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Maximum principle for the fractional N-Laplacian flow

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Abstract. We deal with a family of the fractional N-Laplacian heat flows with variable exponent time-derivative on the Orlicz-Sobolev spaces. We get the maximum principle for these problems. We use the approximating method to get this result: We first show existence of a unique family of the approximating weak solutions from the variable exponent difference fractional N-Laplacian problems. We next show the maximum principle for the family of the approximating weak solution from the variable exponent difference fractional N-Laplacian problem, show the convergence of a family of the approximating weak solutions to the limits, and then obtain the maximum principle for the weak solution of a family of the fractional N-Laplacian heat flows with the variable exponent time-derivative on the Orlicz-Sobolev spaces.

Contents

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1. Introduction

We consider a family of the nonlinear fractional N-Laplacian heat flows with variable exponent time derivative on the fractional Orlicz-Sobolev spaces (See Section 2 for the definitions) under the initial-boundary conditions

$$
(1.1)\qquad \begin{cases} \partial_t(|u|^{r(x)-1}u(x)) + (-\Delta)^s_g u &= 0, \qquad \text{in } \ \Omega\times(0,T) \\ u(x,t) &= 0 \qquad \text{on } \ \partial\Omega\times(0,T), \\ u(x,0) &= u_0 \qquad \text{in } \ \Omega, \end{cases}
$$

where Ω is a bounded domain of R^N, $N \geq 3$, with smooth boundary $\partial\Omega$, $0 < s < 1$, $T > 0$ is an arbitrary number, g is an odd, increasing homeomorphism and C^1 function from $[0,\infty)$ onto $[0,\infty)$ with $g(|x|)\frac{x}{|x|} = G'(x)$, $G(x) = \int_0^{|x|} g(t)dt$ for all $x \in \mathrm{R}$ is an even function, a Young function and also an N-function (see Section 2 for the definitions of a Young function and an N-function), $u = (u^i) = (u^i(x,t))$ is a vector-valued measurable function defined on $\Omega \times [0,T]$ with values into R^k, $r : \bar{\Omega} \to (1,\infty)$ is a continuous function such that $1 < r(x) < \infty$ and $(-\Delta)^s_g$ is the fractional N-Laplacian operator defined on the fractional Orlicz-Sobolev space defined as follows: for each $x \in \Omega \subset \mathrm{R}^N$, $s \in (0,1)$ and any u in the fractional Orlicz-Sobolev space,

$$
(-\Delta)^s_g u(x) = 2\mathrm{P.V.} \int_\Omega g\left(\frac{|u(x)-u(y)|}{|x-y|^s}\right) \frac{u(x)-u(y)}{|u(x)-u(y)|} \frac{dy}{|x-y|^{N+s}}.
$$

Here P.V. denotes the Cauchy principle value. For $0 < s < 1$, $(-\Delta)^s_g$ is called as the fractional N-Laplacian operator. From now on we call (1.1) as the fractional N-Laplacian heat flow.

In this paper we obtain the maximum principle for (1.1). We first obtain the maximum principle for the family of the approximating weak solutions obtained from the variable exponent difference fractional N-Laplacian problem which is a crucial role for the boundedness of the family of the approximating weak solutions and next we show the weak and strong convergence of the family of the approximating weak solutions to the limit.

When $g(\tau) = |\tau|^{p-2}\tau$ and $r(x) = q$ in (1.1), (1.1) becomes

$$
\partial_t(|u|^{q-1}u) + (-\Delta)^s_p u = 0,
$$
$$
u(x,t) = 0 \qquad \text{on } \ \partial\Omega\times(0,T),
$$
$$
u(x,0) = u_0 \qquad \text{in } \ \Omega,
$$

where $(-\Delta)^s_p u(x)$ is the fractional $p-$Laplacian operator defined as

$$
(-\Delta)^s_p u(x) = 2\mathrm{P.V.} \int_\Omega \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{N+s(p+1)}} dy, \qquad x \in \Omega.
$$

In [11], Nakamura and Misawa considered existence of weak solution for the $p-$Sobolev flow

$$
(1.2)\qquad \begin{cases} \partial_t(u|^{q-1}|u) &= \mathrm{div}\left(|\nabla u|^{p-2}\nabla u\right) \qquad \text{in } \ \Omega\times(0,T), \\ u(x,t) &= 0 \qquad \text{on } \ \partial\Omega\times(0,T), \\ u(x,0) &= u_0 \qquad \text{in } \ \Omega, \end{cases}
$$

where $2 \leq p < N$, $q = \frac{Np}{N-p} - 1$. They proved existence of a weak solution of (1.2) by using the approximation method. For the case that $p > 2$, (1.2) is called as $p-$Laplace flow, and for the case that $p = 2$, (1.2) is called as Yamabe flow. Collectively, for the case that $p \geq 2$, (1.2) is called as $p-$Sobolev flow. The Yamabe flow was originally introduced by Hamilton in [8] in his study of the Yamabe problem, the existence of a conformal metric of constant curvature on $N(\geq 3)$-dimensional closed Riemannian manifolds. we refer the readers to [1, 8, 11, 17, 18] for some results about the $p-$Laplace flow and Yamabe flow.

The fractional $N-$Laplacian problems arise in applications of nonlinear elasticity theory, electro rheological fluids, non-Newtonian fluid theory in a porous medium (cf. [4, 14, 19]).

In this paper we deal with the Orlicz spaces, the fractional Orlicz-Sobolev spaces and the fractional $N-$Laplacian operators as more generalized spaces and the corresponding operators than the $p-$Lebesgue Sobolev spaces and their corresponding operators on the growth condition. We refer the readers to [6, 12] and references therein for the theory of Orlicz and Orlicz-Sobolev spaces. We also refer the readers to [3, 13, 15] for some results about the fractional Orlicz-Sobolev spaces and the fractional $N-$Laplacian operator. In [3], the authors provide the connection between the fractional order theories and the Orlicz-Sobolev ones, and define the fractional Orlicz-Sobolev space associated to a Young function and a fractional parameter.

The weak solution of (1.1) is a solution $u \in L^\infty(0,T; W_0^s L_G(\Omega))$ in the weak sense satisfying: for fixed $s \in (0,1)$ and G,

$$\int_0^T \int_\Omega \partial_t\big(|u|^{r(x)-1}u\big)\phi dxdt + \int_0^T \int_\Omega (-\Delta)_g^s u \cdot \phi dxdt = 0 \qquad \text{for any } \phi \in W_0^s L_G(\Omega_T),$$

i.e.,

$$\begin{aligned}&\int_0^T \int_\Omega \partial_t\big(|u|^{r(x)-1}u\big)\phi dxdt \\ &+ \int_0^T \int_\Omega \int_\Omega g\left(\frac{|u(x)-u(y)|}{|x-y|^s}\right) \frac{u(x)-u(y)}{|u(x)-u(y)|} \frac{\phi(x)-\phi(y)}{|x-y|^s} \frac{dxdy}{|x-y|^N} dt = 0.\end{aligned}$$

Let us set

$$g_0 = \inf_{t>0} \frac{tg(t)}{G(t)} \qquad g^0 = \sup_{t>0} \frac{tg(t)}{G(t)}.$$

We assume that

$$1 < g_0 \leq \frac{tg(t)}{G(t)} \leq g^0 < \infty \qquad \forall t \geq 0. \tag{1.3}$$

By Proposition 2.3 of [11], it implies that each G satisfies the Δ_2-condition, i.e., there exists a constant $C > 0$ such that

$$G(2t) \leq CG(t), \qquad t \geq 0.$$

We also assume that G is a function such that

$$G : t \in [0,\infty) \mapsto G(\sqrt{t}) \text{ is convex.} \tag{1.4}$$

Let

$$< (-\Delta)_g^s u, \phi >= \int_\Omega \int_\Omega g\left(\frac{|u(x)-u(y)|}{|x-y|^s}\right) \frac{u(x)-u(y)}{|u(x)-u(y)|} \frac{\phi(x)-\phi(y)}{|x-y|^s} \frac{dxdy}{|x-y|^N},$$

which is provided in [16]. Let us set the N-energy defined on the $W_0^s L_G(\Omega)$ as

$$I_{(s,G)}(u) = \int_\Omega \int_\Omega G\left(\frac{|u(x)-u(y)|}{|x-y|^s}\right) \frac{dxdy}{|x-y|^N},$$

where $g(|u|)\frac{u}{|u|} = G'(|u|)$. Then the functional $I_{(s,G)}$ defined on the $W^sL_G(\Omega)$ is of class $C^1(W^sL_G(\Omega), \mathrm{R})$ and

$$\begin{aligned} < I'_{(s,G)}(u), v > &= \int_\Omega\int_\Omega g\left(\frac{|u(x)-u(y)|}{|x-y|^s}\right)\frac{u(x)-u(y)}{|u(x)-u(y)|}\frac{v(x)-v(y)}{|x-y|^s}\frac{dxdy}{|x-y|^N} \\ &= < (-\Delta)^s_g u, v >, \end{aligned}$$

which is proved in Proposition 3.3 in [15].

Our main result is as follows:

Theorem 1.1. *(Maximum Principle)*
Assume that $s \in (0,1)$, G is an N-function, $u_0 \in W^s_0L_G(\Omega) \cap L_\infty(\Omega)$, (1.3) and (1.4) hold, and $r : \bar\Omega \to (1,\infty)$ is a continuous function such that $1 < r(x) < g_0^* = \frac{Ng_0}{N-sg_0}$, $N > sg_0$. Let u be a weak solution of problem (1.1). Then

$$sup_{0<t<T}\|u(x,t)\|_{L^\infty(\Omega)} \le \|u_0\|_{L^\infty(\Omega)}.$$

For the proof of Theorem 1.1 we use the approximation method. The organization of the paper is as follows: In Section 1 and Section 2, we introduce some notations and definitions, give some properties for the Orlicz space $L_G(\Omega)$, the fractional Orlicz-Sobolev space $W^s_0L_G(\Omega)$, the fractional N-Laplacian operators. In Section 3, we first show the existence of a unique sequence of the approximating weak solutions from the variable exponent difference fractional N-Laplacian problems defined on the fractional Orlicz-Sobolev space $W^s_0L_G(\Omega)$. We next show the maximum principle for the family of the approximating weak solution from the variable exponent difference fractional N-Laplacian problem, show the convergence of a family of the approximating weak solutions to the limits, and then obtain the maximum principle for the weak solution of a family of the fractional N-Laplacian heat flows with the variable exponent time-derivative on the Orlicz-Sobolev spaces.

2. Preliminaries

Let $0 < s < 1$ and $T > 0$ be an arbitrary number. Let g be an odd, increasing homeomorphism and C^1 function from $[0,\infty)$ onto $[0,\infty)$ and let G be the function defined by

$$G(x) = \int_0^{|x|} g(t)dt \qquad \text{for all } \; x \in R.$$

Then G is an even function with respect to the variable $x \in R$, a Young function and also an N-function. A continuous, convex function, $G : R^+ \to R^+$ is called as a Young function if $G(0) = 0$, $\lim_{x\to+\infty} G(x) = +\infty$ and G is convex. A continuous, convex function, $G : R \to R^+$ is called as an N-function if it is even and if $G(x) = 0$ if and only if $x = 0$, $\lim_{x\to 0}\frac{G(x)}{x} = 0$, $\lim_{x\to\infty}\frac{G(x)}{x} = +\infty$). Equivalently, G is an N-function if and only if there exists a nondecreasing, right continuous function $g : [0,\infty) \to R^+$ such that $g(0) = 0$, $g(t)$ is positive for all $t \in (0,\infty)$, $\lim_{t\to+\infty} g(t) = +\infty$ and $G(x) = \int_0^{|x|} g(t)dt$, $\forall x \in R$. The difference between a Young function and an N-function is that a Young function admits the integral formulation $\int_0^x g(t)dt$, $\forall x > 0$, and an N-function is an even function admitting $\int_0^{|x|} g(t)dt$, $\forall x \in R$.

Let $r : \bar{\Omega} \to (1, \infty)$ be a continuous function such that $1 < r(x) < g_0^* = \frac{Ng_0}{N - sg_0}$. The $r(x)-$Lebesgue space $L^{r(x)}(\Omega)$ is the Banach space defined as

$$L^{r(x)}(\Omega) = \Big\{u|\ u : \Omega \subset R^N \to R^k : \text{ is a vector-valued measurable function with } \int_\Omega |u(x)|^{r(x)} dx < \infty\Big\}$$

with the norm

$$\|u\|_{L^{r(x)}(\Omega)} = \left(\int_\Omega |u(x)|^{r(x)} dx\right)^{\frac{1}{r(x)}}$$

and for $r(x) = \infty$, $L^\infty(\Omega)$ is the Banach space of essentially bounded vector-valued function with the norm

$$\|u\|_{L^\infty(\Omega)} = ess\ sup_{x\in\Omega}|u(x)|.$$

For $1 \le r(x) < \infty$, the $r(x)-$Sobolev space $W^{1,r(x)}(\Omega)$ is the Banach space defined as

$$W^{1,r(x)}(\Omega) = \Big\{u \in L^{r(x)}(\Omega)|\ \int_\Omega |\nabla u(x)|^{r(x)} dx < \infty\Big\}$$

with the norm

$$\|u\|_{W^{1,r(x)}(\Omega)} = \left(\int_\Omega [|u(x)|^{r(x)} + |\nabla u(x)|^{r(x)}] dx\right)^{\frac{1}{r(x)}},$$

where $\nabla u = grad\ u = (\nabla u^1, \dots, \nabla u^k)$. Let $W_0^{1,r(x)}(\Omega)$ be the closure of $C_0^\infty(\Omega)$ with respect to the norm $\|\cdot\|_{W^{1,r(x)}(\Omega)}$. For $1 \le q \le \infty$, $L^q(0, T; L^{r(x)}(\Omega))$ is the Banach space defined as

$$L^q(0, T; L^{r(x)}(\Omega)) = \Big\{u|\ u \text{ is a vector-valued functions defined on } \Omega \times (0, T) \text{ with } \int_0^T \|u(t)\|^q_{L^{r(x)}(\Omega)} dt < \infty\Big\}$$

with the norm

$$\|u\|_{L^q(0,T;L^{r(x)}(\Omega))} = \begin{cases} \left(\int_0^T \|u(t)\|^q_{L^{r(x)}(\Omega)} dt\right)^{\frac{1}{q}}, & 1 \le q < \infty, \\ ess\ sup_{0\le t\le T}\|u(t)\|_{L^{r(x)}(\Omega)} & q = \infty \end{cases}$$

and $L^q(0, T; W_0^{1,r(x)}(\Omega))$ be the Banach space with the finite norm

$$\|u\|_{L^q(0,T;W_0^{1,r(x)}(\Omega))} = \left(\int_0^T \|u(t)\|^q_{W_0^{1,r(x)}(\Omega)} dt\right)^{\frac{1}{q}}.$$

Moreover, let $L^{r(x)}(\Omega_T)$ be the Banach space defined as

$$L^{r(x)}(\Omega_T) = \Big\{u|\ u : \Omega \times (0, T) \to R^k : \text{ is a vector-valued measurable function with } \int\int_{\Omega_T} |u(x, t)|^{r(x)} dxdt < \infty\Big\}.$$

with the norm

$$\|u\|_{L^{r(x)}(\Omega_T)} = \left(\int\int_{\Omega_T} |u(x, t)|^{r(x)} dxdt\right)^{\frac{1}{r(x)}}.$$

Let $s \in (0,1)$, $r : \Omega \times \Omega \to (1, \infty)$ *be continuous functions with* $1 < r(x,y) < \infty$. *The fractional Sobolev space with variable exponent* $W^{s,r(x,y)}(\Omega)$ *is the Banach space defined as*

$$W^{s,r(x,y)}(\Omega) = \Big\{ u \in L^{r(x,y)}(\Omega) | \ \int_\Omega \int_\Omega \frac{|u(x)-u(y)|^{r(x,y)}}{|x-y|^{N+sr(x,y)}} dxdy < \infty \Big\}$$

with the norm

$$\|u\|_{s,r(x,y)} = \Big(\|u\|^{r(x,y)}_{L^{r(x,y)}(\Omega)} + [u]^{r(x,y)}_{s,r(x,y)} \Big)^{\frac{1}{r(x,y)}},$$

where

$$[u]_{s,r(x,y)} = \inf \Big\{ \lambda > 0 | \ \int_\Omega \int_\Omega \frac{|u(x)-u(y)|^{r(x,y)}}{\lambda^{r(x,y)}|x-y|^{N+sr(x,y)}} dxdy \le 1 \Big\}.$$

Let G *be an N-function such that* $G'(x) = g(x)$ *for all* $x > 0$ *and let* G^* *be the functions defined by*

$$G^*(x) = \int_0^x g^{-1}(t)dt \qquad \text{for all} \ \ x \ge 0.$$

The function G^* *is called the complementary function of* G *and satisfies*

$$G^*(x) = \sup\{yx - G(y) | \ y \ge 0\} \qquad \text{for all} \ \ x \ge 0.$$

Then G^* *satisfies that*

$$\lim_{x \to 0} \frac{G^*(x)}{x} = 0, \qquad \lim_{x \to \infty} \frac{G^*(x)}{x} = +\infty,$$

i.e., G^* *is an N-function. Moreover, by Young's inequality,*

$$xy \le G(x) + G^*(y), \qquad \text{for all} \ \ x, \ y \ge 0.. \tag{2.1}$$

The Orlicz space $L_G(\Omega)$ *defined by N-function* G *is the space defined as*

$$L_G(\Omega) = \Big\{ u | \ u : \Omega \to R^k \ \text{is a measurable function with}$$
$$\|u\|_{L_G} = \sup\{ \int_\Omega uvdx | \ \int_\Omega G^*(|v|)dx \le 1\} < \infty \ \ \Big\}.$$

Then $L_G(\Omega)$ *is a Banach space with a norm* $\|u\|_{L_G}$. *We note that the norm* $\|u\|_{L_G}$ *is equivalent to the Luxemburg norm*

$$\|u\|_G = \inf \Big\{ \lambda > 0 | \ \int_\Omega G\left(|\frac{u(x)}{\lambda}| \right) \le 1 \Big\}.$$

In the Orlicz space $L_G(\Omega)$, *Hölder inequality is valid (see [13]): for all* $u \in L_G(\Omega)$, $v \in L_{G^*}(\Omega)$, *we have*

$$\int_\Omega |uv| dx \le 2 \|u\|_{L_G} \|v\|_{L_{G^*}}. \tag{2.2}$$

In [3], the Orlicz-Sobolev space $W^1 L_G(\Omega)$ *is defined by*

$$W^1 L_G(\Omega) = \Big\{ u \in L_G(\Omega) | \ \frac{\partial u}{\partial x_i} \in L_G(\Omega), \ i = 1, \ldots, N \Big\}$$

with the norm

$$\|u\|_{1,G} = \|u\|_G + \|\nabla u\|_G.$$

Then $W^1 L_G(\Omega)$ *is a reflexive Banach space. The Orlicz-Sobolev space* $W^1_0 L_G(\Omega)$ *is defined by the closure of* $C_0^\infty(\Omega)$ *in* $W^1 L_G(\Omega)$. *The space* $W^1 L_G(\Omega)$ *is also a* *reflexive Banach space. By Lemma 5.7 in [7], the norm* $\|\nabla u\|_G$ *is an equivalent to the norm* $\|u\|_{1,G}$ *in* $W_0^1 L_G(\Omega)$*. For any given* $0 < s < 1$ *and* G *an N-function, the fractional Orlicz-Sobolev space* $W^s L_G(\Omega)$ *is the space defined by*

$$W^s L_G(\Omega) = \Big\{ u \in L_G(\Omega) | \int_\Omega \int_\Omega G\left(\frac{|u(x)-u(y)|}{|x-y|^s}\right) \frac{dxdy}{|x-y|^N} < \infty \Big\}$$

endowed with the norm

$$\|u\|_{s,G} = \|u\|_G + [u]_{s,G},$$

where $[u]_{s,G}$ *is the Gagliardo semi-norm defined by*

$$[u]_{s,G} = \inf \Big\{ \lambda > 0 | \int_\Omega \int_\Omega G\left(\frac{|u(x)-u(y)|}{\lambda |x-y|^s}\right) \frac{dxdy}{|x-y|^N} \le 1 \Big\}.$$

By [3], for any $0 < s < 1$ *and* G *a Young function such that* G *and* G^* *satisfy that*

$$G(2t) \le C_1 G(t) \text{ and } G^*(2t) \le C_2 G^*(t), \qquad \forall t \ge 0,\ C_1,\ C_2 > 0,$$

$W^s L_G(R^N)$ *is a reflexive and separable Banach space. Furthermore* $C_0^\infty(R^N)$ *is dense in* $W^s L_G(R^N)$ *in the norm* $\|\cdot\|_{s,G}$*. Let us denote* $W_0^s L_G(\Omega)$ *as the closure of* $C_0^\infty(\Omega)$ *in* $W^s L_G(\Omega)$ *in the norm* $\|\cdot\|_{s,G}$ *such that*

$$W_0^s L_G(\Omega) = \Big\{ u \in W^s L_G(\Omega) | \ u = 0 \ a.e., \ on \ \ \partial\Omega \Big\}.$$

Lemma 2.1 (15). *(Generalized Poincaré inequality on the Orlicz-Sobolev space) Let* Ω *be a bounded open subset of* R^N, $0 < s < 1$ *and* G *be an N-function. Then there exists a positive constant* $C_{(s,G)} > 0$ *such that*

$$\|u\|_G \le C_{(s,G)} [u]_{s,G}, \qquad \forall u \in W_0^s L_G(\Omega).$$

Moreover the embedding

$$W_0^s L_G(\Omega) \hookrightarrow L_G(\Omega)$$

is continuous and compact. Furthermore $[u]_{s,G}$ *is a norm of* $W_0^s L_G(\Omega)$ *equivalent to* $\|\cdot\|_{s,G}$.

Lemma 2.2 (2). *Let* $u \in W^s L_G(\Omega)$*. Then*

$$[u]_{s,G}^{g_0} \le I_{(s,G)}(u) \le [u]_{s,G}^{g^0}, \qquad if \ \ [u]_{s,G} > 1,$$

$$[u]_{s,G}^{g^0} \le I_{(s,G)}(u) \le [u]_{s,G}^{g_0}, \qquad if \ \ [u]_{s,G} < 1.$$

It follows that

$$\min\{[u]_{s,G}^{g_0}, [u]_{s,G}^{g^0}\} \le I_{(s,G)}(u) \le \max\{[u]_{s,G}^{g_0}, [u]_{s,G}^{g^0}\}. \tag{2.3}$$

Moreover, there exist constants $\eta > 0$ *such that*

$$\|u\|_{s,g_0} \le \eta \|u\|_{s,G}.$$

Thus the embedding

$$W^s L_G(\Omega) \hookrightarrow W^{s,g_0}(\Omega)$$

is continuous.

Proof. The proof is given by Proposition 3.10 and Theorem 3.11 of [2].

Lemma 2.3. *([16]) Let Ω be a bounded open subset of R^n, $0 < s < 1$ and G be an N-function, $g = G'$ for $x > 0$. Let $1 \le r(x) < g_0^* = \frac{Ng_0}{N - sg_0}$, $N > sg_0$. Then the embedding*

$$W^{s,g_0} \hookrightarrow L^{r(x)}$$

is continuous and compact for all $1 \le r(x) < g_0^*$. Moreover the embedding

$$W^s L_G(\Omega) \hookrightarrow L^{r(x)}$$

is continuous and compact for all $1 \le r(x) < g_0^*$. Furthermore there exists a positive constant $C_{(s,G)}$ such that

$$\|u\|_{L^{r(x)}} \le C_{(s,G)}[u]_{s,G}. \tag{2.4}$$

Proof. Since the embedding $W^{s,g_0} \hookrightarrow L^{r(x)}$ is continuous and compact for all $1 \le r(x) < g_0^*$ and by Lemma 2.2, the embedding $W^s L_G(\Omega) \hookrightarrow W^{s,g_0}$ is continuous, it follows that for all $1 \le r(x) < g_0^*$, the embedding $W^s L_G(\Omega) \hookrightarrow L^{r(x)}$ is continuous and compact.

Lemma 2.4. *If u, $u_n \in W^s L_G(\Omega)$, $n = 1, 2, \ldots$ for fixed s and G, then the following statement are equivalent to each other*
(i) $\lim_{n\to\infty} \|u_n - u\|_{s,G} = 0$, $i = 1, 2$,
(ii) $\lim_{n\to\infty} \int_\Omega G(u_n(x) - u(x))dx = 0$ *and* $\lim_{n\to\infty}[u_n - u]_{s,G} = 0$,
(iii) $u_n \to u$ *in measure in* $W^s L_G(\Omega)$ *and* $\lim_{n\to\infty} \int_\Omega G(u_n(x))dx = \int_\Omega G(u(x))dx$.

Proof. By the definition of $\| \cdot \|_{s,G}$, (i)⇔(ii) holds. We shall show that (i) implies (iii). We assume that (i) holds. Then

$$\begin{aligned} \int_\Omega [G(u_n) - G(x)]dx &\le \int_\Omega g(u + \theta(u_n - u))(u_n - u)dx \\ &\le \|g(u + \theta(u_n - u))\|_{G^*}\|u_n - u\|_G \\ &\le \|g(u + \theta(u_n - u))\|_{G^*}\|u_n - u\|_{s,G} \to 0 \end{aligned}$$

for $0 < \theta < 1$. It follows that (iii) holds. Assume that (iii) holds. Since $\lim_{n\to\infty} \int_\Omega G(u_n(x))dx = \int_\Omega G(u(x))dx$, u_n converges weakly to u in $L_G(\Omega)$. By Lemma 2.1, since $u_n \to u$ in measure in $W^s L_G(\Omega)$ and the embedding $W^s L_G(\Omega) \hookrightarrow L_G(\Omega)$ is continuous and compact, $u_n \to u$ strongly in $W^s L_G(\Omega)$. Thus (i) holds.

3. Proof of Theorem 1.1

Let $0 < s < 1$, $T > 0$ be an arbitrary number, G be an N-function, $u = (u^i) = (u^i(x,t))$ be a vector-valued measurable function defined on $\Omega \times [0, T]$ with values into R^k, $r : \bar{\Omega} \to (1, \infty)$ be a continuous function such that $1 < r(x) < g_0^* = \frac{Ng_0}{N - sg_0}$.

We shall show the maximum principle for (1.1) by using the existence of a unique sequence of the weak solutions for a family of the variable exponent difference fractional N-Laplacian equations on the fractional Orlicz-Sobolev space under the boundary and the initial conditions

$$\begin{cases} (-\Delta)_g^s u_n + \frac{|u_n|^{r(x)-1}u_n - |u_{n-1}|^{r(x)-1}u_{n-1}}{\sigma} = 0 \text{ in } \ \Omega, \ n = 1, 2, \ldots, \\ u_n = 0 \text{ on } \partial\Omega, \\ u_n(0) = u_0 \text{ in } \Omega, \end{cases} \tag{3.1}$$

i. e.,

$$(3.2)\qquad \begin{cases} \int_\Omega \int_\Omega g\left(\frac{|u_n(x)-u_n(y)|}{|x-y|^s}\right) \frac{u_n(x)-u_n(y)}{|u_n(x)-u_n(y)|} \frac{\phi(x)-\phi(y)}{|x-y|^s} \frac{dxdy}{|x-y|N} \\ + \int_\Omega \frac{|u_n|^{r(x)-1}u_n - |u_{n-1}|^{r(x)-1}u_{n-1}}{\sigma} \cdot \phi dx = 0 \end{cases}$$

for any $\phi \in W_0^s L_G(\Omega)$.

Lemma 3.1. *(A unique family of the weak solutions for the variable exponent difference fractional N-Laplacian problems)*
Assume that $s \in (0,1)$, G is an N-function, $g = G'$ for $x > 0$, (1.3) and (1.4) hold, $r : \bar{\Omega} \to (1,\infty)$ be a continuous function such that $1 < r(x) < g_0^* = \frac{Ng_0}{N-sg_0}$, $N > sg_0$, and $u_0 \in W_0^s L_G(\Omega) \cap L^\infty(\Omega)$. Then, for any $T > 0$ there exists a family of weak solutions $u_n \in W_0^s L_G(\Omega)$, $n = 1, 2, \ldots$ of (3.1).

Proof. For the proof we refer the reader to [9].

Let us introduce four families from a family of the weak solutions u_n, $n = 1, 2, \ldots$, of (3.1) as follows:
Let $T > 0$ be arbitrary fixed given number. Let $l \geq 2$ be an approximation number, which will be sent to $+\infty$ and $\sigma = \frac{T}{l}$.

$$(3.3)\begin{cases} t_0 = 0,\ t_n = n\sigma,\ n = 1, 2, \ldots, l, \\ u_\sigma(x,t) = \frac{t-t_{n-1}}{\sigma} u_n(x) + \frac{t_n - t}{\sigma} u_{n-1}(x), \qquad (x,t) \in \Omega \times (t_{n-1}, t_n] \\ u_\sigma(x,t) = u_0(x), \qquad (x,t) \in \Omega \times (-\sigma, 0], \\ \bar{u}_\sigma = u_n(x), \qquad (x,t) \in \Omega \times (t_{n-1}, t_n], \\ v_\sigma(x,t) = \frac{t-t_{n-1}}{\sigma} |u_n|^{r(x)-1} u_n + \frac{t_n - t}{\sigma} |u_{n-1}|^{r(x)-1} u_{n-1}, \\ \qquad\qquad (x,t) \in \Omega \times (t_{n-1}, t_n], \\ \bar{v}_\sigma = |\bar{u}_\sigma|^{r(x)-1} \bar{u}_\sigma. \end{cases}$$

(3.1) $(-\Delta)_g^s u_n + \frac{|u_n|^{r(x)-1}u_n - |u_{n-1}|^{r(x)-1}u_{n-1}}{\sigma} = 0$ can be rewritten in terms of these functions as

$$(3.4)\qquad (-\Delta)_g^s \bar{u}_\sigma + \partial_t v_\sigma = 0,$$

i.e.,

$$(-\Delta)_g^s \bar{u}_\sigma + \partial_t(\frac{t - t_{n-1}}{h} |u_n|^{r(x)-1} u_n + \frac{t_n - t}{h} |u_{n-1}|^{r(x)-1} u_{n-1}) = 0,$$

i.e.,

$$\int_0^T \int_\Omega (-\Delta)_g^s \bar{u}_\sigma \cdot \phi dxdt + \int_0^T \int_\Omega \partial_t v_\sigma \cdot \phi dxdt = 0$$

for any $\phi \in L^\infty(0, T; W_0^s L_G(\Omega))$. Since, by Lemma 3.1, $\{u_n\}$, $n = 1, 2$, is a unique sequence of the weak solutions for (3.1), $\bar{u}_\sigma$ is a unique sequence of the weak solutions for (3.4).

Lemma 3.2. *(Boundedness of some estimates)*
Assume that $s \in (0,1)$, G is an N-function, $g = G'$ for $x > 0$, $u_0 \in W_0^s L_G(\Omega) \cap L_\infty(\Omega)$, (1.3) and (1.4) hold and $r : \bar{\Omega} \to (1,\infty)$ is a continuous function such that $1 < r(x) < g_0^* = \frac{Ng_0}{N-sg_0}$, $N > sg_0$. Let $\{u_n\}$ be a unique sequence of weak solutions for (3.1) and let $\{u_\sigma\}$, $\{\bar{u}_\sigma\}$, $\{v_\sigma\}$ and $\{\bar{v}_\sigma\}$ be the sequences in (3.3). Then the following inequalities hold.
(i)

$$2\int_0^T \int_\Omega (-\Delta)_g^s \bar{u}_\sigma \cdot \bar{u}_\sigma dxdt + sup_{0<t<T} \int_\Omega |\bar{u}_\sigma|^{r(x)-1} \bar{u}_\sigma \cdot \bar{u}_\sigma dx + C$$

$$\leq \int_\Omega |u_0|^{r(x)-1}u_0 \cdot u_0 dx,$$

where $\lim_{\sigma\to 0} C = \lim_{\sigma\to 0}\sum_{n=1}^q \int_\Omega [|u_n|^{r(x)-1}u_n \cdot u_n - |u_{n-1}|^{r(x)-1}u_{n-1}\cdot u_n dx = 0.$
(ii)

$$(3.5)\qquad \begin{cases} sup_{0<t<T}\int_\Omega (-\Delta)^s_g \bar{u}_\sigma \cdot \bar{u}_\sigma dx + \int_0^T\int_\Omega [\int_0^1 \big[|s\bar{u}_\sigma(t) \\ \qquad +(1-s)\bar{u}_\sigma(t-h)|^{r(x)-1}\big] ds]\cdot |\partial_t \bar{u}_\sigma|^2 dxdt + D \\ \leq \int_\Omega (-\Delta)^s_g u_0 \cdot u_0 dx, \end{cases}$$

where $\lim_{\sigma\to 0} D = \lim_{\sigma\to 0}[-\sum_{n=1}^q \int_\Omega [(-\Delta)^s_g u_n \cdot u_{n-1}(x) - (-\Delta)^s_g u_{n-1}\cdot u_{n-1}]dx] = 0.$

Proof. (i) Let $\{u_n\}$, $n = 1, 2, \ldots, l$ be the unique sequence of the weak solutions of (3.1). We note that $\{u_n\}$, $n = 1, 2, \ldots, l$ is in $W^s_0 L_G(\Omega)$. Taking the test function $\phi = u_n$ in (3.1), we have

$$(3.6)\quad \Big\{\ \int_\Omega (-\Delta)^s_g u_n \cdot u_n dx + \int_\Omega \frac{|u_n|^{r(x)-1}u_n - |u_{n-1}|^{r(x)-1}u_{n-1}}{\sigma}\cdot u_n dx = 0,$$

i.e.,

$$(3.7)\qquad \begin{cases} \int_\Omega\int_\Omega g\left(\frac{|u_n(x)-u_n(y)|}{|x-y|^s}\right)\frac{u_n(x)-u_n(y)}{|u_n(x)-u_n(y)|}\frac{u_n(x)-u_n(y)}{|x-y|^s}\frac{dxdy}{|x-y|^N} \\ \qquad + \int_\Omega \frac{|u_n|^{r(x)-1}u_n - |u_{n-1}|^{r(x)-1}u_{n-1}}{\sigma}\cdot u_n dx = 0. \end{cases}$$

Summating over $n = 1, 2, \ldots, q$, for any $q = 1, 2, \ldots, l$, in (3.6), we have

$$(3.8)\begin{cases} \sum_{n=1}^q \int_\Omega [(-\Delta)^s_g u_n \cdot u_n dx \\ \qquad +\sum_{n=1}^q \frac{1}{\sigma}\int_\Omega \left(|u_n|^{r(x)-1}u_n - |u_{n-1}|^{r(x)-1}u_{n-1}\right)\cdot u_n dx = 0, \end{cases}$$

i.e.,

$$(3.9)\begin{cases} \sum_{n=1}^q \int_\Omega\int_\Omega g\left(\frac{|u_n(x)-u_n(y)|}{|x-y|^s}\right)\frac{u_n(x)-u_n(y)}{|u_n(x)-u_n(y)|}\frac{u_n(x)-u_n(y)}{|x-y|^s}\frac{dxdy}{|x-y|^N} \\ \qquad +\sum_{n=1}^q \frac{1}{\sigma}\int_\Omega \left(|u_n|^{r(x)-1}u_n - |u_{n-1}|^{r(x)-1}u_{n-1}\right)\cdot u_n dx = 0. \end{cases}$$

By (1.3), the first term of the left-hand side of (3.9) has the inequality

$$(3.10)\begin{cases} \sum_{n=1}^q \int_\Omega\int_\Omega g\left(\frac{|u_n(x)-u_n(y)|}{|x-y|^s}\right)\frac{u_n(x)-u_n(y)}{|u_n(x)-u_n(y)|}\frac{u_n(x)-u_n(y)}{|x-y|^s}\frac{dxdy}{|x-y|^N} \\ \geq g_0 \sum_{n=1}^q \int_\Omega\int_\Omega G\big(\frac{|u_n(x)-u_n(y)|}{|x-y|^s}\big)\frac{dxdy}{|x-y|^N} = g_0\sum_{n=1}^q I_{(s,G)}(u_n). \end{cases}$$

On the other hand, by Young's inequality,

$$u_{n-1}\cdot u_n \leq |u_{n-1}||u_n| \leq \frac{1}{2}|u_{n-1}|^2 + \frac{1}{2}|u_n|^2,$$

it follows that

$$|u_{n-1}|^{r(x)-1}u_{n-1}\cdot u_n \leq \frac{1}{2}|u_{n-1}|^{r(x)+1} + \frac{1}{2}|u_{n-1}|^{r(x)-1}u_n\cdot u_n.$$

Thus the second term of the left-hand side of (3.8) has the inequality

$$(3.11)\quad \begin{cases} \sum_{n=1}^q \frac{1}{\sigma}\int_\Omega \left(|u_n|^{r(x)-1}u_n - |u_{n-1}|^{r(x)-1}u_{n-1}\right)\cdot u_n dx \\ \geq \frac{1}{2\sigma}\sum_{n=1}^q \int_\Omega \left(|u_n|^{r(x)+1} - |u_{n-1}|^{r(x)+1}\right)dx \\ \qquad + \frac{1}{2\sigma}\sum_{n=1}^q \int_\Omega [|u_n|^{r(x)+1} - |u_{n-1}|^{r(x)-1}u_n^2]dx \\ = \frac{1}{2\sigma}\big[\int_\Omega |u_q|^{r(x)+1}dx - \int_\Omega |u_0|^{r(x)+1}dx\big] \\ \qquad + \frac{1}{2\sigma}\sum_{n=1}^q \int_\Omega [|u_n|^{r(x)+1} - |u_{n-1}|^{r(x)-1}u_n^2]dx. \end{cases}$$

Combining (3.8) and (3.11), we have

$$(3.12)\begin{cases} \sum_{n=1}^{q}\int_{\Omega}[(-\Delta)_g^s u_n\cdot u_n dx \\ \qquad +\sum_{n=1}^{q}\frac{1}{\sigma}\int_{\Omega}\left(|u_n|^{r(x)-1}u_n-|u_{n-1}|^{r(x)-1}u_{n-1}\right)\cdot u_n dx=0 \\ \geq \sum_{n=1}^{q}\int_{\Omega}[(-\Delta)_g^s u_n\cdot u_n dx \\ \qquad +\frac{1}{2\sigma}\left[\int_{\Omega}|u_q|^{r(x)+1}dx-\int_{\Omega}|u_0|^{r(x)+1}dx\right] \\ \qquad +\frac{1}{2\sigma}\sum_{n=1}^{q}\int_{\Omega}[|u_n|^{r(x)+1}-|u_{n-1}|^{r(x)-1}u_n^2]dx. \end{cases}$$

It follows that

$$(3.13)\begin{cases} 2\sigma\sum_{n=1}^{q}\int_{\Omega}[(-\Delta)_g^s u_n\cdot u_n dx+\max_{1\leq q\leq l}\int_{\Omega}|u_q|^{r(x)+1}dx+C \\ \leq \int_{\Omega}|u_0|^{r(x)+1}, \end{cases}$$

where $\lim_{\sigma\to 0}C=\lim_{\sigma\to 0}\sum_{n=1}^{q}\int_{\Omega}[|u_n|^{r(x)+1}-|u_{n-1}|^{r(x)-1}u_n^2]dx=0$ by the definition of $\bar{u}_\sigma$. Rewriting (3.13) by $\bar{u}_\sigma$, we have

$$(3.14)\quad\begin{cases} 2\int_0^T\int_{\Omega}(-\Delta)_g^s\bar{u}_\sigma\cdot\bar{u}_\sigma dxdt+\sup_{0<t<T}\int_{\Omega}|\bar{u}_\sigma|^{r(x)+1}dx+C \\ \qquad\leq\int_{\Omega}|u_0|^{r(x)+1}dx. \end{cases}$$

Thus (i) is proved.

(ii) Taking the test function $\phi=\frac{u_n-u_{n-1}}{\sigma}$ in (3.1), we have

$$(3.15)\quad\begin{cases} \int_{\Omega}[(-\Delta)_g^s u_n\cdot\frac{u_n-u_{n-1}}{\sigma}dx \\ \qquad+\int_{\Omega}\frac{|u_n|^{r(x)-1}u_n-|u_{n-1}|^{r(x)-1}u_{n-1}}{\sigma}\cdot\frac{u_n-u_{n-1}}{\sigma}dx=0, \end{cases}$$

i.e.,

$$(3.16)\quad\begin{cases} \frac{1}{\sigma}\int_{\Omega}\int_{\Omega}g\left(\frac{|u_n(x)-u_n(y)|}{|x-y|^s}\right)\frac{u_n(x)-u_n(y)}{|u_n(x)-u_n(y)|} \\ \qquad\cdot\frac{(u_n(x)-u_{n-1}(x))-(u_n(y)-u_{n-1}(y))}{|x-y|^s}\frac{dxdy}{|x-y|^N} \\ \qquad+\int_{\Omega}\frac{|u_n|^{r(x)-1}u_n-|u_{n-1}|^{r(x)-1}u_{n-1}}{\sigma}\cdot\frac{u_n-u_{n-1}}{\sigma}dx=0. \end{cases}$$

Summating over $n=1,2,\ldots,q$, for any $q=1,2,\ldots,l$, in (3.15) and (3.16), respectively, we have

$$(3.17)\begin{cases} \sum_{n=1}^{q}\int_{\Omega}[(-\Delta)_g^s u_n\cdot\frac{u_n(x)-u_{n-1}(x)}{\sigma}dx \\ \qquad+\sum_{n=1}^{q}\int_{\Omega}\frac{|u_n|^{r(x)-1}u_n-|u_{n-1}|^{r(x)-1}u_{n-1}}{\sigma}\cdot\frac{u_n-u_{n-1}}{\sigma}dx=0, \end{cases}$$

i.e.,

$$\begin{aligned} &\frac{1}{\sigma}\sum_{n=1}^{q}\int_{\Omega}\int_{\Omega}g\left(\frac{|u_n(x)-u_n(y)|}{|x-y|^s}\right) \\ &\qquad\cdot\frac{u_n(x)-u_n(y)}{|u_n(x)-u_n(y)|}\frac{(u_n(x)-u_{n-1}(x))-(u_n(y)-u_{n-1}(y))}{|x-y|^s}\frac{dxdy}{|x-y|^N} \\ &+\sum_{n=1}^{q}\int_{\Omega}\frac{|u_n|^{r(x)-1}u_n-|u_{n-1}|^{r(x)-1}u_{n-1}}{\sigma}\cdot\frac{u_n-u_{n-1}}{\sigma}dx=0. \end{aligned}$$

The first term of the left-hand side of (3.17) has the equalities

$$(3.18)\begin{cases} \frac{1}{\sigma}\sum_{n=1}^{q}\int_{\Omega}[(-\Delta)_g^s u_n\cdot(u_n(x)-u_{n-1}(x))dx \\ \qquad=\frac{1}{\sigma}\sum_{n=1}^{q}[\int_{\Omega}[(-\Delta)_g^s u_n\cdot u_n(x)dx \\ \qquad\qquad-\frac{1}{\sigma}\int_{\Omega}[(-\Delta)_g^s u_{n-1}\cdot u_{n-1}dx]+\frac{D}{\sigma} \\ \qquad=\frac{1}{\sigma}\int_{\Omega}[(-\Delta)_g^s u_q\cdot u_q(x)dx-\frac{1}{\sigma}\int_{\Omega}[(-\Delta)_g^s u_0\cdot u_0 dx+\frac{D}{\sigma}, \end{cases}$$

where $D = -\sum_{n=1}^{q}\int_\Omega[(-\Delta)_g^s u_n\cdot u_{n-1}(x)-(-\Delta)_g^s u_{n-1}\cdot u_{n-1}]dx$ and $\lim_{\sigma\to 0} D = 0$. Combing (1.3) and (3.18), we have

$$(3.19)\quad \begin{cases} \frac{1}{\sigma}\sum_{n=1}^{q}\int_\Omega(-\Delta)_g^s u_n\cdot(u_n(x)-u_{n-1}(x))dx \\ \quad = \frac{1}{\sigma}\int_\Omega(-\Delta)_g^s u_p\cdot u_p(x)dx - \frac{1}{\sigma}\int_\Omega(-\Delta)_g^s u_0\cdot u_0 dx + \frac{D}{\sigma} \\ \quad \ge \frac{1}{\sigma}g_0\int_\Omega G(\frac{|u_p(x)-u_p(y)|}{|x-y|^s})\frac{dxdy}{|x-y|^N} \\ \qquad -\frac{1}{\sigma}g^0\int_\Omega G(\frac{|u_0(x)-u_0(y)|}{|x-y|^s})\frac{dxdy}{|x-y|^N}+\frac{D}{\sigma}. \end{cases}$$

On the other hand, by Mean Value Theorem, the integrand of the second integral of the left-hand side of (3.17) has the equality

$$(3.20)\begin{cases} \frac{|u_n|^{r(x)-1}u_n-|u_{n-1}|^{r(x)-1}u_{n-1}}{\sigma}\cdot\frac{u_n-u_{n-1}}{\sigma} \\ = \int_0^1\big[(r(x)-1)|su_n+(1-s)u_{n-1}|^{r(x)-3}(su_n+(1-s)u_{n-1})^2 \\ \qquad +|su_n+(1-s)u_{n-1}|^{r(x)-1}\big]ds\cdot|\frac{u_n-u_{n-1}}{\sigma}|^2 \\ \ge \int_0^1\big[|su_n+(1-s)u_{n-1}|^{r(x)-1}\big]ds\cdot|\frac{u_n-u_{n-1}}{\sigma}|^2 \end{cases}$$

because $1<r(x)<\infty$. Combining (3.17), (3.18) and (3.20), we have

$$(3.21)\quad \begin{cases} \sum_{n=1}^{q}\int_\Omega[(-\Delta)_g^s u_n\cdot\frac{u_n-u_{n-1}}{\sigma}dx \\ \quad +\sum_{n=1}^{q}\int_\Omega\frac{|u_n|^{r(x)-1}u_n-|u_{n-1}|^{r(x)-1}u_{n-1}}{\sigma}\cdot\frac{u_n-u_{n-1}}{\sigma}dx=0 \\ \ge\frac{1}{\sigma}\max_{1\le q\le l}\int_\Omega[(-\Delta)_g^s u_q\cdot u_q(x)dx-\frac{1}{\sigma}\int_\Omega[(-\Delta)_g^s u_0\cdot u_0 dx \\ \quad +\frac{D}{\sigma}+\sum_{n=1}^{q}\int_\Omega[\int_0^1\big[|su_n+(1-s)u_{n-1}|^{r(x)-1}\big]ds] \\ \qquad \cdot|\frac{u_n(x)-u_{n-1}(x)}{\sigma}|^2dx. \end{cases}$$

From (3.21), we have

$$(3.22)\quad \begin{cases} \sum_{n=1}^{q}\int_\Omega[(-\Delta)_g^s u_n\cdot\frac{u_n-u_{n-1}}{\sigma}dx \\ \quad +\sum_{n=1}^{q}\int_\Omega\frac{|u_n|^{r(x)-1}u_n-|u_{n-1}|^{r(x)-1}u_{n-1}}{\sigma}\cdot\frac{u_n-u_{n-1}}{\sigma}dx=0 \\ \ge\frac{1}{\sigma}\max_{1\le q\le l}g_0\int_\Omega G(|u_q|)\frac{dxdy}{|x-y|^N}-\frac{1}{\sigma}g^0\int_\Omega G(|u_0|)\frac{dxdy}{|x-y|^N} \\ \quad +\frac{D}{\sigma}+\sum_{n=1}^{q}\int_\Omega[\int_0^1\big[|su_n+(1-s)u_{n-1}|^{r(x)-1}\big]ds] \\ \qquad \cdot|\frac{u_n(x)-u_{n-1}(x)}{\sigma}|^2dx. \end{cases}$$

Rewriting (3.21) by v_σ and $\bar u_\sigma$, we have

$$(3.23)\quad \begin{cases} \sup_{0<t<T}\int_\Omega(-\Delta)_g^s\bar u_\sigma\cdot\bar u_\sigma dx \\ \qquad +\int_0^T\int_\Omega[\int_0^1\big[|s\bar u_\sigma(t)+(1-s)\bar u_\sigma(t-\sigma)|^{r(x)-1}\big]ds] \\ \qquad\qquad \cdot|\partial_t\bar u_\sigma|^2dxdt+D \\ \le\int_\Omega(-\Delta)_g^s u_0\cdot u_0 dx. \end{cases}$$

It follows from (3.23) that

$$\begin{aligned} &\sup_{0<t<T}\int_\Omega(-\Delta)_g^s\bar u_\sigma\cdot\bar u_\sigma dx \\ &\qquad +\int_0^T\int_\Omega\int_0^1\big[|s\bar u_\sigma(t)+(1-s)\bar u_\sigma(t-\sigma)|^{r(x)-1}\big]ds\cdot|\partial_t\bar u_\sigma|^2dxdt+D \\ &\le g^0\int_\Omega\int_\Omega G(\frac{|u_0(x)-u_0(y)|}{|x-y^s|})\frac{dxdy}{|x-y|^N}=g^0 I_{(s,G)}(u_0) \\ &\le g^0\max\{[u_0]_{s,G}^{g_0},[u_0]_{s,G}^{g^0}\}. \end{aligned}$$

Thus (ii) is proved.

LEMMA 3.3. *(Maximum Principle in terms of $\bar{u}_\sigma$)*
Let $\{\bar{u}_\sigma\}$ be the sequence in (3.3). Then

$$\sup_{0<t<T} \|\bar{u}_\sigma\|_{L^\infty(\Omega)} \leq \|u_0\|_{L^\infty(\Omega)}.$$

PROOF. Taking the test function

$$\phi = \frac{u_n}{|u_n|}\min\{1, \frac{(|u_n| - \|u_0\|_{L^\infty(\Omega)})_+}{k}\}$$

for $k > 0$ in (3.2), we have

$$(3.24)\quad \begin{cases} \int_\Omega \int_\Omega g\left(\frac{|u_n(x)-u_n(y)|}{|x-y|^s}\right) \frac{u_n(x)-u_n(y)}{|u_n(x)-u_n(y)|} \\ \qquad \cdot [\frac{\frac{u_n}{|u_n|}\min\{1, \frac{(|u_n|-\|u_0\|_{L^\infty(\Omega)})_+}{k}\}(x)}{|x-y|^{N+s}} \\ \qquad\qquad - \frac{\frac{u_n}{|u_n|}\min\{1, \frac{(|u_n|-\|u_0\|_{L^\infty(\Omega)})_+}{k}\}(y)}{|x-y|^{N+s}}]dxdy \\ + \int_\Omega \frac{|u_n|^{r(x)-1}u_n - |u_{n-1}|^{r(x)-1}u_{n-1}}{\sigma} \\ \qquad\qquad \cdot \frac{u_n}{|u_n|}\min\{1, \frac{(|u_n|-\|u_0\|_{L^\infty(\Omega)})_+}{k}\}dx = 0. \end{cases}$$

The first integral of the left-hand side of (3.24) is calculated as

$$(3.25)\quad \begin{cases} \int_\Omega \int_\Omega g\left(\frac{|u_n(x)-u_n(y)|}{|x-y|^s}\right) \frac{u_n(x)-u_n(y)}{|u_n(x)-u_n(y)|} \\ \qquad \cdot [\frac{\frac{u_n}{|u_n|}\min\{1, \frac{(|u_n|-\|u_0\|_{L^\infty(\Omega)})_+}{k}\}(x)}{|x-y|^{N+s}} \\ \qquad\qquad - \frac{\frac{u_n}{|u_n|}\min\{1, \frac{(|u_n|-\|u_0\|_{L^\infty(\Omega)})_+}{k}\}(y)}{|x-y|^{N+s}}]dxdy \geq 0 \end{cases}$$

since $g(|u_n|) \geq 0$, $(u_n(x) - u_n(y))^2 \geq 0$ and $\min\{1, \frac{(|u_n|-\|u_0\|_{L^\infty(\Omega)})_+}{k}\} \geq 0$. Thus the second integral of (3.25) is estimated as

$$(3.26)\quad \begin{cases} \int_\Omega \frac{|u_n|^{r(x)-1}u_n - |u_{n-1}|^{r(x)-1}u_{n-1}}{\sigma} \\ \qquad \cdot \frac{u_n}{|u_n|}\min\{1, \frac{(|u_n|-\|u_0\|_{L^\infty(\Omega)})_+}{k}\}dx \leq 0. \end{cases}$$

By Schwarz's inequality, (3.26) is estimated as

$$(3.27)\quad \frac{1}{\sigma}\int_\Omega [|u_n|^{r(x)} - |u_{n-1}|^{r(x)}]\min\{1, \frac{(|u_n| - \|u_0\|_{L^\infty(\Omega)})_+}{k}\}dx \leq 0.$$

Using Lebesgue's convergence theorem and taking the limit as $k \to 0$ in (3.27), we have

$$(3.28)\qquad \frac{1}{\sigma}\int_{\Omega\cap\{x\in\Omega|\ |u_n|>\|u_0\|_{L^\infty(\Omega)}\}} [|u_n|^{r(x)} - |u_{n-1}|^{r(x)}]dx \leq 0.$$

Let us set

$$\Omega_{n1} = \{x \in \Omega|\ |u_n| > \|u_0\|_{L^\infty(\Omega)}\},$$
$$\Omega_{n2} = \{x \in \Omega|\ |u_n| \leq \|u_0\|_{L^\infty(\Omega)}\}.$$

Thus we have

$$\begin{aligned} 0 &\geq \int_{\Omega\cap\Omega_{n1}} [(|u_n|^{r(x)} - \|u_0\|^{r(x)}_{L^\infty(\Omega)}) - (|u_{n-1}|^{r(x)} - \|u_0\|^{r(x)}_{L^\infty(\Omega)})]dx \\ &\geq \int_{\Omega\cap\Omega_{n1}} (|u_n|^{r(x)} - \|u_0\|^{r(x)}_{L^\infty(\Omega)})_+ dx - \int_{\Omega\cap\Omega_{n-1_1}} (|u_{n-1}|^{r(x)} - \|u_0\|^{r(x)}_{L^\infty(\Omega)})_+ dx \end{aligned}$$

since $\int_{\Omega\cap\Omega_{n1}\cap\Omega_{n-1 2}}\big(|u_{n-1}|^{r(x)}-\|u_0\|^{r(x)}_{L^\infty(\Omega)}\big)dx\le 0$. Thus we have for each $n=1,2,\dots,l$,

$$\int_\Omega\big((|u_n|^{r(x)}-\|u_0\|^{r(x)}_{L^\infty(\Omega)}\big)_+dx\le\int_\Omega\big(|u_0|^{r(x)}-\|u_0\|^{r(x)}_{L^\infty(\Omega)}\big)_+dx=0,$$

it follows that $|u_n|\le\|u_0\|_{L^\infty(\Omega)}$ in Ω for each $n=1,2,\dots,l$. Thus we have

$$\sup_{0<t<T}\|\bar{u}_\sigma\|_{L^\infty(\Omega)}\le\|u_0\|_{L^\infty(\Omega)}.$$

Thus the lemma is proved.

We next show that for a given weak solution $u\in L^\infty(0,T;W_0^sL_G(\Omega))$ of (1.1), there exist the sequences $\{\bar{u}_\sigma\}$, $\{v_\sigma\}$ and $\{\bar{v}_\sigma\}$ in (3.3) such that the subsequences, up to the subsequences, $\{\bar{u}_\sigma\}$, $\{v_\sigma\}$ and $\{\bar{v}_\sigma\}$ converge to the limits u, $v=|u|^{r(x)-1}u$, where $\{\bar{u}_\sigma\}$, $\{v_\sigma\}$ and $\{\bar{v}_\sigma\}$ are the subsequences obtained from the unique sequence $\{u_n\}$ of the weak solutions of (3.1).

LEMMA 3.4. *(Estimate for Time-derivative)* *Let $\{u_\sigma\}$, $\{\bar{u}_\sigma\}$ and $\{v_\sigma\}$ be the sequences in (3.3). Then*

$$\lim_{\sigma\to 0}\|\partial_t v_\sigma\|^2_{L^2(\Omega_T)}\le b\ max\{[u_0]^{g_0}_{s,G},[u_0]^{g^0}_{s,G}\}\tag{3.29}$$

for some $b>0$.

PROOF. We note that

$$\begin{aligned}\partial_t v_\sigma &= \partial_t\Big(\frac{t-t_{n-1}}{\sigma}|u_n|^{r(x)-1}u_n+\frac{t_n-t}{\sigma}|u_{n-1}|^{r(x)-1}u_{n-1}\Big)\\ &= \frac{|u_n|^{r(x)-1}u_n-|u_{n-1}|^{r(x)-1}u_{n-1}}{\sigma}.\end{aligned}$$

By Mean value theorem,

$$\begin{cases}\left|\frac{|u_n|^{r(x)-1}u_n-|u_{n-1}|^{r(x)-1}u_{n-1}}{\sigma}\right|\\ \le\|r(x)\|_{L^\infty(\Omega)}[\int_0^1\big[|su_n+(1-s)u_{n-1}|^{r(x)-1}\big]ds]\cdot\left|\frac{u_n-u_{n-1}}{\sigma}\right|.\end{cases}\tag{3.30}$$

By Lemma 3.3, $\{\bar{u}_\sigma\}$ is bounded in $W_0^sL_G(\Omega_T)$. By (2.3), (3.30), Lemma 3.2, we have

$$\begin{aligned}&\sum_{n=1}^q\sigma\int_\Omega|\partial_t v_\sigma|^2dx\\ &\le\sum_{n=1}^q\sigma\|r(x)\|^2_{L^\infty(\Omega)}\int_\Omega\frac{1}{\sigma^2}[\int_0^1\big[|su_n+(1-s)u_{n-1}|^{r(x)-1}ds\big)^2|u_n-u_{n-1}|^2dx\\ &\le\|r(x)\|^2_{L^\infty(\Omega)}\|\int_0^1|su_n+(1-s)u_{n-1}|^{r(x)-1}ds\|_{L^\infty(\Omega)}\\ &\qquad\cdot\int_0^T\int_\Omega[\int_0^1|su_n+(1-s)u_{n-1}|^{r(x)-1}ds]|\partial_t u_\sigma|^2dxdt\\ &\le ag^0\int_\Omega\int_\Omega G(\frac{|u_0(x)-u_0(y)|}{|x-y|^s})\frac{dxdy}{|x-y|^N}-D=ag^0I_{(s,G)}(u_0)-aD\\ &\le b\max\{[u_0]^{g_0}_{s,G},[u_0]^{g^0}_{s,G}\}-aD\end{aligned}$$

for some $b = ag^0 > 0$, where $\lim_{\sigma\to 0} D = \lim_{\sigma\to 0}[-\sum_{n=1}^{q}\int_\Omega[(-\Delta)_g^s u_n \cdot u_{n-1}(x) - (-\Delta)_g^s u_{n-1}\cdot u_{n-1}]dx = 0$ by the definition of $\bar{u}_\sigma$. It follows that

$$\lim_{\sigma\to 0}\|\partial_t v_\sigma\|^2_{L^2(\Omega_T)} \le b\,\max\{[u_0]^{g_0}_{s,G}, [u_0]^{g^0}_{s,G}\} \tag{3.31}$$

for some $b > 0$.

LEMMA 3.5. *Let $\{u_n\}$, $n = 1, 2, \ldots, l$ be the unique sequence of the weal solutions for (3.1) and $\{v_\sigma\}$ be the sequence in (3.3). Then, for each $t_{n-1} < t \le t_n$, $n = 1, \ldots, l$, we have*

$$|\partial_x v_\sigma| \le C\big(|u_n|^{r(x)-1}|\nabla u_n| + |u_{n-1}|^{r(x)-1}|\nabla u_{n-1}|\big).$$

PROOF. We note that

$$|\nabla(|u_n|^{r(x)-1}u_n)| \le C(1+|u_n|)|u_n|^{r(x)-1}|\nabla u_n|.$$

Thus, by Lemma 3.3,

$$|\nabla(|u_n|^{r(x)-1}u_n)| \le C|u_n|^{r(x)-1}|\nabla u_n|.$$

Similarly, it holds that

$$|\nabla(|u_{n-1}|^{r(x)-1}u_{n-1})| \le C|u_{n-1}|^{r(x)-1}|\nabla u_{n-1}|.$$

It follows that for each $t_{n-1} < t \le t_n$,

$$\begin{aligned} |\nabla v_\sigma| &\le |\frac{t-t_{n-1}}{\sigma}||\nabla(|u_n|^{r(x)-1}u_n))| + |\frac{t_n - t}{\sigma}||\nabla(|u_{n-1}|^{r(x)-1}u_{n-1}))| \\ &\le C\big(|u_n|^{r(x)-1}|\nabla u_n| + |u_{n-1}|^{r(x)-1}|\nabla u_{n-1}|. \end{aligned}$$

LEMMA 3.6. *Let $u_0 \in W_0^s L_G(\Omega)\cap L^\infty(\Omega)$ and u be a weak solution of (1.1). Then there exist the sequences $\{\bar{u}_\sigma\}$, $\{v_\sigma\}$ and $\{\bar{v}_\sigma\}$ in (3.3) such that the subsequences, up to the subsequences, $\{u_\sigma\}$, $\{\bar{u}_\sigma\}$, $\{v_\sigma\}$ and $\{\bar{v}_\sigma\}$ converging to the limits $u \in L^\infty(0,T; W_0^s L_G(\Omega))$ and $v \in L^2(\Omega_T)$ such that*
(i) $\bar{u}_\sigma \rightharpoonup u$ weakly in $L^\infty(0,T; W_0^s L_G(\Omega)) \subset L_G(\Omega))$.
(ii) $\bar{u}_\sigma \to u$ strongly in $L^\infty(0,T; W_0^s L_G(\Omega) \subset L_G(\Omega))$.
(iii) $(-\Delta)_g^s\bar{u}_\sigma\cdot\bar{u}_\sigma \to (-\Delta)_g^s u\cdot u$ as $\sigma\to 0$.
(iv) $\bar{u}_\sigma \to u$ $*-$weakly in $L^\infty(0,T; W_0^s L_G(\Omega)) \subset L_G(\Omega))$ and $(-\Delta)_g^s\bar{u}_\sigma \to (-\Delta)_g^s u$ $*-$weakly in $L^\infty(0,T; W_0^{-s}L_{G^*}(\Omega))$.
(v) $v_\sigma \to v$ strongly in $L^2(\Omega_T)$ as $\sigma\to 0$. Moreover,
(vi) $|\bar{u}_\sigma|^{r(x)-1}\bar{u}_\sigma \to |u|^{r(x)-1}u$ strongly in $L^2(\Omega_T)$ as $\sigma \to 0$.
(vii) $\bar{v}_\sigma \to v$ strongly in $L^2(\Omega_T)$ as $\sigma\to 0$ and $v = |u|^{r(x)-1}u$.

PROOF. (i) By lemma 3.3, $\{\bar{u}_\sigma\}$ is bounded in $L^\infty(0,T; W_0^s L_G(\Omega) \subset L_G(\Omega))$. Thus $\{\bar{u}_\sigma\} \rightharpoonup u$ weakly for some $u \in L^\infty(0,T; W_0^s L_G(\Omega)) \subset L_G(\Omega))$.
(ii) By (i), $\bar{u}_\sigma \rightharpoonup u$ weakly to some $u \in L^\infty(0,T; W_0^s L_G(\Omega) \subset L_G(\Omega))$ for almost every $t \in (0,T)$. Since, by the generalized *Poincaré* inequality on the Orlicz-Sobolev space of Lemma 2.1, the embedding $W_0^s L_G(\Omega) \hookrightarrow L_G(\Omega)$ is continuous and compact, $\bar{u}_\sigma \to u$ strongly to u in $L^\infty(0,T; W_0^s L_G(\Omega) \subset L_G(\Omega))$ for almost every $t \in (0,T)$ as $\sigma\to 0$.
(iii) By (ii), $\bar{u}_\sigma \to u$ strongly for some $u \in L^\infty(0,T; W_0^s L_G(\Omega) \subset L_G(\Omega))$ for almost

every $t \in (0,T)$ as $\sigma \to 0$. Thus we have

$$
\begin{aligned}
&(-\Delta)^s_g \bar{u}_\sigma \cdot \bar{u}_\sigma \\
&= \int_0^T \int_\Omega \int_\Omega g(\frac{|\bar{u}_\sigma(x) - \bar{u}_\sigma(y)|}{|x-y|^s}) \frac{\bar{u}_\sigma(x) - \bar{u}_\sigma(y)}{|\bar{u}_\sigma(x) - \bar{u}_\sigma(y)|} \frac{\bar{u}_\sigma(x) - \bar{u}_\sigma(y)}{|x-y|^s} \frac{dxdy}{|x-y|^{N+1+s}} dt \\
&\longrightarrow \int_0^T \int_\Omega \int_\Omega g(\frac{|u(x) - u(y)|}{|x-y|^s}) \frac{u(x) - u(y)}{|u(x) - u(y)|} \frac{u(x) - u(y)}{|x-y|^s} \frac{dxdy}{|x-y|^{N+1+s}} dt \\
&\quad = (-\Delta)^s_g u \cdot u.
\end{aligned}
$$

Thus $(-\Delta)^s_g \bar{u}_\sigma \cdot \bar{u}_\sigma \to (-\Delta)^s_g u \cdot u$ as $\sigma \to 0$.
(iv) By Lemma 3.3, $\{\bar{u}_\sigma\}$ is bounded in $L^\infty(0,T; W^s_0 L_G(\Omega) \subset L_G(\Omega))$. By Lemma 3.2,
$\sup_{0<t<T} \int_{\Omega_T} (-\Delta)^s_g \bar{u}_\sigma \cdot \bar{u}_\sigma dx$ is bounded as $\sigma \to 0$. Thus $\{(-\Delta)^s_g \tilde{u}_\sigma\}$ is bounded in $L^\infty(0,T; W^{-s}_0 L_{G^*}(\Omega))$ as $\sigma \to 0$. Thus there exists a subsequence, up to a subsequence, $\{\bar{u}_\sigma\}$ converging $*-$weakly to the limit $\eta \in L^\infty(0,T; W^s_0 L_G(\Omega) \subset L_G(\Omega))$ and $\{(-\Delta)^s_g \bar{u}_\sigma\}$ converging $*-$weakly to the limit $(-\Delta)^s_g \eta$ as $\sigma \to 0$, respectively. By (ii), $\bar{u}_\sigma \to u$ strongly, it follows that $\eta = u$, i. e.,

$$\bar{u}_\sigma \to u \ *-\text{weakly in } \ L^\infty(0,T; W^s_0 L_G(\Omega) \subset L_G(\Omega)),$$

$$(-\Delta)^s_g \bar{u}_\sigma \to (-\Delta)^s_g u \ *-\text{weakly in } \ L^\infty(0,T; W^{-s}_0 L_{G^*}(\Omega)) \text{ as } \ \sigma \to 0.$$

(v) By Lemma 3.3 and Lemma 3.5, $\{\bar{u}_\sigma\}$ is bounded in $L^\infty(0,T; W^s_0 L_G(\Omega))$ and $\{\partial_x v_\sigma\}$ is bounded in $L^\infty(0,T; L_G(\Omega))$ for $(x,t) \in \Omega_T$. By Lemma 3.4, $\{\partial_t v_\sigma\}$ is bounded in $L^2(\Omega_T)$ as $\sigma \to 0$. Thus $\{v_\sigma\}$ is bounded in $W^{1,2}_0(\Omega_T)$ as $\sigma \to 0$. Since $2 < \frac{2(N+1)}{N+1-2}$, by Rellich-Kondrachov compactness theorem, the embedding $W^{1,2}_0(\Omega_T) \hookrightarrow L^2(\Omega_T)$ is continuus and compact. Thus $\{v_\sigma\}$ has a subsequence, up to a subsequence, $\{v_\sigma\}$ such that

$$v_\sigma \to v \text{ strongly in } \ L^2(\Omega_T) \text{ as } \ \sigma \to 0. \tag{3.32}$$

(vi) Since $\{\bar{u}_\sigma\}$ is bounded and, by (ii), $\bar{u}_\sigma \to u$ strongly in $L^\infty(0,T; L_G(\Omega))$, $\int_0^1 |s\bar{u}_\sigma + (1-s)u|^{r(x)-1} ds$ is bounded. Combining these facts and (2.2), we have

$$
\begin{aligned}
&\int_0^T \int_\Omega ||\bar{u}_\sigma|^{r(x)-1} \bar{u}_\sigma - |u|^{r(x)-1} u|^2 dxdt \\
&\le \int_0^T \int_\Omega r(x)^2 \big(\int_0^1 |s\bar{u}_\sigma + (1-s)u|^{r(x)-1} ds \big)^2 \cdot |\bar{u}_\sigma - u|^2 dxdt \\
&\le 2\|r(x)\|^2_{L^\infty(\Omega)} \| \big(\int_0^1 |s\bar{u}_\sigma + (1-s)u|^{r(x)-1} ds \big)^2 \|_{L_{G^*}(\Omega_T)} \| (\bar{u}_\sigma - u)^2 \|_{L_G(\Omega_T)} \\
&\quad \longrightarrow 0
\end{aligned}
$$

as $\sigma \to 0$ since $\|r(x)\|_{L^\infty(\Omega)} < \infty$. Thus $|\bar{u}_\sigma|^{r(x)-1} \bar{u}_\sigma \to |u|^{r(x)-1} u$ strongly in $L^2(\Omega_T)$ as $\sigma \to 0$.

(vii) For $t_{n-1} < t \le t_n$, we have

$$\begin{aligned}
&\bar{v}_\sigma - v \\
= \quad & (\bar{v}_\sigma - v_\sigma) + (v_\sigma - v) \\
= \quad & |u_n|^{r(x)-1}u_n - \frac{t-t_{n-1}}{\sigma}|u_n|^{r(x)-1}u_n - \frac{t_n - t}{\sigma}|u_{n-1}|^{r(x)-1}u_{n-1} + (v_\sigma - v) \\
= \quad & (t_n - t)\frac{|u_n|^{r(x)-1}u_n - |u_{n-1}|^{r(x)-1}u_{n-1}}{\sigma} + (v_\sigma - v) \\
= \quad & (t_n - t)\partial_t v_\sigma + (v_\sigma - v).
\end{aligned}$$

Thus we have

$$\begin{aligned}
&\|\bar{v}_\sigma - v\|^2_{L^2(\Omega_T)} \\
= \quad & \int_0^T \int_\Omega |\bar{v}_\sigma - v|^2 dxdt \\
= \quad & \int_0^T \int_\Omega [(t_n - t)\partial_t v_\sigma + (v_\sigma - v)]^2 dxdt \\
= \quad & \int_0^T \int_\Omega (t_n - t)^2(\partial_t v_\sigma)^2 dxdt + 2\int_0^T \int_\Omega (t_n - t)\partial_t v_\sigma \cdot (v_\sigma - v)dxdt \\
& \qquad + \int_0^T \int_\Omega (v_\sigma - v)^2 dxdt.
\end{aligned}$$

By Young's inequality,

$$|(t_n - t)\partial_t v_\sigma \cdot (v_\sigma - v)dxdt| \le \frac{1}{2}|(t_n - t)\partial_t v_\sigma|^2 + \frac{1}{2}(v_\sigma - v)^2.$$

It follows that

$$\begin{aligned}
\|\bar{v}_\sigma - v\|^2_{L^2(\Omega_T)} \quad & \le \quad 2\int_0^T \int_\Omega (t_n - t)^2(\partial_t v_\sigma)^2 dxdt + 2\int_0^T \int_\Omega (v_\sigma - v)^2 dxdt \\
& \le \quad 2\sigma^2\|\partial_t v_\sigma\|^2_{L^2(\Omega_T)} + 2\|v_\sigma - v\|^2_{L^2(\Omega_T)}.
\end{aligned}$$

Thus

$$\|\bar{v}_\sigma - v\|^2_{L^2(\Omega_T)} \to 0 \qquad \text{as} \;\; \sigma \to 0$$

since $\|\partial_t v_\sigma\|^2_{L^2(\Omega_T)}$ is bounded and $v_\sigma \to v$ strongly in $L^2(\Omega_T)$ as $\sigma \to 0$ in (3.32). Thus

$$\bar{v}_\sigma = |\bar{u}_\sigma|^{r(x)-1}\bar{u}_\sigma \to v \qquad \text{strongly in} \;\; L^2(\Omega_T) \text{ as } \;\sigma \to 0.$$

By (vi), $v = |u|^{r(x)-1}u$.

Proof of Theorem 1.1

Let u be a weak solution of problem (1.1). Then by Lemma 3.1, Lemma 3.6 and (3.3), there exists a unique sequence $\{\bar{u}_\sigma\}$ in (3.3) such that the sequences $\{\bar{u}_\sigma\}$, $\{v_\sigma\}$ and $\{\bar{v}_\sigma\}$ in (3.3) have the subsequences, up to the subsequences, $\{\bar{u}_\sigma\}$, $\{v_\sigma\}$ and $\{\bar{v}_\sigma\}$ converging to the limits $u \in L^\infty(0, T; W_0^s L_G(\Omega))$ and $v \in L^2(\Omega_T)$ satisfying

$$(-\Delta)_g^s \bar{u}_\sigma + \partial_t v_\sigma = 0, \tag{3.33}$$

i.e.,

$$(-\Delta)_g^s \bar{u}_\sigma + \partial_t(\frac{t - t_{n-1}}{h}|u_n|^{r(x)-1}u_n + \frac{t_n - t}{h}|u_{n-1}|^{r(x)-1}u_{n-1}) = 0,$$

i.e.,

$$(-\Delta)_g^s u_n + \frac{|u_n|^{r(x)-1}u_n - |u_{n-1}|^{r(x)-1}u_{n-1}}{\sigma} = 0.$$

By Lemma 3.3, the maximum principle for (3.1) in terms of $\{\bar{u}_\sigma\}$

$$\sup_{0<t<T} \|\bar{u}_\sigma\|_{L^\infty(\Omega)} \le \|u_0\|_{L^\infty(\Omega)}.$$

holds. By Lemma 3.6, the subsequences $\{\bar{u}_\sigma\}$, $\{v_\sigma\}$ and $\{\bar{v}_\sigma\}$ converge to the limits $u \in L^\infty(0,T; W_0^s L_G(\Omega))$ and $v \in L^2(\Omega_T)$. Thus we have

$$0 = (-\Delta)_g^s \bar{u}_\sigma + \partial_t v_\sigma \longrightarrow (-\Delta)_g^s u + \partial_t(|u|^{r(x)-1}u) = 0 \qquad \text{as} \ \ \sigma \to 0$$

and

$$\lim_{\sigma\to 0} \sup_{0<t<T} \|\bar{u}_\sigma\|_{L^\infty(\Omega)} \le \|u_0\|_{L^\infty(\Omega)}.$$

That is,

$$\sup_{0<t<T} \|u(x,t)\|_{L^\infty(\Omega)} \le \|u_0\|_{L^\infty(\Omega)},$$

which is the maximum principle for (1.1). Thus Theorem 1.1 is proved.

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