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A PERIODISATION OF SEMISIMPLE LIE ALGEBRAS

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Abstract

In this text we study classical Lie algebras. We prove that a periodisation of such Lie algebras without $\mathfrak{sl}_2$-component can be presented as a free graded Lie algebra modulo quadratic relations only. Our approach will be through a Chevalley basis and our method relies on elementary tools only.

To Jan–Erik Roos on his sixty–fifth birthday

1. Introduction

The aim with this text is to study the periodisation of finite dimensional semisimple Lie algebras over an algebraically closed field F of characteristic 0 without $\mathfrak{sl}_2$-component. Every finite dimensional semisimple Lie algebra can be written as the direct sum of its simple ideals. The condition that there is no $\mathfrak{sl}_2$-component means that no simple ideal in this sum is of $\mathfrak{sl}_2$-type.

From now on, let L be a finite dimensional semisimple Lie algebra over a algebraically closed field F of characteristic 0. We will call such Lie algebras "classical". Furthermore, we always assume that L has no $\mathfrak{sl}_2$-component. Let L_{per} denote the periodisation of L, i.e., let L_{per} be the graded Lie algebra with L in each positive degree. (A formal definition of L_{per} is given in section 3.) Moreover, let $\mathcal{F}(G)$ be the free Lie algebra generated by G, which we consider as graded letting the elements of G have degree 1. We call the homogeneous elements in $\mathcal{F}(G)_2$ quadratic. We now state our main result:

Theorem 1. *Let L be a finite dimensional semisimple Lie algebra over an algebraically closed field F of characteristic 0 without $\mathfrak{sl}_2$-component. Then there is a set of generators G and an ideal R of $\mathcal{F}(G)$, generated by quadratic expressions only, such that $L_{per} \cong \mathcal{F}(G)/R$.*

Periodisations of Lie superalgebras, that can be presented as free graded Lie superalgebras modulo some ideal generated by quadratic expressions only, appears in the theory of local rings (see the article [**7**] by Löfwall-Roos). After the publication of this article, the study of such Lie superalgebras was natural. As a first step in this study, we look at ordinary Lie algebras and develop a method to show Theorem 1.

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Let $\mathfrak{g} = \bigoplus_{i \geqslant 1} \mathfrak{g}_i$ be a positively graded Lie algebra over a field F. In terms of Lie algebra homology, a minimal set of generators for $\mathfrak{g}$ is in one-to-one correspondence with a basis for the homology group $H_1(\mathfrak{g}, F)$. Also, given a minimal set of generators for $\mathfrak{g}$, then a minimal set of generators for the relations of $\mathfrak{g}$ is in one-to-one correspondence with a basis for the homology group $H_2(\mathfrak{g}, F)$ (see [**5**]). Hence, the result in Theorem 1 may be expressed as $H_1(L_{per}, F)$ is concentrated in degree 1 and $H_2(L_{per}, F)$ is concentrated in degree 2. A result due to H.Garland and J.Lepowski [**6**] deal with all homology groups. It is possible that Theorem 1 could be derived from their result. However, our approach to Theorem 1 will be through a Chevalley basis of L and we use only elementary methods. A short description of root systems and a Chevalley basis will be given in section 2. We construct $\mathcal{F}(G)$ and an ideal R generated by quadratic expressions only in accordance with this Chevalley basis (section 3.1). We use induction to show that there is a set of generators for the vector space $\mathcal{F}(G)/R$ which corresponds to a basis of L_{per} (Proposition 3.20). Then one easily finds a graded Lie algebra isomorphism between L_{per} and $\mathcal{F}(G)/R$ (section 3.3).

Even if Theorem 1 could be derived from the result by H.Garland and J.Lepowski, we believe that our method is of interest in itself and in a forthcoming paper, we will generalize this to show similar results about Lie superalgebras (cf. also [**9**]).

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2. Root system and Chevalley basis

Let L be a classical Lie algebra without $\mathfrak{sl}_2$-component. Then L has a Chevalley basis and, as we mentioned in the Introduction, this basis is an important part of our proof of Theorem 1. In this section a description of the Chevalley basis will be given and some useful properties will be emphasized. These properties are well-known results (cf. e.g. [**1**]). However, Lemma 2.7 is a consequence of the assumption that L has no $\mathfrak{sl}_2$-component.

Let $\mathfrak{h}$ be a Cartan subalgebra of L, Φ the set of roots of L relative to $\mathfrak{h}$; that is

$$\Phi = \{\alpha \in \mathfrak{h}^*; \alpha \neq 0 \text{ and } L_\alpha \neq 0\}$$

where

$$L_\alpha = \{x \in L; [H, x] = \alpha(H)x \text{ for all } H \in \mathfrak{h}\}.$$

Then Φ spans $\mathfrak{h}^*$. For every $\alpha \in \Phi$, there is a unique element $H_\alpha \in \mathfrak{h}$ such that $\alpha(H_\alpha) = 2$. In particular, $\alpha(H_\alpha) \neq 0$.

Observation 2.1. Let $\alpha, \beta \in \Phi$. Then, $\beta(H_\alpha)$ is an integer.

Observation 2.2. If $\alpha, \beta \in \Phi$, then $\beta - \beta(H_\alpha)\alpha \in \Phi$.

Observation 2.3. If $\alpha \in \Phi$, then $-\alpha \in \Phi$. If $\alpha \in \Phi$ and $m\alpha \in \Phi$ for some scalar m, then $m = \pm 1$.

Notice that Observation 2.3 implies that $\alpha, \beta \in \Phi$ are linearly dependent if and only if $\alpha = \pm\beta$.

Observation 2.4. If $\alpha, \beta \in \Phi$ are linearly independent, then all roots of the form $\beta + i\alpha$ ($i \in \mathbb{Z}$) forms a string

$$\beta - r\alpha, \beta - (r-1)\alpha, \dots, \beta + (q-1)\alpha, \beta + q\alpha$$

where $r, q \geqslant 0$. This string is called the α-string through β.

Observation 2.5. Let $\alpha, \beta \in \Phi$ such that $\alpha \neq \pm\beta$. Then

$$\alpha(H_\beta)\beta(H_\alpha) \neq \alpha(H_\alpha)\beta(H_\beta).$$

Let $\Pi = \{\alpha_1, \dots, \alpha_l\}$ be the set of fundamental or simple roots, that is; $\alpha_1, \dots, \alpha_l$ is a basis of $\mathfrak{h}^*$ and each root $\alpha \in \Phi$ can be written as $\alpha = \sum_{i=1}^{l} c_i\alpha_i$ where c_i are integral coefficients, either all positive (α is a positive root) or all negative (α is a negative root). Let Φ^+ and Φ^- be the sets of positive and negative roots respectively. Then $\Phi = \Phi^+ \cup \Phi^-$.

Observation 2.6. If $\alpha \in \Phi^+$ such that $\alpha \notin \Pi$, then $\alpha - \beta \in \Phi^+$ for some $\beta \in \Pi$ (see [**1**], Lemma A in section 10.2).

To abbreviate, let $H_i = H_{\alpha_i}$ for all $1 \leqslant i \leqslant l$. Then $\{H_i; 1 \leqslant i \leqslant l\}$ is a basis of $\mathfrak{h}$. From now on, let $\{X_\alpha; \alpha \in \Phi\} \cup \{H_i; 1 \leqslant i \leqslant l\}$ be a Chevalley basis of L. This means that:

1. $[H_\alpha, H_\beta] = 0$ for all $\alpha, \beta \in \Phi$
2. $[H_\beta, X_\alpha] = \alpha(H_\beta)X_\alpha$ for all $\alpha, \beta \in \Phi$
3. $[X_\alpha, X_{-\alpha}] = H_\alpha$ for all $\alpha \in \Phi$
4. Let $\alpha, \beta \in \Phi$ such that $\alpha \neq -\beta$. Then

$$[X_\alpha, X_\beta] = \begin{cases} N_{\alpha,\beta}X_{\alpha+\beta} & \text{if } \alpha + \beta \in \Phi \\ 0 & \text{if } \alpha + \beta \notin \Phi \end{cases}$$

   where $N_{\alpha,\beta}$ is a non-zero integer.
5. If $\alpha, \beta, \alpha + \beta \in \Phi$ then $N_{\alpha,\beta} = -N_{-\alpha,-\beta}$

To simplify calculations, let $N_{\alpha,\beta} = 0$ if α, β or $\alpha + \beta \notin \Phi$. The following lemma is a consequence of the fact that L has no $\mathfrak{sl}_2$-component. Let rank(L) denote the dimension of $\mathfrak{h}^*$. This is well defined when L is semisimple.

Lemma 2.7. *If $\alpha \in \Pi$, then $\alpha + \beta \in \Phi$ for some $\beta \in \Phi$.*

Proof. L is a direct sum of simple Lie algebras, none of $\mathfrak{sl}_2$-type, i.e., all of rank $\geqslant 2$. The union of the root systems of these simple Lie algebras gives the decomposition of Φ into its irreducible components. If $\alpha \in \Pi$, let Φ' be the component such that $\alpha \in \Phi'$ and let L' be the corresponding simple Lie algebra. Since α is a fundamental root of L, α is a fundamental root of L'. Now, Φ' is irreducible and hence, the Dynkin diagram of Φ' is connected. Since $\dim(\Phi') \geqslant 2$, there is a fundamental root $\beta \in \Phi'$ such that $\alpha \neq \beta$. Choose β to be a neighbour of α in the diagram, i.e., $\alpha(H_\beta) \neq 0$. In fact, then $\alpha(H_\beta) < 0$ since α and β are fundamental roots. Furthermore, $\alpha - \alpha(H_\beta)\beta \in \Phi'$. In view of this and of the β-string through α, we have that $\alpha + \beta \in \Phi$. □

We end this section with a lemma about the structure constants.

Lemma 2.8. *Let $\alpha, \beta \in \Phi$ such that $\alpha \neq \pm\beta$. Then*

$$-\alpha(H_\beta) = N_{\alpha,-\beta} N_{\beta,\alpha-\beta} + N_{\beta,\alpha} N_{-\beta,\beta+\alpha}.$$

Proof.

$$\begin{aligned}
[X_\alpha, H_\beta] &= [X_\alpha, [X_\beta, X_{-\beta}]] = [X_\beta, [X_\alpha, X_{-\beta}]] + [X_{-\beta}, [X_\beta, X_\alpha]] \\
&= N_{\alpha,-\beta} N_{\beta,\alpha-\beta} X_\alpha + N_{\beta,\alpha} N_{-\beta,\beta+\alpha} X_\alpha \\
&= (N_{\alpha,-\beta} N_{\beta,\alpha-\beta} + N_{\beta,\alpha} N_{-\beta,\beta+\alpha}) X_\alpha \\
[X_\alpha, H_\beta] &= -\alpha(H_\beta) X_\alpha
\end{aligned}$$

Hence, $-\alpha(H_\beta) = N_{\alpha,-\beta} N_{\beta,\alpha-\beta} + N_{\beta,\alpha} N_{-\beta,\beta+\alpha}$. □

3. The free graded Lie algebra and proof of Theorem 1

The notation follows the previous sections, L is a classical Lie algebra over a field F without $\mathfrak{sl}_2$-component. Furthermore, $\mathfrak{h}$ is a Cartan subalgebra of L, Φ the root system relative to $\mathfrak{h}$, $\Pi = \{\alpha_1, \ldots \alpha_l\}$ a set of fundamental roots of Φ and $\{X_\alpha \ ; \ \alpha \in \Phi\} \cup \{H_i \ ; \ 1 \leqslant i \leqslant l\}$ a Chevalley basis of L. Also, let $\mathcal{F}(G)$ be the free graded Lie algebra letting the elements in G have degree 1. Define the periodisation of L, L_{per}, as follows:

Definition 3.1. Let L_{per} be the positive graded Lie algebra defined by

$$L_{per} = L \otimes tF[t] = \bigoplus_{i \geqslant 1} Lt^i$$

with $[x \otimes a, y \otimes b] = [x, y] \otimes ab$ for all $x, y \in L$ and $a, b \in tF[t]$. If $x \in L$, let $x^{(i)}$ denote the homogeneous element $x \otimes t^i$.

The Lie algebra L_{per} can be considered as the graded Lie algebra with L in each degree. We write $(L_{per})_i$ for the homogeneous part in L_{per} of degree i (i.e., $(L_{per})_i = Lt^i$). Then, if $x \in L$, $x^{(i)}$ is the corresponding element in $(L_{per})_i$. Observe that the set $\{X_\alpha^{(i)} \ ; \ \alpha \in \Phi\} \cup \{H_i^{(i)} \ ; \ 1 \leqslant i \leqslant l\}$ is a basis of the vector space $(L_{per})_i$ for all $i \geqslant 1$. Furthermore, $[x^{(i)}, y^{(j)}] = [x, y]^{(i+j)}$ for all $x, y \in L$ and $i, j \geqslant 1$.

Remark 3.2. Let $\mathbb{C}[t, t^{-1}]$ be the algebra of Laurent polynomials in t and let $\mathfrak{g}$ be a Lie algebra. The well known loop algebra, $\mathfrak{L}(\mathfrak{g}) = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] = \bigoplus_{-\infty}^{\infty} \mathfrak{g}t^i$, is constructed in the same manner as L_{per}, L_{per} is the positive part of the loop algebra of L. Given a matrix A, let $\mathfrak{g}(A)$ be the Kac-Moody algebra. Let $\overset{\circ}{A}$ be a Cartan matrix of finite type and let A be the affine extended Cartan matrix of $\overset{\circ}{A}$. Then $\mathfrak{L}(\mathfrak{g}(\overset{\circ}{A}))$ is isomorphic to $[\mathfrak{g}(A), \mathfrak{g}(A)]/\mathfrak{c}$ where $\mathfrak{c}$ is the center of $[\mathfrak{g}(A), \mathfrak{g}(A)]$ (see Theorem 7.4 in [**8**]). For further information about this subject, we refer to [**8**]. It is probably possible to deduce Theorem 1 from this isomorphism and the Serre relations.

In this section, the object is to prove Theorem 1; i.e., to show that there exists a set G and an ideal R of $\mathcal{F}(G)$, generated by quadratic expressions only, such that $L_{per} \cong \mathcal{F}(G)/R$. In section 3.1 we construct G and R, in 3.2 we state and prove an important proposition, and in section 3.3 we restate and prove Theorem 1.

3.1. Construction of generators and relations

Recall that L_{per} is the graded Lie algebra with L in each degree and if $x \in L$, we denote $x \in (L_{per})_i$ by $x^{(i)}$. Also, to abbreviate, let $x^{(1)} = x$. The object is to construct G and R with the Chevalley basis in mind.

Let $G = \{x_\alpha; \alpha \in \Phi\} \cup \{h_i; 1 \leqslant i \leqslant l\}$ and $\deg(x) = 1$ for all $x \in G$. Let $\mathcal{F} = \mathcal{F}(G)$.

Definition 3.3. Define a Lie algebra homomorphism $\phi : \mathcal{F} \longrightarrow L_{per}$ by induction and linear extension:

1. $\phi : x_\alpha \mapsto X_\alpha, \quad \phi : h_i \mapsto H_i, \quad \alpha \in \Phi, 1 \leqslant i \leqslant l$
2. $\phi : [x, y] \mapsto [\phi(x), \phi(y)], \quad x, y \in \mathcal{F}$

Remark 3.4. An easy induction on i shows that $\phi : \mathcal{F}_i \longrightarrow (L_{per})_i$ for all $i \geqslant 1$. Hence, ϕ is in fact a graded Lie algebra homomorphism. Since $[L, L] = L$, induction on i shows that ϕ maps $\mathcal{F}$ onto L_{per}.

Recall that, for all $\alpha \in \Phi$, there is $H_\alpha \in \mathfrak{h}$ such that $\alpha(H_\alpha) = 2$. If $H_\alpha = \sum_{i=1}^{l} c_i H_i$, let $h_\alpha = \sum_{i=1}^{l} c_i h_i$. Then $\phi(h_\alpha) = H_\alpha$ for all $\alpha \in \Phi$. To construct R, consider the restriction of ϕ to $\mathcal{F}_2$:

$$\phi|_{\mathcal{F}_2} : \mathcal{F}_2 \longrightarrow L_{per} : \begin{cases} [x_\alpha, x_\beta] \mapsto [X_\alpha, X_\beta], & \alpha, \beta \in \Phi \\ [h_i, x_\alpha] \mapsto [H_i, X_\alpha], & \alpha \in \Phi, 1 \leqslant i \leqslant l \\ [h_i, h_j] \mapsto [H_i, H_j], & 1 \leqslant i, j \leqslant l \end{cases}$$

Let R be the ideal in $\mathcal{F}$ generated by $\ker(\phi|_{\mathcal{F}_2})$. Since $\ker(\phi|_{\mathcal{F}_2}) \subseteq \mathcal{F}_2$, R is generated by quadratic expressions. Next is an example of an expression in this kernel:

Example 3.5. If $\alpha, \beta, \alpha + \beta \in \Phi$, then we have the following relation in L

$$\begin{aligned} N_{\alpha,\beta}[X_{\alpha+\beta}, X_{-\alpha-\beta}] &= -N_{-\alpha,-\beta}[X_{\alpha+\beta}, X_{-\alpha-\beta}] = -[X_{\alpha+\beta}, [X_{-\alpha}, X_{-\beta}]] \\ &= -[X_{-\alpha}, [X_{\alpha+\beta}, X_{-\beta}]] - [X_{-\beta}, [X_{-\alpha}, X_{\alpha+\beta}]] \\ &= -N_{\alpha+\beta,-\beta}[X_{-\alpha}, X_\alpha] + N_{-\alpha,\alpha+\beta}[X_\beta, X_{-\beta}]. \end{aligned}$$

Using this relation in L, we get the following expression in $\ker(\phi|_{\mathcal{F}_2}) \subset \mathcal{F}$

$$N_{\alpha,\beta}[x_{\alpha+\beta}, x_{-\alpha-\beta}] + N_{\alpha+\beta,-\beta}[x_{-\alpha}, x_\alpha] - N_{-\alpha,\alpha+\beta}[x_\beta, x_{-\beta}].$$

Even though we do not need it, the following proposition gives an explicit list of quadratic expressions generating R.

Proposition 3.6. *Let L be a classical Lie algebra, $\mathfrak{h}$ a Cartan subalgebra of L, Φ the root system of L relative to $\mathfrak{h}$ with a set $\Pi = \{\alpha_1, \ldots, \alpha_l\}$ of fundamental roots. Let G be the set given above, ϕ the graded Lie algebra epimorphism given by Definition 3.3 and let R be the ideal in $\mathcal{F} = \mathcal{F}(G)$ generated by* $\ker(\phi|_{\mathcal{F}_2})$. *Then R is generated by the following quadratic expressions:*

1. $\alpha(H_\alpha)[h_i, x_\alpha] - \alpha(H_i)[h_\alpha, x_\alpha]$ *for all* $\alpha \in \Phi$, $1 \leqslant i \leqslant l$
2. $(\alpha+\beta)(H_{\alpha+\beta})[x_\alpha, x_\beta] - N_{\alpha,\beta}[h_{\alpha+\beta}, x_{\alpha+\beta}]$ *for all* $\alpha, \beta \in \Phi$ *such that* $\alpha + \beta \in \Phi$
3. $[x_\alpha, x_\beta]$ *for all* $\alpha, \beta \in \Phi$ *such that* $\alpha+\beta \notin \Phi$, $\alpha \neq -\beta$
4. $[h_i, h_j]$ *for all* $1 \leqslant i, j \leqslant l$
5. $[x_\alpha, x_{-\alpha}] - \sum_{i=1}^{l} c_i [x_{\alpha_i}, x_{-\alpha_i}]$ *for all* $\alpha \in \Phi$ *where* $H_\alpha = \sum_{i=1}^{l} c_i H_i$

Proof. First to abbreviate, let $\phi_2 = \phi|_{\mathcal{F}_2}$. Let R' be the ideal in $\mathcal{F}$ generated by the quadratic expressions in *1-5*. We must show that $R' = R$. Let $R'_2 = \mathcal{F}_2 \cap R'$. Clearly, R' is generated by R'_2. Since R by definition is generated by $\ker(\phi_2)$, it suffices to show that $R'_2 = \ker(\phi_2)$.

To take ϕ_2 on an expression in $\mathcal{F}_2$ is to change small letters in the expression to capital ones. In view of the list of properties of the Chevalley basis given in section 2, it is easy to see that the quadratic expressions given in Proposition 3.6 are in $\ker(\phi_2)$. Hence, $R'_2 \subset \ker(\phi_2)$. Consider the vector space epimorphism:

$$\varphi : \mathcal{F}_2/R'_2 \longrightarrow \mathcal{F}_2/\ker(\phi_2) : x + R'_2 \mapsto x + \ker(\phi_2)$$

If we can show that φ is an isomorphism, then we know that $R' = \ker(\phi_2)$ and we are done.

Now, $\mathcal{F}_2/R'_2$ and $\mathcal{F}_2/\ker(\phi_2)$ are finite dimensional vector spaces. Thus, to show that φ is an isomorphism, it suffices to show that $\dim \mathcal{F}_2/R'_2 = \dim \mathcal{F}_2/\ker(\phi_2)$. Since φ is an epimorphism, we have that $\dim \mathcal{F}_2/R'_2 \geqslant \dim \mathcal{F}_2/\ker(\phi_2)$. Furthermore, $\mathrm{Im}(\phi_2) = (L_{per})_2 \cong L$. Hence, $\mathcal{F}_2/\ker(\phi_2) \cong L$, and we must show that $\dim \mathcal{F}_2/R'_2 \leqslant \dim L$.

If $x \in \mathcal{F}_2$, let $\bar{x}$ denote the image of x in $\mathcal{F}_2/R'_2$. $\mathcal{F}_2$ is generated by

$$\{[x_\alpha, x_\beta]; \alpha, \beta \in \Phi\} \cup \{[x_\alpha, h_i]; \alpha \in \Phi, 1 \leqslant i \leqslant l\} \cup \{[h_i, h_j; 1 \leqslant i, j \leqslant l]\}$$

and then, $\mathcal{F}_2/R'_2$ is generated by

$$\{[\bar{x}_\alpha, \bar{x}_\beta]; \alpha, \beta \in \Phi\} \cup \{[\bar{h}_i, \bar{x}_\alpha]; \alpha \in \Phi, 1 \leqslant i \leqslant l\} \cup \{[\bar{h}_i, \bar{h}_j; 1 \leqslant i, j \leqslant l]\}.$$

From *1-5* in Proposition 3.6 we have that:

$$\begin{aligned}
[\bar{h}_i, \bar{x}_\alpha] &= \frac{\alpha(H_i)}{\alpha(H_\alpha)}[\bar{h}_\alpha, \bar{x}_\alpha] \quad \text{for all } \alpha \in \Phi, 1 \leqslant i \leqslant l \\
[\bar{x}_\alpha, \bar{x}_\beta] &= \frac{N_{\alpha,\beta}}{(\alpha+\beta)(H_{\alpha+\beta})}[\bar{h}_{\alpha+\beta}, \bar{x}_{\alpha+\beta}] \quad \text{for all } \alpha, \beta \in \Phi \text{ such} \\
&\quad \text{that } \alpha+\beta \in \Phi \\
[\bar{x}_\alpha, \bar{x}_\beta] &= 0 \quad \text{for all } \alpha, \beta \in \Phi \text{ such that } \alpha+\beta \notin \Phi, \alpha \neq -\beta \\
[\bar{h}_i, \bar{h}_j] &= 0 \quad \text{for all } 1 \leqslant i \neq j \leqslant l \\
[\bar{x}_\alpha, \bar{x}_{-\alpha}] &= \sum_{i=1}^{l} c_i [\bar{x}_{\alpha_i}, \bar{x}_{-\alpha_i}] \quad \text{for all } \alpha \in \Phi \text{ where } H_\alpha = \sum_{i=1}^{l} c_i H_i
\end{aligned}$$

Hence, $\mathcal{F}_2/R'_2$ is generated by

$$\{[\bar{h}_\alpha, \bar{x}_\alpha]; \alpha \in \Phi\} \cup \{[\bar{x}_{\alpha_i}, \bar{x}_{-\alpha_i}]; 1 \leqslant i \leqslant l\}.$$

Since $\{X_\alpha; \alpha \in \Phi\} \cup \{H_i; 1 \leqslant i \leqslant l\}$ is a basis of L, we have that $\dim \mathcal{F}_2/R'_2 \leqslant \dim L$. □

Remark 3.7. If L is a simple Lie algebra, there is a unique maximal root $\gamma \in \Phi$ (the root such that $\gamma + \alpha \notin \Phi$ for any $\alpha \in \Phi^+$), and X_γ is the unique maximal vektor up to nonzero scalar multiples in L viewed as an L-module (i.e., the vector such that $X_\alpha . X_\gamma = 0$ for all $\alpha \in \Phi^+$). Let V be the subspace in $\mathcal{F}_2$ generated by the expressions in Proposition 3.6. Then V is an L-module and the vectors

$$[x_{\gamma-\alpha_i}, x_\gamma] \quad \text{where} \quad \alpha_i \in \Pi \quad \text{such that} \quad \gamma - \alpha_i \in \Phi \tag{1}$$

are the maximal vectors up to nonzero scalar multiples. Hence, V is generated by the vectors (1) as an L-module. Table 1 shows the maximal root and the maximal vectors, or generators, for the different types of L. We see that if L is of type A_l

Table 1: Generators for V

Type	Maximal root γ	Generators for V
A_l	$\alpha_1 + \ldots + \alpha_l$	$[x_{\gamma-\alpha_1}, x_\gamma]$, $[x_{\gamma-\alpha_l}, x_\gamma]$
B_l	$\alpha_1 + 2\alpha_2 + \ldots + 2\alpha_l$	$[x_{\gamma-\alpha_2}, x_\gamma]$
C_l	$2\alpha_1 + \ldots + 2\alpha_{l-1} + \alpha_l$	$[x_{\gamma-\alpha_1}, x_\gamma]$
D_l	$\alpha_1 + 2\alpha_2 + \ldots + 2\alpha_{l-2} + \alpha_{l-1} + \alpha_l$	$[x_{\gamma-\alpha_2}, x_\gamma]$
E_6	$\alpha_1 + 2\alpha_2 + \ldots + 2\alpha_5 + \alpha_6$	$[x_{\gamma-\alpha_2}, x_\gamma]$
E_7	$2\alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7$	$[x_{\gamma-\alpha_1}, x_\gamma]$
E_8	$2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 6\alpha_4 + 5\alpha_5 + 4\alpha_6 + 3\alpha_7 + 2\alpha_8$	$[x_{\gamma-\alpha_8}, x_\gamma]$
F_4	$2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 2\alpha_4$	$[x_{\gamma-\alpha_1}, x_\gamma]$
G_2	$3\alpha_1 + 2\alpha_2$	$[x_{\gamma-\alpha_2}, x_\gamma]$

($l \geqslant 2$), V has two generators and else, V is generated by one vector only.

From now on, we mainly do calculations in $\mathcal{F}/R$. To simplify the notations, we therefore let $\{x_\alpha, \alpha \in \Phi\} \cup \{h_i, 1 \leqslant i \leqslant l\}$ denote the images of the generators for $\mathcal{F}$ in $\mathcal{F}/R$. Hopefully, no confusion will arise. To make later calculations more understandable, we do the following abbreviations. Note that $\alpha(H_\alpha) \neq 0$.

Definition 3.8. Let $\{x_\alpha, \alpha \in \Phi\} \cup \{h_i, 1 \leqslant i \leqslant l\}$ be the images of the generators for F in $\mathcal{F}/R$. Define by induction:

1. $x_\alpha^{(2)} = \frac{1}{\alpha(H_\alpha)}[h_\alpha, x_\alpha], \quad x_\alpha^{(n)} = \frac{1}{\alpha(H_\alpha)}[h_\alpha, x_\alpha^{(n-1)}]$
2. $h_\alpha^{(2)} = [x_\alpha, x_{-\alpha}], \quad h_\alpha^{(n)} = [x_\alpha, x_{-\alpha}^{(n-1)}]$

Furthermore, let $x_\alpha^{(1)} = x_\alpha$, $h_i^{(1)} = h_i$

Example 3.9. In this example, we derive two relations in $\mathcal{F}(G)/R$ that will be used many times later on. Take any $\alpha, \beta \in \Phi$ such that $\alpha + \beta \in \Phi$.

1. Using the properties of the Chevalley basis listed in section 2, the relation in L given in Example 3.5 can be written as

$$N_{\alpha,\beta} H_{\alpha+\beta} + N_{\alpha+\beta,-\beta} H_{-\alpha} - N_{-\alpha,\alpha+\beta} H_\beta = 0.$$

By the definition of h_δ when $\delta \in \Phi$, we have that

$$N_{\alpha,\beta}h_{\alpha+\beta} + N_{\alpha+\beta,-\beta}h_{-\alpha} - N_{-\alpha,\alpha+\beta}h_\beta = 0 \quad \text{in } \mathcal{F}(G)/R.$$

2. Take the expression in $\ker(\phi|_{\mathcal{F}_2})$ given in Example 3.5 and use Definition 3.8. Then we have that

$$N_{\alpha,\beta}h^{(2)}_{\alpha+\beta} + N_{\alpha+\beta,-\beta}h^{(2)}_{-\alpha} - N_{-\alpha,\alpha+\beta}h^{(2)}_\beta = 0 \quad \text{in } \mathcal{F}(G)/R.$$

3.2. Statement and proof of a proposition

In this section the object is to state and prove a proposition concerning relations between elements in $\mathcal{F}/R$. This is an important result, since these relations reduce the dimension in each homogeneous part of $\mathcal{F}/R$ (cf. Proposition 3.20). In fact, as we will see in section 3.3, this proposition does all the hard work in proving Theorem 1.

Proposition 3.10. *Let L be a classical Lie algebra without $\mathfrak{sl}_2$-component, $\mathfrak{h}$ a Cartan subalgebra of L and Φ the root system of L relative to $\mathfrak{h}$ with a set $\Pi = \{\alpha_1, \dots \alpha_l\}$ of fundamental roots. Let $\{X_\alpha\ ;\ \alpha \in \Phi\} \cup \{H_i\ ;\ 1 \leqslant i \leqslant l\}$ be a Chevalley basis of L. Moreover, let G be the set of generators and R the ideal of $\mathcal{F}(G)$ given in section 3.1. Take any $\alpha, \beta \in \Phi$ and $1 \leqslant i, j \leqslant l$. Then, for every $k \geqslant 1$ we have the following relations in $\mathcal{F}(G)/R$:*

1. $[h_i, x^{(k)}_\alpha] = \alpha(H_i)x^{(k+1)}_\alpha$
2. *If $\alpha \neq -\beta$, then* $[x_\alpha, x^{(k)}_\beta] = N_{\alpha,\beta}x^{(k+1)}_{\alpha+\beta}$
3. $[x_\alpha, h^{(k)}_i] = -\alpha(H_i)x^{(k+1)}_\alpha$
4. $[h_i, h^{(k)}_j] = 0$
5. $[x_\alpha, x^{(k)}_{-\alpha}]$ *is a linear combination of* $h^{(k+1)}_1, \dots, h^{(k+1)}_l$

Notice that the statement in *2.* is a bit unclear since $x_{\alpha+\beta}$ is not defined when $\alpha + \beta \notin \Phi$. However, $N_{\alpha,\beta} = 0$ if $\alpha + \beta \notin \Phi$, so what we actually mean is that

$$[x_\alpha, x^{(k)}_\beta] = \begin{cases} N_{\alpha,\beta}x^{(k+1)}_{\alpha+\beta} & \text{if } \alpha+\beta \in \Phi \\ 0 & \text{if } \alpha+\beta \notin \Phi \end{cases}$$

Hence, let $x^{(k)}_\delta = 0$ for all $\delta \notin \Phi$ and all $k \geqslant 1$ and we get the statement in *2.* The advantage in writing like this is the simplicity in the calculations.

The proof of Proposition 3.10 is done by induction and depends on several lemmas. Since these lemmas depend on the induction hypothesis, they are included in the proof.

Proof of Proposition 3.10. We use induction on k. For $k = 1$, *1-5* of Proposition 3.10 are just consequences of the quadratic expressions in R and Definition 3.8 (cf. Proposition 3.6). Assume that Proposition 3.10 is true for all $k < n$ for some $n \geqslant 2$. The induction step is given in the lemmas below. Observe that the induction hypothesis is assumed to be valid in the lemmas.

Lemma 3.11. *Let $i \in \{1, \dots, l\}$. Then for any $\alpha \in \Phi$:*

$$[h_i, x_\alpha^{(n)}] = \alpha(H_i) x_\alpha^{(n+1)}$$

Proof of Lemma 3.11.

$$\begin{aligned}[h_i, x_\alpha^{(n)}] &= \frac{1}{\alpha(H_\alpha)}[h_i, [h_\alpha, x_\alpha^{(n-1)}]] \\ &= \frac{1}{\alpha(H_\alpha)}[h_\alpha, [h_i, x_\alpha^{(n-1)}]] + \frac{1}{\alpha(H_\alpha)}[x_\alpha^{(n-1)}, [h_\alpha, h_i]] \\ &= \frac{\alpha(H_i)}{\alpha(H_\alpha)}[h_\alpha, x_\alpha^{(n)}] = \alpha(H_i) x_\alpha^{(n+1)}\end{aligned}$$

The first and the last equality follows from Definition 3.8, the second from the Jacobi identity and the third equality follows from the fact h_α is a linear combination of $h_1, \dots, h_l$ and the quadratic relations $[h_j, h_i] = 0$ for all $1 \leqslant i , j \leqslant l$. □

Lemma 3.12. *Let $H = \sum_{i=1}^l c_i H_i$ and let $h = \sum_{i=1}^l c_i h_i$ be the corresponding element in $\mathcal{F}/R$. Then*

$$[h, x_\alpha^{(k)}] = \alpha(H) x_\alpha^{(k+1)}$$

for all $k \leqslant n$.

Proof of Lemma 3.12. According to the induction hypothesis and Lemma 3.11 we have that

$$[h_i, x_\alpha^{(k)}] = \alpha(H_i) x_\alpha^{(k+1)}$$

for all $1 \leqslant i \leqslant l$ and $k \leqslant n$. Then, by linearity of the bracket-operator and of α, we have the desired equality. □

Lemma 3.13. *Let $\alpha, \beta \in \Phi$ such that $\alpha \neq -\beta$. Then*

$$[x_\alpha, x_\beta^{(n)}] = N_{\alpha,\beta} x_{\alpha+\beta}^{(n+1)}.$$

Proof of Lemma 3.13. We proceed in steps:
(i) If α, β are linearly independent ($\alpha, \beta \in \mathfrak{h}^*$), there is $H \in \mathfrak{h}$ such that $\alpha(H) = 0$ and $\beta(H) \neq 0$. Let h denote the corresponding element in $\mathcal{F}/R$. Then:

$$\begin{aligned}[x_\alpha, x_\beta^{(n)}] &= \frac{1}{\beta(H)}[x_\alpha, [h, x_\beta^{(n-1)}]] \\ &= \frac{1}{\beta(H)}[h, [x_\alpha, x_\beta^{(n-1)}]] + \frac{1}{\beta(H)}[x_\beta^{(n-1)}, [h, x_\alpha]] \\ &= \frac{N_{\alpha,\beta}}{\beta(H)}[h, x_{\alpha+\beta}^{(n)}] + \frac{\alpha(H)}{\beta(H)}[x_\beta^{(n-1)}, x_\alpha^{(2)}] = N_{\alpha,\beta} x_{\alpha+\beta}^{(n+1)}\end{aligned}$$

The first equation follows from Lemma 3.12, the third from the induction hypothesis and Lemma 3.12 and the last equality follows from Lemma 3.12 and the fact that $\alpha(H) = 0$.
(ii) If α and β are linearly dependent then $\alpha = \beta$ according to Observation 2.3 and the assumption that $\alpha \neq -\beta$. Hence, we must show that $[x_\alpha, x_\alpha^{(n)}] = 0$. By combining Observation 2.6 and Lemma 2.7 and considering the different cases when

$\pm\alpha \in \Pi$ and $\pm\alpha \in \Phi^+$ but $\pm\alpha \notin \Pi$ respectively, we have that there is a $\gamma \in \Phi$ such that $\alpha \neq \pm\gamma$ and $\alpha + \gamma \in \Phi$. Then $N_{\alpha+\gamma,-\gamma} \neq 0$ and we have that:

$$\begin{aligned}
[x_\alpha, x_\alpha^{(n)}] &= \frac{1}{N_{\alpha+\gamma,-\gamma}}[x_\alpha, [x_{\alpha+\gamma}, x_{-\gamma}^{(n-1)}]] \\
&= \frac{1}{N_{\alpha+\gamma,-\gamma}}[x_{\alpha+\gamma}, [x_\alpha, x_{-\gamma}^{(n-1)}]] + \frac{1}{N_{\alpha+\gamma,-\gamma}}[x_{-\gamma}^{(n-1)}, [x_{\alpha+\gamma}, x_\alpha]] \\
&= \frac{N_{\alpha,-\gamma}}{N_{\alpha+\gamma,-\gamma}}[x_{\alpha+\gamma}, x_{\alpha-\gamma}^{(n)}] + \frac{N_{\alpha+\gamma,\alpha}}{N_{\alpha+\gamma,-\gamma}}[x_{-\gamma}^{(n-1)}, x_{2\alpha+\gamma}^{(2)}]
\end{aligned}$$

The first and the last equality follows from the induction hypothesis. We will show that both terms in this sum are zero. If $\alpha - \gamma \notin \Phi$ then $N_{\alpha,-\gamma} = 0$. If $\alpha - \gamma \in \Phi$ then $\alpha + \gamma$ and $\alpha - \gamma$ must be linear independent. By the first part of the proof we have that $[x_{\alpha+\gamma}, x_{\alpha-\gamma}^{(n)}] = 0$ since $2\alpha \notin \Phi$ according to Observation 2.3. Let $\delta = 2\alpha + \gamma$. If $\delta \notin \Phi$ then $N_{\alpha+\gamma,\alpha} = 0$. If $\delta \in \Phi$ then

$$\begin{aligned}
[x_{-\gamma}^{(n-1)}, x_{2\alpha+\gamma}^{(2)}] &= \frac{1}{\delta(H_\delta)}[x_{-\gamma}^{(n-1)}, [h_\delta, x_{2\alpha+\gamma}]] \\
&= \frac{1}{\delta(H_\delta)}(-[h_\delta, [x_{2\alpha+\gamma}, x_{-\gamma}^{(n-1)}]] + [x_{2\alpha+\gamma}, [h_\delta, x_{-\gamma}^{(n-1)}]]) \\
&= \frac{-\gamma(H_\delta)}{\delta(H_\delta)}[x_{2\alpha+\gamma}, x_{-\gamma}^{(n)}]
\end{aligned}$$

The first equation follows from Lemma 3.12 and the last equality follows from Lemma 3.12, the induction hypothesis and the fact that $2\alpha+\gamma-\gamma = 2\alpha \notin \Phi$. Now, $-\gamma$ and $2\alpha + \gamma$ are linearly independent. Hence, according to the first part of the proof $[x_{2\alpha+\gamma}, x_{-\gamma}^{(n)}] = 0$. □

Lemma 3.14. *Let $\alpha, \beta \in \Phi$ such that $\alpha \neq -\beta$. Then*

$$[x_\alpha^{(2)}, x_\beta^{(k-1)}] = N_{\alpha,\beta} x_{\alpha+\beta}^{(k+1)} \quad \textit{for all } k \leqslant n.$$

Proof of Lemma 3.14. According to the induction hypothesis and Lemma 3.13, we have that $[x_\alpha, x_\beta^{(k)}] = N_{\alpha,\beta} x_{\alpha+\beta}^{(k+1)}$ for all $k \leqslant n$. Hence:

$$\begin{aligned}
[x_\alpha^{(2)}, x_\beta^{(k-1)}] &= -\frac{1}{\alpha(H_\alpha)}[x_\beta^{(k-1)}, [h_\alpha, x_\alpha]] \\
&= -\frac{1}{\alpha(H_\alpha)}[h_\alpha, [x_\beta^{(k-1)}, x_\alpha]] - \frac{1}{\alpha(H_\alpha)}[x_\alpha, [h_\alpha, x_\beta^{(k-1)}]] \\
&= \frac{(\alpha+\beta)(H_\alpha)}{\alpha(H_\alpha)} N_{\alpha,\beta} x_{\alpha+\beta}^{(k+1)} - \frac{\beta(H_\alpha)}{\alpha(H_\alpha)} N_{\alpha,\beta} x_{\alpha+\beta}^{(k+1)} \\
&= N_{\alpha,\beta} x_{\alpha+\beta}^{(k+1)}
\end{aligned}$$

□

We have now done the induction step for statement *1 - 2* in Proposition 3.10. Before proceeding, we give a useful lemma that generalizes the relations given in Example 3.9.

Lemma 3.15. *Let $\alpha, \beta, \alpha+\beta \in \Phi$. Then*

$$N_{\alpha,\beta} h^{(n)}_{\alpha+\beta} + N_{\alpha+\beta,-\beta} h^{(n)}_{-\alpha} - N_{-\alpha,\alpha+\beta} h^{(n)}_{\beta} = 0$$

Proof of Lemma 3.15. If $n = 2$, Lemma 3.15 follows from Example 3.9. Assume that $n > 2$. Since $\alpha+\beta \in \Phi$, it follows that $-\alpha-\beta \in \Phi$ and $N_{-\alpha,-\beta} \neq 0$. Then:

$$\begin{aligned} N_{\alpha,\beta} h^{(n)}_{\alpha+\beta} &= -N_{-\alpha,-\beta}[x_{\alpha+\beta}, x^{(n-1)}_{-\alpha-\beta}] = -[x_{\alpha+\beta}, [x_{-\alpha}, x^{(n-2)}_{-\beta}]] \\ &= -[x_{-\alpha}, [x_{\alpha+\beta}, x^{(n-2)}_{-\beta}]] - [x^{(n-2)}_{-\beta}, [x_{-\alpha}, x_{\alpha+\beta}]] \\ &= -N_{\alpha+\beta,-\beta}[x_{-\alpha}, x^{(n-1)}_{\alpha}] - N_{-\alpha,\alpha+\beta}[x^{(n-2)}_{-\beta}, x^{(2)}_{\beta}] \\ &= -N_{\alpha+\beta,-\beta} h^{(n)}_{-\alpha}] - N_{-\alpha,\alpha+\beta}[x^{(n-2)}_{-\beta}, x^{(2)}_{\beta}] \end{aligned}$$

$$\begin{aligned} [x^{(n-2)}_{-\beta}, x^{(2)}_{\beta}] &= \frac{1}{\beta(H_\beta)}[x^{(n-2)}_{-\beta}, [h_\beta, x_\beta]] \\ &= \frac{1}{\beta(H_\beta)}[h_\beta, [x^{(n-2)}_{-\beta}, x_\beta]] + \frac{1}{\beta(H_\beta)}[x_\beta, [h_\beta, x^{(n-2)}_{-\beta}]] \\ &= -\frac{1}{\beta(H_\beta)}[h_\beta, h^{(n-1)}_{\beta}] - [x_\beta, x^{(n-1)}_{-\beta}]] = -h^{(n)}_{\beta} \end{aligned}$$

Hence, $N_{\alpha,\beta} h^{(n)}_{\alpha+\beta} = -N_{\alpha+\beta,-\beta} h^{(n)}_{-\alpha} + N_{-\alpha,\alpha+\beta} h^{(n)}_{\beta}$ □

Lemma 3.16. *Let $i \in \{1, \ldots, l\}$ and let $\alpha \in \Phi$. Then*

$$[x_\alpha, h^{(n)}_i] = -\alpha(H_i) x^{(n+1)}_{\alpha}$$

Proof of Lemma 3.16. First, take any $\delta \in \Phi$ such that $\alpha \neq \pm\delta$. Then:

$$\begin{aligned} [x_\alpha, h^{(n)}_{\delta}] &= [x_\alpha, [x_\delta, x^{(n-1)}_{-\delta}]] = [x_\delta, [x_\alpha, x^{(n-1)}_{-\delta}]] + [x^{(n-1)}_{-\delta}, [x_\delta, x_\alpha]] \\ &= N_{\alpha,-\delta}[x_\delta, x^{(n)}_{\alpha-\delta}] + N_{\delta,\alpha}[x^{(n-1)}_{-\delta}, x^{(2)}_{\delta+\alpha}] \\ &= N_{\alpha,-\delta} N_{\delta,\alpha-\delta} x^{(n+1)}_{\alpha} + N_{\delta,\alpha} N_{-\delta,\delta+\alpha} x^{(n+1)}_{\alpha} \\ &= (N_{\alpha,-\delta} N_{\delta,\alpha-\delta} + N_{\delta,\alpha} N_{-\delta,\delta+\alpha}) x^{(n+1)}_{\alpha} = -\alpha(H_\delta) x^{(n+1)}_{\alpha} \end{aligned}$$

The third equality follows from Lemma 3.13 and Lemma 3.14 and the last equality follows from Lemma 2.8. Hence:

$$[x_\alpha, h^{(n)}_{\delta}] = -\alpha(H_\delta) x^{(n+1)}_{\alpha} \quad \text{for all } \delta \in \Phi \text{ such that } \alpha \neq \pm\delta \tag{2}$$

If $\alpha \neq \pm\alpha_i$, take $\delta = \alpha_i$ in equation (2) and we are done. Assume that $\alpha = \pm\alpha_i$. According to Lemma 2.7 there is a $\beta \in \Phi$ such that $\alpha_i \neq \pm\beta$ and $\alpha_i + \beta \in \Phi$. Now, use the following relation given in Lemma 3.15:

$$N_{\beta,\alpha_i} h^{(n)}_{\beta+\alpha_i} + N_{\beta+\alpha_i,-\alpha_i} h^{(n)}_{-\beta} - N_{-\beta,\beta+\alpha_i} h^{(n)}_{\alpha_i} = 0$$

Then $N_{-\beta,\beta+\alpha_i} \neq 0$ and we have that:

$$\begin{aligned}
[x_\alpha, h_i^{(n)}] &= \frac{N_{\beta,\alpha_i}}{N_{-\beta,\beta+\alpha_i}}[x_\alpha, h_{\beta+\alpha_i}^{(n)}] + \frac{N_{\beta+\alpha_i,-\alpha_i}}{N_{-\beta,\beta+\alpha_i}}[x_\alpha, h_{-\beta}^{(n)}] \\
&= \frac{N_{\beta,\alpha_i}}{N_{-\beta,\beta+\alpha_i}}[x_\alpha^{(n)}, h_{\beta+\alpha_i}] + \frac{N_{\beta+\alpha_i,-\alpha_i}}{N_{-\beta,\beta+\alpha_i}}[x_\alpha^{(n)}, h_{-\beta}] \\
&= [x_\alpha^{(n)}, h_i] = -\alpha(H_i)x_\alpha^{(n+1)}
\end{aligned}$$

The second equality follows from the fact that $\alpha \neq \pm(\beta+\alpha_i)$ and $\alpha \neq \pm\beta$, equation (2) and Lemma 3.12. The third equality follows from Example 3.9 and the last equality follows from Lemma 3.11. □

Before doing the induction step for the last two statements in Proposition 3.10, we need the following lemma.

Lemma 3.17. *Let $\alpha \in \Pi$. Then:*

$$\begin{aligned}
&[x_\alpha, x_{-\alpha}^{(n)}] + [x_{-\alpha}^{(n-1)}, x_\alpha^{(2)}] = 0 \\
&[x_{-\alpha}, x_\alpha^{(n)}] + [x_\alpha^{(n-1)}, x_{-\alpha}^{(2)}] = 0 \\
&[x_\alpha, x_{-\alpha}^{(n)}] + [x_{-\alpha}, x_\alpha^{(n)}] = 0
\end{aligned}$$

Proof of Lemma 3.17. By Lemma 2.7, there is $\beta \in \Phi$ such that $\alpha+\beta \in \Phi$. Furthermore, according to Observation 2.3 and the fact that $0 \notin \Phi$, we have that $\alpha \neq \pm\beta$. Consider the relation given in Lemma 3.15:

$$N_{\alpha,\beta}h_{\alpha+\beta}^{(n)} + N_{\alpha+\beta,-\beta}h_{-\alpha}^{(n)} - N_{-\alpha,\alpha+\beta}h_\beta^{(n)} = 0$$

Taking the Lie bracket with this relation and, on one hand h_α and, on the other hand h_β, we get the following system of equations:

$$\begin{cases} N_{\alpha,\beta}[h_\alpha, h_{\alpha+\beta}^{(n)}] + N_{\alpha+\beta,-\beta}[h_\alpha, h_{-\alpha}^{(n)}] - N_{-\alpha,\alpha+\beta}[h_\alpha, h_\beta^{(n)}] = 0 \\ N_{\alpha,\beta}[h_\beta, h_{\alpha+\beta}^{(n)}] + N_{\alpha+\beta,-\beta}[h_\beta, h_{-\alpha}^{(n)}] - N_{-\alpha,\alpha+\beta}[h_\beta, h_\beta^{(n)}] = 0 \end{cases} \tag{3}$$

Now, take any $\delta, \gamma \in \Phi$. Then we have that:

$$\begin{aligned}
[h_\gamma, h_\delta^{(n)}] &= [h_\gamma, [x_\delta, x_{-\delta}^{(n-1)}]] \\
&= [x_\delta, [h_\gamma, x_{-\delta}^{(n-1)}]] + [x_{-\delta}^{(n-1)}, [x_\delta, h_\gamma]] \\
&= -\delta(H_\gamma)[x_\delta, x_{-\delta}^{(n)}] - \delta(H_\gamma)[x_{-\delta}^{(n-1)}, x_\delta^{(2)}]
\end{aligned} \tag{4}$$

The first equality follows from Definition 3.8, the second from the Jacobi identity. Recall that if $H_\gamma = \sum_{i=1}^{l} c_i H_i$, we defined $h_\gamma = \sum_{i=1}^{l} c_i h_i$ and the last equality follows from Lemma 3.12. Apply equation (4) to each bracket in system (3). To simplify, put:

$$\begin{cases} x = N_{\alpha,\beta}([x_{\alpha+\beta}, x_{-\alpha-\beta}^{(n)}] + [x_{-\alpha-\beta}^{(n-1)}, x_{\alpha+\beta}^{(2)}]) \\ y = N_{\alpha+\beta,-\beta}([x_{-\alpha}, x_\alpha^{(n)}] + [x_\alpha^{(n-1)}, x_{-\alpha}^{(2)}]) \\ z = N_{-\alpha,\alpha+\beta}([x_\beta, x_{-\beta}^{(n)}] + [x_{-\beta}^{(n-1)}, x_\beta^{(2)}]) \end{cases}$$

System (3) then becomes:

$$\begin{cases} -(\alpha+\beta)(H_\alpha)x + \alpha(H_\alpha)y + \beta(H_\alpha)z = 0 \\ -(\alpha+\beta)(H_\beta)x + \alpha(H_\beta)y + \beta(H_\beta)z = 0 \end{cases}$$

Adding the multiple $-\beta(H_\alpha)$ of the second row to the multiple $\beta(H_\beta)$ of the first row, and adding the multiple $-\alpha(H_\alpha)$ of the second row to the multiple $\alpha(H_\beta)$ of the first row we obtain:

$$\begin{cases} \big(\alpha(H_\beta)\beta(H_\alpha) - \alpha(H_\alpha)\beta(H_\beta)\big)(x-y) = 0 \\ \big(\alpha(H_\alpha)\beta(H_\beta) - \alpha(H_\beta)\beta(H_\alpha)\big)(x-z) = 0 \end{cases} \tag{5}$$

According to Observation 2.5 (and the fact that $\alpha \neq \pm\beta$), we have that $\alpha(H_\beta)\beta(H_\alpha) \neq \alpha(H_\alpha)\beta(H_\beta)$. Hence, dividing the equations in system (5) by $\alpha(H_\beta)\beta(H_\alpha) - \alpha(H_\alpha)\beta(H_\beta)$ and $\alpha(H_\alpha)\beta(H_\beta) - \alpha(H_\beta)\beta(H_\alpha)$ respectively, we get

$$\begin{cases} x - y = 0 \\ x - z = 0. \end{cases} \tag{6}$$

The idea is to find a third relation between x, y and z which is linearly independent of the first two.

Consider the following relations:

$$\begin{aligned} N_{\alpha,\beta}[x_{\alpha+\beta}, x^{(n)}_{-\alpha-\beta}] &= -N_{-\alpha,-\beta}[x_{\alpha+\beta}, x^{(n)}_{-\alpha-\beta}] = -[x_{\alpha+\beta}, [x_{-\alpha}, x^{(n-1)}_{-\beta}]] \\ &= -[x_{-\alpha}, [x_{\alpha+\beta}, x^{(n-1)}_{-\beta}]] - [x^{(n-1)}_{-\beta}, [x_{-\alpha}, x_{\alpha+\beta}]] \\ &= -N_{\alpha+\beta,-\beta}[x_{-\alpha}, x^{(n)}_{\alpha}] - N_{-\alpha,\alpha+\beta}[x^{(n-1)}_{-\beta}, x^{(2)}_{\beta}] \\ N_{\alpha,\beta}[x^{(n-1)}_{-\alpha-\beta}, x^{(2)}_{\alpha+\beta}] &= [x^{(n-1)}_{-\alpha-\beta}, [x_\alpha, x_\beta]] \\ &= [x_\alpha, [x^{(n-1)}_{-\alpha-\beta}, x_\beta]] + [x_\beta, [x_\alpha, x^{(n-1)}_{-\alpha-\beta}]] \\ &= N_{-\alpha-\beta,\beta}[x_\alpha, x^{(n)}_{-\alpha}] + N_{\alpha,-\alpha-\beta}[x_\beta, x^{(n)}_{-\beta}] \\ &= -N_{\alpha+\beta,-\beta}[x_\alpha, x^{(n)}_{-\alpha}] - N_{-\alpha,\alpha+\beta}[x_\beta, x^{(n)}_{-\beta}] \end{aligned}$$

Adding these two equalities we get

$$x + N_{\alpha+\beta,-\beta}([x_\alpha, x^{(n)}_{-\alpha}] + [x_{-\alpha}, x^{(n)}_{\alpha}]) + z = 0 \tag{7}$$

We proceed in steps:

(i) When $n = 2$, we must show that $[x_\alpha, x^{(2)}_{-\alpha}] + [x_{-\alpha}, x^{(2)}_{\alpha}] = 0$ for all $\alpha \in \Pi$. If $n = 2$, then

$$y = N_{\alpha+\beta,-\beta}([x_\alpha, x^{(2)}_{-\alpha}] + [x_{-\alpha}, x^{(2)}_{\alpha}])$$

and equation (7) becomes $x + y + z = 0$. This equation is linearly independent with the two in system (6). Hence, $x = y = z = 0$ and particularly, $[x_\alpha, x^{(2)}_{-\alpha}] + [x_{-\alpha}, x^{(2)}_{\alpha}] = 0$.

(ii) For $n > 2$, we first show that

$$[x_\alpha, x^{(k)}_{-\alpha}] = [x^{(m_1)}_{\alpha}, x^{(m_2)}_{-\alpha}] \tag{8}$$

for all $1 \leqslant k \leqslant n-1$ and all $m_1, m_2 \geqslant 1$ such that $m_1 + m_2 = k+1$. Observe that the induction hypothesis gives that $[h_\alpha, h_\alpha^{(k)}] = 0$ for all $k < n$. If $k = 2$ equation (8) follows from (i). Assume that equation (8) is true for all $k < l$ where $2 < l \leqslant n-1$. Take any $1 \leqslant m \leqslant l-1$. Then:

$$\begin{aligned}
[x_\alpha^{(m)}, x_{-\alpha}^{(l-m+1)}] &= -\frac{1}{\alpha(H_\alpha)}[x_\alpha^{(m)}, [h_\alpha, x_{-\alpha}^{(l-m)}]] \\
&= -\frac{1}{\alpha(H_\alpha)}[h_\alpha, [x_\alpha^{(m)}, x_{-\alpha}^{(l-m)}]] - \frac{1}{\alpha(H_\alpha)}[x_{-\alpha}^{(l-m)}, [h_\alpha, x_\alpha^{(m)}]] \\
&= -\frac{1}{\alpha(H_\alpha)}[h_\alpha, [x_\alpha, x_{-\alpha}^{(l-1)}]] - [x_{-\alpha}^{(l-m)}, x_\alpha^{(m+1)}] \\
&= -\frac{1}{\alpha(H_\alpha)}[h_\alpha, h_\alpha^{(l)}] + [x_\alpha^{(m+1)}, x_{-\alpha}^{(l-m)}] = [x_\alpha^{(m+1)}, x_{-\alpha}^{(l-m)}]
\end{aligned}$$

The third equality follows from the fact that $l-1 < l$ and the assumption that equation (8) is true for all $k < l$. The last equality follows from the fact that $l \leqslant n-1$ and the induction hypothesis. Now, let m run from 1 to $l-1$ and we are done. Hence (8) is proved.

Consider the following relation for all $1 \leqslant m \leqslant n-1$:

$$\begin{aligned}
[h_\alpha, h_\alpha^{(n)}] &= [h_\alpha, [x_\alpha, x_{-\alpha}^{(n-1)}]] = [h_\alpha, [x_\alpha^{(m)}, x_{-\alpha}^{(n-m)}]] \\
&= [x_\alpha^{(m)}, [h_\alpha, x_{-\alpha}^{(n-m)}]] + [x_{-\alpha}^{(n-m)}, [x_\alpha^{(m)}, h_\alpha]] \\
&= -\alpha(h_\alpha)[x_\alpha^{(m)}, x_{-\alpha}^{(n-m+1)}] - \alpha(h_\alpha)[x_{-\alpha}^{(n-m)}, x_\alpha^{(m+1)}] \\
&= -\alpha(h_\alpha)([x_\alpha^{(m)}, x_{-\alpha}^{(n-m+1)}] + [x_{-\alpha}^{(n-m)}, x_\alpha^{(m+1)}])
\end{aligned}$$

The second equality follows from equation (8) for $k = n-1$. By letting m run from 1 to $n-1$ we get:

$$\begin{aligned}
&[x_\alpha, x_{-\alpha}^{(n)}] + [x_{-\alpha}^{(n-1)}, x_\alpha^{(2)}] = -[x_{-\alpha}^{(n-1)}, x_\alpha^{(2)}] + [x_{-\alpha}^{(n-2)}, x_\alpha^{(3)}] = \\
&-[x_{-\alpha}^{(n-2)}, x_\alpha^{(3)}] + [x_{-\alpha}^{(n-3)}, x_\alpha^{(4)}] = \cdots = -[x_{-\alpha}^{(2)}, x_\alpha^{(n-1)}] + [x_{-\alpha}, x_\alpha^{(n)}]
\end{aligned} \tag{9}$$

Adding all the equations in (9) gives:

$$(n-1)([x_{-\alpha}, x_\alpha^{(n)}] + [x_\alpha^{(n-1)}, x_{-\alpha}^{(2)}]) = [x_\alpha, x_{-\alpha}^{(n)}] + [x_{-\alpha}, x_\alpha^{(n)}] \tag{10}$$

This inserted in equation (7) gives $x + (n-1)y + z = 0$. Since the characteristic is 0, this equation is linearly independent with the two in system (6) and hence, $x = y = z = 0$. Particularly, $[x_{-\alpha}, x_\alpha^{(n)}] + [x_\alpha^{(n-1)}, x_{-\alpha}^{(2)}] = 0$. By equation (9), we also have that

$$[x_\alpha, x_{-\alpha}^{(n)}] + [x_{-\alpha}^{(n-1)}, x_\alpha^{(2)}] = [x_{-\alpha}, x_\alpha^{(n)}] + [x_\alpha^{(n-1)}, x_{-\alpha}^{(2)}] = 0.$$

Finally, by equation (10) we have that $[x_\alpha, x_{-\alpha}^{(n)}] + [x_{-\alpha}, x_\alpha^{(n)}] = 0$. □

Lemma 3.18. *Let $i, j \in \{1, \ldots, l\}$. Then for all $n \geqslant 1$:*

$$[h_i, h_j^{(n)}] = 0$$

Proof of Lemma 3.18. Take any $1 \leqslant i, j \leqslant l$. Then:

$$\begin{aligned}[h_i, h_j^{(n)}] &= [h_i, [x_{\alpha_j}, x_{-\alpha_j}^{(n-1)}]] \\ &= [x_{\alpha_j}, [h_i, x_{-\alpha_j}^{(n-1)}]] + [x_{-\alpha_j}^{(n-1)}, [x_{\alpha_j}, h_i]] \\ &= -\alpha_j(h_i)[x_{\alpha_j}, x_{-\alpha_j}^{(n)}] - \alpha_j(h_i)[x_{-\alpha_j}^{(n-1)}, x_{\alpha_j}^{(2)}] = 0\end{aligned}$$

The last equality follows from Lemma 3.17. □

Lemma 3.19. *Let $\alpha \in \Phi$. Then $[x_\alpha, x_{-\alpha}^{(n)}]$ is a linear combination of $h_1^{(n+1)}, \ldots, h_l^{(n+1)}$.*

Proof of Lemma 3.19. Recall from section 2 that every positive root can be written uniquely as a sum of fundamental roots. Hence, we can talk about the length of a root. If $\alpha \in \Phi$, let $l(\alpha)$ denote the length of α. Clearly, $l(\alpha) = l(-\alpha)$. We use induction over this length to prove Lemma 3.19.

If $l(\alpha) = 1$, then $\alpha = \pm\alpha_i$ for some $i \in \{1, \ldots l\}$. Clearly, $[x_{\alpha_i}, x_{-\alpha_i}^{(n)}] = h_i^{(n+1)}$ by Definition 3.8. According to Lemma 3.17, we have that $[x_{-\alpha_i}, x_{\alpha_i}^{(n)}] = -[x_{\alpha_i}, x_{-\alpha_i}^{(n)}] = h_i^{(n+1)}$. Now, assume that Lemma 3.19 is true for all roots of length $< m$ for some $m > 1$, and let $\alpha \in \Phi$ such that $l(\alpha) = m$. Since $m > 1$, we have that $\pm\alpha$ is not fundamental. We proceed in steps:
(i) If $\alpha \in \Phi^+$ there is, according to Observation 2.6, $\alpha_i \in \Pi$ such that $\alpha - \alpha_i \in \Phi^+$. Clearly, $l(\alpha - \alpha_i) < m$. Furthermore, $N_{-\alpha+\alpha_i,-\alpha_i} \neq 0$ and

$$\begin{aligned}[x_\alpha, x_{-\alpha}^{(n)}] &= \frac{1}{N_{-\alpha+\alpha_i,-\alpha_i}}[x_\alpha, [x_{-\alpha+\alpha_i}, x_{-\alpha_i}^{(n-1)}]] \\ &= \frac{1}{N_{-\alpha+\alpha_i,-\alpha_i}}[x_{-\alpha+\alpha_i}, [x_\alpha, x_{-\alpha_i}^{(n-1)}]] \\ &\quad + \frac{1}{N_{-\alpha+\alpha_i,-\alpha_i}}[x_{-\alpha_i}^{(n-1)}, [x_{-\alpha+\alpha_i}, x_\alpha]] \\ &= \frac{N_{\alpha,-\alpha_i}}{N_{-\alpha+\alpha_i,-\alpha_i}}[x_{-\alpha+\alpha_i}, x_{\alpha-\alpha_i}^{(n)}] + \frac{N_{-\alpha+\alpha_i,\alpha}}{N_{-\alpha+\alpha_i,-\alpha_i}}[x_{-\alpha_i}^{(n-1)}, x_{\alpha_i}^{(2)}] \\ &= \frac{N_{\alpha,-\alpha_i}}{N_{-\alpha+\alpha_i,-\alpha_i}}[x_{-\alpha+\alpha_i}, x_{\alpha-\alpha_i}^{(n)}] - \frac{N_{-\alpha+\alpha_i,\alpha}}{N_{-\alpha+\alpha_i,-\alpha_i}}[x_{\alpha_i}, x_{-\alpha_i}^{(n)}]\end{aligned}$$

The last equality follows from Lemma 3.17. Now, since $l(\alpha_i) < m$, and $l(-\alpha+\alpha_i) = l(\alpha-\alpha_i) < m$ we have that $[x_{\alpha_i}, x_{-\alpha_i}^{(n)}]$ and $[x_{-\alpha+\alpha_i}, x_{\alpha-\alpha_i}^{(n)}]$ are linear combinations of $h_1^{(n+1)}, \ldots, h_l^{(n+1)}$. Hence, $[x_\alpha, x_{-\alpha}^{(n)}]$ is a linear combination of $h_1^{(n+1)}, \ldots, h_l^{(n+1)}$.
(ii) If $\alpha \in \Phi^-$ then $-\alpha \in \Phi^+$. Since $-\alpha \notin \Pi$, there is, according to Observation 2.6, $\alpha_i \in \Pi$ such that $-\alpha - \alpha_i \in \Phi^+$. Hence, there is $\alpha_i \in \Pi$ such that $\alpha + \alpha_i \in \Phi$.

Clearly, $l(\alpha+\alpha_i) < m$. Furthermore $N_{-\alpha-\alpha_i,\alpha_i} \neq 0$ and

$$\begin{aligned}
[x_\alpha, x_{-\alpha}^{(n)}] &= \frac{1}{N_{-\alpha-\alpha_i,\alpha_i}}[x_\alpha,[x_{-\alpha-\alpha_i}, x_{\alpha_i}^{(n-1)}]] \\
&= \frac{1}{N_{-\alpha-\alpha_i,\alpha_i}}[x_{-\alpha-\alpha_i},[x_\alpha, x_{\alpha_i}^{(n-1)}]] \\
&\quad + \frac{1}{N_{-\alpha-\alpha_i,\alpha_i}}[x_{\alpha_i}^{(n-1)},[x_{-\alpha-\alpha_i}, x_\alpha]] \\
&= \frac{N_{\alpha,\alpha_i}}{N_{-\alpha-\alpha_i,\alpha_i}}[x_{-\alpha-\alpha_i}, x_{\alpha+\alpha_i}^{(n)}] + \frac{N_{-\alpha-\alpha_i,\alpha}}{N_{-\alpha-\alpha_i,\alpha_i}}[x_{\alpha_i}^{(n-1)}, x_{-\alpha_i}^{(2)}] \\
&= \frac{N_{\alpha,\alpha_i}}{N_{-\alpha+\alpha_i,\alpha_i}}[x_{-\alpha-\alpha_i}, x_{\alpha+\alpha_i}^{(n)}] + \frac{N_{-\alpha-\alpha_i,\alpha}}{N_{-\alpha-\alpha_i,\alpha_i}}[x_{\alpha_i}, x_{-\alpha_i}^{(n)}]
\end{aligned}$$

The last equality follows from Lemma 3.17. Since $l(\alpha_i) < m$ and $l(-\alpha-\alpha_i) = l(\alpha+\alpha_i) < m$, we have that $[x_{\alpha_i}, x_{-\alpha_i}^{(n)}]$ and $[x_{-\alpha-\alpha_i}, x_{\alpha+\alpha_i}^{(n)}]$ are linear combinations of $h_1^{(n+1)}, \dots, h_l^{(n+1)}$. Hence, $[x_\alpha, x_{-\alpha}^{(n)}]$ is a linear combination of $h_1^{(n+1)}, \dots, h_l^{(n+1)}$. □

In Lemma 3.11, 3.13, 3.16, 3.18 and 3.19 we have done the induction step for statement *1-5* in Proposition 3.10. Hence Proposition 3.10 is true for all $k \geqslant 1$. □

3.3. Proof of Theorem 1

In this section we restate and prove our main result, Theorem 1, and end with an example of a Lie algebra with $\mathfrak{sl}_2$-component. Before doing so, we prove a result that is a consequence of Proposition 3.10.

Proposition 3.20. *Let L be a classical Lie algebra without $\mathfrak{sl}_2$-component . Let $\{X_\alpha \; ; \; \alpha \in \Phi\} \cup \{H_i \; ; \; 1 \leqslant i \leqslant l\}$ be a Chevalley basis of L, where Φ is the root system of L relative to a Cartan subalgebra $\mathfrak{h}$ and $\Pi = \{\alpha_1, \dots \alpha_l\}$ is a set of fundamental roots. Take G to be the set of generators and R to be the ideal of $\mathcal{F} = \mathcal{F}(G)$ given in section 3.1. Then the set*

$$\{x_\alpha^{(k)}; \alpha \in \Phi\} \cup \{h_i^{(k)}; 1 \leqslant i \leqslant l\}$$

(see Definition 3.8) generates $(\mathcal{F}/R)_k$ for all $k \geqslant 1$.

Proof. This is clear for $k = 1$. Assume that $\{x_\alpha^{(k)} \; ; \; \alpha \in \Phi\} \cup \{h_i^{(k)}; \; 1 \leqslant i \leqslant l\}$ generates $(\mathcal{F}/R)_k$ for all $k \leqslant n$, for some $n \geqslant 1$. Take any $y \in (\mathcal{F}/R)_1$, $z \in (\mathcal{F}/R)_n$. Then

$$y = \sum_{i=1}^{l} a_i h_i + \sum_{\alpha\in\Phi} a_\alpha x_\alpha, \quad z = \sum_{i=1}^{l} b_i h_i^{(n)} + \sum_{\alpha\in\Phi} b_\alpha x_\alpha^{(n)}$$

for some coefficients $a_i, a_\alpha, b_i, b_\alpha$. Hence,

$$\begin{aligned}
[y,z] &= \Big[\sum_{i=1}^{l} a_i h_i + \sum_{\alpha\in\Phi} a_\alpha x_\alpha, \sum_{i=1}^{l} b_i h_i^{(n)} + \sum_{\alpha\in\Phi} b_\alpha x_\alpha^{(n)}\Big] \\
&= \sum_{i=1}^{l}\sum_{j=1}^{l} a_i b_j [h_i, h_j^{(n)}] + \sum_{\alpha\in\Phi}\sum_{i=1}^{l} a_i b_\alpha [h_i, x_\alpha^{(n)}] + \sum_{\alpha\in\Phi}\sum_{i=1}^{l} a_\alpha b_i [x_\alpha, h_i^{(n)}] \\
&\quad + \sum_{\alpha\in\Phi}\sum_{\beta\in\Phi} a_\alpha b_\beta [x_\alpha, x_\beta^{(n)}] \\
&= \sum_{\alpha\in\Phi}\sum_{i=1}^{l} (a_i b_\alpha - a_\alpha b_i)\alpha(H_i) x_\alpha^{(n+1)} + \sum_{\alpha\in\Phi}\sum_{\substack{\beta\in\Phi \\ \beta\neq-\alpha}} a_\alpha b_\beta N_{\alpha,\beta} x_{\alpha+\beta}^{(n+1)} \\
&\quad + \sum_{\alpha\in\Phi} a_\alpha b_{-\alpha} h_\alpha^{(n+1)}.
\end{aligned}$$

The third equality follows from Proposition 3.10. Moreover, according to Proposition 3.10 we have that $h_\alpha^{(n+1)}$ is a linear combination of $h_1^{(n+1)}, \dots, h_l^{(n+1)}$. Since every element in $(\mathcal{F}/R)_{n+1}$ can be written as a finite sum of terms of the form $[y,z]$ where $y \in (\mathcal{F}/R)_1$ and $z \in (\mathcal{F}/R)_n$ we are done. $\square$

Now we restate our main theorem:

Theorem 1. *Let L be a finite dimensional semisimple Lie algebra over an algebraically closed field F of characteristic 0 without $\mathfrak{sl}_2$-component. Then there is a set of generators G and an ideal R of $\mathcal{F}(\mathcal{G})$, generated by quadratic relations only, such that $L_{per} \cong \mathcal{F}(G)/R$.*

Proof. Let G be the set of generators and R the ideal of $\mathcal{F}(G)$ given in section 3.1. To abbreviate, let $\mathcal{F} = \mathcal{F}(G)$. Consider the graded Lie algebra epimorphism $\phi : \mathcal{F} \to L_{per}$ also given in Section 3.1. Now, $R \subset \ker(\phi)$ and let φ be the graded Lie algebra epimorphism such that

$$\varphi : \mathcal{F}/R \longrightarrow L_{per} : \varphi(x+R) = \phi(x) \quad \text{for all } x \in \mathcal{F}.$$

To show that $L_{per} \cong \mathcal{F}/R$ it suffices to show that φ is one-to-one.

The set

$$\bigcup_{k\geqslant 1} \Big(\{X_\alpha^{(k)}; \alpha\in\Phi\} \cup \{H_i^{(k)}; 1\leqslant i\leqslant l\}\Big)$$

is a basis of L_{per} (see the paragraph after Definition 3.1) and, according to Proposition 3.20, the set

$$\bigcup_{k\geqslant 1} \Big(\{x_\alpha^{(k)}; \alpha\in\Phi\} \cup \{h_i^{(k)}; 1\leqslant i\leqslant l\}\Big)$$

generates $\mathcal{F}/R$. Furthermore, by induction over k, we see that

$$\varphi(x_\alpha^{(k)}) = X_\alpha^{(k)}, \qquad \varphi(h_i^{(k)}) = H_i^{(k)}$$

for all $\alpha \in \Phi$, $1 \leqslant i \leqslant l$ and all $k \geqslant 1$. Hence, φ maps a set of generators to a basis and thus, φ is one-to-one. $\square$

We end with an example of a Lie algebra with a $\mathfrak{sl}_2$-component, namely $\mathfrak{sl}_2$ itself, and for which Theorem 1 not is fulfilled.

Example 3.21. Consider the Lie algebra $\mathfrak{sl}_2$ over an algebraically closed field F of characteristic 0. For a Chevalley basis, take

$$X = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

where $[X, Y] = H$, $[H, X] = 2X$ and $[H, Y] = -2Y$. Since $\mathrm{char}(F) = 0$, there are no quadratic relations in $(\mathfrak{sl}_2)_{per}$. Hence, no presentation of $(\mathfrak{sl}_2)_{per}$ with generators and relations can have any quadratic relations. On the other hand, there are cubic relations in $(\mathfrak{sl}_2)_{per}$, e.g. $[Y, [H, Y]]$. In fact, $(\mathfrak{sl}_2)_{per}$ can be presentated by cubic relations. Let $G = \{x, y, h\}$ and let R be generated by $[x, [h, x]]$, $[h, [h, x]] + 2[x, [x, y]]$, $[h, [x, y]]$, $[h, [h, y]] + 2[y, [y, x]]$ and $[y, [h, y]]$. Then $(\mathfrak{sl}_2)_{per} \cong \mathcal{F}(G)/R$. Furthermore, the vectorspace $R \cap \mathcal{F}(G)_3$ is generated by $[x, [h, x]]$ as an $\mathfrak{sl}_2$-module (c.f. Remark 3.7).

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