changes in the energy and enstrophy of the perturbation $\psi(t,x)$ are caused only by changes in the projections $\left\langle \widetilde{\psi}, \widehat{\psi} \right\rangle_1$ and $\left\langle \widetilde{\psi}, \widehat{\psi} \right\rangle_2$ of the solution $\widehat{\psi}(t,x)$ onto the solution $\widetilde{\psi}(t,x)$ (see (2.6)). According to the solvability theorem by Szeptycki [**23**], the enstrophy $\eta(t)$ is the most suitable functional for evaluating the perturbations.

Let us refine formulas (5.1) and (5.2) for the case when $\widetilde{\psi}$ is a dipole modon (4.1) whose center moves along the latitudinal circle $\mu = \mu_0 = b$, and

$$\widehat{\psi}(\lambda_1, \mu_1) = \widehat{R}(\mu_1) \cdot \cos\lambda_1 + \widehat{G}(\mu_1) \tag{5.3}$$

is another dipole modon, the center of which moves along the circle $\mu = \mu_0 = a$. The modons $\widetilde{\psi}(\lambda', \mu')$ and $\widehat{\psi}(\lambda_1, \mu_1)$ are written in the systems (λ', μ') and (λ_1, μ_1). Let the poles $N\prime$ and N_1 of these systems belong to the axes Oz' and Oz_1 of two Cartesian coordinate systems $(x\prime, y\prime, z\prime)$ and (x_1, y_1, z_1), respectively (Fig.1).

Let $\widetilde{C}$ and $\widehat{C}$ be the velocities (4.7) of the modons $\widetilde{\psi}$ and $\widehat{\psi}$, respectively, and at the initial moment $t = 0$ the longitudinal angle $\gamma(t)$ between the poles $N\prime$ and N_1 is equal to γ_0. Then at any time $t > 0$,

$$\gamma(t) = \gamma_0 + (\widetilde{C} - \widehat{C})t \tag{5.4}$$

Also denote as β, ρ and ϑ the angles $N_1ON\prime$, AOB and BOC, respectively.

To calculate the inner product in (5.2) at time t, we rewrite modon (5.3) in the system (λ', μ') . For this, the system (x_1, y_1, z_1) must be rotated so that it coincides with the system $(x\prime, y\prime, z\prime)$ (Fig.1). This rotation can be represented by the matrix $D(\rho, -\beta, \vartheta)$ [**16**] and consists of three successive rotations by the Euler angles ρ, $-\beta$ and ϑ which depend on t (see Appendix A). The rotation $D(\rho, -\beta, \vartheta)$ leads to the relations

$$Y_n^m(\lambda_1, \mu_1) = \sum_{k=-n}^{n} D_{mk}^n(\rho, -\beta, \vartheta) Y_n^k(\lambda', \mu') \ , \tag{5.5}$$

$$D_{mk}^n(\rho, -\beta, \vartheta) = \exp\{i(m+k)\pi\} \exp\{i(m\rho + k\vartheta)\} d_{mk}^n(u) \tag{5.6}$$

[**16**] with $u = u(t)$ defined below by (A.1). Here

(5.7)

$$d_{mk}^n(u) = C_{mk}^n \, (1-u)^{(k-m)/2} \, (1+u)^{-(k+m)/2} \, \frac{d^{n-m}}{d\mu^{n-m}} \left[(1-u)^{n-k} \, (1+u)^{n+k} \right],$$

$$C_{mk}^n = \frac{(-1)^{n-m}}{2^n (n-m)!} \sqrt{\frac{(n+m)!(n-m)!}{(n+k)!(n-k)!}}$$

Since solution (5.3) can be written as

$$\widehat{\psi}(\lambda_1, \mu_1) = \sum_{n=1}^{\infty} \widehat{R}_n Q_n^1(\mu_1) \cdot \cos\lambda_1 + \sum_{n=1}^{\infty} \widehat{G}_n Q_n^0(\mu_1) \tag{5.8}$$

we will need the formulas

$$Q_n^0(\mu_1) = d_{00}^n(u)Q_n^0(\mu') + 2\cos\vartheta d_{10}^n(u)Q_n^1(\mu')\cos\lambda' + U_n^0(\lambda',\mu') \ , \tag{5.9}$$

$$Q_n^1(\mu_1)\cos\lambda_1 = -d_{10}^n(u)Q_n^0(\mu')\cos\rho + h_n(u)Q_n^1(\mu')\cos\lambda' + U_n^1(\lambda',\mu') \tag{5.10}$$

obtained from (5.5) and (5.6). Here $U_n^0(\lambda',\mu')$ and $U_n^1(\lambda',\mu')$ are the functions orthogonal to the spherical harmonics $Q_n^0(\mu')$ and $Q_n^1(\mu')\cdot\cos\lambda'$, $n \geq 1$. Note that $d_{00}^n(u) = P_n(u)$, $d_{10}^n(u) = \chi_n^{-1/2}P_n^1(u)$, and

$$h_n(u) = \cos(\rho+\vartheta)d_{11}^n(u) + (-1)^n\cos(\rho-\vartheta)d_{11}^n(-u)$$

Writing both modons in the form (5.8), using (5.9), (5.10) and relations (4.8) of Lemma 4.1 for G_n and $\widehat{G}_n$, we obtain

$$\left\langle \widetilde{\psi}, \widehat{\psi} \right\rangle_2 = \left\langle \Delta\widetilde{\psi}, \Delta\widehat{\psi} \right\rangle = \sum_{n=1}^{\infty} w_n(u,a,b)R_n\widehat{R}_n \ , \tag{5.11}$$

$$w_n(u,a,b) = \chi_n\left\{\frac{\chi_n}{2}h_n(u) + (\cos\vartheta - \cos\rho)P_n^1(u) + \frac{ab\,P_n(u)}{\sqrt{(1-a^2)\,(1-b^2)}}\right\} \tag{5.12}$$

Substitution of (5.11) in (5.2) leads to the following assertion:

Lemma 5.1. *The enstrophy $\eta(t)$ and energy $K(t)$ of the difference between the dipole modons* (4.1) *and* (5.3) *are calculated by the formulas*

$$\eta(t) = \frac{1}{2}\sum_{n=1}^{\infty}\left\{w_n(1,a,a)\widehat{R}_n^2 + w_n(1,b,b)R_n^2 - 2w_n(u,a,b)R_n\widehat{R}_n\right\} \ , \tag{5.13}$$

$$K(t) = \frac{1}{2}\sum_{n=1}^{\infty}\chi_n^{-1}\left\{w_n(1,a,a)\widehat{R}_n^2 + w_n(1,b,b)R_n^2 - 2w_n(u,a,b)R_n\widehat{R}_n\right\} \ . \tag{5.14}$$

Formulas (5.1), (5.2), (5.13) and (5.14) show that the enstrophy $\eta(t)$ and energy $K(t)$ of the difference between dipole modons (4.1) and (5.3) remain unchanged if and only if $\widehat{C} = C$ (modon velocities are the same; see (A.1)) or $R_n\widehat{R}_n = 0$ for all n (modons are orthogonal).

6. A special case of the movement of two modons

We now consider a particular case when the north poles of the dipole modons (4.1) and (5.3) move along the same latitudinal circle $\mu_0 = a = b$ (Fig.2). The results (especially, Lemma 6.3) will be used in Section 11 to prove the Liapunov instability of dipole modons. Since $a = b$, the enstrophy (5.13) of the difference between these two modons is

$$\eta(t) = \frac{1}{2}\sum_{n=1}^{\infty}\left\{w_n(1,a,a)(\widehat{R}_n^2 + R_n^2) - 2w_n(u,a,a)R_n\widehat{R}_n\right\} \tag{6.1}$$

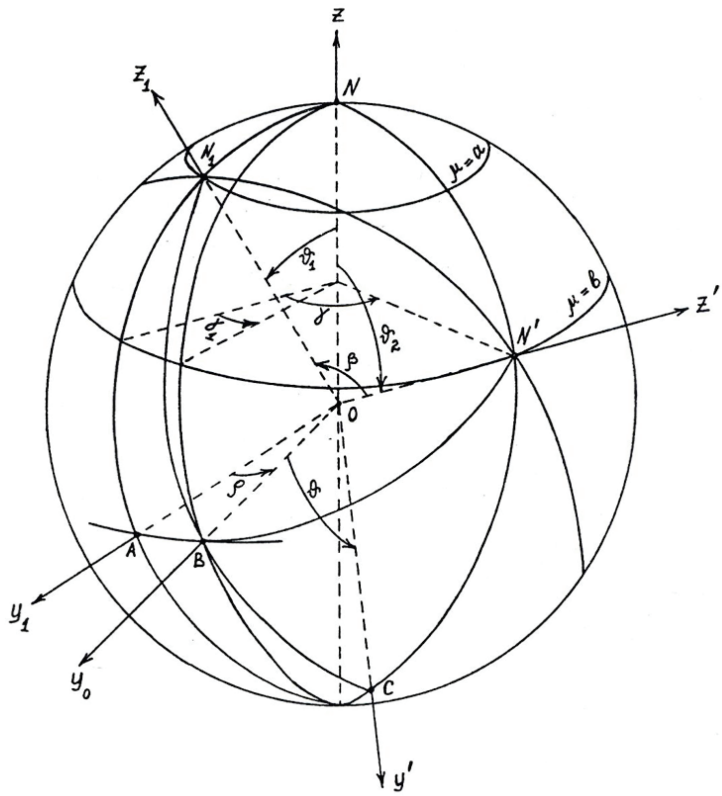

FIGURE 1. Spherical coordinate systems (λ', μ') and (λ_1, μ_1) whose poles N' and N_1 move along the latitudinal circles $\mu = b$ and $\mu = a$, respectively. The curves AB and BC are parts of the equators $\mu_1 = 0$ and $\mu' = 0$, respectively. The axis Oy_0 is orthogonal to the axes Oz' and Oz_1

where, due to (A.6), (A.8) and (5.12),

$$w_n(u, a, a) \equiv w_n(u) = \chi_n \left\{ \frac{\chi_n}{2} h_n(u) + \frac{a^2}{1 - a^2} P_n(u) \right\} , \tag{6.2}$$

$$h_n(u) = \frac{1}{2} \left\{ \cos 2\rho \; (1 + u) P_{n-1}^{(0,2)}(u) - (1 - u) P_{n-1}^{(2,0)}(u) \right\} \tag{6.3}$$

Here $P_n(u) = P_n^0(u)$ is the Legendre polynomial (2.4) and

$$P_n^{(\alpha,\beta)}(u) = \frac{(-1)^n}{2^n n!} (1 - u)^{-\alpha} (1 + u)^{-\beta} \frac{d^n}{du^n} [(1 - u)^{n+\alpha} (1 + u)^{n+\beta}]$$

is the Jacobi polynomial [**16**]. In deriving (6.3), we used relation (5.7) which leads to $d_{11}^n(u) = \frac{1}{2}(1 + u) P_{n-1}^{(0,2)}(u)$. Also, we used the relation $P_{n-1}^{(0,2)}(-u) = P_{n-1}^{(2,0)}(u)$. The Jacobi polynomial can be written as

$$P_n^{(\alpha,\beta)}(u) = \frac{\Gamma(n + \alpha + 1)}{n! \Gamma(\alpha + 1)} F \left(-n, n + \alpha + \beta + 1; \alpha + 1; \frac{1 - u}{2} \right) \tag{6.4}$$

where

$$F(p, r, s, v) = \sum_{n=0}^{\infty} \frac{(p)_n (r)_n}{(s)_n} \frac{v^n}{n!},$$

$|v| < 1$ is the hypergeometric function [**16**], $\Gamma(s)$ is the gamma function, $\Gamma(1) = 1$, $\Gamma(1+s) = s\Gamma(s)$, and $(s)_n = \Gamma(s+n)/\Gamma(s)$ is the Pochhammer symbol [**18**].

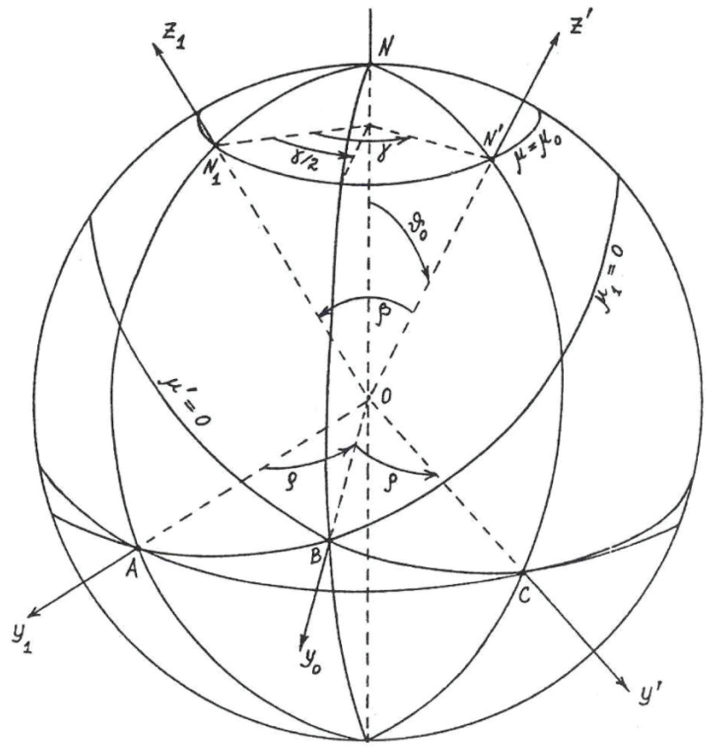

FIGURE 2. The case when the poles $N\prime$ and N_1 of modons move along the same latitudinal circle $\mu_0 = a = b$. Spherical triangles ANB and BNC are symmetric about the line NB. The curves AB and BC belong to the equators $\mu_1 = 0$ and $\mu' = 0$ of the systems (λ_1, μ_1) and (λ', μ'), respectively.

LEMMA 6.1. *If* $\gamma = \pi/2$ *then* $u = a^2$ *and* $\cos 2\rho = (1-u)/(1+u)$.

Proof. It follows from (A.3) that $a\mu_B = -\sqrt{1-a^2}\sqrt{1-\mu_B^2}\sin\gamma_1$ and by (A.7), $\sin\rho = -a\mu_B/\sqrt{1-a^2}$. Since $\cos\rho = -\mu_B/\sqrt{1-a^2}$ (see (A.6)), $\tan\rho = a$, and

$$\cos 2\rho = \frac{1-\tan^2\rho}{1+\tan^2\rho} = \frac{1-a^2}{1+a^2} = \frac{1-u}{1+u}. \quad Q.E.D. \tag{6.5}$$

LEMMA 6.2. [**16**]. *Let* $\alpha + \frac{1}{2} > 0$, $\beta + \frac{1}{2} > 0$, *and* $-1 < u < 1$. *Then*

$$\left|P_n^{(\alpha,\beta)}(u)\right| < \max\{\left|P_n^{(\alpha,\beta)}(-1)\right|,\ \left|P_n^{(\alpha,\beta)}(1)\right|\} \tag{6.6}$$

where $P_n^{(\alpha,\beta)}(1) = \Gamma(n+\alpha+1)/\{n!\Gamma(\alpha+1)\}$ *and* $P_n^{(\alpha,\beta)}(-1) = (-1)^n\,\Gamma(n+\beta+1)/\{n!\Gamma(\beta+1)\}$.

LEMMA 6.3. *Let* $u = a^2$, $-1 < a^2 < 1$. *Then* $|h_n(a^2)| < 1$ *for any natural number* n.

Proof. Since $u = a^2$, $\gamma = \pi/2$, formulas (6.3) and (6.5) lead to

$$h_n(a^2) = \frac{1}{2}(1-a^2)\left\{P_{n-1}^{(0,2)}(a^2) - P_{n-1}^{(2,0)}(a^2)\right\}.$$

Due to (6.4), one can rewrite the last formula as

$$h_n(a^2) = v\left\{F(-n+1, n+2; 1; v) - \frac{\chi_n}{2}F(-n+1, n+2; 3; v)\right\}$$

where $v = (1-a^2)/2$. Using the functional relation

$$\frac{v-1}{\Gamma(s-1)}F(p,r;s-1;v) + \frac{1}{\Gamma(s)}\left\{s-1-(2s-p-r-1)v\right\}F(p,r;s;v)$$
$$+\frac{v}{\Gamma(s+1)}(s-p)(s-r)F(p,r;s+1;v) = 0$$

(see [**18**]) we obtain

$$h_n(a^2) = F(-n+1, n+2; 1; v) - F(-n+1, n+2; 2; v)\ . \tag{6.7}$$

We now use one more relation for hypergeometric functions [**16**]:

$$s(s+1)\left\{F(p,r;s;v) - F(p,r;s+1;v)\right\} - prvF(p+1, r+1; s+2; v) = 0$$

Then (6.7) can be rewritten as $h_n(a^2) = \frac{v}{2}(-n+1)(n+2)F(-(n-2), n+3; 3; v)$. Applying formula (6.4) to the last equation again, we obtain

$$h_n(a^2) = -\frac{n+2}{2n}(1-a^2)P_{n-2}^{(2,2)}(a^2) \tag{6.8}$$

On the other hand, $P_{n-2}^{(2,2)}(1) = \Gamma(n+1)/\{n!\Gamma(3)\} = \frac{1}{2}$ and $P_{n-2}^{(2,2)}(-1) = (-1)^{n-2}\frac{1}{2}$, and therefore, due to (6.6), $\left|P_{n-2}^{(2,2)}(u)\right| < \frac{1}{2}$ for $-1 < u < 1$. Finally, (6.8) and the last inequality lead to $\left|h_n(a^2)\right| \leq \frac{n+2}{4n}(1-a^2) < 1$. *Q.E.D.*

7. Stationary solutions

Let us now consider two particular types of stationary solutions to vorticity equation (3.1):

1) Antisymmetric stationary WV wave

$$\widetilde{\psi}(\lambda,\mu) = \begin{cases} X_i(\lambda,\mu) - \omega_i\mu + D_i, & \text{in } S_{in} \\ X_o(\lambda,\mu) - \omega_o\mu + D_o, & \text{in } S_{out} \end{cases} \tag{7.1}$$

where $S_{in} = \{(\lambda,\mu) \in S : \mu \in (-\mu_0, \mu_0)\}$ is the inner region of sphere, symmetrically located relative to the equator $\mu = 0$, and $S_{out} = \{(\lambda,\mu) \in S : \mu \in (\mu_0, 1] \cup [-1, -\mu_0)\}$ is the outer region which is the union of two polar zones,

$0 < \mu_0 < 1$, and ω_i, ω_o, D_i and D_o are constants [**28**]. The wave (7.1) is antisymmetric with respect to the equator $\mu = 0$. Note that both X_i and X_o are eigenfunctions of the spherical Laplace operator:

$$-\Delta X_i = \chi_\alpha X_i \ , \quad \text{and} \quad -\Delta X_o = \chi_\sigma X_o, \tag{7.2}$$

where $\chi_\alpha = \alpha(\alpha+1)$, $\chi_\sigma = \sigma(\sigma+1)$, besides, $\omega_i = 2/\left(\chi_\alpha - 2\right)$ and $\omega_o = 2/\left(\chi_\sigma - 2\right)$ for the steady WV wave. We also note that by construction the boundaries $\mu = \pm \mu_0$ between the regions S_{in} and S_{out} are streamlines:

$$\widetilde{\psi}_\lambda(\lambda, \mu_0) = 0, \quad \text{and} \quad \widetilde{\psi}_\lambda(\lambda, -\mu_0) = 0 \tag{7.3}$$

2) The second solution is one of the two stationary dipole modons by Verkley [**24, 25**], which can also be written in the form (7.1) as

$$\widetilde{\psi}(\lambda, \mu) = \widetilde{\psi}(\lambda', \mu') = \begin{cases} X_i(\lambda', \mu') - \omega_i \mu + D_i, & \text{in } S_{in} \\ X_o(\lambda', \mu') - \omega_o \mu + D_o, & \text{in } S_{out} \end{cases} \tag{7.4}$$

where X_i and X_o are the eigenfunctions of the eigenvalue problems (7.2), and the pole $\mu' = 1$ of the system (λ', μ') is the center of the inner region $S_{in} = \{(\lambda', \mu') \in S : \mu' > \mu_a\}$ of the modon. The circle $\mu' = \mu_a$ separates the inner region S_{in} from the outer region $S_{out} = \{(\lambda', \mu') \in S : \mu' < \mu_a\}$. It is easy to show that $\widetilde{\psi}_{\lambda'}(\lambda', \mu_a) = C\sqrt{1-\mu_a^2}\sqrt{1-\mu_0^2}\sin\lambda'$. Thus

$$\widetilde{\psi}_{\lambda'}(\lambda', \mu_a) = 0 \tag{7.5}$$

for stationary dipole modons ($C = 0$), or monopole modons ($\mu_0^2 = 1$), i.e. the circle $\mu' = \mu_a$ is the streamline for such modons.

Let $g\,(\lambda, \mu)$ and $h\,(\lambda, \mu) = h_r\,(\lambda, \mu) + i\,h_i\,(\lambda, \mu)$ be, respectively, real and complex smooth functions defined on S. We denote by $G = \{(\lambda, \mu) \in S : \mu \in (\mu_a, 1]\}$ the part of the sphere S bounded by the latitudinal circle $\mu = \mu_a$ $(-1 < \mu_a < 1)$, and let

$$g_\lambda(\lambda, \mu_a) = 0 \tag{7.6}$$

Then in addition to (3.5), condition (7.6) makes it possible to obtain one more integral formula:

$$\mathrm{Re}\int_G J(g, h)\overline{h} dS = 0 \tag{7.7}$$

Indeed, condition (7.6) leads to

$$\int_G J(g, h) dS = 0 \tag{7.8}$$

Integrating the equality $\mathrm{Re} J(g, h)\overline{h} = 0.5\, J(g, h_r^2) + 0.5\, J(g, h_i^2)$ over G and using (7.8), we obtain (7.7). Formula (7.7) is also valid if $G = \{(\lambda, \mu) \in S : \mu \in [\mu_a, \mu_b]\}$ is a periodic channel on S and

$$g_\lambda(\lambda, \mu_a) = g_\lambda(\lambda, \mu_b) = 0 \tag{7.9}$$

Of course, (7.7) also holds if, instead of G, the integral in (7.7) is taken over S/G (the relative complement of the set G in S).

8. Conservation law for infinitesimal perturbations

We now obtain a conservation law for infinitesimal perturbations of the stationary solutions (7.1) and (7.4). Let, for definiteness, $\widetilde{\psi}$ be a steady modon (7.4). For the stationary WV wave (7.1), the proof does not change. The relative vorticity

$$\Delta\widetilde{\psi}(\lambda',\mu') = \begin{cases} -\chi_\alpha\widetilde{\psi}(\lambda',\mu') + (2-\chi_\alpha)\omega_i\mu + \chi_\alpha D_i, & \text{if } (\lambda',\mu') \in S_{in} \\ -\chi_\sigma\widetilde{\psi}(\lambda',\mu') + (2-\chi_\sigma)\omega_o\mu + \chi_\sigma D_o, & \text{if } (\lambda',\mu') \in S_{out} \end{cases} \tag{8.1}$$

is a continuous function on S [**24**]. Since $(2-\chi_\alpha)\,\omega_i = (2-\chi_\sigma)\,\omega_o = -2$ for a steady modon, the absolute vorticity

$$\Delta\widetilde{\psi} + 2\mu = \begin{cases} -\chi_\alpha\left\{\widetilde{\psi}(\lambda',\mu') - D_i\right\}, & \text{in } S_{in} \\ -\chi_\sigma\left\{\widetilde{\psi}(\lambda',\mu') - D_o\right\}, & \text{in } S_{out} \end{cases} \tag{8.2}$$

is continuous on S as the sum of continuous functions $\Delta\widetilde{\psi}$ and 2μ, that is, $(\chi_\alpha - \chi_\sigma)\widetilde{\psi}(\lambda',\mu_a) = D_o - D_i$. Due to (3.4) and (8.2), $J(\widetilde{\psi},\Delta\widetilde{\psi}+2\mu) = 0$ both in the regions S_{in} and S_{out}. Further, by ((8.1)), $\Delta\widetilde{\psi} + \chi_\alpha\widetilde{\psi} = -2\mu + \chi_\alpha D_i$ in S_{in}, and $\Delta\widetilde{\psi} + \chi_\sigma\widetilde{\psi} = -2\mu + \chi_\sigma D_o$ in S_{out}. Since the last terms in these equations are constants, the partial derivatives of functions $\Delta\widetilde{\psi}+\chi_\alpha\widetilde{\psi}$ and $\Delta\widetilde{\psi}+\chi_\sigma\widetilde{\psi}$ with respect to λ' and μ' coincide to each other at the boundary $\mu' = \mu_a$. Using formula (8.2), the linearized equation

$$\Delta\psi_t + J(\widetilde{\psi},\Delta\psi) + J(\psi,\Delta\widetilde{\psi}+2\mu) = 0 \tag{8.3}$$

for an infinitesimal perturbation ψ of the modon can be written as

$$\Delta\psi_t + J(\widetilde{\psi},\Delta\psi+\chi_\alpha\psi) = 0 \ \text{ in } S_{in}\ , \text{ and } \ \Delta\psi_t + J(\widetilde{\psi},\Delta\psi+\chi_\sigma\psi) = 0 \ \text{ in } S_{out} \tag{8.4}$$

We now multiply the first equation (8.4) by $\overline{\Delta\psi + \chi_\alpha\psi}$ and integrate the result over region S_{in}. Then we multiply the second equation (8.4) by $\overline{\Delta\psi + \chi_\sigma\psi}$ and integrate the result over region S_{out}. As a result, we obtain

$$\int_{S_{in}} \Delta\psi_t(\overline{\Delta\psi} + \chi_\alpha\overline{\psi})dS + \int_{S_{in}} J(\widetilde{\psi},\Delta\psi+\chi_\alpha\psi)(\overline{\Delta\psi+\chi_\alpha\psi})dS = 0 \tag{8.5}$$

$$\int_{S_{out}} \Delta\psi_t(\overline{\Delta\psi} + \chi_\sigma\overline{\psi})dS + \int_{S_{out}} J(\widetilde{\psi},\Delta\psi+\chi_\sigma\psi)(\overline{\Delta\psi+\chi_\sigma\psi})dS = 0\ . \tag{8.6}$$

Since the stationary modon $\widetilde{\psi}$ satisfies condition (7.5), formula (7.7) is valid both for $G = S_{in}$ with $h = \Delta\psi + \chi_\alpha\psi$ and for $G = S_{out}$ with $h = \Delta\psi + \chi_\sigma\psi$. Thus, the second integrals in (8.5) and (8.6) vanish, which reduces these equations to

$$\chi_\alpha^{-1}\eta_t^{(i)} = -\mathrm{Re}\int_{S_{in}} \Delta\psi_t\overline{\psi}dS \quad \text{and} \quad \chi_\sigma^{-1}\eta_t^{(o)} = -\mathrm{Re}\int_{S_{out}} \Delta\psi_t\overline{\psi}dS \tag{8.7}$$

where

$$\eta^{(i)}(t) = \frac{1}{2}\int_{S_{in}} |\Delta\psi|^2\, dS \quad \text{and} \quad \eta^{(o)}(t) = \frac{1}{2}\int_{S_{out}} |\Delta\psi|^2\, dS$$

are the parts of perturbation enstrophy $\eta(t) = \frac{1}{2}\left\|\Delta\psi(t)\right\|^2$ corresponding to the modon regions S_{in} and S_{out}, respectively. The right-hand sides of equations (8.7), when summed, give the kinetic energy $K(t) = \frac{1}{2}\left\|\nabla\psi(t)\right\|^2 = -\frac{1}{2}\left\langle \Delta\psi_t, \psi\right\rangle$ of the perturbation ψ, so we proved the following assertion:

THEOREM 8.1. *Infinitesimal perturbations to modon* (7.4) *satisfy the conservation law*

$$\frac{d}{dt}U\left[\psi(t)\right] \equiv \frac{d}{dt}\left\{K - \chi_\alpha^{-1}\eta^{(i)} - \chi_\sigma^{-1}\eta^{(o)}\right\} = 0\ . \tag{8.8}$$

This law is similar to the law obtained earlier by Laedke and Spatschek [**8**] for infinitesimal quasi geostrophic perturbations of the modon on the beta-plane. Using (7.3) (see also (7.9) instead of (7.6)), it is easy to show that law (8.8) is also valid for infinitesimal perturbations of the stationary WV wave (7.1). Note that, in contrast to weak solutions (7.1) and (7.4), a similar law holds for arbitrary perturbations of the Rossby-Haurwitz wave [**22**].

Let $\psi(t)$ be an infinitesimal perturbation of the stationary modon (7.4), and let

$$\eta^{(i)}(t) \equiv \chi_i(t)K(t) \quad \text{and} \quad \eta^{(o)}(t) \equiv \chi_0(t)K(t) \tag{8.9}$$

Then $\chi(t) = \chi_i(t) + \chi_0(t)$ is the mean spectral number of perturbation, $\eta(t) = \chi(t)K(t)$, and all infinitesimal perturbations of modon (7.4) can be divided into three subsets

$$\mathbf{M}_- = \{\psi\,:\,p(\psi) < 1\}\,, \quad \mathbf{M}_0 = \{\psi\,:\,p(\psi) = 1\}\,, \quad \mathbf{M}_+ = \{\psi\,:\,p(\psi) > 1\}$$

where $p(\psi(t)) = \chi_\alpha^{-1}\chi_i(t) + \chi_\sigma^{-1}\chi_0(t)$ is the non-dimensional number characterizing the spectral distribution of perturbation ψ. Substituting (8.9) in (8.8) and using $p(\psi(t))$, we obtain

$$\frac{d}{dt}U\left[\psi(t)\right] \equiv \frac{d}{dt}\left\{K(t)[1 - p(\psi(t))]\right\} = 0 \tag{8.10}$$

and

$$\frac{d}{dt}p(\psi) = \frac{1}{K}\left\{1 - p(\psi)\right\}\frac{d}{dt}K\ . \tag{8.11}$$

According to (8.10), the closer $p(\psi)$ is to 1, the greater the perturbation energy. By (8.11), the energy cascade of growing disturbances of the modon has opposite directions in the sets $\mathbf{M}_-$ and $\mathbf{M}_+$. Thus there is a certain similarity in the behavior of infinitesimal perturbations of the stationary modon in the sets $\mathbf{M}_-$ and $\mathbf{M}_+$ and arbitrary perturbations of the stationary RH wave in the invariant sets $\mathbf{M}_-^n$ and $\mathbf{M}_+^n$ [**22**].

9. Instability conditions

We now obtain the necessary conditions for normal mode instability of stationary modons and WV waves. Similarly to (3.3), we define by

$$\chi(\psi) \equiv \chi(t) = \eta(t)/K(t) \tag{9.1}$$

the mean spectral number of the perturbation ψ, and by $\delta(t) = \eta^{(o)}(t)/\eta(t)$ and $1 - \delta(t) = \eta^{(i)}(t)/\eta(t)$ - the perturbation enstrophy fractions corresponding to the regions S_{out} and S_{in}, respectively ($0 \le \delta \le 1$). Then the dimensionless number $p(\psi)$ in the perturbation law (8.10) can be written as

$$p(\psi) = \chi(\psi)\left\{\delta\chi_\sigma^{-1} + (1-\delta)\chi_\alpha^{-1}\right\} \tag{9.2}$$

This number characterizes the spectral distribution of the perturbation $\psi(\lambda, \mu, t)$ on the sphere.

Let $\widetilde{\psi}$ be the stationary BVE solution (7.1) or (7.4), and let

$$\psi(t, \lambda, \mu) = \Psi(\lambda, \mu)\exp\{\nu t\} \tag{9.3}$$

be its infinitesimal perturbation (normal mode) where Ψ is the perturbation amplitude and $\nu = \nu_r + i\nu_i$. The values ν_r and ν_i characterize, respectively, the growth rate and the frequency of the perturbation. Thus, mode (9.3) is unstable if $\nu_r > 0$, damped if $\nu_r < 0$, and neutral if $\nu_r = 0$. The energy and enstrophy of the mode (9.3) can be written as

$$K(t) = K_\Psi \exp(2\nu_r t) \quad \text{and} \quad \eta(t) = \eta_\Psi \exp(2\nu_r t) \tag{9.4}$$

where $K_\Psi = \frac{1}{2}\|\nabla\Psi\|^2$ and $\eta_\Psi = \frac{1}{2}\|\Delta\Psi\|^2$ are the energy and enstrophy of the mode amplitude $\Psi(\lambda, \mu)$, respectively. Then

$$\chi_\Psi = \eta_\Psi/K_\Psi \tag{9.5}$$

is the mean spectral number of the mode amplitude. Note that $\chi(\psi) \equiv \chi_\Psi$ for perturbation (9.3), that is, $\chi(\psi)$ does not depend on time. Moreover, the characteristics δ and $p(\psi)$ of each mode are also independent of time, and therefore it follows from (8.10) and (9.4) that $\nu_r\,[p(\psi) - 1]\,K_\Psi = 0$. Since $\nu_r > 0$ for the growing mode, we obtain that

$$p(\psi) = 1 \tag{9.6}$$

is a necessary condition for the exponential instability of the mode. Formulas (9.2) and (9.5) lead to the following assertion:

THEOREM 9.1. *Let $\widetilde{\psi}$ be a stationary WV wave* (7.1) *or a modon* (7.4)*. Then the mean spectral number* (9.5) *of the amplitude $\Psi(\lambda, \mu)$ of each unstable perturbation* (9.3) *of the flow $\widetilde{\psi}$ must satisfy the condition*

$$\chi_\Psi = \left\{\delta\chi_\sigma^{-1} + (1-\delta)\chi_\alpha^{-1}\right\}^{-1} \ . \tag{9.7}$$

Thus, the disturbance (9.3) may exponentially grow (or decay) with time only if its amplitude $\Psi(\lambda,\mu)$ belongs to the hypersurface (9.6) in the perturbation space. Now let us refine the instability condition (9.7) for various BVE solutions.

Case 1. *Non-local BVE solutions.* Let $\widetilde{\psi}$ be the WV wave (7.1), or a non-local modon [**25**]. Then $\chi_\sigma > 0$ and $\chi_\alpha > 0$, and therefore $\chi_\Psi^{-1} = \delta\chi_\sigma^{-1} + (1-\delta)\chi_\alpha^{-1}$ is a linear interpolation of χ_σ^{-1} and χ_α^{-1}. In particular, if $\eta^{(o)} = 0$ then $\delta = 0$ and $\chi_\Psi = \chi_\alpha$. And if $\eta^{(i)} = 0$ then $\delta = 1$ and $\chi_\Psi = \chi_\sigma$.

Case 2. *Modons with uniform absolute vorticity.* Let $\widetilde{\psi}$ be a modon with uniform absolute vorticity in the inner region S_{in} [**26**]. Then, for each unstable mode, $\Delta\Psi(\lambda,\mu) = 0$ in the region S_{in} (see [**26**], Appendix B), and due to (9.3), $\Delta\psi = 0$, $\Delta\psi_t = 0$ and $J(\widetilde{\psi},\Delta\psi) = 0$ in S_{in}. Therefore, (8.3) means that $J(\psi,\Delta\widetilde{\psi} + 2\mu) = 0$ also in S_{in}, and the only nonzero equation is the second equation (8.7), in which the integration over S_{out} can be extended to the entire sphere S since $\Delta\psi = 0$ in S_{in}. This leads to

$$\chi_\sigma^{-1}\eta_t = -\mathrm{Re}\int_S \Delta\psi_t\overline{\psi}dS = K_t \ , \quad \text{or} \quad \left\{(\chi - \chi_\sigma)K\right\}_t = 0$$

Consequently, the necessary condition $\chi_\Psi = \chi_\sigma = \sigma(\sigma+1)$ for the instability of this modon depends only on the degree of the spherical function used to construct the modon in its outer region S_{out}, and thus coincides with the necessary condition for the instability of the LP flows and RH waves [**22**].

Case 3. *Localized modons.* Let $\widetilde{\psi}$ be a localized modon [**24**] with $\sigma = -0.5 + ik$, $i = \sqrt{-1}$, $\chi_\sigma = \sigma(\sigma+1) = -(k^2 + 0.25) < 0$ and $\chi_\alpha = \alpha(\alpha+1) > 0$. By Theorem 9.1, the mode (9.3) of such a modon can be unstable only if $\chi_\Psi^{-1} = (1-\delta)\chi_\alpha^{-1} - \delta\left|\chi_\sigma\right|^{-1}$. Since χ_Ψ^{-1} must be positive, we obtain a constraint on the fraction of perturbation enstrophy δ corresponding to the region S_{out}: $\delta < \delta_{cr} = \left|\chi_\sigma\right|\left(\chi_\alpha + \left|\chi_\sigma\right|\right)^{-1} < 1$. Obviously, the critical value δ_{cr} decreases with an increase in the degree α of the spherical harmonic used to construct the modon in its inner region. Further, it follows from (9.7) that $\chi_\Psi = \chi_\alpha\left(1 - \delta/\delta_{cr}\right)^{-1} \geq \chi_\alpha$ for the unstable modes. The minimum $\chi_\Psi = \chi_\alpha$ corresponds to the case $\delta = 0$ when the enstrophy $\Delta\Psi$ of the mode amplitude is equal to zero in the outer region S_{out} of the modon.

10. Growth rate of unstable modes and their orthogonality to the basic solution

Let $\widetilde{\psi}$ be a steady WV wave (7.1) or modon (7.4). We now estimate the maximum growth rate of unstable modes (9.3) of these solutions. It will also be shown that the amplitude Ψ of each unstable, decaying, or nonstationary mode is orthogonal to the solution $\widetilde{\psi}$ in the inner product of the space $\mathbb{H}^1$ (see (2.6)). Substituting the infinitesimal perturbation (9.3) into equation (8.3) we obtain

$$\nu\Delta\Psi + J(\widetilde{\psi},\Delta\Psi) + J(\Psi,\Delta\widetilde{\psi} + 2\mu) = 0 \tag{10.1}$$

Let us consider the inner product (2.2) of Eq. (10.1) with $\Delta\Psi$. Taking the real part of the result, we get

$$\nu_r \left\|\Delta\Psi\right\|^2 = -\mathrm{Re}\left\langle qJ(\widetilde{\psi},\Psi),\Delta\Psi\right\rangle \tag{10.2}$$

where $q(x)=\chi_\alpha$ if $x\in S_{in}$, and $q(x)=\chi_\sigma$ if $x\in S_{out}$. The last formula directly follows from (8.1) if we take into account that $(2-\chi_\alpha)\,\omega_i=(2-\chi_\sigma)\,\omega_o=-2$ for a stationary solution. In deriving (10.2) we also used the norm (2.3) and the first formula (3.6), according to which $\mathrm{Re}\left\langle J(\widetilde{\psi},\Delta\Psi),\Delta\Psi\right\rangle=0$.

Let $\vec{U}=\vec{k}\times\nabla\widetilde{\psi}$ be the velocity of the flow generated by the streamfunction of the basic solution ($\vec{k}$ is the unit vector normal to the sphere), and let $C=\max_S|\nabla\widetilde{\psi}|=\max_S|\vec{U}|$. Since $\|\nabla\Psi\|=\sqrt{2K_\Psi}$ we have

$$\left\|qJ(\widetilde{\psi},\Psi)\right\| \le \max_S\left|\nabla\widetilde{\psi}\right|\ \{\max_S q\}\ \|\nabla\Psi\| \le C\sqrt{2K_\Psi}\max\{\chi_\alpha,|\chi_\sigma|\}\ . \tag{10.3}$$

Applying the Schwarz inequality to the inner product in equation (10.2), and then using the estimate (10.3) and the formula $\sqrt{2K_\Psi}/\left\|\Delta\Psi\right\|=\chi_\Psi^{-1/2}$, which is the modified instability condition (9.7), we obtain the following result:

THEOREM 10.1. *The maximum growth (decay) rate of normal modes to a stationary WV wave* (7.1) *or modon* (7.4) *is limited:*

$$|\nu_r| \le C\max\{\chi_\alpha,|\chi_\sigma|\}\ \left\{\delta\chi_\sigma^{-1}+(1-\delta)\chi_\alpha^{-1}\right\}^{1/2}\ . \tag{10.4}$$

Thus, the growth rate of normal modes depends on velocity $\vec{U}$ and degrees α and σ of the basic flow, as well as on the fraction δ of the mode enstrophy which corresponds to the region S_{out}. In particular, (10.4) reduces to $|\nu_r|\le C\max\{\chi_\alpha,\chi_\sigma\}\,\chi_\sigma^{-1/2}$ for a modon with uniform absolute vorticity (see Case 2), and to $|\nu_r|\le C\max\{\chi_\alpha,|\chi_\sigma|\}\,\chi_\alpha^{-1/2}\,(1-\delta/\delta_{cr})^{1/2}$ for a localized modon (Case 3).

Let us consider the inner product (2.2) of Eq. (10.1) with $\widetilde{\psi}$. Using the second formula (3.6), we obtain

$$\nu\left\langle\Delta\Psi,\widetilde{\psi}\right\rangle+\chi_\alpha\int_{S_{in}}J(\widetilde{\psi},\Psi)\widetilde{\psi}dS+\chi_\sigma\int_{S_{out}}J(\widetilde{\psi},\Psi)\widetilde{\psi}dS=0\ .$$

According to (7.7), the integrals over S_{in} and S_{out} vanish and $\nu\left\langle\Delta\Psi,\widetilde{\psi}\right\rangle=0$. Therefore,

$$\left\langle\Psi,\widetilde{\psi}\right\rangle_1 \equiv \left\langle\nabla\Psi,\nabla\widetilde{\psi}\right\rangle = -\left\langle\Delta\Psi,\widetilde{\psi}\right\rangle = 0 \tag{10.5}$$

for any unstable (or decaying) mode (since $\nu_r\neq0$), as well as for any nonstationary mode (since $\nu_i\neq0$). We summarize this result as

THEOREM 10.2. *Let $\widetilde{\psi}$ be a stationary WV wave* (7.1) *or modon* (7.4). *Then amplitude Ψ of each unstable, decaying, or nonstationary mode is orthogonal to the basic solution $\widetilde{\psi}$ in the inner product of the space $\mathbb{H}^1$ (that is, in the inner product associated with kinetic energy).*

Equation (10.5) means that velocity field $\vec{U} = \vec{k} \times \nabla\widetilde{\psi}$ of the basic solution is orthogonal to the velocity field $\vec{u} = \vec{k} \times \nabla\Psi$ of the normal mode:

$$\left\langle \vec{u}, \vec{U} \right\rangle \equiv \int_S \vec{u} \cdot \vec{U} ds = \int_S \nabla\Psi \cdot \nabla\widetilde{\psi} ds = \left\langle \Psi, \widetilde{\psi} \right\rangle_1 = 0$$

Example 1. The normal mode instability of the antisymmetric WV wave (7.1) is analyzed by the spectral method provided that $\mu_0 = \sin 29.99$, $\alpha = 5.7701$, $\sigma = 4.5419$, $\chi_\alpha = 39.0642$, $\chi_\sigma = 25.1708$. The wave is shown in Fig.3a, and the real parts of the amplitudes of the first three most unstable modes (calculated using a series of spherical harmonics at triangular truncation with maximum index $N = 42$) are shown in Fig.3b-d. It is seen that the first and third modes are antisymmetric about the equator, and the second mode is symmetric about the equator. The values of the mean spectral number χ_Ψ, the e-folding time τ_e and the period T of the modes are given in Table 1.

TABLE 1. Characteristics of the most unstable modes to WV wave (7.1).

Mode	χ_Ψ	τ_e	T
mode 1	32.37	2.24	8.80
mode 2	33.81	2.80	7.60
mode 3	36.45	3.20	4.80

11. Liapunov instability of dipole modons

Many works are devoted to the proof of the stability of solitary eddies (solitons and modons) using analytical methods [**6, 8, 9, 21**] or numerical and laboratory experiments [**2, 5, 11**]. However, Nycander [**7**] refuted the five most serious proofs of the stability of dipole vortices (modons) presented by various authors. We now prove the Liapunov instability of dipole modons in the norm associated with the enstrophy.

Taking account of (4.4)-(4.7) and Verkley's formula

$$\omega_i - \omega_0 = (C - \omega_0) \left\{ 1 - \frac{b_0(\sigma)}{b_i(\alpha)} \right\}$$

[**24**], we can write the function $R(\mu)$ of modon (4.1) as

$$R(\mu) = (C - \omega_0) T(\mu) + \omega_0 \sqrt{1 - \mu_0^2} \sqrt{1 - \mu^2} \tag{11.1}$$

where

$$T(\mu) = \begin{cases} a_i P_\alpha^1(\mu) + \left\{ 1 - \frac{b_0(\sigma)}{b_i(\alpha)} \right\} \sqrt{1 - \mu_0^2} \sqrt{1 - \mu^2}, & \text{if } \mu \geq \mu_a \\ a_0 P_\sigma^1(-\mu), & \text{if } \mu \leq \mu_a \end{cases}$$

is an infinitely differentiable function in the intervals $[-1, \mu_a)$ and $(\mu_a, 1]$. However at $\mu = \mu_a$ it has continuous derivatives only up to the second order.

THEOREM 11.1. *Any dipole modon* (4.1) *moving along a latitudinal circle* $\mu_0 = a$ *is Liapunov unstable in the norm* $\|\psi\|_\eta = \sqrt{\eta}$ *where* η *is the perturbation enstrophy* (5.2).

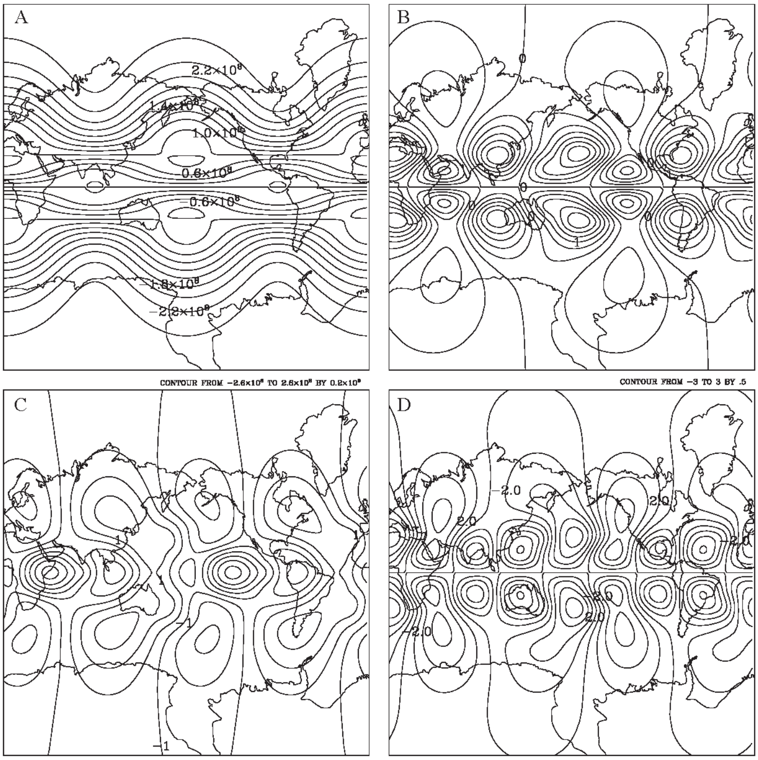

FIGURE 3. WV wave (a), and isolines of the real parts of the amplitudes of the first three most unstable modes: $\nu_r = 0.0708$, (b), $\nu_r = 0.0564$ (c), $\nu_r = 0.0495$ (d)

Proof. Since the norm $\|\psi\|_\eta = 2^{-1/2}\|\Delta\psi\| = 2^{-1/2}\|\psi\|_2$ associated with the perturbation enstrophy $\eta(t)$ is stronger than the norm $\|\psi\|_K = \sqrt{K} = 2^{-1/2}\|\nabla\psi\| = 2^{-1/2}\|\psi\|_1$ associated with the perturbation energy $K(t)$, we prove the theorem in the norm $\|\psi\|_\eta$.

We now consider the special case when two dipole modons (4.1) and (5.3) move along the same latitude circle $\mu_0 = a = b$ (Fig.2) and have the same parameters α, σ, μ_a, but different velocities ω_0 and $\widehat{\omega}_0$. Then, according to (4.7),

$$C - \widehat{C} = \left(1 - \frac{2}{\chi_\sigma}\right)(\omega_0 - \widehat{\omega}_0) \tag{11.2}$$

Further, by virtue of (4.2)-(4.6), the functions $R(\mu)$ and $\widehat{R}(\mu)$ of modons have the form (11.1) with the same function $T(\mu)$, and hence $R(\mu) - \widehat{R}(\mu) = (\omega_0 - \widehat{\omega}_0)F(\mu)$ where

$$F(\mu) = -\frac{2}{\chi_\sigma}T(\mu) + \sqrt{1-\mu_0^2}\sqrt{1-\mu^2} \tag{11.3}$$

We will use the first Liapunov method. According to [**13**], a solution $\widetilde{\psi}(t,x)$ is called Liapunov stable in the norm $\|\cdot\|$ if for any $\epsilon > 0$ and any initial moment t_0 there exists a number $\delta = \delta(\epsilon, t_0) > 0$ and such a time $t_1 \geq t_0$ that

$$\left\| \widehat{\psi}(t,x) - \widetilde{\psi}(t,x) \right\| < \epsilon \tag{11.4}$$

for all $t \geq t_1$ and for any solution $\widehat{\psi}(t,x)$ which satisfies at $t = t_0$ the inequality

$$\left\| \widehat{\psi}(t_0,x) - \widetilde{\psi}(t_0,x) \right\| < \delta \tag{11.5}$$

We now show that in the norm $\|\cdot\|_\eta$, the behavior of the distance $\psi = \widetilde{\psi} - \widehat{\psi}$ between modons (4.1) and (5.3) does not satisfy the definition of Liapunov stability. According to (A.1) and (A.6)-(A.8),

$$u = \cos\beta = a^2 + (1-a^2)\cos\gamma(t)$$

and $\rho = \vartheta = \gamma/2$. Note that if the poles N_1 and $N\prime$ of two modons coincide then $\rho = \vartheta = \gamma = 0$ and $u = 1$. In this case, due to (6.3), $h_n(1) = P_{n-1}^{(0,2)}(1) = 1$ (see Lemma 6.2).

Let $t_0 = 0$ be the initial moment, and let

$$\epsilon = \left\{ \frac{1}{2} \sum_{n=1}^{\infty} \left[w_n(1) - \left| w_n(a^2) \right| \right] R_n^2 \right\}^{1/2} \tag{11.6}$$

where $w_n(u)$ is determined by formula (6.2), the Fourier coefficient R_n of the zonal function (11.1) is determined by formula (4.9), and $\mu_0 = a$ is the latitude of the pole $N\prime$ (Fig.2). By virtue of (6.2)-(6.4),

$$w_n(1) = \chi_n \left\{ \frac{\chi_n}{2} + \frac{a^2}{1-a^2} \right\} > 0 \tag{11.7}$$

Taking into account formulas (6.2), (11.7), 6.3 and estimate $|P_n(u)| \leq 1$, we obtain

$$\left| w_n(a^2) \right| < w_n(1) \tag{11.8}$$

and therefore $\epsilon > 0$.

Let δ be an arbitrary small positive number. Let us show that for ϵ defined by formula (11.6) and any chosen δ, there always exists a solution $\widehat{\psi}(t) = \widehat{\psi}(t,\delta)$ of equation (3.1) such that condition (11.5) is satisfied, but it is impossible to find an instant t_1 such that inequality (11.4) holds or all $t \geq t_1$. This indicates that the dipole modon $\widetilde{\psi}(t,x)$ is Liapunov unstable.

As solutions $\widetilde{\psi}(t,x)$ and $\widehat{\psi}(t,\delta)$ we take modons (4.1) and (5.3), respectively. Recall that the parameters α, σ, μ_a and $\mu_0 = a$ of the two modons are the same, but

$$\omega_0 - \widehat{\omega}_0 = \delta\sqrt{2} / \left\{ \sum_{n=1}^{\infty} w_n(1) F_n^2 \right\}^{1/2} \tag{11.9}$$

where F_n is the coefficient of the Fourier series of the function (11.3). Since the functions $F(\mu)$ and $T(\mu)$ and all their derivatives up to the second order inclusive are continuous in the interval $[-1, 1]$, the series in formula (11.9) converges to a finite value (see the norm (2.7)). Moreover, since δ is small, it follows from (4.4), (4.7), (11.2) and (11.9) that the two modons have slightly different velocities (C and $\widehat{C}$) and amplitudes (A_0, A_i and $\widehat{A}_0$, $\widehat{A}_i$).

Suppose that at the initial moment $t_0 = 0$ the poles $N\prime$ and N_1 of two modons coincide, i.e. $\gamma(0) = \gamma_0 = 0$, $\rho(0) = \vartheta(0) = 0$ and $u(0) = 1$. Then, by (6.1) and (11.9),

$$\eta(0) = \frac{1}{2}\sum_{n=1}^{\infty} w_n(1)\left\{R_n - \widehat{R}_n\right\}^2 = \frac{1}{2}(\omega_0 - \widehat{\omega}_0)^2 \sum_{n=1}^{\infty} w_n(1)F_n^2 = \delta^2$$

i.e. $\left\|\widehat{\psi}(t_0, x) - \widetilde{\psi}(t_0, x)\right\|_\eta < \delta$, and hence condition (11.5) is satisfied. However, if $t = \tau_j$ where

$$\tau_j = \frac{(2j+1)\pi}{2(\omega_0 - \tilde{\omega}_0)(1 - 2/\chi_\sigma)}\ , \quad (j = 0, 1, 2, \dots)$$

then $\gamma(\tau_j) = \pi/2$ according to formula $\gamma(t) = \gamma_0 + (C - \widehat{C})t$ (see (5.4)) and (11.2). Consequently, $u = a^2$ by Lemma 6.1, and formulas (6.1) and (11.6) lead to

(11.10)

$$\eta(\tau_j) = \frac{1}{2}\sum_{n=1}^{\infty} w_n(1)\left\{R_n^2 + \widehat{R}_n^2\right\}^2 - \sum_{n=1}^{\infty} w_n(a^2)R_n\widehat{R}_n$$
$$\geq \frac{1}{2}\sum_{n=1}^{\infty}\left\{w_n(1) - \left|w_n(a^2)\right|\right\}\left\{R_n^2 + \widehat{R}_n^2\right\}^2 = \epsilon^2 + \frac{1}{2}\sum_{n=1}^{\infty}\left\{w_n(1) - \left|w_n(a^2)\right|\right\}\widehat{R}_n^2 .$$

In view of (11.10) and (11.8), $\eta(\tau_j) \geq \varepsilon^2$, and hence $\left\|\widehat{\psi}(\tau_j, x) - \widetilde{\psi}(\tau_j, x)\right\|_\eta > \epsilon$, i.e. inequality (11.4) fails at $t = \tau_j$ for all j. Since the sequence $\{\tau_j\}$ tends to ∞ as $j \to \infty$, it is impossible to find a moment t_1 that satisfies the definition of Liapunov stability. *Q.E.D.*

The described mechanism of the Liapunov instability of dipole modons has nothing to do with the orbital (Poincaré) instability [**13**] and coincides with the mechanism of instability of non-zonal RH waves [**22**] and periodic solutions of the nonlinear pendulum equation. This instability is caused by asynchronous oscillations of two dipole modons due to the difference in their velocities C and $\widehat{C}$ (see (11.2)). Indeed, if we consider in the phase space of solutions a thin tube (of small radius δ) around the orbit of the basic modon $\widetilde{\psi}$ then the orbit of the modon $\widehat{\psi}$ will always be inside this tube. It means that for any time t_1 there is a time t_2 such that $\left\|\widehat{\psi}(t_1, x) - \widetilde{\psi}(t_2, x)\right\|_1 < \delta$.

12. Conclusions

The motion of a nondivergent ideal fluid on a rotating sphere is considered. The linear and nonlinear stability of the infinitely differentiable BVE solutions (Rossby-Haurwitz waves and Legendre polynomial flows) were analyzed in [**22**]. This work analyzes the stability of modons and WV waves. The conservation law for infinitesimal perturbations of these solutions is derived, which is subsequently used to obtain the necessary conditions for their normal mode instability. The new conditions impose restrictions on the spectral distribution of the energy of unstable modes. The maximum growth rate of unstable modes is estimated, and the orthogonality of the amplitude of any unstable mode to the basic solution is shown in the inner product associated with kinetic energy. The results obtained are useful for testing the computational algorithms and software packages developed for the numerical study of linear stability. Note that both the instability conditions for the BVE solutions and the estimates of the maximum growth rate of unstable modes use the average spectral Fjörtoft number [**4, 22**]. Thus, this parameter is of paramount importance when studying the linear stability of ideal flows on a sphere. The Liapunov instability of dipole modons is proved, and the mechanism of instability is explained.

Appendix A. Euler angles

As mentioned in Section 5, in order to bring the system (x_1, y_1, z_1) into coincidence with the system $(x\prime, y\prime, z\prime)$ (see Fig.1) it is necessary to perform three successive rotations by the Euler angles ρ , $-\beta$ and ϑ. The first rotation of the system (x_1, y_1, z_1) is performed around the Oz_1 axis by the angle ρ. After this rotation, the Oy_1 axis will coincide with the Oy_0 axis. The second rotation is carried out around the Oy_0 axis by the negative angle $-\beta$ until the Oz_1 and $Oz\prime$ axes coincide. The last rotation is performed around the $Oz\prime$ axis by the angle ϑ. Note that two rotations defined by matrices $D(\rho, -\beta, \vartheta)$ and $D(\pi+\rho, \beta, \pi+\vartheta)$ are equivalent [**16**].

We now show how to find the angles ρ, β and ϑ if the values a, b and γ are known. Since the angles N_1OB and $BON\prime$ are direct, the cosine theorem [**1**] being applied successively to the spherical triangles N_1NN', N_1NB and BNN' gives

$$u = u(t) \equiv \cos\beta = \mathrm{ab} + \sqrt{1-a^2}\sqrt{1-b^2}\cos\gamma \tag{A.1}$$

$$0 = a\mu_B + \sqrt{1-a^2}\sqrt{1-\mu_B^2}\cos\gamma_1 \tag{A.2}$$

$$0 = b\mu_B + \sqrt{1-b^2}\sqrt{1-\mu_B^2}\cos(\gamma-\gamma_1) \tag{A.3}$$

where γ_1 is the longitudinal angle between the points A and B, and μ_B is the μ-coordinate of point B. It follows from (A.2) and (A.3) that

$$\tan\gamma_1 = (b\sqrt{1-a^2} - a\sqrt{1-b^2}\cos\gamma)/(a\sqrt{1-b^2}\sin\gamma) \tag{A.4}$$

(A.5) $$\mu_B = -\sqrt{1-a^2}\cos\gamma_1/\sqrt{a^2+(1-a^2)\cos^2\gamma_1}\ .$$

The cosine theorem used for the spherical triangle ANB and formula (A.2) give

(A.6) $$\cos\rho = -\mu_B/\sqrt{1-a^2}$$

Relating the coordinates of the point B in the systems (λ,μ) and (λ_1,μ_1) using the sine theorem [**1**] we obtain

(A.7) $$\sin\rho = \sqrt{1-\mu_B^2}\sin\gamma_1$$

The angle ρ is uniquely defined by equations (A.6) and (A.7). A similar analysis for the spherical triangle $BN\prime C$ gives

(A.8) $$\cos\vartheta = -\mu_B/\sqrt{1-b^2} \quad \text{and} \quad \sin\vartheta = \sqrt{1-\mu_B^2}\sin(\gamma-\gamma_1)$$

Since β is in interval $[0,\pi]$, the angle γ_1 is uniquely determined by formulas (A.4) and (A.5).

Appendix B. Proof of Lemma 4.1

Indeed, using the recurrence formulas

$$Q_n^1(\mu) = \chi_n^{-1/2}\sqrt{1-\mu^2}\frac{d}{d\mu}Q_n^0(\mu) \quad \text{and} \quad P_\alpha^1(\mu) = \sqrt{1-\mu^2}\frac{d}{d\mu}P_\alpha^0(\mu)$$

for the associated Legendre functions, as well as equations (4.9) we obtain

(B.1) $$R_n = -\chi_n^{-1/2}\int_{-1}^{1}\frac{d}{d\mu}\left\{\sqrt{1-\mu^2}R(\mu)\right\}Q_n^0(\mu)d\mu$$

and

(B.2) $$\frac{d}{d\mu}\left\{\sqrt{1-\mu^2}P_\alpha^1(\mu)\right\} = \Delta P_\alpha^0(\mu) = -\chi_\alpha P_\alpha^0(\mu)$$

It is easy to see from (4.2)-(4.4) and (B.2) that

(B.3) $$\mu_0\,(1-\mu_0^2)^{-1/2}\,\frac{d}{d\mu}\left\{\sqrt{1-\mu^2}R(\mu)\right\} = -\Delta G(\mu)$$

Substituting (B.3) in (B.1) we obtain

$$\begin{aligned}\chi_n^{-1/2}\mu_0\,(1-\mu_0^2)^{-1/2}\,R_n &= \chi_n^{-1}\int_{-1}^{1}\Delta G(\mu)Q_n^0(\mu)d\mu \\ &= \chi_n^{-1}\int_{-1}^{1}G(\mu)\Delta Q_n^0(\mu)d\mu = -G_n\ . \quad \text{Q.E.D.}\end{aligned}$$

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