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Low Mach number limit of the full compressible Navier-Stokes-Korteweg equations with general initial data

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Abstract. In this paper, the low Mach number limit for the three-dimensional full compressible Navier-Stokes-Korteweg equations with general initial data is rigorously justified within the framework of local smooth solution. Under the assumption of large temperature variations, we first obtain the uniform-in-Mach-number estimates of the solutions in a ε-weighted Sobolev space, which establishes the local existence theorem of the three-dimensional full compressible Navier-Stokes-Korteweg equations on a finite time interval independent of Mach number. Then, the low mach limit is proved by combining the uniform estimates and a strong convergence theorem of the solution for the acoustic wave equations. This result improves that of [K.-J. Sha and Y.-P. Li, Z. Angew. Math. Phys., 70(2019), 169] for well-prepared initial data.

Contents

1. Introduction

The compressible Navier-Stokes-Korteweg (denoted as NSK in the sequel) equations govern the motions of compressible viscous fluids with internal capillarity. It is well known that, capillary phenomena are ubiquitous in nature, and they are complex, very diverse, and are involved in many industrial processes ranging from off-shore engineering to automotive (e.g., carburant injection) and chemical engineering. The formulation of the theory of capillary with diffusive interface can be

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traced back to Van der Waals [**41**] and Korteweg [**23**], and a modern version was proposed by Dunn and Serrin [**13**], and Heida and Málek [**17**], respectively. The full (non-isentropic) compressible NSK equations in the whole space $\mathbb{R}^3$ reads as

$$\begin{cases} \rho_t + \mathrm{div}\,(\rho u) = 0, \\ (\rho u)_t + \mathrm{div}\,(\rho u \otimes u) = \mathrm{div}\,(S + K), \\ (\rho(\frac{1}{2}|u|^2 + e))_t + \mathrm{div}\,(\rho u(\frac{1}{2}|u|^2 + e)) = \lambda \Delta\theta + \mathrm{div}\,((S + K)u). \end{cases} \tag{1.1}$$

The unknown functions $\rho, u = (u_1, u_2, u_3) \in \mathbb{R}^3$ and θ denote the density, the velocity, the absolute temperature of the fluids, respectively. e is the internal energy of the fluids. $\lambda > 0$ is the heat conductivity coefficient. The viscous stress tensor $S = (S_{ij})_{3\times3}$ and the Korteweg tensor $K = (K_{ij})_{3\times3}$ are given by

$$S_{ij} = 2\mu\mathbb{D}_{ij}(u) + (\nu \mathrm{div} u - p)\delta_{ij},\ K_{ij} = \frac{\kappa}{2}(\Delta\rho^2 - |\nabla\rho|^2)\delta_{ij} - \kappa\frac{\partial\rho}{\partial x_i}\frac{\partial\rho}{\partial x_j},$$

where $\mathbb{D}_{ij}(u) = \frac{1}{2}(\frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j})$ is the strain tensor, p is the pressure, μ and ν are the viscosity coefficients of the flow satisfying $2\mu + 3\nu > 0$ and $\mu > 0$, $\delta_{ij} = 1(i = j)$ and $\delta_{ij} = 0(i \neq j)$, and $\kappa > 0$ is the capillary coefficient. The equations of state $p = p(\rho, \theta)$ and $e = e(\rho, \theta)$ relate the pressure p and the internal energy e to the density ρ and the temperature θ of the flow. Note that when $\kappa = 0$, the system (1.1) reduces to the classical full compressible Navier-Stokes equations.

The compressible NSK equations have attracted a lot of attention of physicists and mathematicians because of its physical importance, complexity, rich phenomena, and mathematical challenges. Here we only refer to some study results about the full compressible NSK equations (1.1). Hattori and Li [**16**] considered the local existence and global existence of smooth solution for three-dimensional non-isentropic compressible NSK equations in Sobolev space. Haspot [**15**] showed the existence results of strong solutions to the nonisothermal compressible NSK equations in $\mathbb{R}^3$. Chen, He and Zhao [**5**] obtained the global classical solutions of the one dimensional full compressible NSK equations with large initial data. Hou, Peng and Zhu [**18**] got the global classical solutions for the three-dimensional non-isentropic compressible NSK equations with small initial energy. Chen and Zhao [**9**] discussed the existence, uniqueness and nonlinear stability of stationary solutions to the Cauchy problem of the three-dimensional full compressible NSK equations. Cai, Tan and Xu [**4**], and Tsuda [**40**] established the existence of the time periodic solution to the three-dimensional full compressible NSK equations with a sufficiently small external force which is periodic in the time variable, respectively. Zhang and Tan [**42**] showed decay estimates of smooth solutions for the non-isentropic compressible fluid models of Korteweg type in $\mathbb{R}^3$. Kotschote [**24, 25, 26**] established the local existence, global existence and time-asymptotics of strong solution for the non-isentropic compressible NSK equations in a bounded domain with C^3-boundary. About the stability of basic nonlinear wave patterns such as the discontinuity wave, viscous contact wave and the rarefaction wave of one dimensional compressible NSK equations, we can refer to [**6, 7, 8, 31, 37**] and the references therein.

To our knowledge, the incompressible limit (low Mach number limit) of compressible fluid dynamical equations such as the compressible Euler equations and Navier-Stokes equations is an important and challenging mathematical problem. The first work can be traced back to Klainerman and Majda [**27, 28**], in which they proved the incompressible limit of the isentropic Euler equations to the incompressible Euler equations for local smooth solutions with well-prepared data.

For the isentropic Navier-Stokes equations, the low Mach limit of the global weak solutions with general initial data have been well studied under various boundary conditions, see [**11, 12, 33, 34**] and some references therein. Recently, these results have been extended or improved by many others, and the interesting reader can refer to the monograph [**10**] and the survey paper [**2, 14, 35, 38**] for more related results on the low Mach number limit for the compressible fluid models. Since the compressible NSK equations is the capillarity approximation of the classical full compressible Navier-Stokes equations (see [**3, 19**]), one of the important topics about the equation (1.1) is to study its low Mach number limit. For the isentropic compressible NSK equations, the low Mach number limit has been rigorously proved in [**22, 30, 32**]. Nevertheless, it is more significant and difficult to study the limit for the full compressible NSK equations (1.1) from both physical and mathematical points of view. Sha and Li [**39**] investigated low Mach number limit of local smooth solution for the three-dimensional full compressible NSK equations (1.1) with well-prepared initial data. In this paper, we consider the low Mach number limit of the local smooth solutions for the full compressible NSK equations (1.1) with general data.

To show the low Mach number limit for the full compressible NSK equations (1.1), here we shall focus on the ideal fluids obeying the following perfect gas relations, that is

$$p = \Re\rho\theta,\ e = c_V\theta, \tag{1.2}$$

where the parameters $\Re$ and $c_V > 0$ are the gas constant and the heat capacity, respectively, which will be normalized to be 1 for simplicity of presentation. And we point out our results and energy estimates in this paper still hold for any given $c_V > 0$ (equivalently for any given $\gamma > 1$), the only difference is that some constants of this paper may depend on c_V. Then we can rewrite the system (1.1) by using (1.2) as:

$$\begin{cases} \rho_t + \operatorname{div}(\rho u) = 0, \\ \rho(u_t + u\cdot\nabla u) + \nabla p = \operatorname{div}\Psi(u) + \kappa\rho\nabla\Delta\rho, \\ \rho(\theta_t + u\cdot\nabla\theta) + p\operatorname{div} u = \Psi(u) : \nabla u + \kappa[(\rho\Delta\rho + \frac{|\nabla\rho|^2}{2})\operatorname{div} u \\ \qquad\qquad -\nabla\rho\otimes\nabla\rho : \nabla u] + \lambda\Delta\theta. \end{cases} \tag{1.3}$$

where $\Psi(u) = 2\mu\mathbb{D}(u) + \nu\operatorname{div} u\mathbf{I}_{3\times 3}$, and the scalar product of two 3×3 matrices is defined as $A : B = \sum_{i,j=1}^{3} a_{ij}b_{ij}$. Let $\varepsilon > 0$ be the Mach number, which is a dimensionless number. To understand the role of the thermodynamics, we rewrite the equations (1.3) by the pressure fluctuations $p^\varepsilon(x,t)$, velocity $u^\varepsilon(x,t)$ and temperature fluctuations $\theta^\varepsilon(x,t)$ as [**1**], where

$$p(x,t) = e^{\varepsilon p^\varepsilon(x,\varepsilon t)},\ \theta(x,t) = e^{\theta^\varepsilon(x,\varepsilon t)},\ u(x,t) = \varepsilon u^\varepsilon(x,\varepsilon t), \tag{1.4}$$

where a longer time scale $t = \tau/\varepsilon$ (still denote τ by t later for simplicity) is introduced in order to seize the evolution of the fluctuations. Note that (1.2) and (1.4) imply that $\rho(x,t) = e^{\varepsilon p^\varepsilon(x,\varepsilon t)-\theta^\varepsilon(x,\varepsilon t)}$ since $\Re = 1$. In the meantime, the coefficients μ,ν,λ,κ are scaled as follow:

$$\mu = \varepsilon\mu^\varepsilon,\ \nu = \varepsilon\nu^\varepsilon,\ \lambda = \varepsilon\lambda^\varepsilon,\ \kappa = \varepsilon\kappa^\varepsilon.$$

Under these changes of variables and coefficients, the full compressible NSK equations (1.3) take the following form:

$$
\left\{
\begin{aligned}
& p_t^\varepsilon + u^\varepsilon \cdot \nabla p^\varepsilon + \tfrac{1}{\varepsilon}\mathrm{div}\,(2u^\varepsilon - \lambda^\varepsilon e^{-\varepsilon p^\epsilon + \theta^\varepsilon}\nabla\theta^\varepsilon) \\
& \ = \varepsilon e^{-\varepsilon p^\epsilon}\Psi'(u^\varepsilon) : \nabla u^\varepsilon + \kappa^\varepsilon e^{\varepsilon p^\epsilon - 2\theta^\varepsilon}(\Delta(\varepsilon p^\epsilon - \theta^\varepsilon) \\
& \quad\ + \tfrac{3}{2}|\nabla(\varepsilon p^\epsilon - \theta^\varepsilon)|^2)\mathrm{div}\,u^\varepsilon \\
& \quad\ - \kappa^\varepsilon e^{\varepsilon p^\epsilon - 2\theta^\varepsilon}\nabla(\varepsilon p^\epsilon - \theta^\varepsilon) \otimes \nabla(\varepsilon p^\epsilon - \theta^\varepsilon) : \nabla u^\varepsilon \\
& \quad\ + \lambda^\varepsilon e^{-\varepsilon p^\epsilon + \theta^\varepsilon}\nabla p^\varepsilon \cdot \nabla\theta^\varepsilon, \\
& e^{-\theta^\varepsilon}(u_t^\varepsilon + u^\varepsilon \cdot \nabla u^\varepsilon) + \tfrac{\nabla p^\epsilon}{\varepsilon} \\
& \ = e^{-\varepsilon p^\epsilon}\mathrm{div}\,\Psi'(u^\varepsilon) + \kappa^\varepsilon e^{-\theta^\varepsilon}\nabla\Delta e^{\varepsilon p^\epsilon - \theta^\varepsilon}, \\
& \theta_t^\varepsilon + u^\varepsilon \cdot \nabla\theta^\varepsilon + \mathrm{div}\,u^\varepsilon \\
& \ = \varepsilon^2 e^{-\varepsilon p^\epsilon}\Psi'(u^\varepsilon) : \nabla u^\varepsilon + \kappa^\varepsilon \varepsilon e^{\varepsilon p^\epsilon - 2\theta^\varepsilon}(\Delta(\varepsilon p^\epsilon - \theta^\varepsilon) \\
& \quad\ + \tfrac{3}{2}|\nabla(\varepsilon p^\epsilon - \theta^\varepsilon)|^2)\mathrm{div}\,u^\varepsilon \\
& \quad\ - \kappa^\varepsilon \varepsilon e^{\varepsilon p^\epsilon - 2\theta^\varepsilon}\nabla(\varepsilon p^\epsilon - \theta^\varepsilon) \otimes \nabla(\varepsilon p^\epsilon - \theta^\varepsilon) : \nabla u^\varepsilon \\
& \quad\ + \lambda^\varepsilon e^{-\varepsilon p^\epsilon}\Delta e^{\theta^\varepsilon},
\end{aligned}
\right. \tag{1.5}
$$

where $\Psi'(u^\varepsilon) = 2\mu^\varepsilon \mathbb{D}(u^\varepsilon) + \nu^\varepsilon \mathrm{div}\,u^\varepsilon \mathbf{I}_{3\times 3}$.

Formally, as ε goes to zero, if the solution $(p^\varepsilon, u^\varepsilon, \theta^\varepsilon)$ converges to a limit $(0, w, \vartheta)$ in some sense as $\varepsilon \to 0$, and $(\mu^\varepsilon, \nu^\varepsilon, \lambda^\varepsilon, \kappa^\varepsilon)$ goes to a constant vector $(\mu', \nu', \lambda', \kappa')$, where ν', μ', λ' and κ' are scaled coefficients independent of ε, then taking the limit of the system (1.5) formally, we have

$$
\left\{
\begin{aligned}
& \mathrm{div}\,(2w - \lambda' e^{-\vartheta}\nabla\vartheta) = 0, \\
& e^{-\vartheta}(w_t + w \cdot \nabla w) + \nabla\pi = (\mu' + \nu')\nabla\mathrm{div}\,w + \mu'\Delta w \\
& \qquad\qquad\qquad\qquad\qquad + \kappa' e^{-\vartheta}\nabla\Delta e^{-\vartheta}, \\
& \vartheta_t + w \cdot \nabla\vartheta + \mathrm{div}\,w = \lambda'\Delta e^{\vartheta},
\end{aligned}
\right. \tag{1.6}
$$

for some function π. Since we will prove the above limit rigorously with general initial data, we now supply the system (1.5) with the following initial data:

$$
(p^\varepsilon, u^\varepsilon, \theta^\varepsilon)|_{t=0} = (p_{in}^\varepsilon(x), u_{in}^\varepsilon(x), \theta_{in}^\varepsilon(x)). \tag{1.7}
$$

For simplicity of presentation, we shall assume $\mu^\varepsilon = \mu', \nu^\varepsilon = \nu', \lambda^\varepsilon = \lambda'$ and $\kappa^\varepsilon = \kappa'$. The general case $\mu^\varepsilon \to \mu', \nu^\varepsilon \to \nu', \lambda^\varepsilon \to \lambda'$ and $\kappa^\varepsilon \to \kappa'$ as $\varepsilon \to 0$ can be treated by slightly modifying the arguments presented in this paper.

Finally, as in [**1, 20**], we introduce

$$
\|(p^\varepsilon, u^\varepsilon, \theta^\varepsilon - \bar{\theta})(t)\|_{H^{s,\varepsilon}} := \sup_{\tau\in[0,t]} Q(\tau) + \left(\int_0^t S(\tau)^2 \mathrm{d}\tau\right)^{\frac{1}{2}},
$$

where

$$
Q(t) = \|(p^\varepsilon, u^\varepsilon)(t)\|_s + \|(\varepsilon p^\varepsilon, \varepsilon u^\varepsilon, \theta^\varepsilon - \bar{\theta})(t)\|_{s+1} + \|\nabla(\varepsilon^2 p^\varepsilon - \varepsilon\theta^\varepsilon)(t)\|_{s+1},
$$

and

$$
S(t) = \|\nabla(p^\varepsilon, u^\varepsilon)(t)\|_s + \|\nabla(\varepsilon p^\varepsilon, \varepsilon u^\varepsilon, \theta^\varepsilon)(t)\|_{s+1} + \|\nabla^2(\varepsilon^2 p^\varepsilon - \varepsilon\theta^\varepsilon)(t)\|_{s+1}.
$$

Then the main result of this paper is stated by the following theorem.

THEOREM 1.1. *Let $s \geq 4$ be an integer. Assume that the initial data $(p_{in}^\varepsilon, u_{in}^\varepsilon, \theta_{in}^\varepsilon)$ satisfies*

$$
\begin{aligned}
& \|(p_{in}^\varepsilon, u_{in}^\varepsilon)\|_s + \|(\varepsilon p_{in}^\varepsilon, \varepsilon u_{in}^\varepsilon, \theta_{in}^\varepsilon - \bar{\theta})\|_{s+1} + \|\nabla(\varepsilon^2 p_{in}^\varepsilon - \varepsilon\theta_{in}^\varepsilon)\|_{s+1} \\
\leq\ & M_0,
\end{aligned} \tag{1.8}
$$

for all $\varepsilon \in (0,1]$ and two given positive constants $\bar{\theta}$ and M_0 independent of ε. Then there exist positive constants T_0 and $\varepsilon_0 < 1$, depending only on M_0 and $\bar{\theta}$, such that the Cauchy problem (1.5) and (1.7) has a unique solution $(p^\varepsilon, u^\varepsilon, \theta^\varepsilon)$ satisfying

$$\|(p^\varepsilon, u^\varepsilon, \theta^\varepsilon - \bar{\theta})(t)\|_{H^{s,\varepsilon}} \le M, \tag{1.9}$$

for all $t \in [0, T_0]$ and $\varepsilon \in (0, \varepsilon_0]$, where M is a positive constant depending only on $M_0, \bar{\theta}$ and T_0. Moreover, assume further that the initial data satisfy the following conditions:

$$(p^\varepsilon_{in}, u^\varepsilon_{in}) \to (0, u_0) \text{ in } H^s(\mathbb{R}^3),\ \theta^\varepsilon_{in} - \bar{\theta} \to \vartheta_0 - \bar{\theta} \text{ in } H^{s+1}(\mathbb{R}^3) \text{ as } \varepsilon \to 0, \tag{1.10}$$

$$|\theta^\varepsilon_{in} - \bar{\theta}| \le N_0 |x|^{-1-\zeta},\ |\nabla \theta^\varepsilon_{in}| \le N_0 |x|^{-2-\zeta}, \tag{1.11}$$

for all $\varepsilon \in (0,1]$ and two fixed positive constants N_0 and ζ. Then $(p^\varepsilon, u^\varepsilon)$ converges weakly-$*$ in $L^\infty(0, T_0, H^s(\mathbb{R}^3))$ and strongly in $L^2(0, T_0; H^{s'}_{loc}(\mathbb{R}^3))$ to $(0, w)$, and $\theta^\varepsilon - \bar{\theta}$ converges weakly-$*$ in $L^\infty(0, T_0, H^{s+1}(\mathbb{R}^3))$and strongly in $L^2(0, T_0; H^{s'+1}_{loc}(\mathbb{R}^3))$ to $\vartheta - \bar{\theta}$, for all $0 \le s' < s$, where (w, ϑ) is the solution to the system (1.6) with initial data $(w, \vartheta)|_{t=0} = (w_0, \vartheta_0)$, where w_0 is determined by

$$\operatorname{div}(2w_0 - \kappa' e^{\vartheta_0} \nabla \vartheta_0) = 0,\ \operatorname{curl}(e^{-\vartheta_0} w_0) = \operatorname{curl}(e^{-\vartheta_0} u_0). \tag{1.12}$$

REMARK 1.2. *Since we consider the low Mach number limit of the full compressible NSK equations, our limit system (1.6) is different from that in* [**2, 20**]*. We also need the more regularity of the pressure and the temperature due to the Korteweg stress term. Compared to* [**39**]*, we show the low Mach number limit of the local smooth solutions for the full compressible NSK equations (1.5) with general data. Moreover, it is more interesting to consider the low Mach number limit of the full compressible NSK equations when the background is not constant state in the one dimensional case as in* [**20**]*. That is what our effort should aim at in the forthcoming future.*

Our work is inspired by recent woks of [**1, 29**] on the full compressible Navier-Stokes equations. Let us outline the ideas of the proof as follows now. First, we try to establish the uniform-in-Mach-number estimates of the solutions to the three-dimensional full compressible NSK equations in a ε-weighted Sobolev space. In this procedure, we first show the estimates of $\|(\varepsilon p^\varepsilon, \varepsilon u^\varepsilon, \theta^\varepsilon - \bar{\theta}, \nabla(\varepsilon^2 p^\varepsilon - \varepsilon \theta^\varepsilon))(t)\|_{s+1}$, then control $\|\operatorname{curl} u^\varepsilon(t)\|_{s-1}$, $\|(p^\varepsilon, u^\varepsilon)\|$ and $\|(D^\beta \varepsilon \partial_t (p,u)(t)\|(|\beta| \le s-1)$, by which we can obtain the norm of $\|(\nabla p^\varepsilon, \operatorname{div} u^\varepsilon)\|_{s-1}$ through the density and the momentum equations. It should be pointed out that the analysis for the Cauchy problem (1.5) and (1.7) is very complicated due to the strong coupling of the hydrodynamic motion and the Korteweg stress term. More efforts should be paid on the estimates involving these coupling terms, in particular, on the estimate of higher order spatial derivatives. We shall need the more regularity of p^ε and θ^ε, and exploit the special structure of the system to obtain the tamed estimate on higher order derivatives, so that we can close our estimates on the uniform boundedness of the solutions. Once the uniform boundedness of the solutions has been established, then we can show the local existence of the three-dimensional full compressible NSK equations on a finite time interval independent of Mach number as in [**1, 36**]. Further, the low mach limit is proved by combining the uniform estimates and the local energy decay of the acoustic wave equation in [**36**]. Of cause, It is worth while to note that this approach has been used to study the low Mach number limit of the full magnetohydrodynamic equations with general initial data [**21**].

Notations. We shall use the following notations. For a multi-index $\alpha = (\alpha_1, \alpha_2, \alpha_3)$, we denote $D^\alpha = \partial_{x_1}^{\alpha_1}\partial_{x_2}^{\alpha_2}\partial_{x_3}^{\alpha_3}$ with $|\alpha| = \sum_{i=1}^{3} |\alpha_i|$. Further, we use the notation of communicator by defining $[D^\alpha, u]v := D^\alpha(uv) - uD^\alpha v$. Next, $L^p(\mathbb{R}^3)$ with $1 \le p < \infty$ denotes the space of measurable functions whose p-powers are integrable on $\mathbb{R}^3$, with the norm $\|\cdot\|_{L^p} = (\int_{\mathbb{R}^3} |\cdot|^p dx)^{\frac{1}{p}}$, and $L^\infty(\mathbb{R}^3)$ is the space of bounded measurable functions on $\mathbb{R}^3$, with the norm $\|\cdot\|_{L^\infty} = \mathrm{esssup}_{x\in\mathbb{R}^3}|\cdot|$. Without confusion, we also denote the norm of $L^2(\mathbb{R}^3)$ by $\|\cdot\|$ for brevity. We denote by $\langle\cdot,\cdot\rangle$ the standard inner product in $L^2(\mathbb{R}^3)$ with the norm $\|f\| = \langle f, f\rangle^{\frac{1}{2}}$. Furthermore, for a non-negative integer k, $W^{k,p}(\mathbb{R}^3)$ denotes the standard Sobolev spaces of order k with the norm $\|\cdot\|_{W^{k,p}}$. When $p = 2$, we abbreviately set $W^{k,2}(\mathbb{R}^3) = H^k(\mathbb{R}^3)$ with the norm $\|\cdot\|_k$. In addition, we denote by $C^l([0,T]; H^k(\mathbb{R}^3))$(resp. $L^2(0,T; H^k(\mathbb{R}^3))$) the space of l-time continuous differentiable (resp. square integrable) functions on $[0,T]$ taking values in the space $H^k(\mathbb{R}^3)$. Finally, the symbols $c_i (i = 1, 2, \cdots)$ or $C_j (j = 0, 1, 2, \cdots)$ are always used to denote generic positive constants, independent of ε. $C(\cdot)$ denote a positive smooth function which may vary from line to line.

This paper is arranged as follows. In next section we make some preliminaries. That is, we will recall some communicator estimates, then also list the dispersive estimates on the wave equation obtained by Métivier and Schochet in [**36**]. Of cause, we go over the results on local solutions to the Cauchy problem (1.5) and (1.7) for any given ε. In Section 3 we first establish a priori estimates on $(p^\varepsilon, u^\varepsilon, \theta^\varepsilon)$. Then, with the help of these estimates we establish the uniform boundedness of the solutions and prove the existence part of Theorem 1.1. Finally, in Section 4 we study the local energy decay for the acoustic wave equations and prove the convergence part of Theorem 1.1.

2. Preliminary

In this section, we will give some preliminaries. That is, we will recall some commutator estimates, then also list the dispersive estimates on the wave equations. Finally, we go over the results on local solutions to the Cauchy problem (1.5) and (1.7). To begin with, let us recall some results on the commutator estimates.

Lemma 2.1. *(See* [**27**]*) Let $s > \frac{5}{2}$ be an integer and $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ be a multi-index such that $|\alpha| = k$. Then for any $\sigma \ge 0$, there exists a positive constant C_0 such that for all $f, g \in H^{k+\sigma}(\mathbb{R}^3)$, it holds that*

$$\|[D^\alpha, f]g\|_\sigma \le C_0(\|\nabla f\|_{W^{1,\infty}}\|g\|_{k+\sigma-1} + \|f\|_{k+\sigma}\|g\|_{L^\infty}).$$

Lemma 2.2. *(See* [**29**]*) Let $s \ge 4$ be integers. Then there exists a positive constant C_0 such that for all $\varepsilon \in (0,1], T > 0$ and multi-index $\beta = (\beta_1, \beta_2, \beta_3)$ be a satisfying $|\beta| \le s-1$, and $f, g \in C^0([0,T], H^s(\mathbb{R}^3))$, it holds that*

$$\|[D^\beta(\varepsilon\partial_t), f]g\| \le \varepsilon C_0(\|f\|_{s-1}\|\partial_t g\|_{s-2} + \|\partial_t f\|_{s-1}\|g\|_{s-1}). \tag{2.1}$$

Next, we give the following dispersive estimates on the wave equations obtained by Métivier and Schochet in [**36**], which will be used to show the convergence part of Theorem 1.1 in Section 4.

LEMMA 2.3. *Let $T > 0$ and $\{v^\varepsilon\}$ be a bounded sequence in $C([0,T]; H^2(\mathbb{R}^3))$, such that*

$$\varepsilon^2 \partial_t (a^\varepsilon \partial_t v^\varepsilon) - \nabla \cdot (b^\varepsilon \nabla v^\varepsilon) = c^\varepsilon,$$

where the function c^ε converges to 0 strongly in $L^2(0,T; L^2(\mathbb{R}^3))$. Assume further that for some $s > \frac{3}{2} + 1$, the coefficient $(a^\varepsilon, b^\varepsilon)$ is uniformly bounded in $C([0,T]; H^s(\mathbb{R}^3))$ and converges in $C([0,T]; H^s_{loc}(\mathbb{R}^3))$ to a limit (a,b) satisfying the decay estimates

$$|a(x,t) - \hat{a}| \le C_0 |x|^{-1-\zeta}, \ |\nabla_x a(x,t)| \le C_0 |x|^{-2-\zeta},$$
$$|b(x,t) - \hat{b}| \le C_0 |x|^{-1-\zeta}, \ |\nabla_x b(x,t)| \le C_0 |x|^{-2-\zeta},$$

for some positive constants $\hat{a}, \hat{b}, C_0$ and ζ. Then the sequence v^ε converges to 0 strongly in $L^2(0,T; L^2_{loc}(\mathbb{R}^3))$.

Finally, using the dual argument and iteration argument in [**16**], we have the following local existence theory for system (1.3), see also [**9, 18**].

LEMMA 2.4. *Let $s \ge 4$ be an integer. Assume that the initial data (ρ_0, u_0, θ_0) satisfy*

$$\|\rho_0 - \underline{\rho}\|_{s+1} + \|(u_0, \theta_0 - \underline{\theta})\|_s \le C_0$$

for some positive constants $\underline{\rho}, \underline{\theta}$ and C_0. Then there exists a $\tilde{T} > 0$, such that the system (1.3) with the initial data (ρ_0, u_0, θ_0) admits a unique classical solution (ρ, u, θ) satisfying $\rho - \underline{\rho} \in C([0,\tilde{T}]; H^{s+1}(\mathbb{R}^3))$, $\nabla\rho \in L^2(0,\tilde{T}; H^{s+1}(\mathbb{R}^3))$, $(u, \theta - \underline{\theta}) \in C([0,\tilde{T}]; H^s(\mathbb{R}^3))$, $(\nabla u, \nabla\theta) \in L^2(0,\tilde{T}; H^s(\mathbb{R}^3))$ and

$$\sup_{t\in[0,\tilde{T}]} (\|\rho(t) - \underline{\rho}\|_{s+1} + \|(u, \theta - \underline{\theta})(t)\|_s) + \left(\int_0^T \|\nabla\rho(t)\|^2_{s+1} + \|\nabla(u,\theta)(t)\|_s^2 \mathrm{d}t \right)^{\frac{1}{2}} \le 2C_0.$$

Then it follows from Lemma 2.4 and the transformation (1.4) that for any fixed ε and initial data satisfying (1.8), there exists a $T_\varepsilon > 0$, depending on ε and M_0, such that the Cauchy problem (1.5) and (1.7) admits a unique classical solution $(p^\varepsilon, u^\varepsilon, \theta^\varepsilon) \in C([0,T_\varepsilon); H^s(\mathbb{R}^3))$, $p^\varepsilon - \theta^\varepsilon \in C([0,T_\varepsilon); H^{s+1}(\mathbb{R}^3))$, $(\nabla p^\varepsilon, \nabla u^\varepsilon, \nabla\theta^\varepsilon) \in C([0,T_\varepsilon); H^s(\mathbb{R}^3))$ and $\nabla(p^\varepsilon - \theta^\varepsilon) \in C([0,T_\varepsilon); H^{s+1}(\mathbb{R}^3))$. Moreover, let T^*_ε be the maximal time of existence of a smooth solution, then if T^*_ε is finite, one has $\limsup_{t\to T^*_\varepsilon} \|(p^\varepsilon, u^\varepsilon, \theta^\varepsilon)\|_{W^{1,\infty}} = \infty$.

Therefor, we will see by the same arguments in [**36**] that the existence part of Theorem 1.1 is a consequence of the above assertion and the following key a priori estimate which will be shown in the next section.

PROPOSITION 2.5. *For any given integer $s \ge 4$ and fixed $\varepsilon > 0$. Let $(p^\varepsilon, u^\varepsilon, \theta^\varepsilon)$ be the classical solution to the Cauchy problem (1.5) and(1.7). There exist positive constants $\hat{T}, \varepsilon_0 \le 1$, and an increasing positive polynomial $C(\cdot)$, such that for all $T \in [0,\hat{T}]$ and $\varepsilon \in (0,\varepsilon_0]$, it holds that*

$$O(T) \le C(O_0) e^{(\sqrt{T}+\varepsilon)C(O(T))}, \tag{2.2}$$

where

$$O(T) = \|(p^\varepsilon, u^\varepsilon, \theta^\varepsilon - \bar{\theta})(T)\|_{H^{s,\varepsilon}},$$

and

$$O_0 = \|(p^\varepsilon_{in}, u^\varepsilon_{in})\|_s + \|(\varepsilon p^\varepsilon_{in}, \varepsilon u^\varepsilon_{in}, \theta^\varepsilon_{in} - \bar{\theta})\|_{s+1} + \|\nabla(\varepsilon^2 p^\varepsilon_{in} - \varepsilon\theta^\varepsilon_{in})\|_{s+1}.$$

3. Uniform estimates

In this section we shall prove the priori estimate stated in Proposition 2.5. The superscripts ε of the variables will be dropped in this section for simplicity of presentation. Firstly, from the definition of $Q(t)$ and Sobolev inequality, it holds that

$$\|\partial_x^\beta (u,p)\|_{L^\infty} \le C_0 \|(u,p)\|_s \le C(Q), \ |\beta| \le 1, \tag{3.1}$$

$$\|\partial_x^{\beta'} (\varepsilon u, \varepsilon p, \theta - \bar\theta)\|_{L^\infty} \le C\|(\varepsilon u, \varepsilon p, \theta - \bar\theta)\|_{s+1} \le C(Q), \ |\beta'| \le 2. \tag{3.2}$$

Moreover, by the structure of the equations (1.5), we obtain

$$\|\partial_t (\varepsilon p, \varepsilon u, \theta)\|_s \le C(Q)(S+1), \tag{3.3}$$

which together with Sobolev inequality yields

$$\|\partial_t (\varepsilon p, \varepsilon u, \theta)\|_{L^\infty} \le C(Q)(S+1). \tag{3.4}$$

For the sake of clarity, we will divide the proof of Proposition 2.5 into some Lemmas. First, letting $(\hat p, \hat u, \hat\theta) = (\varepsilon p - \theta + \bar\theta, \varepsilon u, \theta - \bar\theta)$, and from (1.5), a straightforward computation implies that $(\hat p, \hat u, \hat\theta)$ satisfies

$$\begin{cases} \hat p_t + u\cdot\nabla\hat p + \frac{1}{\varepsilon}\mathrm{div}\,\hat u = 0, \\ e^{-\varepsilon p+\theta}(\hat u_t + u\cdot\nabla\hat u) + \frac{1}{\varepsilon}e^{-\varepsilon p+2\theta}(\nabla\hat p + \nabla\hat\theta) \\ \quad = \kappa'\varepsilon\left(\nabla\hat p\Delta\hat p + |\nabla\hat p|^2\nabla\hat p + \nabla(|\nabla\hat p|^2)\right) \\ \qquad + e^{-2\varepsilon p+2\theta}\mathrm{div}\,\Psi'(\hat u) + \kappa'\varepsilon\nabla\Delta\hat p, \\ \hat\theta_t + u\cdot\nabla\hat\theta + \frac{1}{\varepsilon}\mathrm{div}\,\hat u \\ \quad = \varepsilon e^{-\varepsilon p}\Psi'(u) : \nabla\hat u + \kappa' e^{\varepsilon p-2\theta}(\Delta\hat p + \frac{3}{2}|\nabla\hat p|^2)\mathrm{div}\,\hat u \\ \qquad - \kappa' e^{\varepsilon p-2\theta}\nabla\hat p\otimes\nabla\hat p : \nabla\hat u + \lambda' e^{-\varepsilon p}\mathrm{div}(e^\theta\nabla\hat\theta). \end{cases} \tag{3.5}$$

Further, for any multi-index $|\alpha| \le s+1$, we denote $D^\alpha(\hat p, \hat u, \hat\theta)$ by $(\hat p^\alpha, \hat u^\alpha, \hat\theta^\alpha)$ and apply the operator D^α to the system (3.5) to find that

$$\begin{cases} \hat p^\alpha_t + u\cdot\nabla\hat p^\alpha + \frac{1}{\varepsilon}\mathrm{div}\,\hat u^\alpha = f_1, \\ e^{-\varepsilon p+\theta}(\hat u^\alpha_t + u\cdot\nabla\hat u^\alpha) + \frac{1}{\varepsilon}e^{-\varepsilon p+2\theta}(\nabla\hat p^\alpha + \nabla\hat\theta^\alpha) \\ \quad = e^{-2\varepsilon p+2\theta}\mathrm{div}\,\Psi'(\hat u^\alpha) \\ \qquad + \kappa'\varepsilon\left(\nabla\Delta\hat p^\alpha + \nabla\hat p\Delta\hat p^\alpha + |\nabla\hat p|^2\nabla\hat p^\alpha + \nabla(\nabla\hat p\cdot\nabla\hat p^\alpha)\right) + f_2, \\ \hat\theta^\alpha_t + u\cdot\nabla\hat\theta^\alpha + \frac{1}{\varepsilon}\mathrm{div}\,\hat u^\alpha \\ \quad = \varepsilon e^{-\varepsilon p}\Psi'(u) : \nabla\hat u^\alpha + \kappa' e^{\varepsilon p-2\theta}(\Delta\hat p + \frac{3}{2}|\nabla\hat p|^2)\mathrm{div}\,\hat u^\alpha \\ \qquad - \kappa' e^{\varepsilon p-2\theta}\nabla\hat p\otimes\nabla\hat p : \nabla\hat u^\alpha \\ \qquad + \lambda' e^{-\varepsilon p+\theta}\nabla\hat\theta\cdot\nabla\hat\theta^\alpha + \lambda' e^{-\varepsilon p+\theta}\Delta\hat\theta^\alpha + f_3, \end{cases} \tag{3.6}$$

where

$$\begin{aligned} f_1 &= -[D^\alpha, u]\cdot\nabla\hat p, \\ f_2 &= -[D^\alpha, e^{-\varepsilon p+\theta}]\hat u_t - [D^\alpha, e^{-\varepsilon p+\theta}u]\cdot\nabla\hat u - \frac{1}{\varepsilon}[D^\alpha, e^{-\varepsilon p+2\theta}](\nabla\hat p + \nabla\hat\theta) \\ &\quad + [D^\alpha, e^{-2\varepsilon p+2\theta}]\mathrm{div}\,\Psi'(\hat u) + [D^\alpha, \kappa'\varepsilon\nabla\hat p]\Delta\hat p + [D^\alpha, \kappa'\varepsilon|\nabla\hat p|^2]\nabla\hat p \\ &\quad + \nabla([D^\alpha, \kappa'\varepsilon\nabla\hat p]\cdot\nabla\hat p), \end{aligned}$$

and

$$
\begin{aligned}
f_3 \quad = \quad & -[D^\alpha, u]\cdot\nabla\hat\theta + [D^\alpha, \varepsilon e^{-\varepsilon p}\Psi'(u)] : \nabla\hat u \\
& +[D^\alpha, \kappa' e^{\varepsilon p-2\theta}(\Delta\hat p + \frac{3}{2}|\nabla\hat p|^2)]\mathrm{div}\,\hat u \\
& -[D^\alpha, \kappa' e^{\varepsilon p-2\theta}\nabla\hat p\otimes\nabla\hat p] : \nabla\hat u \\
& +[D^\alpha, \lambda' e^{-\varepsilon p+\theta}\nabla\hat\theta]\cdot\nabla\hat\theta + [D^\alpha, \lambda' e^{-\varepsilon p+\theta}]\Delta\hat\theta.
\end{aligned}
$$

Then we have

LEMMA 3.1. *Let $s \geq 4$ be an integer, and (p, u, θ) be the classical solution to the Cauchy problem (1.5) and (1.7) on $[0, T_1]$, then there exist an increasing positive polynomial $C(\cdot)$, and some positive constants c_1 and c_2 such that for all $t \in [0, T_1]$ and $\varepsilon \in (0, 1]$, it holds that*

$$
\begin{aligned}
& \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\sum_{|\alpha|\leq s+1}\int_{\mathbb{R}^3}\Big(e^{-\varepsilon p+2\theta}|\hat p^\alpha|^2 + e^{-\varepsilon p+\theta}|\hat u^\alpha|^2 \\
& + e^{-\varepsilon p+2\theta}|\hat\theta^\alpha|^2 + \kappa'|\nabla(\varepsilon\hat p^\alpha)|^2\Big)\mathrm{d}x \\
& + c_1\|\nabla(\varepsilon u)(t)\|_{s+1}^2 + c_2\|\nabla\theta(t)\|_{s+1}^2 \leq C(Q)(1+S). \qquad (3.7)
\end{aligned}
$$

Proof. Taking the inner product between $(3.6)_1$, $(3.6)_2$, $(3.6)_3$ and $e^{-\varepsilon p+2\theta}\hat p^\alpha$, $\hat u^\alpha$, $e^{-\varepsilon p+2\theta}\hat\theta^\alpha$, respectively, the summation of the three equalities yields

$$
\begin{aligned}
& \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3}\Big(e^{-\varepsilon p+2\theta}|\hat p^\alpha|^2 + e^{-\varepsilon p+\theta}|\hat u^\alpha|^2 + e^{-\varepsilon p+2\theta}|\hat\theta^\alpha|^2\Big)\mathrm{d}x \\
& = \langle e^{-2\varepsilon p+2\theta}\mathrm{div}\,\Psi'(\hat u^\alpha), \hat u^\alpha\rangle + \langle\lambda' e^{-\varepsilon p+\theta}\Delta\hat\theta^\alpha, e^{-\varepsilon p+2\theta}\hat\theta^\alpha\rangle \qquad (3.8) \\
& + \langle\kappa'\varepsilon\nabla\Delta\hat p^\alpha, \hat u^\alpha\rangle + \sum_{i=1}^5 H_i,
\end{aligned}
$$

where

$$
\begin{aligned}
H_1 \quad = \quad & \frac{1}{2}\int_{\mathbb{R}^3}(e^{-\varepsilon p+\theta})_t|\hat u^\alpha|^2 + (e^{-\varepsilon p+2\theta})_t|\hat p^\alpha|^2 + (e^{-\varepsilon p+2\theta})_t|\hat\theta^\alpha|^2\mathrm{d}x, \\
H_2 \quad = \quad & -\frac{1}{\varepsilon}\langle e^{-\varepsilon p+2\theta}(\nabla\hat p^\alpha + \nabla\hat\theta^\alpha), \hat u^\alpha\rangle - \frac{1}{\varepsilon}\langle\mathrm{div}\,\hat u^\alpha, e^{-\varepsilon p+2\theta}\hat p^\alpha\rangle \\
& -\frac{1}{\varepsilon}\langle\mathrm{div}\,\hat u^\alpha, e^{-\varepsilon p+2\theta}\hat\theta^\alpha\rangle, \\
H_3 \quad = \quad & -\langle u\cdot\nabla\hat p^\alpha, e^{-\varepsilon p+2\theta}\hat p^\alpha\rangle - \langle e^{-\varepsilon p+\theta}u\cdot\nabla\hat u^\alpha, \hat u^\alpha\rangle - \langle u\cdot\nabla\hat\theta^\alpha, e^{-\varepsilon p+2\theta}\hat\theta^\alpha\rangle, \\
H_4 \quad = \quad & \langle\kappa'\varepsilon\Big(\nabla\hat p\Delta\hat p^\alpha + |\nabla\hat p|^2\nabla\hat p^\alpha + \nabla(\nabla\hat p\cdot\nabla\hat p^\alpha)\Big), \hat u^\alpha\rangle \\
& +\langle\kappa' e^{\varepsilon p-2\theta}\Big((\Delta\hat p + \frac{3}{2}|\nabla\hat p|^2)\mathrm{div}\,\hat u^\alpha + \nabla\hat p\otimes\nabla\hat p : \nabla\hat u^\alpha\Big), e^{-\varepsilon p+2\theta}\hat\theta^\alpha\rangle \\
& +\langle\varepsilon e^{-\varepsilon p}\Psi'(u) : \nabla\hat u^\alpha, e^{-\varepsilon p+2\theta}\hat\theta^\alpha\rangle + \langle\lambda' e^{-\varepsilon p+\theta}\nabla\hat\theta\cdot\nabla\hat\theta^\alpha, e^{-\varepsilon p+2\theta}\hat\theta^\alpha\rangle,
\end{aligned}
$$

and

$$
H_5 = \langle f_1, e^{-\varepsilon p+2\theta}\hat p^\alpha\rangle + \langle f_2, \hat u^\alpha\rangle + \langle f_3, e^{-\varepsilon p+2\theta}\hat\theta^\alpha\rangle.
$$

First, an application of integration by parts and (3.2) yields that

$$\begin{aligned}
&\langle e^{-2\varepsilon p+2\theta}\operatorname{div}\Psi'(\hat{u}^\alpha),\hat{u}^\alpha\rangle\\
=&-\int_{\mathbb{R}^3} e^{-2\varepsilon p+2\theta}((\mu'+\nu')|\operatorname{div}\hat{u}^\alpha|^2+\mu'|\nabla\hat{u}^\alpha|^2)\mathrm{d}x\\
&-\langle\mu'\nabla(e^{-2\varepsilon p+2\theta})\cdot\nabla\hat{u}^\alpha,\hat{u}^\alpha\rangle-\langle(\mu'+\nu')\nabla(e^{-2\varepsilon p+2\theta})\operatorname{div}\hat{u}^\alpha,\hat{u}^\alpha\rangle\\
\le&-c_1\|\nabla(\varepsilon u^\alpha)\|^2+C_0\|\nabla(\varepsilon p,\theta)\|_{L^\infty}\|\varepsilon u^\alpha\|^2\\
\le&-c_1\|\nabla(\varepsilon u^\alpha)\|^2+C(Q)S,
\end{aligned}\tag{3.9}$$

for some constant $c_1>0$. Similarly, there exists a constant $c_2>0$ such that

$$\langle\lambda' e^{-\varepsilon p+\theta}\Delta\hat{\theta}^\alpha, e^{-\varepsilon p+2\theta}\hat{\theta}^\alpha\rangle\le -c_2\|\nabla\theta^\alpha\|^2+C(Q)S.\tag{3.10}$$

Next, using integration by parts, the structure of $(3.6)_1$, (3.1) and (3.2), we have the estimate of the term $\langle\kappa'\varepsilon\nabla\Delta\hat{p}^\alpha,\hat{u}^\alpha\rangle$ as follows:

$$\begin{aligned}
&\langle\kappa'\varepsilon\nabla\Delta\hat{p}^\alpha,\hat{u}^\alpha\rangle\\
=&-\langle\kappa'\varepsilon\Delta\hat{p}^\alpha,\operatorname{div}\hat{u}^\alpha\rangle=\langle\kappa'\varepsilon\Delta\hat{p}^\alpha,\varepsilon\hat{p}^\alpha_t+\varepsilon u\cdot\nabla\hat{p}^\alpha+\varepsilon g_1\rangle\\
=&-\frac{\kappa'}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla(\varepsilon\hat{p}^\alpha)\|^2+\langle\kappa'\varepsilon\Delta\hat{p}^\alpha,\varepsilon u\cdot\nabla\hat{p}^\alpha+[D^\alpha,\varepsilon u]\cdot\nabla\hat{p}\rangle\\
\le&-\frac{\kappa'}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla(\varepsilon\hat{p}^\alpha)\|^2+C_0\|\Delta(\varepsilon\hat{p}^\alpha)\|(\|u\cdot\nabla(\varepsilon\hat{p}^\alpha)\|+\|[D^\alpha,\varepsilon u]\cdot\nabla\hat{p}\|)\\
\le&-\frac{\kappa'}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla(\varepsilon\hat{p}^\alpha)\|^2+C_0\|\Delta(\varepsilon\hat{p})\|_{s+1}(\|u\|_{L^\infty}\|\nabla(\varepsilon\hat{p})\|_{s+1}\\
&+\|\varepsilon u\|_{W^{1,\infty}}\|\nabla\hat{p}\|_s+\|\nabla\hat{p}\|_{L^\infty}\|\varepsilon u\|_{s+1})\\
\le&-\frac{\kappa'}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla(\varepsilon\hat{p}^\alpha)\|^2+C(Q)(S+1).
\end{aligned}\tag{3.11}$$

Finally, let us estimate the terms in $H_i(i=1,2,3,4,5)$. First, by virtue of (3.4), it is trivial that

$$H_1\le C_0\|(\varepsilon\partial_t p,\partial_t\theta)\|_{L^\infty}\|(\hat{p}^\alpha,\hat{u}^\alpha,\hat{\theta}^\alpha)\|^2\le C(Q)(S+1).\tag{3.12}$$

Next, when $|\alpha|=0$, noting $\hat{p}+\hat{\theta}=\varepsilon p$, and using integration by parts, Cauchy inequality and (3.1), one gets

$$H_2=\langle\frac{1}{\varepsilon}\nabla(e^{-\varepsilon p+2\theta})(\hat{p}+\hat{\theta}),\hat{u}\rangle\le C_0\|p\|_{L^\infty}(\|\nabla(\varepsilon p)\|+\|\nabla\theta\|)\|\hat{u}\|\le C(Q)S.$$

For $|\alpha|\ge 1$, noting $\hat{p}^\alpha+\hat{\theta}^\alpha=\varepsilon p^\alpha$, utilizing integrate by parts, Cauchy inequality and (3.2), we also have

$$\begin{aligned}
H_2=&\langle\frac{1}{\varepsilon}\nabla(e^{-\varepsilon p+2\theta})(\hat{p}^\alpha+\hat{\theta}^\alpha),\hat{u}^\alpha\rangle\le C_0\|\nabla(-\varepsilon p+2\theta)\|_{L^\infty}\|\varepsilon p^\alpha\|\|\hat{u}^\alpha\|\\
\le& C_0\|\nabla(-\varepsilon p+2\theta)\|_{L^\infty}\|\varepsilon p\|_{s+1}\|\varepsilon u\|_{s+1}\le C(Q)(S+1).
\end{aligned}$$

In conclusion, we have

$$H_2\le C(Q)(S+1).\tag{3.13}$$

Similarly, we can obtain

$$\begin{aligned}
H_3=&\frac{1}{2}\int_{\mathbb{R}^3}\operatorname{div}(e^{-\varepsilon p+2\theta}u)(|\hat{p}^\alpha|^2+|\hat{u}^\alpha|^2+|\hat{\theta}^\alpha|^2)\mathrm{d}x\\
\le& C_0(\|\operatorname{div}u\|_{L^\infty}+\|\nabla(\varepsilon p,\theta)u\|_{L^\infty})\|(\hat{p}^\alpha,\hat{u}^\alpha,\hat{\theta}^\alpha)\|^2\le C(Q).
\end{aligned}\tag{3.14}$$

Now let us focus on the terms of H_4. Using integration by parts, Cauchy inequality and (3.2), one get

$$\begin{aligned}
& \langle \kappa'\varepsilon \left(\nabla\hat{p}\Delta\hat{p}^\alpha + |\nabla\hat{p}|^2\nabla\hat{p}^\alpha + \nabla(\nabla\hat{p}\cdot\nabla\hat{p}^\alpha)\right), \hat{u}^\alpha\rangle \\
= \; & \langle \kappa'\varepsilon \left(\nabla\hat{p}\Delta\hat{p}^\alpha + |\nabla\hat{p}|^2\nabla\hat{p}^\alpha\right), \hat{u}^\alpha\rangle - \langle \kappa'\varepsilon\nabla\hat{p}\cdot\nabla\hat{p}^\alpha, \operatorname{div}\hat{u}^\alpha\rangle \\
\leq \; & C_0\|\hat{u}^\alpha\|(\|\nabla(\varepsilon p-\theta)\|_{L^\infty}\|\Delta(\varepsilon\hat{p}^\alpha)\| + \|\nabla(\varepsilon p-\theta)\|_{L^\infty}^2\|\nabla(\varepsilon\hat{p}^\alpha)\|) \\
& + C_0\|\nabla(\varepsilon p-\theta)\|_{L^\infty}\|\operatorname{div}\hat{u}^\alpha\|\|\nabla(\varepsilon\hat{p}^\alpha)\| \\
\leq \; & C(Q)(S+1).
\end{aligned}$$

Similarly, we can estimate the other terms in H_4. Hence, we have

$$H_4 \leq C(Q)(S+1). \tag{3.15}$$

Finally, it remains to estimate the terms in H_5. At first, noting that for $|\alpha|=0$, $f_1=f_2=f_3=0$. Then in the following we only need to bound H_5 for $|\alpha|\geq 1$. First, noting $\hat{p}=\varepsilon p-(\theta-\bar{\theta})$, and applying Cauchy inequality, Lemma 2.1 and (3.1), we have

$$\begin{aligned}
\langle f_1, e^{-\varepsilon p+2\theta}\hat{p}^\alpha\rangle & \leq C_0\|[D^\alpha, u]\cdot\nabla\hat{p}\|\|\hat{p}^\alpha\| \\
& \leq C_0(\|u\|_{W^{1,\infty}}\|\nabla\hat{p}\|_s + \|\nabla\hat{p}\|_{L^\infty}\|u\|_{s+1})\|\hat{p}\|_{s+1} \\
& \leq C(Q)(S+1). \qquad (3.16)
\end{aligned}$$

Next, noting the fact that $\hat{p}+\hat{\theta}=\varepsilon p$ and applying integration by parts to the last term in $\langle f_2, \hat{u}^\alpha\rangle$, one has

$$\begin{aligned}
& \langle f_2, \hat{u}^\alpha\rangle \\
= \; & -\langle [D^\alpha, e^{-\varepsilon p+\theta}]\varepsilon\partial_t u, \hat{u}^\alpha\rangle - \langle [D^\alpha, e^{-\varepsilon p+\theta}u]\cdot\nabla\hat{u}, \hat{u}^\alpha\rangle - \langle [D^\alpha, e^{-\varepsilon p+2\theta}]\nabla p, \hat{u}^\alpha\rangle \\
& + \langle [D^\alpha, e^{-2\varepsilon p+2\theta}]\operatorname{div}\Psi'(\hat{u}), \hat{u}^\alpha\rangle + \langle [D^\alpha, \kappa'\varepsilon\nabla\hat{p}]\Delta\hat{p}, \hat{u}^\alpha\rangle \\
& + \langle [D^\alpha, \kappa'\varepsilon|\nabla\hat{p}|^2]\nabla\hat{p}, \hat{u}^\alpha\rangle - \langle [D^\alpha, \kappa'\varepsilon\nabla\hat{p}]\cdot\nabla\hat{p}, \operatorname{div}\hat{u}^\alpha\rangle.
\end{aligned}$$

Moreover, from Lemma 2.1 and (3.2), we obtain the estimates of the following term in $\langle f_2, \hat{u}^\alpha\rangle$:

$$\begin{aligned}
& \langle [D^\alpha, \kappa'\varepsilon\nabla\hat{p}]\Delta\hat{p}), \hat{u}^\alpha\rangle \\
\leq \; & C_0\varepsilon\|\hat{u}^\alpha\|(\|\Delta\hat{p}\|_{W^{1,\infty}}\|\varepsilon\nabla\hat{p}\|_s + \|\nabla\hat{p}\|_{L^\infty}\|\varepsilon\Delta\hat{p}\|_s) \leq C(Q)(S+1).
\end{aligned}$$

The other terms in $\langle f_2, \hat{u}^\alpha\rangle$ can be estimated. Hence, we have

$$\langle f_2, \hat{u}^\alpha\rangle \leq C(Q)(S+1). \tag{3.17}$$

Noting that f_3 is similar to $\langle f_2, \hat{u}^\alpha\rangle$ in structure, then it is easy to bound the terms involving $\langle f_3, e^{-\varepsilon p+2\theta}\hat{\theta}^\alpha\rangle$ in a same way and get

$$\langle f_3, e^{-\varepsilon p+2\theta}\hat{\theta}^\alpha\rangle \leq C(Q)(S+1). \tag{3.18}$$

Then, from (3.16), (3.17) and (3.18), we have

$$H_5 \leq C(Q)(S+1). \tag{3.19}$$

Therefore, putting (3.9)-(3.15) and (3.19) into (3.8) and summing for $|\alpha|\leq s+1$, we obtain (3.7). This completes the proof.

LEMMA 3.2. *Let $s\geq 4$ be an integer, and (p,u,θ) be the classical solution to the Cauchy problem (1.5) and (1.7) on $[0,T_1]$, then there exist an increasing* *positive polynomial $C(\cdot)$, and some positive constants c_3, c_4 and C_1, such that for all $t \in [0, T_1]$ and $\varepsilon \in (0, 1]$, it holds that*

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t} \sum_{|\alpha| \leq s+1} \int_{\mathbb{R}^3} \varepsilon \nabla \hat{p}^\alpha \hat{u}^\alpha \mathrm{d}x + \frac{c_3}{2} \|\nabla \hat{p}\|_{s+1}^2 + \frac{c_4}{2} \|\Delta(\varepsilon \hat{p})\|_{s+1}^2 \\ \leq\ & C(Q)(S+1) + C_1 \|\nabla(\varepsilon u, \theta)\|_{s+1}^2. \end{aligned} \tag{3.20}$$

Proof. From $(3.6)_2$, we have

$$\begin{aligned} &\hat{u}_t^\alpha + u \cdot \nabla \hat{u}^\alpha + \frac{1}{\varepsilon} e^\theta (\nabla \hat{p}^\alpha + \nabla \hat{\theta}^\alpha) = e^{-\varepsilon p + \theta} \mathrm{div}\, \Psi'(\hat{u}^\alpha) + \kappa' \varepsilon e^{\varepsilon p - \theta} \big(\nabla \Delta \hat{p}^\alpha \\ &+ \nabla \hat{p} \Delta \hat{p}^\alpha + |\nabla \hat{p}|^2 \nabla \hat{p}^\alpha + \nabla (\nabla \hat{p} \cdot \nabla \hat{p}^\alpha)\big) + \tilde{f}_2, \end{aligned} \tag{3.21}$$

where

$$\begin{aligned} \tilde{f}_2 \ =\ & -[D^\alpha, u] \cdot \nabla \hat{u} - [D^\alpha, \frac{1}{\varepsilon} e^\theta](\nabla \hat{p} + \nabla \hat{\theta}) + [D^\alpha, e^{-\varepsilon p + \theta}] \mathrm{div}\, \Psi'(\hat{u}^\alpha) \\ & + [D^\alpha, \kappa' \varepsilon e^{\varepsilon p - \theta}] \nabla \Delta \hat{p} + [D^\alpha, \kappa' \varepsilon e^{\varepsilon p - \theta} \nabla \hat{p}] \Delta \hat{p} + [D^\alpha, \kappa' \varepsilon e^{\varepsilon p - \theta} |\nabla \hat{p}|^2] \nabla \hat{p} \\ & + [D^\alpha, \kappa' \varepsilon e^{\varepsilon p - \theta}] \nabla (|\nabla \hat{p}|^2) + \kappa' \varepsilon e^{\varepsilon p - \theta} \nabla ([D^\alpha, \nabla \hat{p}] \cdot \nabla \hat{p}). \end{aligned}$$

Then we take the inner product between (3.21) and $\varepsilon \nabla \hat{p}^\alpha$ to obtain

$$\begin{aligned} & \frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}^3} \varepsilon \nabla \hat{p}^\alpha \hat{u}^\alpha \mathrm{d}x + \langle e^\theta \nabla \hat{p}^\alpha, \nabla \hat{p}^\alpha \rangle \\ =\ & \langle \kappa' \varepsilon e^{\varepsilon p - \theta} \nabla \Delta \hat{p}^\alpha, \varepsilon \nabla \hat{p}^\alpha \rangle + \langle \hat{u}^\alpha, \varepsilon \nabla \hat{p}_t^\alpha \rangle - \langle u \cdot \nabla \hat{u}^\alpha, \varepsilon \nabla \hat{p}^\alpha \rangle \\ & - \langle e^\theta \nabla \hat{\theta}^\alpha, \nabla \hat{p}^\alpha \rangle + \langle e^{-\varepsilon p + \theta} \mathrm{div} \Psi'(\hat{u}^\alpha), \varepsilon \nabla \hat{p}^\alpha \rangle \\ & + \langle \kappa' \varepsilon e^{\varepsilon p - \theta} \left(\nabla \hat{p} \Delta \hat{p}^\alpha + |\nabla \hat{p}|^2 \nabla \hat{p}^\alpha + \nabla (\nabla \hat{p} \cdot \nabla \hat{p}^\alpha) \right), \varepsilon \nabla \hat{p}^\alpha \rangle + \langle \tilde{f}_2, \varepsilon \nabla \hat{p}^\alpha \rangle \\ =\ & : \langle \kappa' \varepsilon e^{\varepsilon p - \theta} \nabla \Delta \hat{p}^\alpha, \varepsilon \nabla \hat{p}^\alpha \rangle + \sum_{i=1}^{6} I_i. \end{aligned} \tag{3.22}$$

Firstly, it is trivial that

$$\langle e^\theta \nabla \hat{p}^\alpha, \nabla \hat{p}^\alpha \rangle \geq c_3 \|\nabla \hat{p}^\alpha\|^2, \tag{3.23}$$

for some constant $c_3 > 0$. Moreover, thanks to the boundedness of $e^{\varepsilon p - \theta}$, and using Cauchy inequality and (3.2), then there exists a constant $c_4 > 0$ such that

$$\begin{aligned} & \langle \kappa' \varepsilon e^{\varepsilon p - \theta} \nabla \Delta \hat{p}^\alpha, \varepsilon \nabla \hat{p}^\alpha \rangle \\ =\ & -\langle \kappa' \varepsilon e^{\varepsilon p - \theta} \Delta \hat{p}^\alpha, \varepsilon \Delta \hat{p}^\alpha \rangle - \langle \kappa' \varepsilon \nabla (e^{\varepsilon p - \theta}) \Delta \hat{p}^\alpha, \varepsilon \nabla \hat{p}^\alpha \rangle \\ \leq\ & -c_4 \|\Delta (\varepsilon \hat{p}^\alpha)\|^2 + C_0 \|\nabla (\varepsilon p - \theta)\|_{L^\infty} \|\Delta \hat{p}\|_{s+1} \|\nabla (\varepsilon \hat{p})\|_{s+1} \\ \leq\ & -c_4 \|\Delta (\varepsilon \hat{p}^\alpha)\|^2 + C(Q)(S+1). \end{aligned} \tag{3.24}$$

In the following, we shall estimate $I_i (i = 1, 2, 3, 4, 5, 6)$. First, integrating by parts and using the structure of $(3.6)_1$, Lemma 2.1 and (3.1), we can control the term I_1 as follow:

$$\begin{aligned} I_1 \ =\ & \langle \hat{u}^\alpha, \varepsilon \nabla \hat{p}_t^\alpha \rangle = -\langle \mathrm{div}\, \hat{u}^\alpha, \varepsilon u \cdot \nabla \hat{p}^\alpha + \varepsilon [D^\alpha, u] \cdot \nabla \hat{p} + \mathrm{div}\, \hat{u}^\alpha \rangle \\ \leq\ & C_0 \|\mathrm{div}\, \hat{u}^\alpha\| \left(\varepsilon \|u \cdot \nabla \hat{p}^\alpha\| + \varepsilon \|[D^\alpha, u] \cdot \nabla \hat{p}\| + \|\mathrm{div}\, \hat{u}^\alpha\| \right) \\ \leq\ & C_0 \|\nabla \hat{u}\|_{s+1}^2 + C_0 (\|u\|_{L^\infty} \|\nabla (\varepsilon \hat{p})\|_{s+1} + \|\varepsilon \nabla \hat{p}\|_{W^{1,\infty}} \|u\|_s) \\ \leq\ & C_0 \|\nabla (\varepsilon u)\|_{s+1}^2 + C(Q)(S+1). \end{aligned} \tag{3.25}$$

Moreover, it directly follows from Cauchy inequality that

$$I_2 = -\langle u \cdot \nabla \hat{u}^\alpha, \varepsilon \nabla \hat{p}^\alpha \rangle \leq \frac{c_3}{4} \|\nabla \hat{p}^\alpha\|^2 + C_0 \|\nabla(\varepsilon u)\|_{s+1}^2. \tag{3.26}$$

and

$$I_3 = -\langle e^\theta \nabla \hat{\theta}^\alpha, \nabla \hat{p}^\alpha \rangle \leq \frac{c_3}{4} \|\nabla \hat{p}^\alpha\|^2 + C_0 \|\nabla \theta\|_{s+1}^2. \tag{3.27}$$

Next, using integration by parts, (3.2) and Cauchy inequality, we also have

$$\begin{aligned}
I_4 &= \langle e^{-\varepsilon p+\theta} \mathrm{div}\Psi'(\hat{u}^\alpha), \varepsilon \nabla \hat{p}^\alpha \rangle \\
&= (\mu' + \nu') \langle e^{-\varepsilon p+\theta} \nabla \mathrm{div} \hat{u}^\alpha, \varepsilon \nabla \hat{p}^\alpha \rangle \\
&\quad + \mu' \sum_{i=1}^{3} \sum_{j=1}^{3} \langle e^{-\varepsilon p+\theta} \partial_{x_i x_i} \hat{u}^{j,\alpha}, \varepsilon \partial_{x_j} \hat{p}^\alpha \rangle \\
&= -(2\mu' + \nu') \langle e^{-\varepsilon p+\theta} \mathrm{div} \hat{u}^\alpha, \varepsilon \Delta \hat{p}^\alpha \rangle \\
&\quad -(2\mu' + \nu') \langle \nabla e^{-\varepsilon p+\theta} \mathrm{div} \hat{u}^\alpha, \varepsilon \nabla \hat{p}^\alpha \rangle \\
&\quad + \mu' \sum_{i=1}^{3} \sum_{j=1}^{3} \langle \partial_{x_j} e^{-\varepsilon p+\theta} \partial_{x_i} \hat{u}^{j,\alpha}, \varepsilon \partial_{x_i} \hat{p}^\alpha \rangle \\
&\quad - \mu' \sum_{i=1}^{3} \sum_{j=1}^{3} \langle \partial_{x_i} e^{-\varepsilon p+\theta} \partial_{x_i} \hat{u}^{j,\alpha}, \varepsilon \partial_{x_j} \hat{p}^\alpha \rangle \\
&\leq C_0 \|\mathrm{div}\, \hat{u}^\alpha\| \|\Delta(\varepsilon \hat{p}^\alpha)\| + C_0 \|\nabla(-\varepsilon p + \theta)\|_{L^\infty} \|\nabla \hat{u}^\alpha\| \|\nabla(\varepsilon \hat{p}^\alpha)\| \\
&\leq \frac{c_4}{2} \|\Delta(\varepsilon \hat{p}^\alpha)\|^2 + C_0 \|\nabla(\varepsilon u)\|_{s+1}^2 + C(Q)(S+1).
\end{aligned} \tag{3.28}$$

Finally, similar as (3.14), (3.15) and (3.19), we have

$$I_5 \leq C(Q)(S+1). \tag{3.29}$$

Therefore, putting (3.23)-(3.29) into (3.22) and summing for all $|\alpha| \leq s+1$, we obtain (3.20). This completes the proof.

Once we have Lemma 3.1 and 3.2, we have the following H^{s+1} estimates on $(\varepsilon p, \varepsilon u, \theta - \bar{\theta})$ and $\nabla(\varepsilon^2 p - \varepsilon \theta)$:

LEMMA 3.3. *Let $s \geq 4$ be an integer, and (p, u, θ) be the classical solution to the Cauchy problem (1.5) and (1.7) on $[0, T_1]$, there exist positive constant $\varepsilon_0 \leq 1$ and an increasing positive polynomial $C(\cdot)$, such that for all $T \in [0, \hat{T}]$ with $\hat{T} = \min\{1, T_1\}$ and for all $\varepsilon \in (0, \varepsilon_0]$, it holds that*

$$\begin{aligned}
&\sup_{t \in [0,T]} \|(\varepsilon p, \varepsilon u, \theta - \bar{\theta}, \nabla(\varepsilon^2 p - \varepsilon \theta))(t)\|_{s+1}^2 \\
&\quad + \int_0^T \|\nabla(\varepsilon p, \varepsilon u, \theta)(t)\|_{s+1}^2 + \|\nabla^2(\varepsilon^2 p - \varepsilon \theta)(t)\|_{s+1}^2 \mathrm{d}t \\
&\leq \; C(O_0) e^{\sqrt{T} C(O(T))}.
\end{aligned} \tag{3.30}$$

Proof. Choosing some proper $\tilde{c} > 0$ such that $\tilde{c} C_1 < \frac{c_1}{2}$, $\tilde{c} C_1 < \frac{c_2}{2}$ and

$$\begin{aligned}
&e^{-\varepsilon p+2\theta} |\hat{p}^\alpha|^2 + e^{-\varepsilon p+\theta} |\hat{u}^\alpha|^2 + e^{-\varepsilon p+2\theta} |\hat{\theta}^\alpha|^2 + \kappa' |\nabla(\varepsilon \hat{p}^\alpha)|^2 + 2\tilde{c} \hat{u}^\alpha \nabla(\varepsilon \hat{p}^\alpha) \\
\sim \quad &|\hat{p}^\alpha|^2 + |\hat{u}^\alpha|^2 + |\hat{\theta}^\alpha|^2 + |\nabla(\varepsilon \hat{p}^\alpha)|^2,
\end{aligned}$$

then the summation of $\tilde{c}$(3.20) and (3.7) yields that

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\sum_{|\alpha|\le s+1}\int_{\mathbb{R}^3}(|\hat{p}^\alpha|^2+|\hat{u}^\alpha|^2+|\hat{\theta}^\alpha|^2+|\nabla(\varepsilon\hat{p}^\alpha)|^2)\mathrm{d}x\\ +\quad&\|\nabla(\varepsilon u,\theta,\hat{p})(t)\|_{s+1}^2+\|\Delta(\varepsilon\hat{p})\|_{s+1}^2\le C(Q)(1+S).\end{aligned}$$

Then integrating the above inequality on $[0,T]$ for any $T\in[0,\hat{T}]$ and using the elliptic estimate $\|\Delta(\varepsilon\hat{p})\|_{s+1}\sim\|\nabla^2(\varepsilon\hat{p})\|_{s+1}$ and $\hat{p}=\varepsilon p-(\theta-\bar{\theta})$, we have

$$\begin{aligned}&\sup_{t\in[0,T]}\|(\varepsilon p,\varepsilon u,\theta-\bar{\theta},\nabla(\varepsilon\hat{p}))(t)\|_{s+1}^2\\ &+\int_0^T\|\nabla(\varepsilon u,\hat{p},\theta)(t)\|_{s+1}^2+\|\nabla^2(\varepsilon\hat{p})(t)\|_{s+1}^2\mathrm{d}t\\ \le\;&C(O_0)+\int_0^T C(Q(t))(S(t)+1)\mathrm{d}t\\ \le\;&C(O_0)+TC(O(T))+\sqrt{T}\Big(\int_0^t S(t)^2\mathrm{d}t\Big)^{\frac{1}{2}}\le C(O_0)e^{\sqrt{T}C(O(T))},\end{aligned}\tag{3.31}$$

which deduces (3.30) immediately. This completes the proof.

Next, we are going to show the following H^{s-1} estimate on $\mathrm{curl}\,u$, which plays an important role in obtaining a uniform bound for u.

LEMMA 3.4. *Let $s\ge 4$ be an integer, and (p,u,θ) be the classical solution to the Cauchy problem (1.5) and (1.7) on $[0,T_1]$, there exist positive constant $\varepsilon_0\le 1$ and an increasing positive polynomial $C(\cdot)$, such that for all $T\in[0,\hat{T}]$ and $\varepsilon\in(0,\varepsilon_0]$, it holds that*

$$\begin{aligned}&\sup_{t\in[0,T]}\|\mathrm{curl}\,(e^{-\theta}u)(t)\|_{s-1}^2+\int_0^T\|\nabla\mathrm{curl}\,(e^{-\theta}u)(t)\|_{s-1}^2\mathrm{d}t\\ \le\;&C(O_0)e^{\sqrt{T}C(O(T))}.\end{aligned}\tag{3.32}$$

Proof. Applying the operator curl to the equation $(1.5)_2$, a direct computation yields

$$\begin{aligned}\partial_t\mathrm{curl}(e^{-\theta}u)+u\cdot\nabla(\mathrm{curl}(e^{-\theta}u))\;=\;&\mu'\mathrm{div}\,(e^{-\epsilon p+\theta}\nabla\mathrm{curl}(e^{-\theta}u))\\ &-\nabla(\kappa'e^{-\theta})\times\nabla\Delta e^{-\varepsilon p+\theta}+g_1,\end{aligned}\tag{3.33}$$

where

$$\begin{aligned}g_1\;=\;&\mu'\mathrm{div}\,(e^{-\varepsilon p}\nabla(e^\theta)\otimes\mathrm{curl}\,(e^{-\theta}u))-\mu'\nabla e^{-\varepsilon p}\cdot\nabla(e^\theta\mathrm{curl}\,(e^{-\theta}u))\\ &-\mu'e^{-\varepsilon p}\Delta(\nabla(e^\theta)\times(e^{-\theta}u))-\nabla e^{-\varepsilon p}\times(\mu'\Delta u+(\mu'+\nu')\nabla\mathrm{div}\,u)\\ &-\mathrm{curl}\,\big(e^{-\theta}u\partial_t\theta\big)-[\mathrm{curl},u]\cdot\nabla(e^{-\theta}u)-\mathrm{curl}\,(e^{-\theta}u(u\cdot\nabla\theta)).\end{aligned}$$

Further, we apply the operator D^α $(0\le|\alpha|\le s-1)$ to the equation (3.33) to obtain

$$\begin{aligned}&\partial_tD^\alpha(\mathrm{curl}(e^{-\theta}u))+u\cdot\nabla D^\alpha(\mathrm{curl}(e^{-\theta}u))\\ =\;&\mu'\mathrm{div}\,(e^{-\epsilon p+\theta}\nabla D^\alpha(\mathrm{curl}(e^{-\theta}u)))-D^\alpha(\nabla(\kappa'e^{-\theta})\times\nabla\Delta e^{-\varepsilon p+\theta})\\ &+D^\alpha g_1+g_2,\end{aligned}\tag{3.34}$$

where

$$g_2=-[D^\alpha,u]\cdot\nabla(\mathrm{curl}\,(e^{-\theta}u))+\mu'\mathrm{div}\,([D^\alpha,e^{-\varepsilon p+\theta}]\nabla\mathrm{curl}\,(e^{-\theta}u)).$$

Then, taking the inner product between (3.34) and $D^\alpha(\mathrm{curl}(e^{-\theta}u))$, we deduce that

$$
\begin{aligned}
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|D^\alpha(\mathrm{curl}(e^{-\theta}u))\|^2 \;=\;& -\langle u\cdot\nabla D^\alpha(\mathrm{curl}(e^{-\theta}u)), D^\alpha(\mathrm{curl}(e^{-\theta}u))\rangle\\
&+\langle \mu'\mathrm{div}\,(e^{-\epsilon p+\theta}\nabla D^\alpha(\mathrm{curl}(e^{-\theta}u)), D^\alpha(\mathrm{curl}(e^{-\theta}u))\rangle\\
&-\langle D^\alpha(\nabla(\kappa' e^{-\theta})\times\nabla\Delta e^{-\varepsilon p+\theta}), D^\alpha(\mathrm{curl}(e^{-\theta}u))\rangle\\
&+\langle D^\alpha g_1+g_2, D^\alpha(\mathrm{curl}(e^{-\theta}u))\rangle =: \sum_{i=1}^4 J_i. \qquad (3.35)
\end{aligned}
$$

Now let us estimate $J_i (i=1,2,3,4)$. First, from integration by parts and (3.1), it is easy to get

$$
J_1 \le C_0\|\mathrm{div}\,u\|_{L^\infty}\|D^\alpha(\mathrm{curl}\,(e^{-\theta}u))\|^2 \le C(Q), \qquad (3.36)
$$

and

$$
\begin{aligned}
J_2 \;=\;& -\langle \mu' e^{-\epsilon p+\theta}\nabla D^\alpha(\mathrm{curl}(e^{-\theta}u)), \nabla D^\alpha(\mathrm{curl}(e^{-\theta}u))\rangle\\
\le\;& -c_5\|\nabla D^\alpha(\mathrm{curl}(e^{-\theta}u))\|^2, \qquad (3.37)
\end{aligned}
$$

for some positive constant c_5.

Next, by virtue of Sobolev inequality, we have

$$
\begin{aligned}
J_3 \;\le\;& C_0\|\mathrm{curl}(e^{-\theta}u)\|_{s-1}\|\nabla\Delta e^{-\varepsilon p+\theta})\|_{s-1}\|\nabla(\kappa' e^{-\theta})\|_{s-1}\\
\le\;& C(Q)(S+1). \qquad (3.38)
\end{aligned}
$$

Finally, noting that

$$
\|\nabla D^\alpha(\mathrm{curl}(e^{-\theta}u))\| \;\le\; \|e^{-\theta}\mathrm{curl}\,u-\nabla(e^{-\theta})u\|_s \le C(Q)(S+1),
$$

and from integration by parts and Sobolev inequality, one has

$$
\begin{aligned}
&\langle \mu' D^\alpha(\mathrm{div}\,(e^{-\varepsilon p}\nabla(e^\theta)\otimes\mathrm{curl}\,(e^{-\theta}u))), D^\alpha(\mathrm{curl}(e^{-\theta}u))\rangle\\
=\;& -\langle \mu' D^\alpha(e^{-\varepsilon p}\nabla(e^\theta)\otimes\mathrm{curl}\,(e^{-\theta}u)), \nabla D^\alpha(\mathrm{curl}(e^{-\theta}u))\rangle\\
\le\;& C_0\|\nabla\theta\|_{s-1}\|\mathrm{curl}(e^{-\theta}u)\|_{s-1}\|\nabla D^\alpha(\mathrm{curl}(e^{-\theta}u))\| \le C(Q)(S+1).
\end{aligned}
$$

Similarly, the other terms in J_4 can be bounded from above by $C(Q)(S+1)$. Therefore, we have

$$
J_4 \le C(Q)(S+1). \qquad (3.39)
$$

Hence, inserting (3.36)-(3.39) into (3.35) and integrating the result inequality on $[0,T]$, we arrive at (3.32). This completes the proof.

Now let us show L^2-estimate on (p,u), by which we can control

$$
\|(D^\beta(\varepsilon\partial_t)p, D^\beta(\varepsilon\partial_t)u)\|
$$

for all $0\le|\beta|\le s-1$. To begin with, setting $(\tilde p,\tilde u)=(p, 2u-\lambda' e^{-\varepsilon p+\theta}\nabla\theta)$, then from (1.5), we see that

$$
\begin{aligned}
\tilde p_t+u\cdot\nabla\tilde p+\frac{1}{\varepsilon}\mathrm{div}\,\tilde u \;=\;& \varepsilon e^{-\varepsilon p}\Psi'(u):\nabla u+\kappa' e^{\varepsilon p-2\theta}(\Delta\hat p+\frac{3}{2}|\nabla\hat p|^2)\mathrm{div}\,u\\
&-\kappa' e^{\varepsilon p-2\theta}\nabla\hat p\otimes\nabla\hat p:\nabla u+\lambda' e^{-\varepsilon p+\theta}\nabla p\cdot\nabla\theta, \qquad (3.40)
\end{aligned}
$$

and

$$
\frac{1}{2}e^{-\theta}(\tilde u_t+u\cdot\nabla\tilde u)+\frac{\nabla\tilde p}{\varepsilon}=\frac{1}{2}e^{-\varepsilon p}\mathrm{div}\,\Psi'(\tilde u)+\kappa' e^{-\theta}\nabla\Delta(e^{\varepsilon p-\theta})+h, \qquad (3.41)
$$

where

$$
\begin{aligned}
h=&\frac{1}{2}e^{-\varepsilon p}\mathrm{div}\,\Psi'(\lambda' e^{-\varepsilon p+\theta}\nabla\theta)+\frac{1}{4}\lambda' e^{-\varepsilon p}\nabla\mathrm{div}\,\tilde{u}+\frac{1}{4}\lambda' e^{-\varepsilon p}\nabla\mathrm{div}\,(\lambda' e^{-\varepsilon p+\theta}\nabla\theta)\\
&-\frac{1}{2}e^{-\theta}\lambda'(e^{-\varepsilon p+\theta})_t\nabla\theta+\frac{1}{2}e^{-\theta}[\nabla,u]\cdot(\lambda' e^{-\varepsilon p+\theta}\nabla\theta)-\frac{1}{2}e^{-\theta}\nabla(\lambda' e^{-\varepsilon p+\theta})u\cdot\nabla\theta\\
&-\frac{1}{2}\lambda' e^{-\varepsilon p}\nabla\Big(\varepsilon^2 e^{-\varepsilon p}\Psi'(u):\nabla u\Big)-\frac{1}{2}\lambda' e^{-\varepsilon p}\nabla\Big(\kappa'\varepsilon e^{\varepsilon p-2\theta}(\Delta\hat{p}+\frac{3}{2}|\nabla\hat{p}|^2)\mathrm{div}\,u\Big)\\
&+\frac{1}{2}\lambda' e^{-\varepsilon p}\nabla\Big(\kappa'\varepsilon e^{\varepsilon p-2\theta}\nabla\hat{p}\otimes\nabla\hat{p}:\nabla u\Big)-\frac{1}{2}\lambda' e^{-\varepsilon p}\nabla\Big(\lambda' e^{-\varepsilon p+\theta}(\nabla\theta\cdot\nabla\theta+\Delta\theta)\Big).
\end{aligned}
$$

Then we have

Lemma 3.5. *Let $s\geq 4$ be an integer, and (p,u,θ) be the classical solution to the Cauchy problem (1.5)and (1.7) on $[0,T_1]$, there exist positive constants $\varepsilon_0\leq 1$ and an increasing positive polynomial $C(\cdot)$, such that for all $T\in[0,\hat{T}]$ and $\varepsilon\in(0,\varepsilon_0]$, it holds that*

$$
\sup_{t\in[0,T]}\|(p,u)(t)\|^2+\int_0^T\|\nabla u(t)\|^2\mathrm{d}t\leq C(O_0)e^{\sqrt{T}C(O(T))}. \tag{3.42}
$$

Proof. Taking the inner product of $(3.40)_1$ and (3.41) with $\tilde{p}$ and $\tilde{u}$, respectively, the summation of the two resultant equalities yields

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3}(\frac{1}{2}|\tilde{p}|^2+\frac{1}{4}e^{-\theta}|\tilde{u}|^2)\mathrm{d}x\\
=&\ \langle\frac{1}{2}e^{-\varepsilon p}\mathrm{div}\,\Psi'(\tilde{u}),\tilde{u}\rangle+\langle\kappa' e^{-\theta}\nabla\Delta(e^{\varepsilon p-\theta}),\tilde{u}\rangle+\sum_{i=1}^4 K_i,
\end{aligned}\tag{3.43}
$$

where

$$
K_1=\frac{1}{4}\int_{\mathbb{R}^3}(e^{-\theta})_t|\tilde{u}|^2\mathrm{d}x,\ K_2=-\langle u\cdot\nabla p,p\rangle-\frac{1}{2}\langle e^{-\theta}u\cdot\nabla\tilde{u},\tilde{u}\rangle,\ K_4=\langle h_4,\tilde{u}\rangle,
$$

$$
\begin{aligned}
K_3\ =\ &\langle\varepsilon e^{-\varepsilon p}\Psi'(u):\nabla u,p\rangle+\langle\kappa' e^{\varepsilon p-2\theta}(\Delta\hat{p}+\frac{3}{2}|\nabla\hat{p}|^2)\mathrm{div}\,u,p\rangle\\
&-\langle\kappa' e^{\varepsilon p-2\theta}\nabla\hat{p}\otimes\nabla\hat{p}:\nabla u,p\rangle+\langle\lambda' e^{-\varepsilon p+\theta}\nabla p\cdot\nabla\theta,p\rangle.
\end{aligned}
$$

Now let us estimate the terms in the right-hand side of (3.43). First, we use integration by parts, Cauchy inequality and (3.2) to obtain

$$
\begin{aligned}
\langle\frac{1}{2}e^{-\varepsilon p}\mathrm{div}\,\Psi'(\tilde{u}),\tilde{u}\rangle\ =&\ -\frac{1}{2}\langle e^{-\varepsilon p}\,\Psi'(\tilde{u}),\nabla\tilde{u}\rangle+\frac{1}{2}\langle e^{-\varepsilon p}\nabla(\varepsilon p)\,\Psi'(\tilde{u}),\tilde{u}\rangle\\
\leq&\ -c_6\|\nabla\tilde{u}\|^2+C_0\|\nabla(\varepsilon p)\|_{L^\infty}\|\|\nabla\tilde{u}\|\|\tilde{u}\|\\
\leq&\ -\frac{3c_6}{4}\|\nabla\tilde{u}\|^2+C(Q),
\end{aligned}\tag{3.44}
$$

for some constant $c_6>0$.

Next, with the aid of integration by parts and Cauchy inequality, the Korteweg term can be estimated as follow:

$$\begin{aligned}
&\langle \kappa' e^{-\theta}\nabla\Delta(e^{\varepsilon p-\theta}),\tilde{u}\rangle = \langle \kappa' e^{-\theta}\nabla(e^{\varepsilon p-\theta}(\Delta\hat{p}+|\nabla\hat{p}|^2)),\tilde{u}\rangle\\
=\ & -\langle \kappa' e^{\varepsilon p-2\theta}(\Delta\hat{p}+|\nabla\hat{p}|^2),\operatorname{div}\tilde{u}\rangle + \langle \kappa' e^{\varepsilon p-2\theta}(\Delta\hat{p}+|\nabla\hat{p}|^2)\nabla\theta,\tilde{u}\rangle\\
\le\ & C_0(\|\operatorname{div}\tilde{u}\|\|\Delta\hat{p}\| + \|\nabla\hat{p}\|_{L^\infty}\|\operatorname{div}\tilde{u}\|\|\nabla\hat{p}\|\\
&\quad + \|\tilde{u}\|_{L^\infty}\|\nabla\theta\|\|\Delta\hat{p}\| + \|\tilde{u}\|_{L^\infty}\|\nabla\hat{p}\|_{L^\infty}\|\nabla\theta\|\|\nabla\hat{p}\|)\\
\le\ & \frac{c_6}{4}\|\nabla\tilde{u}^k\|^2 + C_0\|\nabla(\varepsilon p,\theta)\|_1^2 + C(Q)(1+S).
\end{aligned}\tag{3.45}$$

Moreover, similar to (3.12), (3.13) and (3.15), we can show one by one:

$$K_1+K_2+K_3 \le C(Q)(S+1).\tag{3.46}$$

Finally, let us focus on the terms in K_4. Utilizing integration by parts and Cauchy inequality, we can deal with the first two terms of $\langle h_4,\tilde{u}\rangle$ as follow:

$$\begin{aligned}
&\langle \frac{1}{2}e^{-\varepsilon p}\operatorname{div}\Psi'(\lambda' e^{-\varepsilon p+\theta}\nabla\theta),\tilde{u}\rangle\\
=\ & -\langle \frac{1}{2}e^{-\varepsilon p},(\mu'+\nu')\operatorname{div}(\lambda' e^{-\varepsilon p+\theta}\nabla\theta)\operatorname{div}\tilde{u} + \mu'\nabla(\lambda' e^{-\varepsilon p+\theta}\nabla\theta):\nabla\tilde{u}\rangle\\
& -\frac{1}{2}\langle(\mu'+\nu')\nabla(e^{-\varepsilon p})\operatorname{div}(\lambda' e^{-\varepsilon p+\theta}\nabla\theta) + \mu'\nabla(e^{-\varepsilon p})\cdot\nabla(\lambda' e^{-\varepsilon p+\theta}\nabla\theta),\tilde{u}\rangle\\
\le\ & C_0\|\nabla(\lambda' e^{-\varepsilon p+\theta}\nabla\theta)\|\|\nabla\tilde{u}\| + C_0\|\nabla(\varepsilon p)\|_{L^\infty}\|\nabla(\lambda' e^{-\varepsilon p+\theta}\nabla\theta)\|\|\tilde{u}\|\\
\le\ & \frac{c_6}{8}\|\nabla\tilde{u}\|^2 + C_0\|\nabla\theta\|_1^2 + C(Q)(S+1),
\end{aligned}$$

and

$$\begin{aligned}
&\langle \frac{1}{4}\lambda' e^{-\varepsilon p}\nabla\operatorname{div}\tilde{u},\tilde{u}\rangle\\
=\ & -\langle \frac{1}{4}\lambda' e^{-\varepsilon p}\operatorname{div}\tilde{u},\operatorname{div}\tilde{u}\rangle - \langle \frac{1}{4}\lambda'\nabla(e^{-\varepsilon p})\operatorname{div}\tilde{u},\tilde{u}\rangle\\
\le\ & -\frac{3c_6}{4}\|\nabla\tilde{u}\|^2 + C_0\|\nabla(\varepsilon p)\|_{L^\infty}\|\operatorname{div}\tilde{u}\|\|\tilde{u}\| \le -\frac{3c_6}{4}\|\nabla\tilde{u}\|^2 + C(Q)(S+1).
\end{aligned}$$

Furthermore, a similar argument deduces that

$$\langle h_4,\tilde{u}\rangle \le -\frac{c_6}{4}\|\nabla\tilde{u}\|^2 + C_0\|\nabla\theta\|_1^2 + C(Q)(S+1).\tag{3.47}$$

Therefore, combining (3.44)-(3.47) and (3.43), we arrive at

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3}(\frac{1}{2}|\tilde{p}|^2 + \frac{1}{4}e^{-\theta}|\tilde{u}|^2)\mathrm{d}x + \frac{c_6}{4}\|\nabla\tilde{u}\|^2\\
\le\ & C(Q)(S+1) + C_0\|\nabla(\varepsilon p,\theta)\|_1^2.
\end{aligned}\tag{3.48}$$

Since we have obtained the estimate of $\int_0^T\|\nabla(\varepsilon p,\theta)(t)\|_{s+1}^2\mathrm{d}t$ by (3.30), integrating the inequality (3.48) on $[0,T]$, it is easy to obtain

$$\begin{aligned}
&\sup_{t\in[0,T]}\int_{\mathbb{R}^3}(\frac{1}{2}|\tilde{p}|^2 + \frac{1}{4}e^{-\theta}|\tilde{u}|^2)\mathrm{d}x + \frac{c_6}{4}\int_0^T\|\nabla\tilde{u}(t)\|^2\mathrm{d}t\\
\le\ & C(O_0)e^{\sqrt{T}C(O(T))} + C_0\int_0^T\|\nabla(\varepsilon p,\theta)(t)\|_1^2\mathrm{d}t \le C(O_0)e^{\sqrt{T}C(O(T))}.
\end{aligned}$$

Then we obtain the inequality (3.30) by using the facts $p=\tilde{p}$ and $u=\frac{1}{2}\tilde{u}+\frac{1}{2}\lambda' e^{-\varepsilon p+\theta}\nabla\theta$ again. This completes the proof.

In the next lemma we utilize Lemma 3.5 to control $(D^\beta(\varepsilon\partial_t)p, D^\beta(\varepsilon\partial_t)u)$.

LEMMA 3.6. *Let $s \geq 4$ be an integer, and (p,u,θ) be the classical solution to the Cauchy problem (1.5)and (1.7) on $[0,T_1]$. Set $(p_\beta,u_\beta,\theta_\beta) = D^\beta(\varepsilon\partial_t)(p,u,\theta)$ for $0 \leq |\beta| \leq s-1$. Then there exist positive constants $\varepsilon_0 \leq 1$ and an increasing positive polynomial $C(\cdot)$, such that for all $T \in [0,\hat{T}]$ and $\varepsilon \in (0,\varepsilon_0]$, it holds that*

$$\sup_{t\in[0,T]} \|(p_\beta,u_\beta)(t)\|^2 + \int_0^T \|\nabla u_\beta(t)\|^2 \mathrm{d}t \leq C(O_0)e^{\sqrt{T}C(O(T))}. \tag{3.49}$$

The proof is similar to that of Lemma 3.5 and can be omitted. We also see the detail in [**21**].

Finally, we can control the norm $\|(\nabla p, \mathrm{div} u)\|_{s-1}$ and $\int_0^T \|(\nabla p, \mathrm{div} u)\|_s \mathrm{d}t$ by using Lemma 3.3, 3.5, 3.6 and the structure of the system (1.5). Firstly, the equations $(1.5)_{1,2}$ can be rewritten as follow:

$$\begin{cases} 2\mathrm{div}\, u = -\varepsilon\partial_t p + F_1, \\ \nabla p = -e^{-\theta}\varepsilon\partial_t u + F_2, \end{cases} \tag{3.50}$$

where

$$\begin{aligned} F_1 \quad = \quad & -\varepsilon u\cdot\nabla p + \mathrm{div}\,(\lambda' e^{-\varepsilon p+\theta}\nabla\theta) + \varepsilon^2 e^{-\varepsilon p}\Psi'(u) : \nabla u + \lambda'\varepsilon e^{-\varepsilon p+\theta}\nabla p\cdot\nabla\theta \\ & +\kappa'\varepsilon e^{\varepsilon p-2\theta}(\Delta\hat{p} + \frac{3}{2}|\nabla\hat{p}|^2)\mathrm{div}\, u - \kappa'\varepsilon e^{\varepsilon p-2\theta}\nabla\hat{p}\otimes\nabla\hat{p} : \nabla u, \end{aligned}$$

and

$$F_2 = -\varepsilon e^{-\theta} u\cdot\nabla u + \varepsilon e^{-\varepsilon p}\mathrm{div}\,\Psi'(u) + \kappa'\varepsilon e^{-\theta}\nabla\Delta e^{\varepsilon p-2\theta}.$$

Then, by virtue of Lemma 3.3, 3.5, 3.6 and the definition of $O(T)$, it is easy to obtain F_1. That is

$$\begin{aligned} \|F_1\|_{s-1} \quad \leq \quad & C_0\varepsilon\|u\|_{s-1}\|p\|_s + C(\|(\varepsilon p,\theta-\bar{\theta})\|_{s+1})\|\theta-\bar{\theta}\|_{s+1} + C(\|\varepsilon p\|_{s+1})\|u\|_s^2 \\ & +C(\|(\varepsilon p,\theta-\bar{\theta})\|_{s+1})(\|(\varepsilon p,\theta-\bar{\theta})\|_{s+1} + \|(\varepsilon p,\theta-\bar{\theta})\|_{s+1}^2)\|u\|_{\mathcal{H}^s} \\ & +C(\|(\varepsilon p,\theta-\bar{\theta})\|_{s+1})\|\varepsilon p\|_{s+1}\|\theta-\bar{\theta}\|_{s+1} \\ \leq \quad & \varepsilon C(O(T)) + C(O_0)e^{\sqrt{T}C(O(T))} \leq C(O_0)e^{(\sqrt{T}+\varepsilon)C(O(T))}, \end{aligned} \tag{3.51}$$

and

$$\begin{aligned} \int_0^T \|F_1\|_s^2 \mathrm{d}t \quad \leq \quad & \int_0^T C_0\varepsilon^2\|u\|_s^2\|\nabla p\|_s^2 + C(\|(\varepsilon p,\theta-\bar{\theta})\|_{s+1})(\|\nabla\theta\|_{s+1}^2 + \|\varepsilon u\|_{s+1}^4)\mathrm{d}t \\ & + \int_0^T C(\|(\varepsilon p,\theta-\bar{\theta})\|_{s+1})(\|\nabla\hat{p}\|_{s+1}^2 + \|\hat{p}\|_{s+1}^4)\|\varepsilon u\|_{s+1}^2 \mathrm{d}t \\ & + \int_0^T C(\|(\varepsilon p,\theta-\bar{\theta})\|_{s+1})\|\varepsilon p\|_{s+1}^2\|\theta-\bar{\theta}\|_{s+1}^2 \mathrm{d}t \\ \leq \quad & C(O_0)e^{(\sqrt{T}+\varepsilon)C(O(T))}. \end{aligned} \tag{3.52}$$

In a same way, we can obtain the following estimates of F_2:

$$\|F_2\|_{s-1} \leq C(O_0)e^{(\sqrt{T}+\varepsilon)C(O(T))}, \tag{3.53}$$

and

$$\int_0^T \|F_2\|_s^2 \mathrm{d}t \leq C(O_0)e^{(\sqrt{T}+\varepsilon)C(O(T))}. \tag{3.54}$$

Then we have

LEMMA 3.7. *Let $s \geq 4$ be an integer, and (p, u, θ) be the classical solution to the Cauchy problem (1.5)and (1.7) on $[0, T_1]$, there exist positive constants $\varepsilon_0 \leq 1$ and an increasing positive polynomial $C(\cdot)$, such that for all $T \in [0, \hat{T}]$ with $\hat{T} = \min\{1, T_1\}$ and $\varepsilon \in (0, \varepsilon_0]$, it holds that*

$$\begin{aligned} &\sup_{t\in[0,T]} \|(\mathrm{div} u, \nabla p)(t)\|_{s-1}^2 + \int_0^T \|(\mathrm{div} u, \nabla p)(t)\|_s^2 \mathrm{d}t \\ &\leq C(O_0)e^{(\sqrt{T}+\varepsilon)C(O(T))}. \end{aligned} \tag{3.55}$$

Proof of Proposition 2.5. From Lemma 3.3, 3.4, 3.5 and Lemma 3.7, and using some arguments as in the proof of Lemma 6.28 in [**1**], then Proposition 2.5 immediately follows.

Once Proposition 2.5 is established, the existence part of Theorem 1.1 can be proved by directly applying the same arguments in [**1, 36**], and the similar procedure has been used to study low Mach number limit for the full magnetohydrodynamic equations with general initial data in [**21**]. Hence we can omit the details here.

4. Low mach limit

In this section, we shall prove the convergence part of Theorem 1.1 by modifying the arguments developed by Métivier and Schochet [**36**], see also some extensions [**1, 29**] and some applications in [**21**].

Proof of the convergence part of Theorem 1.1. Firstly, it follows from the proof of the existence part of Theorem 1.1 that

$$\sup_{t\in[0,T_0]} \|(p^\varepsilon, u^\varepsilon)(t)\|_s + \|(\theta^\varepsilon - \bar{\theta})(t)\|_{s+1} < \infty \tag{4.1}$$

Then after extracting a subsequence, one has

$$(p^\varepsilon, u^\varepsilon) \rightharpoonup (\bar{p}, w) \ \text{ weak-}* \text{ in } L^\infty(0, T_0; H^s(\mathbb{R}^3)), \tag{4.2}$$

$$(\theta^\varepsilon - \bar{\theta}) \rightharpoonup \vartheta - \bar{\theta} \ \text{ weak-}* \text{ in } L^\infty(0, T_0; H^{s+1}(\mathbb{R}^3)), \tag{4.3}$$

as $\varepsilon \to 0$ for some $(\bar{p}, w) \in L^\infty(0, T_0; H^s(\mathbb{R}^3))$ and $\vartheta \in L^\infty(0, T_0; H^{s+1}(\mathbb{R}^3))$. What is more, it follows from the equation (1.5) that

$$\partial_t \theta^\varepsilon \in L^\infty(0, T_0; H^{s-2}(\mathbb{R}^3)). \tag{4.4}$$

Then (4.3) and (4.4) together with Aubin-Lions lemma yield that

$$\theta^\varepsilon - \bar{\theta} \to \vartheta - \bar{\theta} \text{ strongly in } C([0, T_0]; H^{s'+1}_{loc}(\mathbb{R}^3)) \tag{4.5}$$

for all $s' < s$, after extracting a subsequence. Similarly, from (3.32), we also have for all $s' < s$,

$$\mathrm{curl}\,(e^{-\theta^\varepsilon} u^\varepsilon) \to \mathrm{curl}\,(e^{-\vartheta} w) \text{ strongly in } C([0, T_0]; H^{s'-1}_{loc}(\mathbb{R}^3)). \tag{4.6}$$

To obtain the limit system, one needs to show the strong convergence of $(p^\varepsilon, u^\varepsilon)$ in $L^2(0, T_0; H^{s'}_{loc}(\mathbb{R}^3))$ for all $s' < s$. To this end, we first show that $\bar{p} = 0$ and $\mathrm{div}\,(2w - \kappa' e^{\vartheta} \nabla \vartheta) = 0$. In fact, the equations $(1.5)_{1,2}$ can be rewritten as

$$\varepsilon \partial_t p^\varepsilon + \mathrm{div}\,(2u^\varepsilon - \kappa' e^{-\varepsilon p^\varepsilon + \theta^\varepsilon} \nabla \theta^\varepsilon) = \varepsilon f^\varepsilon, \tag{4.7}$$

and

$$\varepsilon e^{-\theta^\varepsilon}\partial_t u^\varepsilon + \nabla p^\varepsilon = \varepsilon g^\varepsilon, \tag{4.8}$$

here

$$\begin{aligned} f^\varepsilon \;=\; & -u^\varepsilon\cdot\nabla p^\varepsilon + \varepsilon e^{-\varepsilon p^\epsilon}\Psi'(u^\varepsilon):\nabla u^\varepsilon + \kappa^\varepsilon e^{\varepsilon p^\epsilon - 2\theta^\varepsilon}(\Delta(\varepsilon p^\epsilon - \theta^\varepsilon) \\ & + \frac{3}{2}|\nabla(\varepsilon p^\epsilon - \theta^\varepsilon)|^2)\operatorname{div} u^\varepsilon \\ & -\kappa^\varepsilon e^{\varepsilon p^\epsilon - 2\theta^\varepsilon}\nabla(\varepsilon p^\epsilon - \theta^\varepsilon)\otimes\nabla(\varepsilon p^\epsilon - \theta^\varepsilon):\nabla u^\varepsilon + \lambda' e^{-\varepsilon p^\epsilon + \theta^\varepsilon}\nabla p^\varepsilon\cdot\nabla\theta^\varepsilon, \end{aligned}$$

and

$$g^\varepsilon = -e^{-\theta^\varepsilon}u^\varepsilon\cdot\nabla u^\varepsilon + e^{-\varepsilon p^\epsilon}\operatorname{div}\Psi'(u^\varepsilon) + \kappa^\varepsilon e^{-\theta^\varepsilon}\nabla\Delta e^{\varepsilon p^\epsilon - \theta^\varepsilon}.$$

By virtue of the uniform estimate (1.9), f^ε and g^ε are uniformly bounded in $L^\infty(0,T_0,H^{s-2}(\mathbb{R}^3))$. Passing to the weak limit in (4.7) and (4.8), respectively, we see that $\nabla\bar{p}=0$ and $\operatorname{div}(2w-\kappa' e^{\vartheta}\nabla\vartheta)=0$. Since $\bar{p}\in L^\infty(0,T_0;H^s(\mathbb{R}^3))$, we infer that $\bar{p}=0$.

Notice that we have strong compactness for the incompressible component by (4.6). So it is sufficient to prove the strong convergence of the acoustic part of u^ε. Indeed, we can claim that for all $s'<s$,

$$p^\varepsilon \to 0 \text{ strongly in } L^2(0,T_0;H^{s'}_{loc}(\mathbb{R}^3)), \tag{4.9}$$

$$\operatorname{div}(2u^\varepsilon - \lambda' e^{-\varepsilon p^\varepsilon + \theta^\varepsilon}\nabla\theta^\varepsilon) \to 0 \text{ strongly in } L^2(0,T_0;H^{s'-1}_{loc}(\mathbb{R}^3)). \tag{4.10}$$

Once we have above convergence results at hand, we can continue to finish the proof of the convergence part in Theorem 1.1. First it follows from (4.5) and (4.10) that

$$\operatorname{div} u^\varepsilon \to \operatorname{div} w \text{ strongly in } L^2(0,T_0;H^{s'-1}_{loc}(\mathbb{R}^3)),$$

which together with (4.6) yields that

$$u^\varepsilon \to w \text{ strongly in } L^2(0,T_0;H^{s'}_{loc}(\mathbb{R}^3)).$$

Next, let us drive the limit system for $(0,w,\vartheta)$. Using (4.9) and (4.10), and passing to the limit in the equation $(1.5)_3$, one sees that the limit $(0,w,\vartheta)$ satisfies that

$$\begin{cases} \operatorname{div}(2w-\kappa' e^{\vartheta}\nabla\vartheta)=0, \\ \partial_t\vartheta + w\cdot\nabla\vartheta + \operatorname{div} w = \lambda'\Delta(e^{\vartheta}), \end{cases} \tag{4.11}$$

in the sense of distribution. For the equation for u^ε, we apply the operator curl to the equation $(1.5)_2$ and take to the limit on the result equation to deduce that

$$\operatorname{curl}\left(e^{-\vartheta}(\partial_t w + w\cdot\nabla w) - \operatorname{div}\Psi'(w) - \kappa' e^{-\vartheta}\nabla\Delta e^{-\vartheta}\right) = 0.$$

Thus, it follows that

$$e^{-\vartheta}(\partial_t w + w\cdot\nabla w) + \nabla\pi = \operatorname{div}\Psi'(w) + \kappa' e^{-\vartheta}\nabla\Delta e^{-\vartheta}. \tag{4.12}$$

for some function π. On the other hand, following the arguments in [**36**], one can obtain that (w,ϑ) satisfies the initial condition

$$(w,\vartheta)|_{t=0} = (w_0,\vartheta_0), \tag{4.13}$$

where w_0 is determined by

$$\operatorname{div}(2w_0 - \kappa' e^{\vartheta_0}\nabla\vartheta_0)=0,\ \operatorname{curl}(e^{-\vartheta_0}w_0) = \operatorname{curl}(e^{-\vartheta_0}u_0).$$

Finally, the standard iterative method shows that the systems (4.11) and (4.12) with initial data (4.13) has a unique solution $(w, \vartheta - \bar{\theta}) \in C([0, T_0], H^s(\mathbb{R}^3)) \times C([0, T_0], H^{s+1}(\mathbb{R}^3))$, which implies that the above convergence holds for the full sequence of $(p^\varepsilon, u^\varepsilon, \theta^\varepsilon)$. Therefore the proof is completed.

In the following, we should give the proof of (4.9) and (4.10). Indeed, the proof of them is based on (1.9) and Lemma 2.3, in which the dispersive estimates on the wave equation are obtained by Métivier and Schochet [**36**].

Proof of (4.9) and (4.10). Since p^ε and $\mathrm{div}\,(2u^\varepsilon - \lambda' e^{-\varepsilon p^\varepsilon + \theta^\varepsilon})$ are uniformly bounded in $L^\infty(0, T_0; H^s(\mathbb{R}^3))$ and $L^\infty(0, T_0; H^{s-1}(\mathbb{R}^3))$ respectively, we need only to present the strong convergence in $L^2(0, T_0; L^2_{loc}(\mathbb{R}^3))$ by the interpolation theorem. Firstly, applying $\varepsilon^2 \partial_t$ to the equation $(1.5)_1$, we find that

$$
\begin{aligned}
&\varepsilon^2 \partial_t \Big(\partial_t p^\epsilon + u^\varepsilon \cdot \nabla p^\varepsilon \Big) + \varepsilon \partial_t \mathrm{div}\,(2u^\varepsilon - \lambda' e^{\varepsilon p^\epsilon - \theta^\varepsilon} \nabla \theta^\varepsilon) \\
=&\varepsilon^2 \partial_t \Big(\varepsilon e^{-\varepsilon p^\epsilon} \Psi'(u^\varepsilon) : \nabla u^\varepsilon + \lambda' e^{\varepsilon p^\epsilon - \theta^\varepsilon} \nabla p^\varepsilon \cdot \nabla \theta^\varepsilon \Big) \\
&+ \varepsilon^2 \partial_t \Big(\kappa' e^{\varepsilon p^\epsilon - 2\theta^\varepsilon} (\Delta \hat{p}^\varepsilon + \frac{3}{2} |\nabla \hat{p}^\varepsilon|^2) \mathrm{div}\, u^\varepsilon - \kappa' e^{\varepsilon p^\epsilon - 2\theta^\varepsilon} \nabla \hat{p}^\varepsilon \otimes \nabla \hat{p}^\varepsilon : \nabla u^\varepsilon \Big).
\end{aligned}
\tag{4.14}
$$

On the other hand, multiplying $(1.5)_2$ by e^{θ^ε} and then applying the operator $\varepsilon\mathrm{div}$ to the resulting equation, one gets

$$
\begin{aligned}
\varepsilon \partial_t \mathrm{div}\, u^\varepsilon + \varepsilon \mathrm{div}\,(u^\varepsilon \cdot \nabla u^\varepsilon) + \mathrm{div}\,(e^\theta \nabla p^\epsilon) =&\varepsilon \mathrm{div}\, \Big(e^{-\varepsilon p^\epsilon + \theta^\varpi} \mathrm{div}\, \Psi'(u^\varepsilon) \Big) \\
&+ \varepsilon \mathrm{div}\, \Big(\kappa' \nabla \Delta e^{\varepsilon p^\epsilon - \theta^\varepsilon} \Big).
\end{aligned}
\tag{4.15}
$$

Then, subtracting (4.15) from (4.14) yields that

$$
\varepsilon^2 \partial_t \Big(\frac{1}{2} \partial_t p^\epsilon \Big) - \mathrm{div}\,(e^{\theta^\varepsilon} \nabla p^\epsilon) = \varepsilon F(p^\varepsilon, u^\varepsilon, \theta^\varepsilon),
\tag{4.16}
$$

where

$$
\begin{aligned}
&F(p^\varepsilon, u^\varepsilon, \theta^\varepsilon) = \\
&- \frac{1}{2} \varepsilon \partial_t \Big(u^\varepsilon \cdot \nabla p^\varepsilon \Big) + \partial_t \mathrm{div}\,(\frac{1}{2} \lambda' e^{\varepsilon p^\epsilon - \theta^\varepsilon} \nabla \theta^\varepsilon) \\
&+ \frac{1}{2} \varepsilon \partial_t \Big(\varepsilon e^{-\varepsilon p^\epsilon} \Psi'(u^\varepsilon) : \nabla u^\varepsilon + \lambda' e^{\varepsilon p^\epsilon - \theta^\varepsilon} \nabla p^\varepsilon \cdot \nabla \theta^\varepsilon \Big) \\
&+ \frac{1}{2} \varepsilon \partial_t \Big(\kappa' e^{\varepsilon p^\epsilon - 2\theta^\varepsilon} (\Delta \hat{p}^\varepsilon + \frac{3}{2} |\nabla \hat{p}^\varepsilon|^2) \mathrm{div}\, u^\varepsilon - \kappa' e^{\varepsilon p^\epsilon - 2\theta^\varepsilon} \nabla \hat{p}^\varepsilon \otimes \nabla \hat{p}^\varepsilon : \nabla u^\varepsilon \Big) \\
&+ \mathrm{div}\,(u^\varepsilon \cdot \nabla u^\varepsilon) - \mathrm{div}\, \Big(e^{-\varepsilon p^\epsilon + \theta^\varpi} \mathrm{div}\, \Psi'(u^\varepsilon) \Big) - \mathrm{div}\, \Big(\kappa' \nabla \Delta e^{\varepsilon p^\epsilon - \theta^\varepsilon} \Big).
\end{aligned}
$$

By virtue of the boundedness of $(p^\varepsilon, u^\varepsilon, \theta^\varepsilon - \bar{\theta})$, we have

$$
\varepsilon F(p^\varepsilon, u^\varepsilon, \theta^\varepsilon) \to 0 \text{ strongly in } L^2(0, T_0; L^2(\mathbb{R}^3)).
$$

Noting the strong convergence of θ^ε and the assumption on the initial data, the conditions in Lemma 2.3 can be verified as in Section 8.1 of [**1**]. Then from Lemma 2.3, it follows that

$$
p^\varepsilon \to 0 \text{ strongly in } L^2(0, T_0; L^2_{loc}(\mathbb{R}^3)).
$$

Next, we apply the operator $\varepsilon\partial_t$ to the equation (4.16) to obtain

$$\begin{aligned}\varepsilon^2\partial_t\Big(\frac{1}{2}\partial_t(\varepsilon\partial_t p^\varepsilon)\Big) - \operatorname{div}(e^{\theta^\varepsilon}\nabla(\varepsilon\partial_t p^\varepsilon)) =& \varepsilon^2\partial_t F(p^\varepsilon, u^\varepsilon, \theta^\varepsilon) + \varepsilon\operatorname{div}(e^{\theta^\varepsilon}\partial_t\theta^\varepsilon p^\varepsilon)\\ =&: \varepsilon\tilde{F}(p^\varepsilon, u^\varepsilon, \theta^\varepsilon),\end{aligned} \tag{4.17}$$

then a similar argument deduces that $\varepsilon\partial_t p^\varepsilon \to 0$ strongly in $L^2(0, T_0; L^2_{loc}(\mathbb{R}^3))$. Moreover, we can rewrite $(1.5)_1$ again as follow:

$$\begin{aligned}&\operatorname{div}(2u^\varepsilon - \lambda' e^{\varepsilon p^\epsilon - \theta^\varepsilon}\nabla\theta^\varepsilon)\\ =& -\varepsilon\partial_t p^\epsilon - \varepsilon u^\varepsilon\cdot\nabla p^\varepsilon + \varepsilon^2 e^{-\varepsilon p^\epsilon}\Psi'(u^\varepsilon) : \nabla u^\varepsilon + \lambda'\varepsilon e^{\varepsilon p^\epsilon - \theta^\varepsilon}\nabla p^\varepsilon\cdot\nabla\theta^\varepsilon\\ &+ \kappa'\varepsilon e^{\varepsilon p^\epsilon - 2\theta^\varepsilon}(\Delta\hat{p}^\varepsilon + \frac{3}{2}|\nabla\hat{p}^\varepsilon|^2)\operatorname{div} u^\varepsilon - \kappa'\varepsilon e^{\varepsilon p^\epsilon - 2\theta^\varepsilon}\nabla\hat{p}^\varepsilon\otimes\nabla\hat{p}^\varepsilon : \nabla u^\varepsilon\\ =&: -\varepsilon\partial_t p + \varepsilon F_1.\end{aligned} \tag{4.18}$$

Similarly, from (4.18), (1.9) and Lemma 2.3, we can show the convergence of $\operatorname{div}(2u^\varepsilon - \lambda' e^{\varepsilon p^\epsilon - \theta^\varepsilon}\nabla\theta^\varepsilon) \to 0$ strongly in $L^2(0, T_0; L^2_{loc})$, which completes the proof.

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