Dynamics of PDE, Vol.21, No.3, 235-260, 2024

On the Cauchy problem for a Kadomtsev-Petviashvili hierarchy on non-formal operators and its relation with a group of diffeomorphisms

Jean-Pierre Magnot and Enrique G. Reyes

Communicated by Boris Khesin, received April 26, 2021.

Abstract. We establish a rigorous link between infinite-dimensional regular Frölicher Lie groups built out of non-formal pseudodifferential operators and the Kadomtsev-Petviashvili hierarchy. We introduce a (parameter-depending) version of the Kadomtsev-Petviashvili hierarchy on a regular Frölicher Lie group of series of non-formal odd-class pseudodifferential operators. We solve its corresponding Cauchy problem, and we establish a link between the dressing operator of our hierarchy and the action of diffeomorphisms and non-formal Sato-like operators on jet spaces. In appendix, we describe the group of Fourier integral operators in which this correspondence seems to take place. Also, motivated by Mulase's work on the KP hierarchy, we prove a group factorization theorem for this group of Fourier integral operators.

Contents

2020 *Mathematics Subject Classification.* 35Q51; 37K10; 37K25; 37K30; 58J40. Secondary: 58B25; 47N20.

Key words and phrases. Kadomtsev-Petviashvili hierarchy, Mulase factorization, infinite jets, Fréchet Lie groups, Frölicher Lie groups, Fourier integral operators, odd-class pseudodifferential operators.

1. Introduction

The Kadomtsev-Petviashvili hierarchy (KP hierarchy, for short) is a system of nonlinear differential equations on an infinite number of dependent variables, each of which depends on infinitely many independent variables. It reads as follows:

$$\frac{\partial}{\partial t_n} L = [(L^n)_+, L] = (L^n)_+ \cdot L - L \cdot (L^n)_+ \,, \qquad n = 1, 2, 3, \cdots \tag{1.1}$$

in which

$$L = \frac{\partial}{\partial x} + u_1 \left(\frac{\partial}{\partial x} \right)^{-1} + u_2 \left(\frac{\partial}{\partial x} \right)^{-2} + \cdots \,,$$

u_1, u_2, $\cdots$ are dependent variables, $(L^n)_+$ indicates the projection of the product $L^n = L \ldots L$ on the space of differential operators, and t_1, t_2, $\cdots$ denote independent variables. An object such as L above is a *formal pseudodifferential operator*. It is known that the set of formal pseudodifferential operators can be equipped with an associative algebra structure, see [**8**], and therefore (1.1) makes sense, at least, in an algebraic context. The reader is referred to [**8**, Chapters 1, 5] for a thorough algebraic discussion of KP and other important hierarchies.

The KP hierarchy is related to several soliton equations: for example, it contains the Korteweg-de Vries hierarchy and more generally the Gelfand-Dickey hierarchies, see [**8**]. Moreover, it is *universal*. In Mulase's words, "the KP system is the master equation for the largest possible family of iso-spectral deformations of arbitrary ordinary differential operators", see [**37**, Section 3] and [**38**]; see also [**8**, Corollary 6.2.8] for another expression of this universality. Solutions to KP can be recovered from quantum field theory and algebraic geometry among other fields, see for instance [**20, 35, 37**] and references therein, and (1.1) can be posed for instance in contact geometry, see [**34**].

Can we solve Equation (1.1), in the *Differential Equations* sense of understanding its associated Cauchy problem? Yes. In the 1980's Mulase published several fundamental papers on the algebraic structure and formal integrability properties of the KP hierarchy, see [**36, 37, 38**]. A common theme in these papers was the use of a powerful algebraic theorem on the factorization of a group of formal pseudodifferential operators of infinite order which integrates the algebra of formal pseudodifferential operators: this factorization —a delicate algebraic generalization of the Birkhoff decomposition of loop groups appearing for example in [**45**]— allowed him to *solve* the Cauchy problem for the KP hierarchy in an algebraic setting, see [**38**, Theorem 1.4]. A review of this theorem is in [**12**]. We have re-interpreted Mulase's results and extended them in the context of (generalized) differential geometry on diffeological and Frölicher spaces, and we have used this re-interpretation to prove well-posedness of the KP hierarchy in analytic categories, see [**13, 29, 33**] and our recent review [**32**].

It is important to point out that in the above mentioned papers the operators under consideration are *formal* pseudodifferential operators: they are not understood as operators acting on smooth maps or smooth sections of vector bundles. They differ from non-formal pseudodifferential operators by (unknown) smooth kernel operators, the so-called smoothing operators. As is well-known, any classical non-formal pseudodifferential operator A generates a formal operator (the one obtained from the asymptotic expansion of the symbol of A, see [**1, 2, 15**]), but there is no canonical way to recover a non-formal operator from a formal one.

Can we introduce and discuss a version of the KP hierarchy (1.1) using classical non-formal pseudo-differential operators? Yes. *The aim of this paper is to show that a version of Equation* (1.1) *can indeed be posed and solved on regular Frölicher Lie groups built with the help of a particular class of non-formal pseudo-differential operators.* Our first motivation for considering this problem comes from the following observation: pushing forward equations onto a quotient of a relation of equivalence is easy and unambiguous (up to compatibility conditions), while pulling-back equations from a quotient space to full space can be often performed in very many ways. As explained in the previous paragraph, the KP hierarchy can be understood as being posed on a quotient space of classical pseudodifferential operators, and so it is very natural to aim at proposing a version of (1.1) using the pseudodifferential operators themselves. Our second motivation for considering non-formal pseudo-differential operators comes from our previous work [**33**]. In this reference we use versions of "dressing operators" for equation (1.1), and we obtain solutions to KP with the help of an operator which acts on initial conditions (see [**33**, Section 4]). It is natural to wonder if it is possible to understand these operators in a non-formal setting.

What class of pseudodifferential operators can we use, in order to write down an equation such as (1.1)? *We work with odd-class classical pseudodifferential operators which act on smooth sections of a given trivial (finite rank) vector bundle* $S^1 \times V$. These pseudodifferential operators were first considered by Kontsevich and Vishik in [**21, 22**] in order to deal with spectral functions and renormalized determinants. We use them in two ways:

- We take them as building blocks for our non-formal KP hierarchy. One reason why odd-class pseudodifferential operators are natural to use in this context is the fact that differential operators are all odd-class, and so we can indeed hope to pose Equation (1.1) with their help.
- We build a central extension of $Diff_+(S^1)$ by a group of bounded odd-class classical pseudodifferential operators, in which $Diff_+(S^1)$ is the group of orientation-preserving diffeomorphisms of S^1. We present this construction because the structure of this central extension allows us to prove rather easily a Mulase-type factorization theorem in our non-formal context, an observation we think is interesting of its own[1].

The novelty of this paper, especially compared with our previous works [**29**] and [**32, 33**], is then two-fold: first, we generalize the Kadomtsev-Petviashvili hierarchy by considering non-formal pseudo-differential operators: to the best of our knowledge this is the first time that an integrable hierarchy in such a setting is formulated, studied, and proven to be well-posed; and second, we establish a link between the dressing operator of the hierarchy and the central extension of $Diff_+(S^1)$: such a link was unknown till the present work.

We organize our work as follows. Section 1 is this introduction. Section 2 is a short review on Frölicher Lie groups, mostly inspired by [**32, 33**]. In this

[1]Elements of this central extension are Fourier integral operators called $Diff_+(S^1)-$pseudodifferential operators. To the best of our knowledge, groups of $Diff(S^1)-$pseudodifferential operators were independently described (with S^1 replaced by a compact Riemannian manifold M) in [**30**], in the context of differential geometry of non-parametrized, non-linear grassmannians, and in [**43**] as a possible structure group on which Chern-Weil constructions could be performed.

paper we consider several infinite-dimensional groups built with the help of non-formal pseudodifferential operators. Some of these groups are beyond the reach of traditional analytic means, but they do possess Frölicher structures, and so it is natural to begin with a review of the Frölicher setting. Section 3 is on Frölicher Lie groups of pseudodifferential operators, following mainly [**30**]. References for the analytic tools used therein are [**3, 15, 46**]. Then, in Section 4 we propose our version of KP hierarchy: we consider the Lie algebra $Cl_{h,odd}(S^1, V)$ of formal power series in a parameter h whose coefficients are classical odd-class pseudodifferential operators satisfying some technical conditions. These conditions allow us to find a *regular* (a notion explained in Section 2) Frölicher Lie group which integrates the Lie algebra $Cl_{h,odd}(S^1, V)$. In this extended context we can pose and solve the Cauchy problem for our version of KP. In Section 4 we also highlight a non-formal operator $U_h \in Cl_{h,odd}(S^1, V)$ which depends on the initial condition of our KP hierarchy; this operator generates its solutions very much in the spirit of the standard theory of R-matrices, see [**44**], and also [**12, 33**] and references therein. In Section 5 we show how to recover the operator U_h by analysing the Taylor expansion of functions in the image of the twisted operator $A : f \in C^\infty(S^1; V) \mapsto S_0^{-1}(f) \circ g$, in which $g \in Diff_+(S^1)$ and S_0 is our version of a "dressing operator" as considered for example in [**8**, Chapter 6]. Finally, we include an appendix in which we introduce a group of Fourier integral operators, the central extension of $Diff_+(S^1)$ by the group $Cl^{0,*}_{odd}(S^1, V)$ of all odd-class, invertible and bounded, classical pseudodifferential operators. As mentioned above, working with this central extension we can prove a non-formal analogue of the Mulase decomposition of [**36, 37, 38**].

2. Preliminaries on categories of regular Frölicher Lie groups

In this section we recall briefly the formal setting which allows us to work rigorously with Lie groups built with the help of pseudodifferential operators. No new statements are given here: we follow the expositions appearing in [**29, 32, 33**]. We begin with the notion of a diffeological space:

Definition 2.1. Let X be a set.
- A **p-parametrization** of dimension p on X is a map from an open subset O of $\mathbb{R}^p$ to X.
- A **diffeology** on X is a set $\mathcal{P}$ of parametrizations on X such that:
  - For each $p \in \mathbb{N}$, any constant map $\mathbb{R}^p \to X$ is in $\mathcal{P}$;
  - For each arbitrary set of indexes I and family $\{f_i : O_i \to X\}_{i\in I}$ of compatible maps that extend to a map $f : \bigcup_{i\in I} O_i \to X$, if $\{f_i : O_i \to X\}_{i\in I} \subset \mathcal{P}$, then $f \in \mathcal{P}$.
  - For each $f \in \mathcal{P}$, $f : O \subset \mathbb{R}^p \to X$, and $g : O' \subset \mathbb{R}^q \to O$, in which g is a smooth map (in the usual sense) from an open set $O' \subset \mathbb{R}^q$ to O, we have $f \circ g \in \mathcal{P}$.

If $\mathcal{P}$ is a diffeology on X, then $(X, \mathcal{P})$ is called a **diffeological space** and, if $(X, \mathcal{P})$ and $(X', \mathcal{P}')$ are two diffeological spaces, a map $f : X \to X'$ is **smooth** if and only if $f \circ \mathcal{P} \subset \mathcal{P}'$.

The notion of a diffeological space is due to J.M. Souriau, see [**47**]; see also [**6**] for related constructions, and [**19, 48**] for a contemporary point of view. Of particular interest to us is the following subcategory of the category of diffeological spaces.

Definition 2.2. A **Frölicher** space is a triple $(X, \mathcal{F}, \mathcal{C})$ such that

- $\mathcal{C}$ is a set of paths $\mathbb{R} \to X$,
- $\mathcal{F}$ is the set of functions from X to $\mathbb{R}$, such that a function $f : X \to \mathbb{R}$ is in $\mathcal{F}$ if and only if for any $c \in \mathcal{C}$, $f \circ c \in C^\infty(\mathbb{R}, \mathbb{R})$;
- A path $c : \mathbb{R} \to X$ is in $\mathcal{C}$ (i.e. is a **contour**) if and only if for any $f \in \mathcal{F}$, $f \circ c \in C^\infty(\mathbb{R}, \mathbb{R})$.

If $(X, \mathcal{F}, \mathcal{C})$ and $(X', \mathcal{F}', \mathcal{C}')$ are two Frölicher spaces, a map $f : X \to X'$ is **smooth** if and only if $\mathcal{F}' \circ f \circ \mathcal{C} \subset C^\infty(\mathbb{R}, \mathbb{R})$.

This definition first appeared in [**14**]; we use terminology borrowed from Kriegl and Michor's book [**23**]. A short comparison of the notions of diffeological and Frölicher spaces is in [**28**]; the reader can also see [**29, 31, 33, 50**] for extended expositions. In particular, it is explained in [**33**] that *Frölicher and Gateaux smoothness are the same notion if we restrict ourselves to a Fréchet context.*

Any family of maps $\mathcal{F}_g$ from X to $\mathbb{R}$ generates a Frölicher structure $(X, \mathcal{F}, \mathcal{C})$ by setting, after [**23**]:
- $\mathcal{C} = \{c : \mathbb{R} \to X$ such that $\mathcal{F}_g \circ c \subset C^\infty(\mathbb{R}, \mathbb{R})\}$
- $\mathcal{F} = \{f : X \to \mathbb{R}$ such that $f \circ \mathcal{C} \subset C^\infty(\mathbb{R}, \mathbb{R})\}$.

We call $\mathcal{F}_g$ a **generating set of functions** for the Frölicher structure $(X, \mathcal{F}, \mathcal{C})$. One easily see that $\mathcal{F}_g \subset \mathcal{F}$. A Frölicher space $(X, \mathcal{F}, \mathcal{C})$ carries a natural topology, the pull-back topology of $\mathbb{R}$ via $\mathcal{F}$. In the case of a finite dimensional differentiable manifold X we can take $\mathcal{F}$ as the set of all smooth maps from X to $\mathbb{R}$, and $\mathcal{C}$ the set of all smooth paths from $\mathbb{R}$ to X. Then, the underlying topology of the Frölicher structure is the same as the manifold topology [**23**].

We also remark that if $(X, \mathcal{F}, \mathcal{C})$ is a Frölicher space, we can define a natural diffeology on X by using the following family of maps f defined on open domains $D(f)$ of Euclidean spaces, see [**28**]:

$$\mathcal{P}_\infty(\mathcal{F}) = \coprod_{p \in \mathbb{N}} \{ f : D(f) \to X;\ \mathcal{F} \circ f \in C^\infty(D(f), \mathbb{R}) \quad \text{(in the usual sense)}\}.$$

If X is a finite-dimensional differentiable manifold, this diffeology is called the *nébuleuse diffeology*, see [**47**]. Now, we can easily show the following:

Proposition 2.3. [**28**] *Let $(X, \mathcal{F}, \mathcal{C})$ and $(X', \mathcal{F}', \mathcal{C}')$ be two Frölicher spaces. A map $f : X \to X'$ is smooth in the sense of Frölicher if and only if it is smooth for the underlying diffeologies $\mathcal{P}_\infty(\mathcal{F})$ and $\mathcal{P}_\infty(\mathcal{F}')$.*

Thus, Proposition 2.3 and the foregoing remarks imply that the following implications hold:

$$\text{smooth manifold} \quad \Rightarrow \quad \text{Frölicher space} \quad \Rightarrow \quad \text{diffeological space}$$

These implications can be refined. The reader is referred to the Ph.D. thesis [**50**] for a deeper analysis of them.

Remark 2.4. The set of contours $\mathcal{C}$ of the Frölicher space $(X, \mathcal{F}, \mathcal{C})$ does not give us a diffeology, because a diffeology needs to be stable under restriction of domains. In the case of paths in $\mathcal{C}$ the domain is always $\mathbb{R}$ whereas the domain of 1-plots can (and has to) be any interval of $\mathbb{R}$. However, $\mathcal{C}$ defines a "minimal diffeology" $\mathcal{P}_1(\mathcal{F})$ whose plots are smooth parametrizations which are locally of the type $c \circ g$, in which $g \in \mathcal{P}_\infty(\mathbb{R})$ and $c \in \mathcal{C}$. Within this setting, we can replace $\mathcal{P}_\infty$

by $\mathcal{P}_1$ in Proposition 2.3. The main technical tool needed to discuss this issue is Boman's theorem [**23**, p.26]. Related discussions are in [**28, 50**].

PROPOSITION 2.5. *Let $(X, \mathcal{P})$ and $(X', \mathcal{P}')$ be two diffeological spaces. There exists a diffeology $\mathcal{P} \times \mathcal{P}'$ on $X \times X'$ made of plots $g : O \to X \times X'$ that decompose as $g = f \times f'$, where $f : O \to X \in \mathcal{P}$ and $f' : O \to X' \in \mathcal{P}'$. We call it the* ***product diffeology****, and this construction extends to an infinite product.*

We apply this result to the case of Frölicher spaces and we derive (compare with [**23**]) the following:

PROPOSITION 2.6. *Let $(X, \mathcal{F}, \mathcal{C})$ and $(X', \mathcal{F}', \mathcal{C}')$ be two Frölicher spaces equipped with their natural diffeologies $\mathcal{P}$ and $\mathcal{P}'$. There is a natural structure of Frölicher space on $X \times X'$ which contours $\mathcal{C} \times \mathcal{C}'$ are the 1-plots of $\mathcal{P} \times \mathcal{P}'$.*

We can also state the above result for infinite products; we simply take Cartesian products of the plots, or of the contours.

Now we consider quotients after [**47**] and [**19**, p. 27]: Let $(X, \mathcal{P})$ be a diffeological space, and let X' be a set. Let $f : X \to X'$ be a map. We define the **push-forward diffeology** as the coarsest (i.e. the smallest for inclusion) among the diffologies on X', which contains $f \circ \mathcal{P}$.

PROPOSITION 2.7. *Let $(X, \mathcal{P})$ b a diffeological space and $\mathcal{R}$ an equivalence relation on X. Then, there is a natural diffeology on $X/\mathcal{R}$, noted by $\mathcal{P}/\mathcal{R}$, defined as the push-forward diffeology on $X/\mathcal{R}$ by the quotient projection $X \to X/\mathcal{R}$.*

Given a subset $X_0 \subset X$, where X is a Frölicher space or a diffeological space, we equip X_0 with structures induced by X as follows:

(1) If X is equipped with a diffeology $\mathcal{P}$, we define a diffeology $\mathcal{P}_0$ on X_0 called the **subset or trace diffeology**, see [**47, 19**], by setting
$$\mathcal{P}_0 = \{p \in \mathcal{P} \text{ such that the image of } p \text{ is a subset of } X_0\} .$$
(2) If $(X, \mathcal{F}, \mathcal{C})$ is a Frölicher space, we take as a generating set of maps $\mathcal{F}_g$ on X_0 the restrictions of the maps $f \in \mathcal{F}$. In this case, the contours (resp. the induced diffeology) on X_0 are the contours (resp. the plots) on X whose images are a subset of X_0.

Our last general construction is the so-called functional diffeology. Its existence implies the following crucial fact: the category of diffeological spaces is Cartesian closed, something which is certainly not true in the category of smooth manifolds. Our discussion follows [**19**]. Let $(X, \mathcal{P})$ and $(X', \mathcal{P}')$ be diffeological spaces. Let $M \subset C^\infty(X, X')$ be a set of smooth maps. The **functional diffeology** on S is the diffeology $\mathcal{P}_S$ made of plots
$$\rho : D(\rho) \subset \mathbb{R}^k \to S$$
such that, for each $p \in \mathcal{P}$, the maps $\Phi_{\rho,p} : (x, y) \in D(p) \times D(\rho) \mapsto \rho(y)(x) \in X'$ are plots of $\mathcal{P}'$. We have, see [**19**, Paragraph 1.60]:

PROPOSITION 2.8. *Let X, Y, Z be diffeological spaces. Then,*
$$C^\infty(X \times Y, Z) = C^\infty(X, C^\infty(Y, Z)) = C^\infty(Y, C^\infty(X, Z))$$
as diffeological spaces equipped with functional diffeologies.

Now, given an algebraic structure, we can define a corresponding compatible diffeological (resp. Frölicher) structure, see for instance [**25**]. For example, see [**19**, pp. 66-68], if $\mathbb{R}$ is equipped with its canonical diffeology (resp. Frölicher structure), we say that an $\mathbb{R}-$vector space equipped with a diffeology (resp. Frölicher structure) is a diffeological (resp. Frölicher) vector space if addition and scalar multiplication are smooth. We state:

Definition 2.9. Let G be a group equipped with a diffeology (Frölicher structure). We call it a **diffeological (Frölicher) group** if both multiplication and inversion are smooth.

Since we are interested in infinite-dimensional analogues of Lie groups, we need to consider tangent spaces of diffeological spaces, and we have to deal with Lie algebras and exponential maps. We state, after [**10**] and [**7**] the following definition:

Definition 2.10. (i) For each $x \in X$, we consider

$$C_x = \{c \in C^\infty(\mathbb{R}, X) | c(0) = x\}$$

and we take the equivalence relation $\mathcal{R}$ given by

$$c\mathcal{R}c' \Leftrightarrow \forall f \in C^\infty(X, \mathbb{R}), \partial_t(f \circ c)|_{t=0} = \partial_t(f \circ c')|_{t=0}.$$

Equivalence classes of $\mathcal{R}$ are called **germs** and are denoted by $\partial_t c(0)$ or $\partial_t c(t)|_{t=0}$. The **internal tangent cone** at x is the quotient ${}^iT_xX = C_x/\mathcal{R}$. If $X = \partial_t c(t)|_{t=0} \in {}^iT_X$, we define the derivation $Df(X) = \partial_t(f \circ c)|_{t=0}$.

(ii) The **internal tangent space** at $x \in X$ is the vector space generated by the internal tangent cone.

The reader may compare this definition to the one appearing in [**23**] for manifolds in the "convenient" $c^\infty-$setting. The internal tangent cone at a point x is not a vector space in many examples; this motivates item (ii) above, see [**7, 10**]. Fortunately, the internal tangent cone at $x \in X$ *is* a vector space for the objects under consideration in this work, see Proposition 2.11 below, see also [**24**]; it will be called, simply, the tangent space at $x \in X$.

Following Iglesias-Zemmour, see [**19**], we do not assert that arbitrary diffeological groups have associated Lie algebras; however, the following holds, see [**25**, Proposition 1.6.] and [**33**, Proposition 2.20].

Proposition 2.11. *Let G be a diffeological group. Then the tangent cone at the neutral element T_eG is a diffeological vector space.*

The proof of Proposition 2.11 appearing in [**33**] uses explicitly the diffeologies $\mathcal{P}_1$ and $\mathcal{P}_\infty$ which appear in Proposition 3 and Remark 4 of this work.

Definition 2.12. The diffeological group G is a **diffeological Lie group** if and only if the Adjoint action of G on the diffeological vector space ${}^iT_eG$ defines a Lie bracket. In this case, we call ${}^iT_eG$ the Lie algebra of G and we denote it by $\mathfrak{g}$.

Let us concentrate on Frölicher Lie groups, following [**29**] and [**25**]. If G is a Frölicher Lie group then, after (i) and (ii) above we have that:

$$\mathfrak{g} = \{\partial_t c(0); c \in \mathcal{C} \text{ and } c(0) = e_G\}$$

is the space of germs of paths at e_G. Moreover:

- Let $(X,Y) \in \mathfrak{g}^2$, $X+Y = \partial_t(c.d)(0)$ where $c,d \in \mathcal{C}^2$, $c(0)=d(0)=e_G$, $X = \partial_t c(0)$ and $Y = \partial_t d(0)$.
- Let $(X,g) \in \mathfrak{g} \times G$, $Ad_g(X) = \partial_t(gcg^{-1})(0)$ where $c \in \mathcal{C}$, $c(0)=e_G$, and $X = \partial_t c(0)$.
- Let $(X,Y) \in \mathfrak{g}^2$, $[X,Y] = \partial_t(Ad_{c(t)}Y)$ where $c \in \mathcal{C}$, $c(0)=e_G$, $X = \partial_t c(0)$.

All these operations are smooth and thus well-defined as operations on Frölicher spaces, see [**25, 29, 31, 33**].

The basic properties of adjoint, coadjoint actions, and of Lie brackets, remain globally the same as in the case of finite-dimensional Lie groups, and the proofs are similar: see [**25**] and [**10**] for details.

Definition 2.13. [**25**] A Frölicher Lie group G with Lie algebra $\mathfrak{g}$ is called **regular** if and only if there is a smooth map

$$Exp : C^\infty([0;1],\mathfrak{g}) \to C^\infty([0,1],G)$$

such that $g(t) = Exp(v(t))$ is the unique solution of the differential equation

$$\left\{ \begin{array}{l} g(0) = e \\ \frac{dg(t)}{dt} g(t)^{-1} = v(t) \end{array} \right. \tag{2.1}$$

We define the exponential function as follows:

$$\begin{array}{rcl} exp : \mathfrak{g} & \to & G \\ v & \mapsto & exp(v) = g(1)\ , \end{array}$$

where g is the image by Exp of the constant path v.

When the Lie group G is a vector space V, the notion of regular Lie group specialize to what is called *regular vector space* in [**29**] and *integral vector space* in [**25**]; we follow the latter terminology.

Definition 2.14. [**25**] Let $(V,\mathcal{P})$ be a Frölicher vector space. The space $(V,\mathcal{P})$ is **integral** if there is a smooth map

$$\int_0^{(.)} : C^\infty([0;1];V) \to C^\infty([0;1],V)$$

such that $\int_0^{(.)} v = u$ if and only if u is the unique solution of the differential equation

$$\left\{ \begin{array}{l} u(0) = 0 \\ u'(t) = v(t) \end{array} \right. .$$

This definition applies, for instance, if V is a complete locally convex topological vector space equipped with its natural Frölicher structure given by the Frölicher completion of its nébuleuse diffeology, see [**19, 28, 29**].

Definition 2.15. Let G be a Frölicher Lie group with Lie algebra $\mathfrak{g}$. Then, G is regular with integral Lie algebra if $\mathfrak{g}$ is integral and G is regular in the sense of Definitions 2.13 and 2.14.

We finish this section with two structural results essentially proven in [**29**]. The first one is used in the construction of regular Lie groups of non-formal pseudodifferential and Fourier operators, see [**29, 30**] and Section 3 below, while the second one provides us with examples of Frölicher Lie groups. Applications of the latter theorem appear in the analysis of the Cauchy problem for the Kadomtsev-Petviashvili carried out in [**29, 33**] and in Section 4 below.

Theorem 2.16. *Let*

$$1 \longrightarrow K \xrightarrow{i} G \xrightarrow{p} H \longrightarrow 1$$

be an exact sequence of Frölicher Lie groups, such that there is a smooth section $s : H \to G$, and such that the trace diffeology on $i(K) \subseteq G$ coincides with the push-forward diffeology from K to $i(K)$. We consider also the corresponding sequence of Lie algebras

$$0 \longrightarrow \mathfrak{k} \xrightarrow{i'} \mathfrak{g} \xrightarrow{p} \mathfrak{h} \longrightarrow 0 \,.$$

Then,

- *The Lie algebras $\mathfrak{k}$ and $\mathfrak{h}$ are integral if and only if the Lie algebra $\mathfrak{g}$ is integral;*
- *The Frölicher Lie groups K and H are regular if and only if the Frölicher Lie group G is regular.*

Theorem 2.17. *Let $(A_n)_{n\in\mathbb{N}}$ be a sequence of integral Frölicher vector spaces equipped with a graded smooth multiplication operation on $\bigoplus_{n\in\mathbb{N}^*} A_n$, i.e. a multiplication such that for each $n, m \in \mathbb{N}^*$, $A_n.A_m \subset A_{n+m}$ is smooth with respect to the corresponding Frölicher structures.*

- *Let us define the (non unital) algebra of formal series:*

  $$\mathcal{A} = \left\{ \sum_{n\in\mathbb{N}^*} a_n | \forall n \in \mathbb{N}^*, a_n \in A_n \right\} \,,$$

  equipped with the Frölicher structure of the infinite product. Then, the space

  $$1 + \mathcal{A} = \left\{ 1 + \sum_{n\in\mathbb{N}^*} a_n | \forall n \in \mathbb{N}^*, a_n \in A_n \right\}$$

  is a regular Frölicher Lie group with integral Frölicher Lie algebra $\mathcal{A}$. Moreover, the exponential map defines a smooth bijection $\mathcal{A} \to 1 + \mathcal{A}$.
- *Let A_0^* be the Frölicher group of invertible elements of A_0. Then, the space*

  $$A_0^* + \mathcal{A} = \left\{ \sum_{n\in\mathbb{N}} a_n | a_0 \in A_0^* \text{ and } \forall n \in \mathbb{N}^*, a_n \in A_n \right\}$$

  is a regular Frölicher Lie group with integral Frölicher Lie algebra.

A result similar to Theorem 2.16 is also valid for Fréchet Lie groups, see [**23**].

3. Preliminaries on pseudodifferential operators

We introduce the groups and algebras of non-formal pseudodifferential operators needed to set up our version of the KP hierarchy. Basic definitions are valid for a real or complex finite-dimensional vector bundle E over S^1; below (see paragraph "Notations") we specialize our considerations to the case $E = S^1 \times V$ in which V is a finite-dimensional complex vector space. The following definition appears in [**3**, Section 2.1].

Definition 3.1. The graded algebra of differential operators acting on the space of smooth sections $C^\infty(S^1, E)$ is the algebra $DO(E)$ generated by:

• Elements of $End(E)$, the group of smooth maps $E \to E$ leaving each fibre globally invariant and which restrict to linear maps on each fibre. This group acts on sections of E via (matrix) multiplication;

• The differentiation operators

$$\nabla_X : g \in C^\infty(S^1, E) \mapsto \nabla_X g$$

where ∇ is a connection on E and X is a vector field on S^1.

Multiplication operators are operators of order 0; differentiation operators and vector fields are operators of order 1. In local coordinates, a differential operator of order k has the form $P(u)(x) = \sum p_{i_1 \cdots i_r} \nabla_{x_{i_1}} \cdots \nabla_{x_{i_r}} u(x)$, $\quad r \leq k$, in which the coefficients $p_{i_1 \cdots i_r}$ can be matrix-valued. We note by $DO^k(S^1)$,$k \geq 0$, the differential operators of order less or equal than k.

The algebra $DO(E)$ is graded by order. It is a subalgebra of the algebra of classical pseudodifferential operators$Cl(S^1, E)$, an algebra that contains, for example, the square root of the Laplacian, its inverse, and all trace-class operators on $L^2(S^1, E)$. Basic facts on pseudodifferential operators defined on a vector bundle $E \to S^1$ can be found for instance in [**15**] and in the review [**43**]. A global symbolic calculus for pseudodifferential operators has been defined independently by J. Bokobza-Haggiag, see [**4**] and H. Widom, see [**51**]. In these papers is shown how the geometry of the base manifold M furnishes an obstruction to generalizing local formulas of composition and inversion of symbols; we do not recall these formulas here since they are not involved in our computations.

Following [**26**, Section 1], see also [**30**], we assume henceforth that S^1 is equipped with charts such that the changes of coordinates are translations. We also restrict our considerations to complex vector bundles over S^1. It is well-known that they are trivial, i.e. $E = S^1 \times V$. Taking this fact into account, we use the following notational conventions:

Notations. We note by $PDO(S^1, V)$ (resp. $PDO^o(S^1, V)$, resp. $Cl(S^1, V)$) the space of pseudodifferential operators (resp. pseudodifferential operators of order o, resp. classical pseudodifferential operators) acting on smooth sections of E, and by $Cl^o(S^1, V) = PDO^o(S^1, V) \cap Cl(S^1, V)$ the space of classical pseudodifferential operators of order o. We also denote by $Cl^{o,*}(S^1, V)$ the group of units of $Cl^o(S^1, V)$.

A topology on spaces of classical pseudo differential operators has been described by Kontsevich and Vishik in [**21**]; see also [**5, 42, 46**] for other descriptions. We use all along this work the Kontsevich-Vishik topology. This is a Fréchet topology such that each space $Cl^o(S^1, V)$ is closed in $Cl(S^1, V)$. We set

$$PDO^{-\infty}(S^1, V) = \bigcap_{o \in \mathbb{Z}} PDO^o(S^1, V) \ .$$

It is well-known that $PDO^{-\infty}(S^1, V)$ is a two-sided ideal of $PDO(S^1, V)$, see e.g. [**15, 46**]. Therefore, we can define the quotients

$$\mathcal{F}PDO(S^1, V) = PDO(S^1, V)/PDO^{-\infty}(S^1, V),$$

$$\mathcal{F}Cl(S^1, V) = Cl(S^1, V)/PDO^{-\infty}(S^1, V),$$

$$\mathcal{F}Cl^o(S^1, V) = Cl^o(S^1, V)/PDO^{-\infty}(S^1, V) \ .$$

The script font $\mathcal{F}$ stands for *formal pseudodifferential operators.* The quotient $\mathcal{F}PDO(S^1,V)$ is an algebra isomorphic to the set of formal symbols, see [**4**], and the identification is a morphism of $\mathbb{C}$-algebras for the usual multiplication on formal symbols (see e.g. [**15**]).

We note that these quotient spaces have natural diffeology structures provided by Proposition 2.7. Moreover, we have the following result.

Theorem 3.2. *The groups $Diff_+(S^1)$, $Cl^{0,*}(S^1,V)$, and $\mathcal{F}Cl^{0,*}(S^1,V)$, in which $\mathcal{F}Cl^{0,*}(S^1,V)$ is the group of units of the algebra $\mathcal{F}Cl^0(S^1,V)$, are regular Fréchet Lie groups.*

Indeed, it follows from [**11, 41**] that $Diff_+(S^1)$ is open in the Fréchet manifold $C^\infty(S^1,S^1)$. This fact makes it a Fréchet manifold and, following [**41**], a regular Fréchet Lie group. The same result follows from the discussion appearing in [**40**, Section III.3]. Also, it is noticed in [**27**] that the results of [**16**] imply that the group $Cl^{0,*}(S^1,V)$ (resp. $\mathcal{F}Cl^{0,*}(S^1,V)$) is open in $Cl^0(S^1,V)$ (resp. $\mathcal{F}Cl^0(S^1,V)$) and that it is a regular Fréchet Lie group. This fact is also discussed in [**43**, Proposition 4]. Our comments after Definition 2, see also Remark 2.6 in [**33**], imply that these groups are also regular Frölicher Lie groups.

Definition 3.3. A classical pseudodifferential operator A on S^1 is called odd class if and only if for all $n\in\mathbb{Z}$ and all $(x,\xi)\in T^*S^1$ we have:

$$\sigma_n(A)(x,-\xi)=(-1)^n\sigma_n(A)(x,\xi)\ ,$$

in which σ_n is the symbol of A.

This particular class of pseudodifferential operators has been introduced in [**21, 22**]; it is also called the “even-even class”, see [**46**]. We will follow the terminology of the first two references: hereafter, the notation Cl_{odd} will refer to odd class classical pseudodifferential operators.

We need the following result, essentially present in [**21, 46**]:

Lemma 3.4. *$Cl_{odd}(S^1,V)$ and $Cl^0_{odd}(S^1,V)$ are associative algebras.*

Proof. We work locally. Let A,B be two odd class pseudodifferential operators of order m and m' respectively; then, the homogeneous pieces of the symbols of A,B,AB are related via (see [**46**, Section 1.5.2, Equation(1.5.2.3)])

$$\sigma_{m+m'-j}(AB)(x,\xi)=\sum_{|\mu|+k+l=j}\frac{1}{\mu!}\partial_\xi^\mu\sigma_{m-k}(A)(x,\xi)D_x^\mu\sigma_{m'-l}(B)(x,\xi)\ ,$$

in which $|\mu|$ is the length of the multi-index μ. We have:

$$\partial_\xi^\mu\sigma_{m-k}(A)(x,-\xi)=(-1)^{m-k+|\mu|}\partial_\xi^\mu\sigma_{m-k}(A)(x,\xi)$$

and

$$D_x^\mu\sigma_{m'-l}(B)(x,-\xi)=(-1)^{m'-l}D_x^\mu\sigma_{m'-l}(B)(x,\xi)\ ,$$

and we easily obtain

$$\begin{aligned}&\sigma_{m+m'-j}(AB)(x,-\xi)\\ =\ &(-1)^{m+m'-j}\sum_{|\mu|+k+l=j}\frac{1}{\mu!}\partial_\xi^\mu\sigma_{m-k}(A)(x,\xi)D_x^\mu\sigma_{m'-l}(B)(x,\xi)\\ =\ &(-1)^{m+m'-j}\sigma_{m+m'-j}(AB)(x,\xi)\ .\end{aligned}$$

The first claim is proven.

That $Cl^0_{odd}(S^1,V)$ is an associative algebra follows from the previous result and the standard fact that zero-order classical pseudodifferential operators form an algebra, see for instance [**43**]. □

Now we observe that because of the symmetry property stated in Definition 3.3, an odd-class pseudodifferential operator A has a partial symbol of non-negative order n that reads

$$\sigma_n(A)(x,\xi) = \gamma_n(x)(i\xi)^n\,, \tag{3.1}$$

where $\gamma_n \in C^\infty(S^1, L(V))$. This consequence of Definition 3.3 allows us to check the following result, which is of importance for the upcoming description of our KP hierarchy:

PROPOSITION 3.5. *The space of odd class classical pseudodifferential operators satisfies the direct sum decomposition*

$$Cl_{odd}(S^1,V) = Cl^{-1}_{odd}(S^1,V) \oplus DO(S^1,V)\;. \tag{3.2}$$

We finish this section with a proposition which singles out an interesting Lie group included in $Cl_{odd}(S^1,V)$.

PROPOSITION 3.6. *The algebra $Cl^0_{odd}(S^1,V)$ is a closed subalgebra of $Cl^0(S^1,V)$. Moreover, $Cl^{0,*}_{odd}(S^1,V)$ is*

- *an open subset of $Cl^0_{odd}(S^1,V)$ and,*
- *a regular Fréchet Lie group.*

PROOF. We note by $\sigma(A)(x,\xi)$ the total formal symbol of $A \in Cl^0(S^1,V)$. We let

$$\phi : Cl^0(S^1,V) \to \mathcal{F}Cl^0(S^1,V)$$

defined by

$$\phi(A) = \sum_{n\in\mathbb{N}} \sigma_{-n}(x,\xi) - (-1)^n\sigma_{-n}(x,-\xi).$$

This map is smooth, and

$$Cl^0_{odd}(S^1,V) = Ker(\phi),$$

which shows that $Cl^0_{odd}(S^1,V)$ is a closed subalgebra of $Cl^0(S^1,V)$. Moreover, if $H = L^2(S^1,V)$,

$$Cl^{0,*}_{odd}(S^1,V) = Cl^0_{odd}(S^1,V) \cap GL(H),$$

which proves that $Cl^{0,*}_{odd}(S^1,V)$ is open in the Fréchet algebra $Cl^0(S^1,V)$, and it follows that it is a regular Fréchet Lie group by arguing along the lines of [**16, 40**]. □

4. The h-KP hierarchy with non-formal odd-class operators

First of all let us make some comments on the spaces just introduced. In order to find an analogue to Equation (1.1) we need to consider a space of pseudodifferential operators which is closed with respect to taking powers of operators. Since the space of odd-class pseudodifferential operators is an associative algebra, we can take this space as the arena for the dependent variable appearing in Equation (1.1). Proposition 3.5 implies that we have the diagram of Lie groups and Lie algebras

$$\begin{array}{ccccc} Cl^{-1,*}_{odd}(S^1,V) & \to & Cl^*_{odd}(S^1,V) & \to & (H \quad ?) \\ \downarrow & & \downarrow & & \\ Cl^{-1}_{odd}(S^1,V) & \to & Cl_{odd}(S^1,V) & \to & DO(S^1,V) \\ & & \| & & \\ & & Cl^{-1}_{odd}(S^1,V) \oplus DO(S^1,V) & & \end{array}$$

Is it possible to find a suitable Frölicher Lie group H? If it were possible, we could set up an equation of the form

$$\frac{\partial}{\partial t_n} L = [(L^n)_D, L]$$

for fixed n, where $(.)_D$ denotes projection into the space of differential operators, and try to study its corresponding Cauchy problem with the help of a factorization theorem, as in our previous papers [**29, 32, 33**]. Now, in these articles we find a regular Frölicher Lie group H with Lie algebra the space of differential operators by using formal differential operators of infinite order but, if we proceed in this way in the present context, we would leave the framework of non-formal pseudodifferential operators. Thus, instead of doing this, we use series, motivated by [**29, 32**] and [**33**, Subsection 4.2].

Definition 4.1. Let h be a formal parameter. The set of odd class h-pseudo differential operators is the set of formal series

$$Cl_{h,odd}(S^1,V) = \left\{ \sum_{n\in\mathbb{N}} a_n h^n \mid a_n \in Cl^n_{odd}(S^1,V) \right\} . \tag{4.1}$$

We have the following result on the structure of $Cl_{h,odd}(S^1,V)$:

Theorem 4.2. *The set $Cl_{h,odd}(S^1,V)$ is a Fréchet Lie algebra, and its group of units given by*

$$Cl^*_{h,odd}(S^1,V) = \left\{ \sum_{n\in\mathbb{N}} a_n h^n \mid a_n \in Cl^n_{odd}(S^1,V), a_0 \in Cl^{0,*}_{odd}(S^1,V) \right\} , \tag{4.2}$$

is a regular Fréchet Lie group.

Proof. As we showed in Proposition 20 (and as it also follows from the work [**16**] by Glöckner) the group $Cl^{0,*}_{odd}(S^1,V)$ is a regular Fréchet Lie group since it is open in $Cl^0_{odd}(S^1,V)$. According to classical properties of composition of pseudodifferential operators [**46**], see also [**21**], the natural multiplication on $Cl_{h,odd}(S^1,V)$ is smooth for the product structure inherited from the classical Fréchet topology on classical pseudodifferential operators, and inversion is smooth using the classical formulas of inversion of series. In this way we conclude that $Cl_{h,odd}(S^1,V)$ is a Fréchet algebra.

Moreover, the series $\sum_{n\in\mathbb{N}} a_n h^n \in Cl_{h,odd}(S^1,V)$ is invertible if and only if $a_0 \in Cl^{0,*}_{odd}(S^1,V)$, which shows that $Cl^*_{h,odd}(S^1,V)$ is open in $Cl_{h,odd}(S^1,V)$. Now the results appearing in [**16**] allow us to end the proof. □

Remark 4.3. The assumption $a_n \in Cl^n_{odd}$ in Definition 4.1 and Theorem 4.2 can be relaxed to the condition

$$a_0 \in Cl^{0,*}_{odd} \text{ and } \forall n \in \mathbb{N}^*, a_n \in Cl_{odd} ;$$

this is sufficient for having a regular Lie group. The more stringent growth conditions imposed in (4.1) and (4.2) will ensure regularity *and* they will allow us to use arguments borrowed from [**33**, Subsection 4.1] for proving existence and smoothness of solutions to our KP hierarchy to be introduced next.

Now we need to split the algebra $Cl_{h,odd}(S^1,V)$. We do so in a very straightforward way: since an operator $A \in Cl_{odd}(S^1,V)$ splits into $A = A_S + A_D$, in which $A_S \in Cl^{-1}_{odd}(S^1,V)$ and $A_D \in DO(S^1,V)$ (see Proposition 3.5) we have, for $A = \sum_{n\in\mathbb{N}} a_n h^n \in Cl_{h,odd}(S^1,V)$, the decomposition $A = A_S + A_D$ with

$$A_S = \sum_{n\in\mathbb{N}} (a_n)_S h^n$$

and

$$A_D = \sum_{n\in\mathbb{N}} (a_n)_D h^n \ .$$

We set $DO_h(S^1,V) = \{\sum_{n\in\mathbb{N}} a_n h^n : a_n \in DO(S^1,V)\}$.

We now introduce our version of the KP hierarchy with non-formal pseudodifferential operators. Let us assume that $t_1, t_2, \cdots, t_n, \cdots$, are an infinite number of different formal variables which will become the independent variables of our equation. We make the following definition:

DEFINITION 4.4. Let $L_0 \in h\frac{d}{dx} + h\, Cl^{-1}_{odd}(S^1,V)$. We say that an operator

$$L(t_1,t_2,\cdots) \in Cl_{h,odd}(S^1,V)[[ht_1,...,h^n t_n...]]$$

satisfies the $h-$deformed KP hierarchy if and only if

$$\left\{ \begin{array}{ll} L(0) = & L_0 \\ \frac{d}{dt_n} L = & [(L^n)_D, L] \end{array} \right. . \tag{4.3}$$

Let us make some comments on Definition 4.4. We have followed Mulase, see [**36, 38**], in fixing the "time dependence" of the dependent variable via series. Thus, the dependent variable in our equation (4.3) is a series of the form

$$L(t_1,t_2,\cdots) = \sum_s L_s (h\tau)^s \ , \tag{4.4}$$

in which $L_s \in Cl_{h,odd}(S^1,V)$, $s = (\alpha_1,\alpha_2,\cdots)$, $\alpha_i \in \mathbb{N}$, $\alpha_i \neq 0$ just for finite number of indexes i, and we define $(h\tau)^s = (ht_1)^{\alpha_1}(h^2 t_2)^{\alpha_2}\cdots$. This series can be understood as a smooth function from the algebraic sum

$$T = \bigoplus_{n\in\mathbb{N}^*} (\mathbb{R}t_n) \tag{4.5}$$

equipped with the product topology and product Frölicher structure, to the Fréchet Lie algebra $Cl_{h,odd}(S^1,V)$, see Proposition 2.6 and [**29, 33**]. The "space dependence", on the other hand, is fixed with the help of a derivation on S^1 which in standard coordinates (see Section 3) reads d/dx. We also stress the fact that, in contradistinction to Mulase's papers [**36, 38**], we are using *scaled variables*,

$$\left\{ \begin{array}{lll} t_n & \mapsto & h^n t_n \\ \frac{d}{dx} & \mapsto & h\frac{d}{dx} \end{array} \right. .$$

Our reason to do this is that we need to work with regular Frölicher Lie groups, and this scaling allows us to do so, as we explain in [**29, 33**]. Finally, we remark that it follows from Theorem 17 that $Cl^*_{h,odd}(S^1,V)[[ht_1,...,h^nt_n...]]$ is a regular Frölicher Lie group with Lie algebra $Cl_{h,odd}(S^1,V)[[ht_1,...,h^nt_n...]]$.

In this context, we have a "Mulase factorization", in the spirit of [**36, 38**] and [**29, 33**], which looks schematically as follows:

$$\begin{array}{ccccc} Cl^{-1,*}_{h,odd}(S^1,V) & \rightarrow & Cl^*_{h,odd}(S^1,V) & \rightarrow & DO^*_h(S^1,V) \\ \downarrow & & \downarrow & & \downarrow \\ \mathbb{C}Id \oplus Cl^{-1}_{h,odd}(S^1,V) & \rightarrow & Cl_{h,odd}(S^1,V) & \rightarrow & DO_h(S^1,V) \end{array}$$

in which

$$DO^*_h(S^1,V) = \left\{ \sum_{n\in\mathbb{N}} a_n h^n \,|\, a_n \in DO(S^1,V)\,, a_0 \in DO^{0,*}(S^1,V) \right\} .$$

Now we solve the initial value problem (4.3) for an initial condition L_0 as in Definition 4.4. Below we need to use the operator $U_h = \exp\left(\sum_{n\in\mathbb{N}^*} h^n t_n (L_0)^n\right)$: since $\sum_{n\in\mathbb{N}^*} h^n t_n (L_0)^n$ belongs to $Cl_{h,odd}(S^1,V)[[ht_1,h^2t_2,\cdots]]$, U_h belongs to the regular Frölicher Lie group $Cl^*_{h,odd}(S^1,V)[[ht_1,...,h^nt_n...]]$. Moreover, as we explained after Definition 4.4, we can consider $\sum_{n\in\mathbb{N}^*} h^n t_n (L_0)^n$ as a smooth function with domain T and image contained in $Cl_{h,odd}(S^1,V)$, and so for each fixed $(t_1,...,t_n,...) \in T$ we have $U_h(t_1,...,t_n,...) \in Cl^*_{h,odd}(S^1,V)$.

Theorem 4.5. *Let* $U_h(t_1,...,t_n,...) = \exp\left(\sum_{n\in N^*} h^n t_n (L_0)^n\right) \in Cl^*_{h,odd}(S^1,V)$. *Then:*

- *There exists a unique pair* (S,Y) *such that*
  (1) $U_h = S^{-1}Y$,
  (2) $Y \in Cl^*_{h,odd}(S^1,V)_D$
  (3) $S \in Cl^*_{h,odd}(S^1,V)$ *and* $S-1 \in Cl_{h,odd}(S^1,V)_S$.
  Moreover, the map
$$(L_0,t_1,...,t_n,...) \in \left(h\frac{d}{dx} + h\,Cl^{-1}_{odd}(S^1,V)\right) \times T \mapsto (U_h,Y) \in (Cl^*_{h,odd}(S^1,V))^2$$
  is smooth.
- *The operator* $L \in Cl_{h,odd}(S^1,V)[[ht_1,...,h^nt_n...]]$ *given by* $L = SL_0S^{-1} = YL_0Y^{-1}$, *is the unique solution to the hierarchy of equations*
$$\left\{ \begin{array}{rcl} \frac{d}{dt_n}L & = & [(L^n)_D(t),L(t)] = -\,[(L^n)_S(t),L(t)] \\ L(0) & = & L_0 \end{array} \right. , \tag{4.6}$$
  in which the operators in this infinite system are understood as formal operators, this is, as formal series of the form (4.4).
- *The operator* $L \in Cl_{h,odd}(S^1,V)[[ht_1,...,h^nt_n...]]$ *given by* $L = SL_0S^{-1} = YL_0Y^{-1}$ *is the unique solution of the hierarchy of equations*
$$\left\{ \begin{array}{rcl} \frac{d}{dt_n}L & = & [(L^n)_D(t),L(t)] = -\,[(L^n)_S(t),L(t)] \\ L(0) & = & L_0 \end{array} \right. \tag{4.7}$$
  in which the operators in this infinite system are understood as h-series of odd-class, non-formal operators.

Proof. First of all, we consider U_h. Since

$$U_h(t_1,...,t_n,...) = \exp\left(\sum_{n\in N^*} h^n t_n (L_0)^n\right) \in Cl^*_{h,odd}(S^1,V)[[ht_1,...,h^nt_n...]] ,$$

we can write, as in (4.4),

$$U_h = \sum_s A_s (h\tau)^s \ ,$$

in which here and hereafter s is a multi-index as defined after equation (4.4), and $A_s \in Cl^*_{h,odd}(S^1, V)$. In turn, for each s we can set $A_s = \sum_{n\in\mathbb{N}} a_{sn} h^n$, where $a_{sn} \in Cl^n_{odd}(S^1, V)$, $n \geq 1$, and $a_{s0} \in Cl^{0,*}_{odd}(S^1, V)$. Thus, we have

$$U_h = \sum_s \left(\sum_{n\in\mathbb{N}} a_{sn} h^n \right) (h\tau)^s \ .$$

Now we observe that, since $a_{sn} \in Cl^n_{odd}(S^1, V)$, the total symbol of a_{sn} can be written as

$$\sigma(a_{sn}) = \sum_{-\infty<k\leq n} a_{snk} \xi^k$$

in which $a_{snk} : S^1 \to \mathbb{R} \otimes End(V)$. (We recall that the pass from pseudodifferential operators to symbols is discussed in detail in, e.g., [**1**, Section 2] and [**2**, p. 55]). This means that we can write $\sigma(U_h)$ in two ways:

$$\begin{aligned} \sigma(U_h) &= \sum_s \left(\sum_{n\in\mathbb{N}} \left(\sum_{-\infty<k\leq n} a_{snk}\xi^k \right) h^n \right) (h\tau)^s \\ &= \sum_{n\in\mathbb{N}} \left[\sum_{-\infty<k\leq n} \left(\sum_s a_{snk} (h\tau)^s \right) \xi^k \right] h^n \\ &= \sum_{n\in\mathbb{N}} \left[\sum_{-\infty<k\leq n} \left(\sum_s a_{snk} \tau^s \right) \xi^k \right] h^s \, h^n \ , \end{aligned} \tag{4.8}$$

in which $h^s = h^{\alpha_1} h^{2\alpha_2} h^{3\alpha_3} \cdots$ and $\tau^s = t_1^{\alpha_1} t_2^{\alpha_2} t_3^{\alpha_3} \cdots$ for $s = (\alpha_1, \alpha_2, \alpha_3, \cdots)$, and we also easily check that

$$\sigma(U_h) = \sum_{k\in\mathbb{Z}} \left(\sum_{\frac{k+|k|}{2} \leq n < \infty} \left(\sum_s a_{snk} \tau^s \right) h^n h^s \right) \xi^k \ . \tag{4.9}$$

Equations (4.8) and (4.9) tell us that $\sigma(U_h)$ belongs to the algebra $\Psi_h(R)$ (see Definition 4.3 in [**33**]: in that reference we use q instead of h) in which R is the algebra of power series in τ whose coefficients belong to the differential algebra of smooth functions $C^\infty(S^1) \otimes End(V)$. (Also, we can say that $\sigma(U_h) \in \widetilde{\mathcal{A}}$, where $\widetilde{\mathcal{A}}$ is defined in Section 5.4 of [**29**]). Now we use that $a_{s0} \in Cl^{0,*}_{odd}(S^1, V)$ and that therefore its total symbol is of the form

$$\sigma(a_{s0}) = \sum_{-\infty<k\leq 0} a_{s0k}\xi^k = a_{s00} + \sum_{-\infty<k\leq -1} a_{s0k}\xi^k$$

in which a_{s00} is invertible:

Equation (4.9) implies that the free term of $\sigma(U_h)$ is

$$\sum_{0\leq n<\infty} \left(\sum_s a_{sn0}\tau^s \right) h^n h^s = a_{000} + \sum_{1\leq n<\infty} \left(\sum_s a_{sn0}\tau^s \right) h^n h^s \ ,$$

and since a_{s00} is invertible for each s, we conclude that $\sigma(U_h)$ belongs to $G\Psi_h(R)$ (see again [**33**, Definition 4.3]; we can also say that $\sigma(U_h) \in G_{\mathcal{A}}$ in the notation of [**29**]). Now we recall, see [**33**, Equation (4.14)], that the global factorization $G\Psi_h(R) = G_{R,h} \cdot \mathcal{D}_h(R)$ —in which the subgroups $G_{R,h}$ and $\mathcal{D}_h(R)$ are introduced in [**33**, Definition 4.3]— holds. We find that there exist unique $S_{symb} \in G_{R,h}$ and $Y_{symb} \in \mathcal{D}_h(R)$ such that

$$\sigma(U_h) = S_{symb}^{-1} \cdot Y_{symb} \, .$$

It follows that there exist power series Y and S of non-formal odd class operators, defined up to smoothing operators, such that $S_{symb} = \sigma(S)$ (in which $\sigma(S)$ is defined in an obvious way by using the h valuation) and $Y_{symb} = \sigma(Y)$, and so we can write

$$\sigma(U_h) = \sigma(S)^{-1} \cdot \sigma(Y) \, .$$

The series $\sigma(Y)$ is a formal series in $h, t_1, \cdots t_n, \cdots$ of symbols of differential operators, which are in one-to-one correspondence with a series of (non-formal) differential operators. Thus, the operator Y is uniquely defined, not up to a smoothing operator; it depends smoothly on U_h, and so does $S = YU_h^{-1}$. This ends the proof of the first point.

The second point on the $h-$deformed KP hierarchy is proven along the lines of [**29, 33**], since it corresponds essentially to an existence result for symbols.

Finally, we prove the third point: We have that $L = YL_0Y^{-1}$ is well-defined and, following classical computations which can be found in e.g. [**12, 33**], we have:

(1) $L^k = YL_0^kY^{-1}$

(2) $U_h\, L_0^kU_h^{-1} = L_0^k$ since L_0 commutes with $U_h = \exp(\sum_k h^k t_k\, L_0^k)$.

It follows that

$$L^k = YL_0^kY^{-1} = SS^{-1}YL_0^kY^{-1}SS^{-1} = SU_hL_0^kU_h^{-1}S^{-1} = SL_0^kS^{-1}.$$

We take t_k-derivative of U_h for each $k \geq 1$. We get the equation

$$\frac{dU_h}{dt^k} = -S^{-1}\frac{dS}{dt_k}S^{-1}Y + S^{-1}\frac{dY}{dt_k}$$

and so, using $U_h = S^{-1}\,Y$, we obtain the decomposition

$$SL_0^kS^{-1} = -\frac{dS}{dt_k}S^{-1} + \frac{dY}{dt_k}Y^{-1} \, .$$

Since $\frac{dS}{dt_k}S^{-1} \in Cl_{h,odd}(S^1,V)_S$ and $\frac{dY}{dt_k}Y^{-1} \in Cl_{h,odd}(S^1,V)_D$, we conclude that

$$(L^k)_D = \frac{dY}{dt_k}Y^{-1} \quad \text{and} \quad (L^k)_S = -\frac{dS}{dt_k}S^{-1} \, .$$

Now we take t_k-derivative of L:

$$\begin{aligned}
\frac{dL}{dt_k} &= \frac{dY}{dt_k}L_0Y^{-1} - YL_0Y^{-1}\frac{dY}{dt_k}Y^{-1} \\
&= \frac{dY}{dt_k}Y^{-1}YL_0Y^{-1} - YL_0Y^{-1}\frac{dY}{dt_k}Y^{-1} \\
&= (L^k)_D\,L - L\,(L^k)_D \\
&= [(L^k)_D, L] \, .
\end{aligned}$$

We check the initial condition: We have $L(0) = Y(0)L_0Y(0)^{-1}$, but $Y(0) = 1$ by the definition of U_h.

Smoothness with respect to the variables $(S_0, t_1, ..., t_n, ...)$ is already proved by construction, and we have established smoothness of the map $L_0 \mapsto Y$ at the beginning of the proof. Thus, the map

$$L_0 \mapsto L(t) = Y(t) L_0 Y^{-1}(t)$$

is smooth. The corresponding equation

$$\frac{d}{dt_k} L = -\left[(L^k)_S, L\right]$$

is obtained the same way.

It remains to check that the announced solution is the unique solution to the non-formal hierarchy (4.7). This is still true at the formal level, but two solutions which differ by smoothing operators may appear at this step of the proof. Let $(L+K)(t_1, ...)$ be another solution, in which K is an h series of smoothing operators depending on the variables $t_1, ...,$ and L is the solution derived from U_h. Then, for each $n \in \mathbb{N}^*$ we have

$$(L + K)^n_D = L^n_D \ ,$$

which implies that K satisfies the *linear* equation

$$\frac{dK}{dt_n} = [L^n_D, K]$$

with initial conditions $K|_{t=0} = 0$. We can construct the unique solution K by induction on n, beginning with $n = 1$. Let g_n be such that

$$(g_n^{-1} dg_n)(t_n) = L^n_D(t_1, ...t_{n-1}, t_n, 0, ...) \ .$$

Then we get that

$$K(t_1, ...t_n, 0....) = Ad_{g_n(t_n)} \left(K(t_1, t_{n-1}, 0...)\right) \ ,$$

and hence, by induction,

$$K(0) = 0 \Rightarrow K(t_1, 0...) = 0 \Rightarrow \cdots \Rightarrow K(t_1, ...t_n, 0....) = 0 \Rightarrow \cdots \ ,$$

which implies that $K = 0$. □

Remark 4.6. Usually, in the treatment of (non-$h-$scaled) KP hierarchy, the operator L_0 is assumed to read as $L_0 = S_0 \left(\frac{d}{dx}\right) S_0^{-1}$ where S_0 is addressed as the Sato operator, see [**36**, Section 1]. This decomposition does not fit with the choice of an arbitrary initial value L_0 as stated in our previous work [**33**], see also [**13**]. However, considering the previous proof, we remark that the Mulase decomposition, as well as our computations involving L^k and $\frac{dL}{dt_k}$, depend only on L_0 and not on the existence of S_0. We believe that our construction of the solution L may also be applied to a more standard (non $h-$scaled) version of the KP hierarchy. This is work in progress.

Remark 4.7. The reader may notice that the proof of the third item of Theorem 4.5 is inspired by Reyman and Semenov-Tian-Shansky approach to integrability via R-matrices and factorization theorems, see for instance [**44**, Section 1.12, Theorem 7]. However, our result is not exactly an instance of the Reyman–Semenov-Tian-Shansky theory since in this paper we are not considering the hamiltonian content of Equation (4.3). What we are observing here is that techniques appropriated for the study of integrability of Hamiltonian systems can be adapted to prove well-posedness of the interesting equation (4.3). As in Mulase's papers

[**36, 38**], the crucial point of the proof is the existence of a factorization of an infinite-dimensional Lie group, and not the possible hamiltonian character of the equation being investigated.

5. KP equations and $Diff_+(S^1)$

Let $A_0 \in Cl^{-1}_{odd}(S^1, V)$, and set $S_0 = \exp(A_0)$. The operator $S_0 \in Cl^{-1,*}_{odd}(S^1, V)$ is our version of the dressing operator of standard KP theory, see for instance [**8**, Chapter 6]. We define the operator L_0 by

$$f \mapsto L_0(f) = h\left(S_0 \circ \frac{d}{dx} \circ S_0^{-1}\right)(f)$$

for $f \in C^\infty(S^1, V)$. We note that $L_0^k(f) = h^k S_0 \frac{d^k}{dx^k}(S_0^{-1}(f))$, a formula which we will use presently. Our aim is to connect the operator

$$U_h = \exp\left(\sum_{n\in\mathbb{N}*} h^n t_n L_0^n\right) ,$$

which generates the solutions of the $h-$deformed KP hierarchy described in Theorem 4.5, with the Taylor expansion of functions in the image of the twisted operator

$$A : f \in C^\infty(S^1, V) \mapsto S_0^{-1}(f) \circ g ,$$

in which $g \in Diff_+(S^1)$. We remark that $A \in Cl^{-1,*}_{odd}(S^1, V)$ for each $g \in Diff_+(S^1)$; our decomposition theorem proven in the appendix (see Theorem 5.7) will imply that it is also smooth with respect to g.

For convenience, we identify S^1 with $[0; 2\pi[\sim \mathbb{R}/2\pi\mathbb{Z}$, assuming implicitly that all the values under consideration are up to terms of the form $2k\pi$, for $k \in \mathbb{Z}$. Set $c = S_0^{-1}(f) \circ g \in C^\infty(S^1, V)$. We compute:

$$\begin{aligned}
c(x_0 + h) &= \left(S_0^{-1}(f) \circ g\right)(x_0 + h) \\
&\sim_{x_0} \left(S_0^{-1}(f) \circ g\right)(x_0) + \sum_{n\in\mathbb{N}^*}\left[\frac{h^n}{n!}\frac{d^n}{dx^n}\left(S_0^{-1}(f) \circ g\right)\right](x_0) \\
&= \left(S_0^{-1}(f)\right) \circ g(x_0) + \\
&\quad \sum_{n\in\mathbb{N}^*}\left[\frac{h^n}{n!}\sum_{k=1}^{n} B_{n,k}(u_1(x_0), ..., u_{n-k+1}(x_0))\frac{d^k}{dx^k}\left(S_0^{-1}(f)\right) \circ g(x_0)\right] ,
\end{aligned}$$

in which we have used the classical Faá de Bruno formula for the higher chain rule in terms of Bell's polynomials $B_{n,k}$, and $u_i(x_0) = g^{(i)}(x_0)$ for $i = 1, \cdots n - k + 1$. We can rearrange the last sum and write

$$\begin{aligned}
c(x_0 + h) \sim_{x_0} &\left(S_0^{-1}(f) \circ g\right)(x_0) + \\
&\sum_{k\in\mathbb{N}^*}\sum_{n\geq k}\left[\frac{h^n}{n!}B_{n,k}(u_1(x_0), ..., u_{n-k+1}(x_0))\frac{d^k}{dx^k}\left(S_0^{-1}(f)\right)\right](g(x_0))
\end{aligned}$$

or,

$$c(x_0 + h) \sim_{x_0} \sum_{k\in\mathbb{N}}\left[a_k h^k \frac{d^k}{dx^k}\left(S_0^{-1}(f)\right)\right](g(x_0)) \tag{5.1}$$

in which $a_0 = 1$ and

$$a_k = \sum_{n \geq k} \frac{h^{n-k}}{n!} B_{n,k}(u_1(x_0), ..., u_{n-k+1}(x_0))$$

for $k \geq 1$. In terms of the operator L_0, Equation (5.1) means that

$$c(x_0 + h) \quad \sim_{x_0} \quad S_0^{-1} \sum_{k \in \mathbb{N}} \left[a_k \, L_0^k(f) \right] (g(x_0)) \, . \tag{5.2}$$

We now define the sequence $(t_n)_{n \in \mathbb{N}^*}$ by the formula

$$\log \left(\sum_{k \in \mathbb{N}} a_k X^k \right) = \sum_{n \in \mathbb{N}^*} t_n X^n \; , \tag{5.3}$$

so that both, a_k and t_n, are series in the variable h. We obtain

$$c(x_0 + h) \quad \sim_{x_0} \quad S_0^{-1} \exp \left(\sum_{n \in \mathbb{N}^*} \frac{t_n}{h^n} \, L_0^k(f) \right) (g(x_0)) \, .$$

We state the following theorem:

THEOREM 5.1. *Let $f \in C^\infty(S^1, V)$ and set $c = S_0^{-1}(f) \circ g \in C^\infty(S^1, V)$. The Taylor series at x_0 of the function c is given by*

$$c(x_0 + h) \sim_{x_0} S_0^{-1} \left(U_h(t_1/h, t_2/h^2, ...)(f) \right) (g(x_0)) \; ,$$

in which the times t_i are related to the derivatives of g via Equation (5.3).

The coefficients of the series a_k and t_n appearing in (5.3) depend smoothly on $g \in Diff_+(S^1)$ and $x_0 \in S^1$. Indeed, the map

$$(x, g) \in S^1 \times Diff_+(S^1) \mapsto (g(x), (u_n(x))_{n \in \mathbb{N}^*}) \in S^1 \times \mathbb{R}^{\mathbb{N}^*}$$

is smooth due to Proposition 2.6 (more precisely, due to the generalization of Proposition 2.6 to infinite products); smoothness a_k then follows, while smoothness of t_n is consequence of Equation (5.3).

REMARK 5.2. As a by-product of the foregoing computations, we notice the following relation. If $f \in C^\infty(S^1, V)$, we can write

$$f(x_0 + h) \sim_{x_0} f(x_0) + \sum_{n \in \mathbb{N}^*} \left(\frac{h^n}{n!} \left(\frac{d}{dx} \right)^n f \right) (x_0) = \left(\exp \left(h \frac{d}{dx} \right) f \right) (x_0) \in J^\infty(S^1, V)$$

for $x_0 \in S^1$.Thus, the operator $\exp\left(h \frac{d}{dx}\right)$ belongs to the space $Cl_h(S^1, V)$.

Appendix:the group of $Diff_+(S^1)-$pseudodifferential operators

Now we present a restricted class of groups of Fourier integral operators which we will call $Diff_+(S^1)$-pseudodifferential operators following [**30**]. These groups appear as central extensions of $Diff_+(S^1)$ by groups of (often bounded) pseudodifferential operators. We do not state the basic facts on Fourier integral operators here (they can be found in the classical paper [**18**]), but we recall the following theorem, which was stated in [**30**] for a general base manifold M.

THEOREM 5.3. [**30**, Theorem 4] *Let H be a regular Lie group of pseudodifferential operators acting on smooth sections of a trivial bundle $E \sim V \times S^1 \to S^1$. The group $Diff(S^1)$ acts smoothly on $C^\infty(S^1, V)$, and it is assumed to act smoothly on H by adjoint action. If H is stable under the $Diff(S^1)-$adjoint action, then there exists a regular Lie group G of Fourier integral operators defined through the exact sequence:*

$$1 \to H \to G \to Diff(S^1) \to 1 .$$

If H is a Frölicher Lie group, then G is a Frölicher Lie group.

This result was proven in [**30**] by applying Theorem 2.16. Using the equivalence between Gateaux-smooth and Frölicher-smooth in the Fréchet category stated after Definition 2 and proven in [**33**], we have a Fréchet version of Theorem 5.3: if H is a regular Fréchet Lie group which is stable under $Diff(S^1)-$adjoint action, and G is a smooth Fréchet manifold isomorphic to $H \times Diff(S^1)$ with multiplication and inversion Frölicher (hence Fréchet) smooth, we have the equivalence:

$$H \text{ is a } \textit{regular} \text{ Fréchet Lie group } \Leftrightarrow G \text{ is a } \textit{regular} \text{ Fréchet Lie group } .$$

The pseudodifferential operators considered in Theorem 5.3 can be classical, odd class, or anything else. Applying the formulas of "changes of coordinates" (which can be understood as adjoint actions of diffeomorphisms) of e.g. [**15**], we obtain that odd-class pseudodifferential operators are stable under the adjoint action of $Diff(S^1)$. Thus, we can define the following group:

DEFINITION 5.4. The group $FCl^{0,*}_{Diff(S^1),odd}(S^1, V)$ is the regular Fréchet Lie group G obtained in Theorem 5.3 with $H = Cl^{0,*}_{odd}(S^1, V)$.

Following [**30**], we remark that operators A in this group can be understood as operators in $Cl^{0,*}_{odd}(S^1, V)$ twisted by diffeomorphisms, this is,

$$A = B \circ g \tag{5.4}$$

for unique $g \in Diff(S^1)$ and unique $B \in Cl^{0,*}_{odd}(S^1, V)$, and also that its Lie algebra is isomorphic as a vector space to $Cl^{0}_{odd}(S^1, V) \oplus Vect(S^1)$, in which $Vect(S^1)$ is the space of smooth vector fields on S^1.

REMARK 5.5. The diffeomorphism g appearing in (5.4) is the phase of the operators, but here the phase (and hence the decomposition (5.4)) is unique, which is not the case for general Fourier integral operators, see e.g. [**18**]. This construction of phase functions of $Diff(M)-$pseudodifferential operators differs from the one described by Omori [**41**] and Adams, Ratiu and Schmid [**1**] for the groups of Fourier integral operators; the exact relation among these constructions still needs to be investigated.

Now we note that the group $Diff(S^1)$ decomposes into two connected components $Diff(S^1) = Diff_+(S^1) \cup Diff_-(S^1)$, where the connected component of the identity, $Diff_+(S^1)$, is the group of orientation preserving diffeomorphisms of S^1. We make the following definition:

DEFINITION 5.6. The group $FCl^{0,*}_{Diff_+(S^1),odd}(S^1, V)$ is the regular Fréchet Lie group of all operators in $FCl^{0,*}_{Diff(S^1),odd}(S^1, V)$ whose phase diffeomorphisms lie in the group $Diff_+(S^1)$.

Theorem 5.7. *Let $U \in FCl^{0,*}_{Diff_+(S^1),odd}(S^1,V)$. There exists an unique pair*

$$(S,Y) \in Cl^{-1,*}_{odd}(S^1,V) \times \left(DO^{0,*}(S^1,V) \rtimes Diff_+(S^1)\right)$$

such that

$$U = SY\ .$$

Moreover, the map $U \mapsto (S,Y)$ is smooth and, there is a short exact sequence of Lie groups:

$$1 \to Cl^{-1,*}_{odd}(S^1,V) \to FCl^{0,*}_{Diff_+(S^1),odd}(S^1,V) \to DO^0(S^1,V) \rtimes Diff_+(S^1) \to 1$$

for which the Y-part defines a smooth global section, and which is a morphism of groups.

Proof. We already know that U splits in an unique way as $U = A_0 \,.\, g$, in which $g \in Diff_+(S^1)$ and $A_0 \in Cl^{0,*}_{odd}(S^1,V)$. By Proposition 3.5, the pseudodifferential operator A_0 can be written uniquely as a sum, $A = A_I + A_D$, in which $A_D \in DO^0(S^1,V) \subset Cl_{odd}(S^1,V)$. Since A_0 is invertible, $\sigma_0(A_0) \in C^\infty(S^1, GL(V))$ and hence $A_D \in DO^{0,*}(S^1,V)$. We can write

$$U = A_0.A_D^{-1}.A_D.g.$$

We get $Y = A_D.g \in DO^{0,*}(S^1,V) \rtimes Diff_+(S^1)$ and $S = A_0.A_D^{-1} \in Cl^{0,*}_{odd}(S^1,V)$ (the inverse of an odd class operator is an odd class operator). Let us compute the principal symbol $\sigma_0(S)$:

$$\sigma_0(S) = \sigma_0(A_0)\sigma_0(A_D^{-1}) = \sigma_0(A_0)\sigma_0(A_0)^{-1} = Id_V\ .$$

Thus, $S \in Cl^{-1,*}_{odd}(S^1,V)$. Moreover, the maps $U \mapsto g$ and $A_0 \mapsto A_D$ are smooth, and this observation ends the proof. $\square$

Let us summarize our constructions. The semi-direct product of Fréchet Lie groups

$$FCl^{0,*}_{Diff_+(S^1),odd}(S^1,V) = Cl^{0,*}_{odd}(S^1,V) \rtimes Diff_+(S^1)$$

fully described by the exact sequence

$$1 \quad \to \quad Cl^{0,*}_{odd}(S^1,V) \quad \to \quad FCl^{0,*}_{Diff_+(S^1),odd}(S^1,V) \quad \to \quad Diff_+(S^1) \quad \to \quad 1$$

and by the associated sequence of Lie algebras

$$0 \quad \to \quad Cl^0_{odd}(S^1,V) \quad \to \quad Cl^0(S^1,V) \rtimes Vect(S^1) \quad \to \quad Vect(S^1) \quad \to \quad 0\,,$$

in which we have used (3.1) and (3.2) in order to understand differential operators having symbols of order 1 as elements of $Vect(S^1) \otimes Id_V$, can be completed by the following diagram in which vertical and horizontal lines are short exact sequences of Lie groups:

$$\begin{array}{ccccccccc}
 & & & & 1 & & 1 & & \\
 & & & & \downarrow & & \downarrow & & \\
1 & \to & Cl^{-1,*}_{odd}(S^1,V) & \to & Cl^{0,*}_{odd}(S^1,V) & \to & DO^{0,*}(S^1,V) & \to & 1 \\
 & & \| & & \downarrow & & \downarrow & & \\
1 & \to & Cl^{-1,*}_{odd}(S^1,V) & \to & FCl^{0,*}_{Diff_+(S^1),odd}(S^1,V) & \to & DO^{0,*}(S^1,V) \rtimes Diff_+(S^1) & \to & 1 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & Diff_+(S^1) & = & Diff_+(S^1) & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 1 & & 1 & &
\end{array}$$

The corresponding diagram of Lie algebras, all of them embedded in $Cl_{odd}(S^1,V)$ is:

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \rightarrow & Cl^{-1}_{odd}(S^1,V) & \rightarrow & Cl^0_{odd}(S^1,V) & \rightarrow & DO^0(S^1,V) & \rightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \rightarrow & Cl^{-1}_{odd}(S^1,V) & \rightarrow & Cl^0_{odd}(S^1,V)\rtimes Vect(S^1) & \rightarrow & DO^0(S^1,V)\rtimes Vect(S^1) & \rightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & Vect(S^1) & = & Vect(S^1) & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

We end this appendix by considering exponential mappings. We can do so, since the Lie groups $Cl^{-1,*}_{odd}(S^1,V)$, $FCl^{0,*}_{Diff_+(S^1),odd}(S^1,V)$ and $DO^0(S^1,V)\rtimes Diff_+(S^1)$ are regular (see our discussion at the beginning of this section and Definition 24). Let us consider a curve $L(t)$ in the Lie algebra of $FCl^{0,*}_{Diff_+(S^1),odd}(S^1,V)$ which, thanks to Proposition 25 and the Mulase decomposition, we can identify (as a vector space) with $Cl^{-1}_{odd}(S^1,V)\oplus DO^1(S^1,V)$. Thus, we assume

$$L(t)\in C^\infty([0;1],Cl^{-1}_{odd}(S^1,V)\oplus DO^1(S^1,V))$$

and we write $L(t)=L_D(t)+L_S(t)$. We compare the exponential $\exp(L)(t)\in C^\infty([0;1],FCl^{0,*}_{Diff_+(S^1),odd}(S^1,V))$ with

$$\exp(L_D)(t)\in C^\infty\left([0;1],DO^{0,*}(S^1,V))\rtimes Diff_+(S^1)\right)$$

and

$$\exp(L_S)(t)\in C^\infty([0;1],Cl^{-1,*}_{odd}(S^1,V))\ .$$

On the one hand, we can write

$$\exp(L)(t)=S(t)Y(t)$$

according to Theorem 5.7, and we know that the paths $t\mapsto S(t)$ and $t\mapsto Y(t)$ are smooth. On the other hand, using the definition of the left exponential map, we get

$$\frac{d}{dt}\exp(L)(t)=exp(L)(t).L(t)\ .$$

Thus, gathering the last two expressions we obtain

$$\begin{aligned}
&\frac{d}{dt}\exp(L)(t)\\
=\ &\frac{d}{dt}(S(t)Y(t))\\
=\ &\left(\frac{d}{dt}S(t)\right)S^{-1}(t)S(t)Y(t)+S(t)Y(t)Y^{-1}(t)\left(\frac{d}{dt}Y(t)\right)\\
=\ &\left(\frac{d}{dt}S(t)S^{-1}(t)\right)\exp(L)(t)+\exp(L)(t)Y^{-1}(t)\left(\frac{d}{dt}Y(t)\right)\\
=\ &\exp(L)(t)\left(Ad_{\exp(L)(t)^{-1}}\left(\left(\frac{d}{dt}S(t)S^{-1}(t)\right)\right)+Y^{-1}(t)\left(\frac{d}{dt}Y(t)\right)\right)\ .
\end{aligned}$$

Now, $Y^{-1}(t)\frac{d}{dt}Y(t)$ is a smooth path on the space of differential operators of order 1, and we have

$$Ad_{\exp(L)(t)^{-1}}\left(\left(\frac{d}{dt}S(t)S^{-1}(t)\right)\right) \in Cl^{-1}_{odd}(S^1,V)\ .$$

These calculations allow us to prove the following:

PROPOSITION 5.8. *Let us assume that* $L(t)$ *is a curve in the Lie algebra of the group* $FCl^{0,*}_{Diff_+(S^1),odd}(S^1,V)$, *that* $L(t)=L_S(t)+L_D(t)$ *with* $L_S(t)\in Cl^{-1}_{odd}(S^1,V)$ *and* $L_D(t)\in DO^1(S^1,V)$, *and that* $\exp(L)(t)=S(t)Y(t)$. *Then,*

$$Y(t)=\exp(L_D)(t)$$

and

$$S(t)=\exp\left(Ad_{\exp(L)(t)}\left(L_S\right)\right)(t)\ .$$

PROOF. We have already obtained that

$$L_D = Y(t)^{-1}\frac{d}{dt}Y(t)$$

and that

$$L_S = Ad_{\exp(L)(t)^{-1}}\left(\left(\frac{d}{dt}S(t)S^{-1}(t)\right)\right)$$

because of the uniqueness of the decomposition

$$L=L_S+L_D\ .$$

We obtain the result by passing to the exponential maps on the groups $Cl^{-1,*}_{odd}(S^1,V)$ and $DO^{0,*}(S^1,V)\rtimes Diff_+(S^1)$. □

Acknowledgements: Both authors have been partially supported by ANID (Chile) via the *Fondo Nacional de Desarrollo Científico y Tecnológico* operating grants # 1161691 and # 1201894. The authors would like to thank Saad Baaj for comments leading to Remark 5.5.

References

[1] M. Adams, T. Ratiu, R. Schmidt, A Lie group structure for Fourier integral operators; *Math. Annalen* **1986** *276*, no 1, 19–41.
[2] M. Adams, T. Ratiu, R. Schmidt, The Lie group structure of diffeomorphism groups and invertible Fourier integral operators, with applications. In: "Infinite Dimensional Groups with Applications", V. Kac (Ed.) MSRI Publications, Springer, 1985.
[3] N. Berline, E. Getzler, M. Vergne, Heat Kernels and Dirac Operators, Springer (2004)
[4] J. Bokobza-Haggiag, Opérateurs pseudo-différentiels sur une variété différentiable; *Ann. Inst. Fourier, Grenoble* **19,1** 125-177 (1969).
[5] A. Cardona, C. Ducourtioux, J.-P. Magnot, S. Paycha, Weighted traces on pseudodifferential operators and geometry on loop groups; *Infin. Dimens. Anal. Quantum Probab. Relat. Top.* **5** no4 503-541 (2002)
[6] K. T. Chen, *Iterated Path Integrals.* Bulletin AMS **83** (1977), 831–879.
[7] D. Christensen, E. Wu, Tangent spaces and tangent bundles for diffeological spaces. *Cahiers de Topologie et Géométrie Différentielle* Volume **LVII** (2016), 3–50.
[8] L. A. Dickey, Soliton equations and Hamiltonian systems, second edition (2003). Advanced Series in Mathematical Physics 12, World Scientific Publ. Co., Singapore.
[9] C. Ducourtioux, *Weighted traces on pseudodifferential operators and associated determinants* **Ph.D thesis, Université Blaise Pascal** Clermont-Ferrand, France (2000)
[10] D. Dugmore, P. Ntumba, On tangent cones of Frölicher spaces *Quaetiones mathematicae* **30** no.1 (2007), 67–83.

[11] J. Eells, A setting for global analysis *Bull. Amer. Math. Soc.* **72** 751-807 (1966)
[12] A. Eslami Rad, E. G. Reyes, The Kadomtsev-Petviashvili hierarchy and the Mulase factorization of formal Lie groups *J. Geom. Mech.* **5**, no 3 (2013) 345–363.
[13] A. Eslami Rad, J.-P. Magnot, E. G. Reyes, The Cauchy problem of the Kadomtsev-Petviashvili hierarchy with arbitrary coefficient algebra. *J. Nonlinear Math. Phys.* 24:sup1 (2017), 103–120.
[14] A. Frölicher, A. Kriegl, Linear spaces and differentiation theory, Wiley series in Pure and Applied Mathematics, Wiley Interscience (1988)
[15] P. Gilkey, Invariance theory, the heat equation and the Atiyah-Singer index theorem, Publish or Perish (1984)
[16] H. Glöckner, Algebras whose groups of the units are Lie groups *Studia Math.* **153**, no2 (2002), 147–177.
[17] G. Hector, Géométrie et topologie des espaces difféologiques, in Analysis and Geometry in: Foliated Manifolds (Santiago de Compostela, 1994), World Sci. Publishing (1995), pp. 55–80
[18] Hörmander,L.; Fourier integral operators. I; *Acta Mathematica* **127** 79-189 (1971)
[19] P. Iglesias-Zemmour, Diffeology, Mathematical Surveys and Monographs **185** AMS (2013).
[20] M. Kazarian, KP hierarchy for Hodge integrals; *Advances in Mathematics* 221 (2009) 1–21.
[21] M. Kontsevich, S. Vishik, Determinants of elliptic pseudodifferential operators Max Plank Institut fur Mathematik, Bonn, Germany, preprint n. 94-30 (1994)
[22] M. Kontsevich, S. Vishik, Geometry of determinants of elliptic operators. Functional analysis on the eve of the 21st century, Vol. 1 (New Brunswick, NJ, 1993), *Progr. Math.* **131**,173-197 (1995)
[23] A. Kriegl, P. W. Michor, The convenient setting for global analysis, Math. surveys and monographs **53**, American Mathematical society, Providence, USA. (2000)
[24] M. Laubinger, A Lie algebra for Frölicher groups. *Indag. Math.* **21** (2011) no 3-4, 156–174
[25] J. Leslie, On a Diffeological Group Realization of certain Generalized symmetrizable Kac-Moody Lie Algebras *J. Lie Theory* **13** (2003), 427-442.
[26] J.-P. Magnot, Renormalized traces and cocycles on the algebra of S^1-pseudodifferential operators; *Lett. Math. Phys.* **75** no2, 111-127 (2006)
[27] J.-P. Magnot, Chern forms on mapping spaces, *Acta Appl. Math.* **91**, no. 1, 67-95 (2006).
[28] J.-P. Magnot, Difféologie du fibré d'Holonomie en dimension infinie, *C. R. Math. Soc. Roy. Can.* **28** (2006) 121-127.
[29] J.-P. Magnot, Ambrose-Singer theorem on diffeological bundles and complete integrability of KP equations. *Int. J. Geom. Meth. Mod. Phys.* **10**, no 9 (2013) Article ID 1350043.
[30] J.-P. Magnot, On $Diff(M)-$pseudodifferential operators and the geometry of non linear grassmannians. *Mathematics* **4**, 1; doi:10.3390/math4010001 (2016)
[31] J.-P. Magnot, The group of diffeomorphisms of a non-compact manifold is not regular *Demonstr. Math.* 51, No. 1, 8–16 (2018)
[32] J.-P. Magnot, E. G. Reyes, The Cauchy problem of the Kadomtsev-Petviashvili hierarchy and infinite-dimensional groups. in *Nonlinear Systems and Their Remarkable Mathematical Structures, Volume 2;* Norbert Euler and Maria Clara Nucci Editors, CRC press (2019) section B6
[33] J.-P. Magnot, E. G. Reyes, Well-posedness of the Kadomtsev-Petviashvili hierarchy, Mulase factorization, and Frölicher Lie groups. *Ann. H. Poincaré* (2020). https://doi.org/10.1007/s00023-020-00896-3.
[34] L. Mendoza, E. G. Reyes, Massey products, A_∞-algebras, differential equations, and Chekanov homology. *J. Nonlinear Math. Phys* 22, 342–360 (2015).
[35] J. Mickelsson, Current algebras and groups, Plenum monographs in Nonlinear Physics, Springer (1989).
[36] M. Mulase, Complete integrability of the Kadomtsev-Petvishvili equation. *Advances in Math.* **54** (1984), 57–66.
[37] M. Mulase, Cohomological structure in soliton equations and Jacobian varieties. *J. Diff. Geom.* 19 (1984), 403–430.
[38] M. Mulase, Solvability of the super KP equation and a generalization of the Birkhoff decomposition. *Invent. Math.* **92** (1988), 1–46.
[39] S. Natanzon, A.V. Zabrodin, Formal solutions of the $\hbar-$KP hierarchy. *J. Phys. A: Math. Theor.* **49** ArticleId 145206 (2016)

[40] K.-H. Neeb, Towards a Lie theory of locally convex groups *Japanese J. Math.* **1** (2006), 291-468
[41] H. Omori, Infinite Dimensional Lie Groups, AMS Translations of Mathematical Monographs no **158** Amer. Math. Soc., Providence, R.I. (1997)
[42] S. Paycha, Regularised integrals, sums and traces. An analytic point of view. University Lecture Series **59**, AMS (2012).
[43] S. Paycha, Paths towards an extension of Chern-Weil calculus to a class of infinite dimensional vector bundles. *Geometric and topological methods for quantum field theory*, 81–143, Cambridge Univ. Press, Cambridge, (2013)
[44] A. M. Perelomov, Integrable systems of classical mechanics and Lie algebras, Birkhäuser Verlag, Berlin (1990).
[45] A. Pressley, G. Segal, Loop groups, OUP (1986)
[46] S. Scott, Traces and determinants of pseudodifferential operators, OUP (2010)
[47] J.-M. Souriau, un algorithme générateur de structures quantiques **Astérisque** (hors série) 341-399 (1985)
[48] A. Stacey, Comparative Smootheology. *Theory and Applications of Categories*, **25** (2011) No. 4, pp 64-117.
[49] K. Takasaki, T. Takebe, $\hbar-$expansion of KP hierarchy, Theor. Math. Phys. **171** 683-690 (2012).
[50] J. Watts, Diffeologies, differentiable spaces and symplectic geometry. **University of Toronto, PhD thesis.** arXiv:1208.3634v1.
[51] H. Widom, A complete symbolic calculus for pseudodifferential operators; *Bull. Sc. Math. 2e serie* **104** (1980) 19-63

LAREMA, Université d'Angers, 2 Bd Lavoisier, 49045 Angers cedex 1, France and Lycée Jeanne d'Arc, 40 avenue de Grande Bretagne, 63000 Clermont-Ferrand, France.
Email address: magnot@math.univ-angers.fr; jean-pierr.magnot@ac-clermont.fr

Departamento de Matemática y Ciencia de la Computación, Universidad de Santiago de Chile, Casilla 307 Correo 2, Santiago, Chile.
Email address: enrique.reyes@usach.cl; e_g_reyes@yahoo.ca