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EXPLICIT BRAUER INDUCTION FOR SYMPLECTIC AND ORTHOGONAL REPRESENTATIONS

OLAF NEISSE AND VICTOR P. SNAITH

(*communicated by J.F. Jardine*)

Abstract

Explicit Brauer Induction formulae with certain natural behaviour have been developed for complex representations, for example by work of Boltje, Snaith and Symonds. In this paper we present induction formulae for symplectic and orthogonal representations of finite groups. The problems are motivated by number theoretical and topological questions. We will prove naturality with respect to restriction and inflation. Also we investigate complexification maps and use them to compare the orthogonal and symplectic induction formulae with Boltje's complex induction formula.

Introduction

Motivated by a question on L-functions, Brauer published in 1947 [**10**] his famous induction theorem which states that any complex/unitary representation of a finite group can be expressed as an integral linear combination of representations which are obtained by induction from one-dimensional representations of subgroups. This result had and still has important implications in many mathematical areas like number theory, character theory or topology.

The natural question arose to write down explicitly such a linear combination. During 1986-1989 different explicit Brauer induction formula were developed (e.g. [**35**],[**4**],[**43**]). The key property of all these formulae is naturality with respect to homomorphisms of groups. Here we shall be concerned with generalising one of these formulae, namely the canonical induction formula of [**4**].

Then, in connection with root-numbers, orthogonal and symplectic representations of finite groups became important. Similar induction theorems were proved by Deligne/Serre (for orthogonal) and Martinet (for symplectic) (see [**22**]). Also topological questions like stable decompositions of the classifying spaces of symplectic or orthogonal matrix groups are related to explicit integral induction formulae with natural behaviour for those induction theorems as explained in §1. However, these topological problems are not totally answered yet, since our formulae do not have integer coefficients in general.

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In this paper we introduce corresponding explicit induction formulae by an algebraic approach, where the coefficients of the linear combinations are derived from topological calculations. This enables us to present formulae of the same shape for all three induction theorems:

Main Theorem[1]

For $n \in \mathbb{N}$ let $X(n)$ denote either $U(n)$ (unitary case), $O(2n)$ (orthogonal case) or $Sp(n)$ (symplectic case). Let G a finite group and $\rho : G \to X(n)$ a representation. Then

$$(*)\ \ \rho = \sum_{(H,\Psi)<(H_1,\Psi_1)<..<(H_r,\Psi_r)} (-1)^r \frac{|H|}{|G|} m(\mathrm{Res}^G_{H_r}\rho, \Psi_r)\ \mathrm{Ind}^G_H \Psi$$

where the sum runs over all chains of pairs (H_i, Ψ_i) of subgroups $H_i \leqslant G$ and representations $\Psi_i : H_i \to X(i)$, and $m(\Theta, \Psi)$ is defined by

$$m(\Theta,\Psi) = \begin{cases} \langle\theta,\psi\rangle_H & \text{with } \psi = \Psi \quad \text{unitary case} \\ \langle\theta,\psi\rangle_H \,/\, \langle\psi,\psi\rangle_H & \text{with } \psi = c(\Psi) \quad \text{symplectic case} \\ \left.\begin{array}{ll} \langle\theta,\psi\rangle_H & \text{with } c(\Psi) = \psi \text{ irr.} \\ \langle\theta,\lambda\rangle_H & \text{with } c(\Psi) = \lambda + \overline{\lambda} \\ [\langle\theta,\phi\rangle_H/2] & \text{mit } c(\Psi) = 2\phi \\ 1 & \text{with } c(\Psi) = \phi + \phi' \text{ odd} \\ 0 & \text{otherwise} \end{array}\right\} & \text{orthogonal case} \end{cases}$$

and $c\colon R^O(G) \to R^U G$ resp. $c\colon R^{Sp}_+(G) \to R^U G$ denotes the complexification map from the Grothendieck rings $R^O(G)$ and $R^{Sp}_+(G)$ of orthogonal resp. symplectic representations to those of unitary representations.

Moreover, we discuss how these three formulae are related to each other by the natural complexification operations c on orthogonal resp. symplectic representations.

In section 1 we explain the topological motivation for this research, namely a proof of the existence of an exponential stable decomposition of the classifying space BSp of the symplectic group Sp. The idea was to imitate the proof in the unitary case, where an explicit Brauer induction formula can play an important role.

Section 2 contains a brief summary on the canonical explicit induction formula for unitary representations. We also recall terminology, properties of the formula and the language of Mackey functors.

Then, in section 3, we study symplectic representations. In 3.9 the formula for symplectic representations is described. Its functoriality with respect to restriction, inflation and other naturality properties are proved.

We will show (after a long calculation) that the induction formulae commute with complexification, proving that our symplectic formula gives a natural induction formula. It will turn out that it is not integral for general groups. But, in 3.21, we

[1] for details on notation and interpretation see 2.2, 3.9 and 6.4

prove integrality for representations of Galois groups of certain local number field extensions.

In section 4 we examine the interaction between the symplectic induction formula and symplectic Adams operations. This will be important for section 5, because there we explore another motivation for such explicit formulae, namely the connections to symplectic local root numbers. In §5.2 we formulate a conjecture that our formula $a_G^{Sp}(\rho)$ is 2-adically integral when ρ is a local Galois representation and the residue characteristic is odd. If true, as explained in Remark 5.4, this conjecture would enable one to promote the two-dimensional formulae of [**26**], [**27**] and [**17**] to a formula for all symplectic root numbers of p-adic local fields when $p \neq 2$.

Finally, in section 6, we deal with orthogonal representations. As explained in §6, our motivation for deriving the orthogonal formula comes more from algebraic topology than from algebra. In 6.4 we give details on our explicit orthogonal induction formula. While this formula respects restriction, inflation and conjugation (as we will prove), we lose additivity and integrality. However, after calculating the defect term which obstructs the orthogonal formula from commuting with the unitary formula via complexification, we find that our orthogonal formula is natural as well.

We have developed our formulae combinatorially from scratch, this seemed more directly accessible from a reader's point of view. However, we are grateful to the referee who pointed out that we could have used the methodology of [**7**] in several places.

1. Topological motivation

1.1. Let G_n denote one of the classical compact Lie groups, $\mathrm{U}(n), \mathrm{Sp}(n)$ or $\mathrm{O}(2n)$, of unitary, symplectic or orthogonal matrices. Since G_{n-1} embeds canonically into G_n (by adding $1 \in G_1$ at the bottom right-hand corner) we may form the mapping cone, BG_n/BG_{n-1}, of the induced map between classifying spaces. When $n = 0$ we set $BG_0/BG_{-1} = S^0$, the zero-dimensional sphere. Let X_+ denote the disjoint union of the space X and a base-point. In the stable homotopy category [**1**] there is a homotopy equivalence of the form

$$(BG_\infty)_+ \simeq \vee_{k \geqslant 0} BG_k/BG_{k-1}$$

which was first proved in [**33**]. In fact, from this equivalence one can easily deduce equivalences of the form

$$(BG_n)_+ \simeq \vee_{0 \leqslant k \leqslant n} BG_k/BG_{k-1}.$$

Stable decompositions of classifying spaces are important [**28**] because the factors are much simpler to work with than the whole. For example, $BU(n)/BU(n-1)$ is just the Thom space, $MU(n)$, of the universal n-plane bundle on $BU(n)$.

In this section we shall show how Explicit Brauer Induction may be used to derive these stable decompositions.

1.2. Let $R_+^U(G), R_+^{Sp}(G)$ and $R_+^O(G)$ denote, respectively, the free abelian group on the G-conjugacy classes of *representations* $\phi : H \longrightarrow G_1$ where H is a subgroup of G. Hence $R_+^U(G)$ (denoted by $R_+(G)$ in [**41**]) is the free abelian group on the

G-conjugacy classes of homomorphisms $\phi : H \longrightarrow U(1) = S^1$. However, in the symplectic and orthogonal cases, because a representation into G_1 is a G_1-conjugacy class of a homomorphism, a free generator $\phi : H \longrightarrow G_1$ is equivalent to $X\phi(g - g^{-1})X^{-1} : H \longrightarrow G_1$ for any $g \in G$, $X \in G_1$. The equivalence class of (H, ϕ) will be denoted by $(H, \phi)^G$.

If $J \subseteq G$ we have a restriction homomorphism

$$\mathrm{Res}^G_J : R^Z_+(G) \longrightarrow R^Z_+(J)$$

for $Z = U, Sp, O$ defined by

$$\mathrm{Res}^G_J((H, \phi)^G) = \sum_{z \in J\backslash G/H} (J \bigcap zHz^{-1}, (z^{-1})^*\phi)^J$$

where $(z^{-1})^*\phi(zhz^{-1}) = \phi(h)$. If $\pi : G \longrightarrow K$ is a surjection there is an inflation homomorphism

$$\mathrm{Inf}^G_K : R^Z_+(K) \longrightarrow R^Z_+(G)$$

given by $\mathrm{Inf}^G_K(H, \phi)^K = (\pi^{-1}(H), \phi \cdot \pi)^G$.

By means of these maps $R^Z_+(-)$ gives a functor from finite groups to abelian groups when $Z = U, Sp, O$. When $Z = U$, we even obtain a Mackey functor from finite groups to the category of rings ([**4**], [**5**], [**11**], [**35**], [**38**], [**41**]).

1.3. $R^U_+(G)$ *and stable homomotopy decompositions*

Let $R(G)$ denote the complex representation ring of G, so $R(G) = K_0(\mathbb{C}G)$, and let $IR(G) = Ker(\epsilon : R(G) \longrightarrow \mathbb{Z})$ denote the augmentation ideal given by the kernel of the homomorphism which sends a virtual representation to its dimension. Henceforth, following [**41**], we shall abbreviate $R^U_+(G)$ to $R_+(G)$.

The central result of Explicit Brauer Induction is the existence of natural transformations from representations of G to $R_+(G)$ which are right inverse to the map

$$b_G : R_+(G) \longrightarrow R(G)$$

given by $b_G(H, \phi)^G = \mathrm{Ind}^G_H(\phi) \in R(G)$. The formula of ([**5**], [**41**], [**43**]) gives a natural homomorphism

$$a_G : R(G) \longrightarrow R_+(G)$$

such that $a_G(\phi : G \longrightarrow U(1)) = (G, \phi)^G$. There is only one such homomorphism and it satisfies $b_G a_G = 1$.

Now let p be a prime and consider the case when $G = GL_n\mathbb{F}_q$ with q a power of p. In [**29**] the canonical modular representations of G are used, by means of the Brauer lifting technique of [**19**], to construct a canonical element

$$\sigma_p \in \lim_{\overleftarrow{n,q}} IR(GL_n\mathbb{F}_q) \subset \lim_{\overleftarrow{n,q}} R(GL_n\mathbb{F}_q).$$

By naturality of the homomorphisms, $\{a_{GL_n\mathbb{F}_q}\}$, we obtain

$$a_{GL\overline{\mathbb{F}}_p}(\sigma_p) \in \lim_{\overleftarrow{n,q}} IR_+(GL_n\mathbb{F}_q) \subset \lim_{\overleftarrow{n,q}} R_+(GL_n\mathbb{F}_q).$$

Here $\overline{\mathbb{F}}_p$ is an algebraic closure of $\mathbb{F}_p$ and $IR_+(G)$ is the kernel of the homomorphism to the Burnside ring given by $\epsilon(H, \phi)^G = [G/H]$ ([**35**], [**41**]).

If X and Y are base-pointed spaces let $\{X, Y\}$ denote the stable homotopy classes of maps from X to Y; that is, the morphisms from X to Y in the stable homotopy category [**1**]. If G is a finite group there exists a natural transformation

$$T : IR_+(G) \longrightarrow \{BG_+, BU(1)_+\}$$

given by sending $(H, \phi)^G - (H, 1)^G$ to the composition

$$BG_+ \xrightarrow{\tau} BH_+ \xrightarrow{B\phi_+} BU(1)_+$$

where τ is the stable homotopy transfer ([**37**] pp.163-4; see also [**2**], [**3**], [**21**], [**24**]). In fact, if $IA(G)$ is the augmentation ideal of the Burnside ring, $A(G)$, then the $IA(G)$-adic completion of T is an isomorphism. We shall not need this result, which was first proved (with $U(1)$ replaced by any torus) in ([**37**] Ch.V Theorem 1.17) and was extended to all Lie groups in [**24**].

Hence we have a canonical element

$$T(a_{GL\overline{\mathbb{F}}_p}(\sigma_p)) \in \lim_{\overleftarrow{n,q}} \{(BGL_n\mathbb{F}_q)_+, BU(1)_+\} \cong \{(BGL\overline{\mathbb{F}}_p)_+, BU(1)_+\}.$$

Let $Q(X_+)$ denote the iterated loopspace $Q(X_+) = \lim_n \Omega^n\Sigma^n(X_+)$. Then, if Y is a base-pointed space there is an adjunction isomorphism of the form

$$adj : \{Y, BU(1)_+\} \xrightarrow{\cong} [Y, Q(BU(1)_+)]$$

whose range is the set of based homotopy classes of maps from Y to $Q(BU(1)_+)$. Therefore we obtain a (homotopy class of a) map of the form

$$\widetilde{T}_p = adj(T(a_{GL\overline{\mathbb{F}}_p}(\sigma_p))) : (BGL\overline{\mathbb{F}}_p)_+ \longrightarrow Q(BU(1)_+).$$

Now we shall examine how the properties of a_G translate into useful properties of $\widetilde{T}_p$.

Direct sum of matrices makes $BGL\overline{\mathbb{F}}_p$ into an H-space with multiplication

$$m : BGL\overline{\mathbb{F}}_p \times BGL\overline{\mathbb{F}}_p \longrightarrow BGL\overline{\mathbb{F}}_p.$$

The iterated loopspace $Q(BU(1)_+)$ is also an H-space and additivity of the Brauer lifting together with the fact that a_G is a *homomorphism* implies that $\widetilde{T}_p$ is an H-map so that

$$m(\widetilde{T}_p \times \widetilde{T}_p) \simeq \widetilde{T}_p \cdot m : BGL\overline{\mathbb{F}}_p \times BGL\overline{\mathbb{F}}_p \longrightarrow Q(BU(1)_+).$$

On the other hand, in the stable homotopy category, there is a Snaith splitting ([**32**], [**34**]; see also [**13**], [**14**]) of the form

$$\bigvee_{k \geqslant 0} j_k : \ Q(BU(1)_+) \xrightarrow{\simeq} \bigvee_{k \geqslant 0} (B\Sigma_k \int U(1))/(B\Sigma_{k-1} \int U(1))$$

where $\Sigma_k \int U(1)$ is the wreath product given by the normaliser of the diagonal maximal torus in $U(k)$. As usual, when $k = 0$ we adopt the convention that mapping cone of $B\Sigma_{-1}\int U(1) \longrightarrow B\Sigma_0\int U(1)$ is the zero-sphere, S^0. Composing with the map

$$(B\Sigma_k \int U(1))/(B\Sigma_{k-1} \int U(1)) \longrightarrow BU(k)/BU(k-1) = MU(k)$$

induced by the inclusion of $\Sigma_k \int U(1)$ into $U(k)$ we obtain a stable map of the form

$$\bigvee_{k \geqslant 0} \hat{j}_k : \quad Q(BU(1)_+) \longrightarrow \bigvee_{k \geqslant 0} MU(k).$$

Furthermore, as explained in ([**34**]; see also [**13**], [**14**]), the maps $\bigvee_{k\geqslant 0} j_k$ and $\bigvee_{k\geqslant 0} \hat{j}_k$ are exponential with respect to the pairings

$$\begin{array}{c} (B\Sigma_k \int U(1))/(B\Sigma_{k-1} \int U(1)) \wedge (B\Sigma_l \int U(1))/(B\Sigma_{l-1} \int U(1)) \\ \downarrow \\ (B\Sigma_{k+l} \int U(1))/(B\Sigma_{k+l-1} \int U(1)) \end{array}$$

and

$$MU(k) \wedge MU(l) \longrightarrow MU(k+l)$$

induced by direct sum of matrices. That is, we have a homotopy-commutative diagram of stable maps of the following form

$$\begin{tikzcd} QBU(1)_+ \times QBU(1)_+ \arrow[r, "\sum_k \hat{j}_k \wedge \hat{j}_{t-k}"] \arrow[d, "m"'] & \bigvee_k MU(k) \wedge MU(t-k) \arrow[d] \\ QBU(1)_+ \arrow[r, "\hat{j}_t"] & MU(t) \end{tikzcd}$$

Now we come to our motivating topological result.

Theorem 1.4. *([**34**] Theorem 2.2; [**33**] Theorem 4.3)*
There exists a stable homotopy equivalence of the form

$$\overline{\sigma}_U = \bigvee_{k \geqslant 0} \overline{\sigma}_{U,k} : \quad BU_+ \xrightarrow{\simeq} \bigvee_{k \geqslant 0} MU(k).$$

In addition, $\overline{\sigma}_U$ is an exponential map in the sense that, for each $t \geqslant 0$, the diagram

$$\begin{tikzcd} BU_+ \wedge BU_+ = (BU \times BU)_+ \arrow[r, "\sum_k \overline{\sigma}_{U,k} \wedge \overline{\sigma}_{U,t-k}"] \arrow[d, "m_+"'] & \bigvee_k MU(k) \wedge MU(t-k) \arrow[d] \\ BU_+ \arrow[r, "\hat{j}_t"] & MU(t) \end{tikzcd}$$

commutes in the stable homotopy category.

Proof. It suffices to construct an H-map, τ, from BU to $QBU(1)_+$ which restricts to the canonical map on $BU(1)$. Then we set σ_U equal to the composition with $\sum_{k=0}^{\infty} \hat{j}_k$. The map $\widetilde{T}_p$ has the right properties except that its domain is $(BGL_n\overline{\mathbb{F}}_p)_+$ rather than BU. However, by the technique of localisation and completion in homotopy theory ([**8**], [**9**], [**23**], [**42**]) it suffices to construct τ on the rationalisation of BU and on its completion at each prime, l. Since BU and $QBU(1)_+$ are rationally equivalent we may take the rationalisation of τ to be the identity map. The completion of BU at the prime l is equal to the l-adic completion of $(BGL_n\overline{\mathbb{F}}_p)$ where p is chosen to generate $(\mathbb{Z}/l^2)^*$ if l is odd and $p = 3$ when $l = 2$. Therefore we may choose the l-adic completion of a suitable $\widetilde{T}_p$ as the completion of τ at l, which completes the proof. □

Remark 1.5. We have used the Explicit Brauer Induction map to prove Theorem 1.4. The naturality of the map is required to in order to turn Quillen's element, σ_p of §1.3, into a map from $BGL\overline{\mathbb{F}}_p$ to $QBU(1)_+$. It is the fact that a_G is a *homomorphism* which yields an H-map and hence the exponential property of the splitting.

The Explicit Brauer Induction formulae of [**35**] are natural in the symplectic and orthogonal case, too. Using this one could give a proof of the stable decompositions of [**33**]

$$BSp \simeq \bigvee_{k=0}^{\infty} MSp(k), \quad BO \simeq \bigvee_{k=0}^{\infty} BO(2k)/BO(2k-2)$$

similar to that of Theorem 1.4.

In [**34**] Theorem 2.2 (see also [**33**] Theorem 3.2) it is claimed that BSp admits an *exponential* stable decomposition of the above form. However there is a gap in the proposed proof since the cavalier reference to the existence of "an analogous symplectic vector field" in ([**33**] Example 2.13) is not true. For a time this gap did not seem serious in view of the fact that [**25**] offered an alternative construction of an exponential stable decomposition of BSp.

On Friday, 18 July 1997 one of us (VPS) learned of the argument of [**30**] which showed that none of the stable decompositions of BSp which were then in the literature were exponential. The way around this gap then seemed clear. We believed that one could use the topological approach to symplectic Explicit Brauer Induction described in ([**43**] §6) to construct a natural homomorphism of the form

$$a_G^{Sp} : R^{Sp}(G) \longrightarrow R_+^{Sp}(G)$$

and then imitate the proof of Theorem 1.4. After all, in ([**43**] p.180) one finds the remark that "The symplectic case presents no new difficulties; one just replaces the complex projective space by the quaternionic version." Unfortunately, as we studied the symplectic and orthogonal cases more closely, we discovered that this remark is false. As a result, at the moment, we do not know whether or not there is an exponential stable decomposition for BSp. In fact, our analysis, together with the topological results of [**30**] strongly suggests that no such exponential stable decomposition for BSp exists.

2. Induction formula for unitary representations

2.1. *Brauer Induction formula*

In this section we recall briefly the natural explicit Brauer induction formula in the complex case. This formula is due to Robert Boltje ([**5**]; see also [**4**], [**7**] and [**41**]).

Let G be a finite group. Let $R(G)$ denote the Grothendieck group of the category of finite-dimensional left $\mathbb{C}G$-modules. Every such module yields a matrix representation $G \to \mathrm{Gl}(n, \mathbb{C})$ which is conjugate to a unitary representation $G \to \mathrm{U}(n)$. Since such a representation is determined by its character, we may identify $R(G)$ with the character ring of G, the free abelian group on the set of irreducible characters on G. The subgroup of $R(G)$ generated by the set of linear characters $G \to \mathrm{U}(1) = S^1$ will be denoted by $L(G)$. Brauer proved [**10**] that every unitary representation $\rho : G \to \mathrm{U}(n)$ is an integral sum of representations which are induced from linear characters on subgroups of G, so

$$\rho = \sum_i n_i \mathrm{Ind}_{H_i}^G(\phi_i) \quad \text{with } H_i \leqslant G,\ \phi_i : H_i \to \mathrm{U}(1)\ .$$

Canonical explicit Brauer induction formulae with various properties were given by Boltje [**4**], Snaith [**35**] etc. We are going to use the formula which is trivial on one-dimensional representations and natural with respect to restriction and inflation, namely

$$\rho = \sum_{(H_0,\phi_0)<..<(H_r,\phi_r)} (-1)^r \frac{|H_0|}{|G|} m(\mathrm{Res}_{H_r}^G(\rho), \phi_r)\mathrm{Ind}_{H_0}^G(\phi_0)$$

where $H_i \leqslant G$, $\phi_i : H_i \to \mathrm{U}(1)$ and $m(\theta, \phi) = \langle \theta; \phi \rangle_H$ denotes the multiplicity of $\phi : H \to \mathrm{U}(1)$ in $\theta \in R(H)$.

In order to work with this formula we denote $R_+(G)$ the free abelian group on the G-conjucacy classes of linear characters on subgroups of G. More precisely, let G act on the set $\mathcal{M}(G)$ of pairs (H, ϕ) with $H \leqslant G$ and $\phi : H \to \mathrm{U}(1)$, and let $(H, \phi)^G$ denote the G-orbit of (H, ϕ) in $\mathcal{M}(G)$, then those orbits, collected in a set denoted $\mathcal{M}(G)/_G$, form a $\mathbb{Z}$-basis of $R_+(G)$. This is an example of a general method called the $_+$-construction (see [**6**]). There are, for $J \leqslant G$, homomorphisms

$$\mathrm{Res}_J^G : R_+(G) \to R_+(J)$$

and

$$\mathrm{Ind}_J^G : R_+(J) \to R_+(G)$$

and a natural conjugation map, giving the functor R_+ the structure of a G-Mackey functor.

Theorem 2.2. *[**6**, 3.1.2]; see also [**4**, 2.35] and [**41**, 2.2.15]*

The map

$$a_G : R(G) \ni \rho \mapsto \sum_{(H_0,\phi_0)<..<(H_r,\phi_r)} (-1)^r \frac{|H_0|}{|G|} m(\mathrm{Res}^G_{H_r}(\rho), \phi_r)(H_0, \phi_0)^G$$

is a homomorphism which takes values in $R_+(G)$.

This homomorphism $a_G : R(G) \to R_+(G)$ satisfies the following natural properties:

Proposition 2.3. *[**41**, 2.3.2]*
i) For $J \leqslant G$ the following diagram commutes

$$\begin{tikzcd} R(G) \arrow[r, "a_G"] \arrow[d, "\mathrm{Res}^G_J"'] & R_+(G) \arrow[d, "\mathrm{Res}^G_J"] \\ R(J) \arrow[r, "a_J"] & R_+(J) \end{tikzcd}$$

ii) For $\phi \in L(G)$, $a_G(\phi) = (G, \phi)^G$.
iii) For $N \lhd G$ the following diagram commutes

$$\begin{tikzcd} R(G/N) \arrow[r, "a_{G/N}"] \arrow[d, "\mathrm{Infl}^G_{G/N}"'] & R_+(G/N) \arrow[d, "\mathrm{Infl}^G_{G/N}"] \\ R(G) \arrow[r, "a_G"] & R_+(G) \end{tikzcd}$$

Theorem 2.4. *[**41**, 2.3.2]*
If $b_G : R_+(G) \to R(G)$ is the homomorphism defined by

$$(H, \phi)^G \mapsto \mathrm{Ind}^G_H(\phi)$$

that a_G is a splitting; that is $b_G \circ a_G = \mathrm{id} : R(G) \to R(G)$.

Example 2.5. To prepare the reader for the sort of combinatorial arguments which we shall use later (in §3, for example) and to explain notation we shall derive the following example ab initio. By Remark 2.7, we could have deduced the formula of Proposition 2.6 from the dihedral formula of [**4**].

Let Q_{4n} denote the generalised quaternion group

$$Q_{4n} = \left\langle x, y \mid x^n = y^2,\ y^4 = 1, yxy^{-1} = x^{-1} \right\rangle$$

and let Ψ denote the symplectic representation

$$\Psi : Q_{4n} \longrightarrow Sp(1) \subset \mathbf{H}^*$$

given by $\Psi(x) = \xi_{2n}$ and $\Psi(y) = j$. Here $\xi_n = e^{2\pi\sqrt{-1}/n}$ and j is the usual quaternion.

We wish to evaluate

$a_{Q_{4n}}(c(\Psi))$

$= \sum_{(H,\phi)<(H_1,\phi_1)<\ldots<(H_r,\phi_r)} (-1)^r \frac{|H|}{4n} < c(\Psi), \phi_r >_{H_r} \cdot (H,\phi)^{Q_{4n}} \in R_+(Q_{4n}).$

If the multiplicity $< c(\Psi), \phi_r >_{H_r}$ is non-zero then $\mathrm{Res}^{Q_{4n}}_{H_r}(c(\Psi)) = \phi_r \oplus \overline{\phi}_r$ so that ϕ_r must be injective on H_r, because Ψ is. Hence H_r must be cyclic. The cyclic subgroups contained in $\langle x \rangle \cong \mathbb{Z}/2n$ are given by $\langle x^m \rangle \cong \mathbb{Z}/(2n/m)$ for each m dividing $2n$. Otherwise each $x^i y$ satisfies $(x^i y)^2 = y^2$ and generates a cyclic subgroup of order four. When n is odd, all the subgroups, $\langle x^i y \rangle$, are conjugate in Q_{4n} but when n is even, there are two conjugacy classes, namely $\langle y \rangle$ and $\langle xy \rangle$. Also $\Psi(y^2) = -1$ so that, if χ is the non-trivial character on $\langle y^2 \rangle$, $(y^2, \chi) \leqslant (H, \phi)$ for any $(H,\phi)^{Q_{4n}}$ which has a non-zero coefficient in $a_{Q_{4n}}(c(\Psi))$ ([**41**] Corollary 2.2.40).

Now consider the coefficient of $(H,\phi)^{Q_{4n}}$ when $\langle y^2 \rangle \subseteq H \subseteq \langle x \rangle$ with $|H| = t > 2$. In this case $(H,\phi) \neq (H,\overline{\phi})$ but $(H,\phi)^{Q_{4n}} = (H,\overline{\phi})^{Q_{4n}}$. If $\lambda \in \{\phi, \overline{\phi}\}$ then we have $1 =< c(\Psi), \phi_r >_{H_r}$ for any chain of the form $(H,\lambda) < (H_1,\phi_1) < \ldots < (H_r,\phi_r)$. The chains starting with ϕ are distinct from those starting with $\overline{\phi}$ so that the coefficient of $(H,\phi)^{Q_{4n}}$ is equal to

$$\frac{2 \cdot |H|}{4n} \sum_{\{1\}<A_1<A_2<\ldots<A_r \leqslant <x>/H} (-1)^r.$$

Here the sum of taken over all proper chains of subgroups, $A_i = H_i/H \subseteq \langle x \rangle/H$ or, when $\langle x \rangle = H$, just the trivial chain. By ([**41**] Exercise 2.5.1)

$$\frac{2 \cdot |H|}{4n} \sum_{\{1\}<A_1<A_2<\ldots<A_r \leqslant <x>/H} (-1)^r = \frac{2 \cdot |H|}{4n} \sum_{d|\ |<x>/H|} \mu(d)$$

where $\mu(n)$ denotes the classical Möbius function. This expression is zero unless $H = \langle x \rangle$ in which case it equals one.

Now consider the possibly non-trivial coefficients of the basis elements $(\langle y \rangle, \phi)^{Q_{4n}}$ and $(\langle xy \rangle, \phi)^{Q_{4n}}$. If g has order four let ρ_g denote the character on $< g >$ given by $\rho_g(g) = \sqrt{-1}$. When n is odd we must evaluate the coefficients of $(< y >, \rho_y)^{Q_{4n}}$ and $(\langle y \rangle, \overline{\rho}_y)^{Q_{4n}}$. These coefficients are both equal to one since $< y >$ is a maximal cyclic subgroup of Q_{4n} and, for example, there are $2n$ (H,ϕ)'s which are conjugate to $(\langle y \rangle, \rho_y)$. When n is even the distinct $(H,\phi)^{Q_{4n}}$'s with $H = \langle x^i y \rangle$ are $(\langle y \rangle, \rho_y)^{Q_{4n}}$ and $(\langle xy \rangle, \rho_{xy})^{Q_{4n}}$, each of which has coefficient equal to one.

Finally we must evaluate the coefficient of $(\langle y^2 \rangle, \chi)^{Q_{4n}}$, which is given by

$$\sum_{(\langle y^2 \rangle,\chi)<(H_1,\phi_1)<\ldots<(H_r,\phi_r)} (-1)^r \frac{|H|}{4n} < c(\Psi), \phi_r >_{H_r}.$$

The $2n$ chains of length one of the form $(\langle y^2 \rangle, \chi) < (\langle x^i y \rangle, \phi_1)$ contribute -1 to this coefficient. The remaining terms contribute zero, as is seen by the argument used on $(H,\phi)^{Q_{4n}}$'s with $H \subseteq \langle x \rangle$ together with the observation that, in this case, for the trivial chain, the multiplicity $< c(\Psi), \chi >_{<y^2>} = 2$.

The preceding discussion establishes the following result:

Proposition 2.6.
In the notation of 2.5

$$a_{Q_{4n}}(c(\Psi)) = \begin{cases} (\langle x\rangle, \phi_x)^{Q_{4n}} + (\langle y\rangle, \rho_y)^{Q_{4n}} + (\langle y\rangle, \overline{\rho}_y)^{Q_{4n}} - (\langle y^2\rangle, \chi)^{Q_{4n}} \\ \qquad \qquad if\ n\ is\ odd, \\ (\langle x\rangle, \phi_x)^{Q_{4n}} + (\langle y\rangle, \rho_y)^{Q_{4n}} + (\langle xy\rangle, \rho_{xy})^{Q_{4n}} - (\langle y^2\rangle, \chi)^{Q_{4n}} \\ \qquad \qquad if\ n\ is\ even \end{cases}$$

where $\rho_x(x) = \xi_{2n}$.

Remark 2.7. The formula for $a_{Q_{4n}}(c(\Psi))$ is determined by the projective representation associated to $c(\Psi)$ [**43**] which is the same projective representation as the one associated to the dihedral representation

$$\nu : D_{2n} \longrightarrow GL_2\mathbb{C}$$

given by

$$\nu(x) = \begin{pmatrix} \xi_n & 0 \\ 0 & \overline{\xi}_n \end{pmatrix},\ \nu(y) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

where

$$D_{2n} =< x, y \mid x^n = 1 = y^2, yxy = x^{-1} > .$$

This implies that $a_{D_{2n}}(\nu)$ is also given by the formulae of Proposition 2.6.

3. Induction formula for symplectic representations

3.1. *Symplectic representations*

Let G be a finite group. Let $R^{Sp}(G)$ denote the Grothendieck group of finite-dimensional $\mathbb{H}G$-modules, or in language of matrix representations, the Grothendieck group of equivalence classes of symplectic representations

$$\rho : G \longrightarrow \mathrm{Sp}(n) := \mathrm{Sp}(n, \mathbb{H}) .$$

By an induction theorem of Martinet, every such symplectic representation ρ is a $\mathbb{Z}$-linear combination of representations induced from one-dimensionals; that is $\rho : G \to \mathrm{Sp}(n)$ can be written as

$$\rho = \sum_i n_i \mathrm{Ind}_{H_i}^G(\Psi_i)$$

with $H_i \leqslant G$, $\Psi_i : H_i \to \mathrm{Sp}(1)$ and $n_i \in \mathbb{Z}$. In order to make this formula explicit, we apply the machinery of Mackey functors as described in [**6**]. Endowed with the usual conjugation, restriction and induction maps, $H \mapsto R^{Sp}(H)$ is a G-Mackey functor.

In $R^{Sp}(H)$ (with $H \leqslant G$) we consider the free abelian group $L^{Sp}(H)$ generated by the classes of one-dimensional $\mathbb{H}H$-modules, that is by Sp(1)-conjugacy classes of homomorphisms

$$\Psi : H \longrightarrow \mathrm{Sp}(1) = S^3.$$

Let $H \mapsto R_+^{Sp}(H)$ denote the Mackey functor obtained by the $_+$-construction on the subfunctor $H \mapsto L^{Sp}(H)$. So $R_+^{Sp}(G)$ is the free abelian group on the $G-\mathrm{Sp}(1)$-conjugacy classes of elements in $L^{Sp}(H)$ for $H \leqslant G$; these classes correspond to G-conjugacy classes of a one-dimensional symplectic representation (up to isomorphisms) of the subgroup, H, and will be denoted by $(H, \Psi)^G \in R_+^{Sp}(G)$. For $J \leqslant G$ we define homomorphisms

$$\mathrm{Res}_J^G : R_+^{Sp}(G) \longrightarrow R_+^{Sp}(J)$$

and

$$\mathrm{Ind}_J^G : R_+^{Sp}(J) \longrightarrow R_+^{Sp}(G)$$

in a manner which is analogous to the complex case (or given by the $_+$-construction). For $N \triangleleft G$ we have the inflation map

$$\mathrm{Infl}_{G/N}^G : R_+^{Sp}(G/N) \longrightarrow R_+^{Sp}(G)$$

defined by mapping $(HN/N, \overline{\Psi})^{G/N}$ to $(HN, \Psi)^G$. Let

$$b_G^{Sp} : R_+^{Sp}(G) \longrightarrow R^{Sp}(G)$$

be the homomorphism defined on the basis by $b_G^{Sp}((H,\Psi)^G) = \mathrm{Ind}_H^G(\Psi)$. Our aim is to obtain a map $\hat{a}_G^{sp} : R^{Sp}(G) \to R_+^{Sp}(G)$ which is a splitting of b_G^{Sp}, that is $b_G^{Sp} \circ \hat{a}_G^{sp} = \mathrm{id} : R^{Sp}(G) \to R^{Sp}(G)$, and which behaves naturally with respect to restriction. But this will not be possible in this integral form, as we shall see.

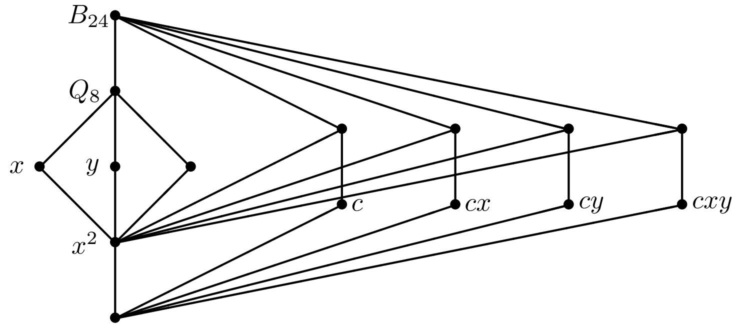
3.2. *One-dimensional symplectic representations*

Besides the cyclic and the quaternion type groups there are three more types of finite subgroups in the unit group Sp(1) of length 1 in $\mathbb{H}^\times$, namely the binary tetrahedral, octahedral and icosahedral groups. They arise from finite groups in $SO(3)$ given by rigid solids centered in the origin. These groups can be pulled back via $\pi : \mathrm{Sp}(1) \to SO(3)$, the map letting Sp(1) act on the pure quaternion space (a 3-dimensional real space with standard inner product) via conjugation. The kernel of this map is $\{\pm 1\}$.

The binary tetrahedral group B_{24} is the preimage of the group of motions of a regular tetrahedron, which is isomorphic to the alternating group A_4. So B_{24} is an extension of a cyclic group of order 2 with A_4. In fact, B_{24} can be expressed as the semidirect product of the quaternion group Q_8 of order 8 and a cyclic group of order 3 acting faithfully on Q_8, so

$$B_{24} = < x, y, c | x^4 = 1, y^2 = x^2, c^3 = 1, x^y = x^{-1}, x^c = y, y^c = xy > \ .$$

The lattice of subgroups can be pictured as follows:



Using this notation, let Ψ denote the representation

$$\Psi : B_{24} \to \mathrm{Sp}(1) \subset \mathbb{H}^\times$$

defined by

$$x \mapsto i \ \ , \ \ y \mapsto j \ \ , \ \ c \mapsto -\frac{1}{2}(1-i-j-k) \ \ .$$

This is, up to Sp(1)-conjugation, the unique faithful symplectic representation of B_{24}.

The binary octahedral group B_{48} is the preimage of the group of motions of a regular cube, which is isomorphic to the symmetric group S_4. So B_{48} is an extension of a cyclic group of order 2 with S_4. This group of order 48 appears as the nonsplit extension of the quaternion group of order 8 with the symmetric group S_3 acting as the full automorphism group. We can describe B_{48} as an extension of the binary tetrahedral group with a cyclic group $\langle d \rangle$ of order 2, acting as described in this group presentation:

$$B_{48} = < x, y, c, d | x^4 = c^3 = 1, d^2 = y^2 = x^2,$$
$$x^y = x^3, x^c = y, y^c = xy, x^d = x^3 y, y^d = y^3, c^d = c^2 > \ \ .$$

The maximal subgroups of B_{48} are the normal subgroup B_{24} of index 2, three conjugate groups of quaternion type Q_{16} of order 16 and four conjugate groups of quaternion type Q_{12} of order 12.

With this notation, let Ψ denote the (up to Sp(1)-conjugation) unique faithful symplectic representation

$$\Psi : B_{48} \to \mathrm{Sp}(1) \subset \mathbb{H}^\times$$

given by

$$x \mapsto i \ \ , \ \ y \mapsto j \ \ , \ \ c \mapsto -\frac{1}{2}(1-i-j-k) \ \ , \ \ d \mapsto \frac{\sqrt{2}}{2}(i-k) \ \ .$$

The binary icosahedral group B_{120} is the preimage of the group of motions of a regular icosahedron, which is isomorphic to the alternation group A_5. In fact, B_{120} is isomorphic to $SL_2(5)$, and can be described as [**12**]

$$B_{120} = < r, s, t | r^2 = s^3 = t^5 = rst > \quad .$$

B_{120} has order 120. The maximal subgroups of B_{120} are six conjugate groups of quaternion type Q_{20} of order 20, five conjugate groups of type binary tetrahedral group of order 24 and ten conjugate groups of quaternion type Q_{12} of order 12.

There are actually two types of faithful symplectic representations

$$\Psi : B_{120} \to \mathrm{Sp}(1) \subset \mathbb{H}^{\times}$$

given by

$$\begin{aligned} r &\mapsto i \\ s &\mapsto \frac{1}{2}(1 - (\zeta_5 + \zeta_5^4)i - (\zeta_5^2 + \zeta_5^3)j) \\ t &\mapsto \frac{1}{2}(-(\zeta_5 + \zeta_5^4) + i - (\zeta_5^2 + \zeta_5^3)k) \ . \end{aligned}$$

depending on the choice of ζ_5 (e.g. ζ_5 is $\exp(\frac{2\pi i}{5})$ or $\exp(\frac{4\pi i}{5})$).

3.3. *Complexification*

We use the usual embedding $\mathrm{Sp}(n) \to \mathrm{U}(2n)$ to define the complexification map

$$c = c_G : R^{Sp}(G) \longrightarrow R(G)\ .$$

So $\rho : G \to \mathrm{Sp}(n)$ (in $R^{Sp}(G)$) maps to

$$c(\rho) : G \longrightarrow \mathrm{Sp}(n) \longrightarrow \mathrm{U}(2n).$$

Of course this is an injective homomorphism. Furthermore we define the homomorphism

$$c_+ = c_{+,G} : R_+^{Sp}(G) \longrightarrow R_+(G)$$

by the formula

$$c_{+,G}((H, \Psi)^G) = \mathrm{Ind}_H^G(a_H(c(\Psi))) \in R_+(G).$$

The homomorphism was first defined in ([**41**] §5.4.40 p.213). There is an a_H missing in the formula of ([**41**] §5.4.41) but not in the proof of the following result, which is part of the proof of Theorem 5.4.42 of [**41**]. Obviously b_G and b_G^{Sp} are naturally connected via complexification, so that $b_G \circ c_+ = c \circ b_G^{Sp} : R_+^{Sp}(G) \to R(G)$.

Proposition 3.4.

The homomorphism $c_{+,G}$ is natural with respect to restriction so that, if $J \leqslant G$,

$$\mathrm{Res}_J^G \circ c_{+,G} = c_{+,J} \circ \mathrm{Res}_J^G : R_+^{Sp}(G) \longrightarrow R_+(J).$$

Proof. We recall that there is a double coset formula for the composition

$$\mathrm{Res}_J^G \circ \mathrm{Ind}_H^G : R_+(H) \longrightarrow R_+(J)$$

if $H, J \leqslant G$. Explicitly we have

$$\mathrm{Res}_J^G(\mathrm{Ind}_H^G((K, \Psi)^H)) = \sum_{w \in J \backslash G / H} \mathrm{Ind}_{J \cap wHw^{-1}}^J(w^*(\mathrm{Res}_{H \cap w^{-1}Jw}^H((K, \Psi)^H)))$$

whose proof is similar to the proof of the product formula of ([**41**] Exercise 2.5.7).

Hence, if $(H, \Psi)^G \in R_+^{Sp}(G)$ with $\Psi : H \longrightarrow \mathrm{Sp}(1)$, then

$$\begin{aligned}
&\mathrm{Res}^G_J(c_{+,G}((H, \Psi)^G)) \\
&= \mathrm{Res}^G_J(\mathrm{Ind}^G_H(a_H(c(\Psi) : H \longrightarrow U(2)))) \\
&= \textstyle\sum_{w \in J\backslash G/H} \mathrm{Ind}^J_{J\cap wHw^{-1}}(w^*(\mathrm{Res}^H_{H\cap w^{-1}Jw}(a_H(c(\Psi))))) \\
&= \textstyle\sum_{w \in J\backslash G/H} \mathrm{Ind}^J_{J\cap wHw^{-1}}(w^*(a_{J\cap wHw^{-1}}(c(\mathrm{Res}^H_{H\cap w^{-1}Jw}(\Psi))))) \\
&= \textstyle\sum_{w \in J\backslash G/H} \mathrm{Ind}^J_{J\cap wHw^{-1}}(a_{J\cap wHw^{-1}}(c(\mathrm{Res}^{wHw^{-1}}_{J\cap wHw^{-1}}((w^{-1})^*(\Psi))))) \\
&= c_{+,J}(\mathrm{Res}^G_J((H, \Psi)^G)),
\end{aligned}$$

which completes the proof. □

Proposition 3.5.
Suppose that

$$x = \sum_i n_i(H_i, \psi_i)^G \in Ker(c_{+,G})$$

and that each image, $\psi_i(H_i)$, is abelian. Then $x = 0$. In particular, $c_{+,G}$ is injective for G abelian.

Proof. Amongst the H_i's which appear in x, choose an H which is maximal in the poset of conjugacy classes of subgroups of G. Then we may write

$$x = \sum_i n_i(H, \psi_i)^G + \sum_j n_j(H_j, \psi_j)^G$$

where the $(H, \psi_i)^G$'s are all distinct and where none of the H_j in the second sum satisfies $H_j \geqslant gHg^{-1}$ for any $g \in G$. Then, if $c(\psi_i) = \phi_i \oplus \overline{\phi}_i$,

$$0 = c_{+,G}(x) = \sum_i n_i((H, \phi_i)^G + (H, \overline{\phi}_i)^G) + \sum_j n_j \mathrm{Ind}^G_{H_j}(a_{H_j}(c(\psi_j))) \in R_+(G)$$

and it is clear that no term from the second sum can cancel any term from the first sum, so that

$$0 = \sum_i n_i((H, \phi_i)^G + (H, \overline{\phi}_i)^G) \in R_+(G).$$

This can only happen if, for distinct i_0, i_1, $(H, \phi_{i_0})^G = (H, \phi_{i_1})^G$ or $(H, \phi_{i_0})^G = (H, \overline{\phi}_{i_1})^G$. In turn, this can only happen if there exists $g \in N_G(H)$ such that $\phi_{i_0} = g^*(\phi_{i_1}) = \phi_{i_1}(g - g^{-1})$ or $\phi_{i_0} = g^*(\overline{\phi}_{i_1})$. Both these relations imply that $\psi_{i_0} = g^*(\psi_{i_1})$ and so $(H, \psi_{i_0})^G = (H, \psi_{i_1})^G$, which is a contradiction. □

3.6. *Example*

We calculate explicitly a_G applied to the complexification of the faithful irreducible symplectic representations Ψ of the binary tetrahedral, octahedral and icosahedral groups (see 3.2).

Using the notation of 3.2, for the binary tetrahedral group B_{24},

$$\begin{aligned}
a_{B_{24}}(c(\Psi)) &= \tfrac{1}{6}(\langle x\rangle,\zeta_4)^{B_{24}} + \tfrac{1}{6}(\langle x\rangle,\overline{\zeta_4})^{B_{24}} + \tfrac{1}{6}(\langle y\rangle,\zeta_4)^{B_{24}} + \tfrac{1}{6}(\langle y\rangle,\overline{\zeta_4})^{B_{24}} \\
&+\tfrac{1}{6}(\langle xy\rangle,\zeta_4)^{B_{24}} + \tfrac{1}{6}(\langle xy\rangle,\overline{\zeta_4})^{B_{24}} + \tfrac{1}{4}(\langle -c\rangle,\zeta_6)^{B_{24}} + \tfrac{1}{4}(\langle -c\rangle,\overline{\zeta_6})^{B_{24}} \\
&+\tfrac{1}{4}(\langle -cx\rangle,\zeta_6)^{B_{24}} + \tfrac{1}{4}(\langle -cx\rangle,\overline{\zeta_6})^{B_{24}} + \tfrac{1}{4}(\langle -cy\rangle,\zeta_6)^{B_{24}} + \tfrac{1}{4}(\langle -cy\rangle,\overline{\zeta_6})^{B_{24}} \\
&+\tfrac{1}{4}(\langle -cxy\rangle,\zeta_6)^{B_{24}} + \tfrac{1}{4}(\langle -cxy\rangle,\overline{\zeta_6})^{B_{24}} + \tfrac{1}{12}(2-(3\cdot 2+4\cdot 2))(\ x^2\ ,\varepsilon)^{B_{24}} \\
&= (C_4,\zeta_4)^{B_{24}} + (C_6,\zeta_6)^{B_{24}} + (C_6,\overline{\zeta_6})^{B_{24}} - (C_2,\varepsilon)^{B_{24}}
\end{aligned}$$

Here, for $n \in \mathbb{N}$, (C_n,ζ_n) denotes a cyclic groups C_n of order n (e.g. $C_4 = \langle x\rangle$) with a faithful unitary representation ζ_n on C_n. We note that all cyclic subgroups are conjugate. For $n = 2$, that is $C_2 = \langle x^2\rangle$, the representation given by $x^2 \mapsto -1$ is denoted ε.

For the binary octahedral group B_{48}

$$a_{B_{48}}(c(\Psi)) = (C_8,\zeta_8)^{B_{48}} + (C_6,\zeta_6)^{B_{48}} + (C_4,\zeta_4)^{B_{48}} - (C_2,\varepsilon)^{B_{48}}\ ,$$

where for example $C_8 = \langle xd\rangle$, $C_6 = \langle x^2c\rangle$, $C_4 = \langle d\rangle$ and $C_2 = \langle x^2\rangle$.

For the two symplectic representations on the binary icosahedral B_{120} we find that

$$a_{B_{120}}(c(\Psi)) = (C_{10},\zeta_{10})^{B_{120}} + (C_6,\zeta_6)^{B_{120}} + (C_4,\zeta_4)^{B_{120}} - (C_2,\varepsilon)^{B_{120}}\ .$$

Here the faithful unitary representation on C_{10} (e.g. $C_{10} = \langle t\rangle$) has to be taken such that its square is the fifth root of unity chosen to define Ψ. For the groups we may pick $C_6 = \langle s\rangle$, $C_4 = \langle r\rangle$ and $C_2 = \langle rst\rangle$.

3.7. *Some elements in $Ker(c_{+,G})$*

As a matter of fact, $c_{+,G}$ is in general not injective. The kernel is generated by elements which we describe now.

Type τ:

If $H \subseteq G$ and $\psi : H \longrightarrow Sp(1)$ has non-abelian image which is isomorphic to a generalised quaternion group, Q_{4n} for some n. Then, by the formula of 2.2,

$$a_H(c(\psi)) = \sum_\alpha m_\alpha (H_\alpha,\phi_\alpha)^H = \sum_\alpha m_\alpha (H_\alpha,\overline{\phi}_\alpha)^H \in R_+(H)$$

in which each image, $\phi_\alpha(H_\alpha)$, is abelian. Also there exists $\psi_\alpha : H_\alpha \longrightarrow Sp(1)$, unique up to $H-Sp(1)$-conjugation, such that $c(\psi_\alpha) = \phi_\alpha \oplus \overline{\phi}_\alpha$. Then, in $R_+(G)$,

$$\begin{aligned}
&c_{+,G}(2(H,\psi)^G - \textstyle\sum_\alpha m_\alpha (H_\alpha,\psi_\alpha)^G) \\
&= 2\textstyle\sum_\alpha m_\alpha (H_\alpha,\phi_\alpha)^G - \sum_\alpha m_\alpha ((H_\alpha,\phi_\alpha)^G + (H_\alpha,\overline{\phi}_\alpha)^G) \\
&= 0.
\end{aligned}$$

Denote by $\tau_G(H,\psi)$ the element

$$\tau_G(H,\psi) = 2(H,\psi)^G - \sum_\alpha m_\alpha(H_\alpha,\psi_\alpha)^G \in Ker(c_{+,G}) \subseteq R_+^{Sp}(G).$$

Notice that each of the images, $\psi_\alpha(H_\alpha)$ is abelian.

Type β:

Let B_{24}, B_{48}, B_{120} denote the binary tetrahedral, octahedral and icosahedral groups, respectively. Let $\Psi_n : B_n \longrightarrow Sp(1)$ denote the faithful representation described in 3.2.

From 3.6, we have an inclusion $Q_8 \subset B_{24}$ under which all C_4's are conjugate. Therefore

$$\mathrm{Ind}_{Q_8}^{B_{24}}(c_{+,Q_8}(Q_8,\Psi)^{Q_8}) = 3(C_4,\zeta_4)^{B_{24}} - (C_2,\epsilon)^{B_{24}}.$$

If $\psi_n : C_n \longrightarrow Sp(1)$ satisfies $c(\psi_n) = \zeta_n \oplus \overline{\zeta_n}$ then, from 3.6,

$$c_{+,B_{24}}((B_{24},\Psi_{24})^{B_{24}}) + c_{+,B_{24}}((C_4,\psi_4)^{B_{24}}) - c_{+,B_{24}}((C_6,\psi_6)^{B_{24}}) =$$

$$\mathrm{Ind}_{Q_8}^{B_{24}}(c_{+,Q_8}(Q_8,\Psi)^{Q_8})$$

and therefore

$$\beta_{24} = (B_{24},\Psi_{24})^{B_{24}} + (C_4,\psi_4)^{B_{24}} - (C_6,\psi_6)^{B_{24}} - (Q_8,\Psi)^{B_{24}} \in Ker(c_{+,B_{24}}).$$

Now consider $B_{48} =< x,y,c,d \mid \ \dots > $ as in 3.2. This case is a little more delicate because there are two conjugacy classes of C_4, namely $C_4 =< d >$ and $C_4' =< y > \subset N =< x,y,c > \triangleleft B_{48}$. We have

$$c_{+,B_{48}}((B_{48},\Psi_{48})^{B_{48}}) = (C_8,\zeta_8)^{B_{48}} + (C_6,\zeta_6)^{B_{48}} + (C_4,\zeta_4)^{B_{48}} - (C_2,\epsilon)^{B_{48}}.$$

Also $Q_{16} =< xd, d >$ and $C_8 =< xd >$ so that

$$c_{+,B_{48}}((Q_{16},\Psi_{16})^{B_{48}}) = (C_8,\zeta_8)^{B_{48}} + (C_4,\zeta_4)^{B_{48}} + (C_4',\zeta_4)^{B_{48}} - (C_2,\epsilon)^{B_{48}}.$$

Furthermore $Q_{12} =< c, d >$ so that

$$c_{+,B_{48}}((Q_{12},\Psi_{12})^{B_{48}}) = (C_6,\zeta_6)^{B_{48}} + 2(C_4,\zeta_4)^{B_{48}} - (C_2,\epsilon)^{B_{48}}.$$

Also we have $Q_8' =< x, y >$ and $Q_8 =< d, y >$ so that

$$c_{+,B_{48}}((Q_8',\Psi_8)^{B_{48}}) = 3(C_4',\zeta_4)^{B_{48}} - (C_2,\epsilon)^{B_{48}}$$

and

$$c_{+,B_{48}}((Q_8,\Psi_8)^{B_{48}}) = 2(C_4,\zeta_4)^{B_{48}} + (C_4',\zeta_4)^{B_{48}} - (C_2,\epsilon)^{B_{48}}.$$

Therefore

$$\beta_{48} = (B_{48},\Psi_{48})^{B_{48}} - (Q_{16},\Psi_{16})^{B_{48}} - (Q_{12},\Psi_{12})^{B_{48}} - (Q_8',\Psi_8)^{B_{48}} \in Ker(c_{+,B_{48}}).$$

Next consider $B_{120} =< r, s, t \mid r^2 = r^3 = t^5 = rst >$. The relations

$$c_{+,B_{120}}((B_{120}, \Psi_{120})^{B_{120}}) \\ = (C_{10}, \zeta_{10})^{B_{120}} + (C_6, \zeta_6)^{B_{120}} + (C_4, \zeta_4)^{B_{120}} - (C_2, \epsilon)^{B_{120}},$$

$$c_{+,B_{120}}((Q_{20} =< r, t >, \Psi_{20})^{B_{120}}) \\ = (C_{10}, \zeta_{10})^{B_{120}} + 2(C_4, \zeta_4)^{B_{120}} - (C_2, \epsilon)^{B_{120}},$$

$$c_{+,B_{120}}((Q_{12} =< r, s >, \Psi_{12})^{B_{120}}) = (C_6, \zeta_6)^{B_{120}} + 2(C_4, \zeta_4)^{B_{120}} - (C_2, \epsilon)^{B_{120}},$$

$$c_{+,B_{120}}((Q_8, \Psi_8)^{B_{120}}) = 3(C_4, \zeta_4)^{B_{120}} - (C_2, \epsilon)^{B_{120}}$$

imply that

$$\beta_{120} = (B_{120}, \Psi_{120})^{B_{120}} - (Q_{20}, \Psi_{20})^{B_{120}} -$$

$$(Q_{12}, \Psi_{12})^{B_{120}} + (Q_8, \Psi_8)^{B_{120}} \in Ker(c_{+,B_{120}}).$$

The elements of $Ker(c_{+,G})$ of Type β are defined to be those of the form $Ind_H^G(\pi^* \beta_n)$ where $\pi : H \longrightarrow B_n$ is a surjective homomorphism and $n = 24, 48$ or 120.

Type σ:

Elements, $\Sigma \in Ker(c_{+,G})$, of Type σ are defined to be those which satisfy a relation of the form

$$2\Sigma = \sum_\alpha \tau_G(H_\alpha, \psi_\alpha).$$

Here are two examples of elements of Type σ.

i) Let

$$Q_{4n} =< x, y \mid x^n = y^2, yxy^{-1} = x^{-1}, y^4 = 1 >$$

denote the generalised quaternion group of order $4n$ and let z have order two. Set $G = Q_{4n} \times \langle z \rangle$ so that G contains four copies of Q_{4n} given by $Q_1 = \langle x, y \rangle$, $Q_2 = \langle xz, y \rangle$, $Q_3 = \langle x, yz \rangle$ and $Q_4 = \langle xz, yz \rangle$. Each of these subgroups has a homomorphism, $\Psi : Q_v \longrightarrow \mathrm{Sp}(1)$, sending the generator xz^s to $e^{\pi i/n}$ and yz^t to j for appropriate s, t. Setting

$$\begin{aligned} \Sigma_n = {} & (Q_1, \Psi)^G + (Q_2, \Psi)^G + (Q_3, \Psi)^G + (Q_4, \Psi)^G + 2(\langle y^2 \rangle, \psi_2)^G \\ & - (\langle x \rangle, \psi_{2n})^G - (\langle y \rangle, \psi_4)^G - (\langle xy \rangle, \psi_4)^G - (\langle xz \rangle, \psi_{2n})^G \\ & - (\langle yz \rangle, \psi_4)^G - (\langle xyz \rangle, \psi_4)^G \end{aligned}$$

we find that $c_+(\Sigma_n) = 0$ and

$$2\Sigma_n = \tau_G(Q_1, \Psi) + \tau_G(Q_2, \Psi) + \tau_G(Q_3, \Psi) + \tau_G(Q_4, \Psi).$$

ii) If z, w have order two set $G = C_2 \ltimes (Q_8 \times < z >) \times < w >$ where $Q_8 =< x, y >$

and the generator, λ, of the left-hand C_2 acts by $\lambda(x) = xz, \lambda(y) = yz$. Then

$$\begin{aligned} \Sigma' &= (\langle x, y\rangle, \Psi)^G + (\langle xz, y\rangle, \Psi)^G + (\langle y^2\rangle, \psi_2)^G - (\langle x\rangle, \psi_4)^G \\ &-(\langle y\rangle, \psi_4)^G - (\langle xy\rangle, \psi_4)^G - (\langle xyz\rangle, \psi_4)^G + (\langle xw, yw\rangle, \Psi)^G \\ &+(\langle xwz, yw\rangle, \Psi)^G + (\langle y^2\rangle, \psi_2)^G - (\langle xw\rangle, \psi_4)^G - (\langle yw\rangle, \psi_4)^G \end{aligned}$$

satisfies $c_+(\Sigma') = 0$ and

$$\begin{aligned} 2\Sigma' &= \tau_G(< \langle x, y\rangle, \Psi) + \tau_G(\langle xz, y\rangle, \Psi) \\ &+\tau_G(\langle xw, yw\rangle, \Psi) + \tau_G(\langle xzw, yw\rangle, \Psi). \end{aligned}$$

Proposition 3.8.
In the notation of 3.7, $Ker(c_{+,G})$ is generated by elements of Types τ, β and σ.

Proof. Suppose that

$$x = \sum_i n_i(H_i, \psi_i)^G \in (Ker(c_{+,G}) : RSp_+(G) \longrightarrow R_+(G)).$$

For all the terms with $n_i \neq 0$ we may subtract a $\mathbb{Z}$-linear combination of the $\tau_G(H, \psi)$'s and $\mathrm{Ind}_H^G(\pi^*(\beta_n))$'s to ensure that either $\psi_i(H_i) \subset Sp(1)$ is abelian or $n_i = 1$ and $\psi_i(H_i)$ is isomorphic to a generalised quaternion group. Under these circumstances write

$$x = \sum_{\psi_i(H_i)\ non-abelian} (H_i, \psi_i)^G + \sum_{\psi_j(H_j)\ abelian} (H_i, \psi_i)^G$$

then

$$2x - \sum_{\psi_i(H_i)\ non-abelian} \tau_G(H_i, \psi_i) = \sum_m a_m(H_m, \psi_m)^G \in Ker(c_{+,G})$$

with every image, $\psi_m(H_m)$ abelian in the right-hand sum. Hence

$$2x - \sum_{\psi_i(H_i)\ non-abelian} \tau_G(H_i, \psi_i) = 0,$$

by Lemma 3.5, and x is of Type σ. □

3.9. *A symplectic induction formula*
Throughout we will identify $R^{Sp}(G) \otimes_{\mathbb{Z}} \mathbb{Q}$ with the $\mathbb{Q}$-vectorspace we get from $R^{Sp}(G)$ by extending scalars, denoted by $\mathbb{Q}R^{Sp}(G)$. Analogously we set $\mathbb{Q}R_+^{Sp}(G) = R_+^{Sp}(G) \otimes_{\mathbb{Z}} \mathbb{Q}$. All homomorphisms on $R^{Sp}(G)$ and $R_+^{Sp}(G)$, especially Res_J^G and Ind_J^G, extend in a natural way to homomorphisms between these $\mathbb{Q}$-vectorspaces. Define the map

$$a_G^{Sp} : \mathbb{Q}R^{Sp}(G) \to \mathbb{Q}R_+^{Sp}(G)$$

by the formula

$$a_G^{Sp}(\rho) = \sum_{(H_0,\Psi_0)<..<(H_r,\Psi_r)} (-1)^r \frac{|H_0|}{|G|} m(\mathrm{Res}^G_{H_r}(\rho), \Psi_r)(H_0, \Psi_0)^G$$

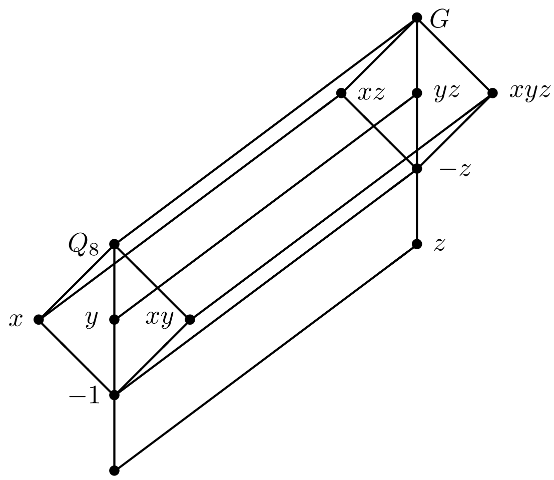
with $m(\theta, \Psi) = \frac{\langle c(\theta);c(\Psi)\rangle_H}{\langle c(\Psi);c(\Psi)\rangle_H}$ for $\theta, \Psi \in R^{Sp}(H)$, $H \leqslant G$. This is a homomorphism since m is linear in the first argument because both, the restriction map and the scalar product on characters, are linear.

3.10. *Example:* $G = C_p \times Q_8$

Let p be an odd prime and let G denote the group

$$C_p \times Q_8 =< x, y, z | x^4 = y^4 = z^p = 1, zx = xz, zy = yz, x^2 = y^2, yx = x^3y > \ .$$

So $G = \{x^i y^j z^k | 0 \leqslant i \leqslant 3, 0 \leqslant j \leqslant 1, 0 \leqslant k \leqslant p-1\}$ is a group of order $8p$. To shorten notation set $-1 := x^2$. The lattice of subgroups of G is as pictured.



Let ζ denote a primitive root of unity of order p. We calculate the explicit induction formula for the symplectic irreducible representation $\rho : G \to \mathrm{Sp}(2)$ given by sending x to $\begin{matrix} i & 0 \\ 0 & i \end{matrix}$, y to $\begin{matrix} j & 0 \\ 0 & j \end{matrix}$ and z to $\begin{matrix} \frac{\zeta+\zeta^{-1}}{2} & -i\frac{\zeta-\zeta^{-1}}{2} \\ i\frac{\zeta-\zeta^{-1}}{2} & \frac{\zeta+\zeta^{-1}}{2} \end{matrix}\Bigg)$. Furthermore, ρ is irreducible because $c(\rho)$ is a sum of two 2-dimensional representations, whose characters χ and χ' are not real-valued.

To calculate the formula we have to study the restriction of ρ on the subgroups H of G. In the following table we give all the nonzero multiplicities $m = m(\mathrm{Res}^G_H(\rho), \Psi)$ for pairs $H, c(\Psi)$:

$\langle xz\rangle, i\zeta+\overline{i\zeta}$	$\langle xz\rangle, \text{-}i\zeta+\overline{\text{-}i\zeta}$	$\langle\text{-}z\rangle, \text{-}\zeta+\overline{\text{-}\zeta}$	$\langle z\rangle, \zeta+\overline{\zeta}$	Q_8, χ	$\langle x\rangle, i+\overline{i}$	$\langle\text{-}1\rangle, 2\varepsilon$	$1, 2\mathbb{1}$
$\frac{2}{2}$	$\frac{2}{2}$	$\frac{4}{2}$	$\frac{4}{2}$	$\frac{2}{1}$	$\frac{4}{2}$	$\frac{8}{4}$	$\frac{8}{4}$

Here $\pm i$ (resp. $\pm\zeta$) denotes a complex linear representation sending the generator to $\pm i$ (resp. $\pm\zeta$), χ the faithful irreducible representation on Q_8 and ε the faithful linear representation on a group of order 2. In the list we have taken $\langle xz\rangle$ resp. $\langle x\rangle$ as representatives for the groups of order $4p$ resp. 4, as the other groups behave in the same way. Observe furthermore that the representations $i\zeta{+}\overline{i\zeta}$ and $\bar{i}\zeta{+}\overline{\bar{i}\zeta}$ on $\langle xz\rangle$ are conjugate by G. Now

$$\begin{aligned} a_G^{Sp}(\rho) = {} & \tfrac{1}{2}(\tfrac{2}{2})(\langle xz\rangle, i\zeta{+}\overline{i\zeta})^G + \tfrac{1}{2}(\tfrac{2}{2})(\langle xz\rangle, \text{-}i\zeta{+}\overline{\text{-}i\zeta})^G + \tfrac{1}{2}(\tfrac{2}{2})(\langle yz\rangle, i\zeta{+}\overline{i\zeta})^G \\ & + \tfrac{1}{2}(\langle yz\rangle, \text{-}i\zeta{+}\overline{\text{-}i\zeta})^G \tfrac{2}{2} + \tfrac{1}{2}(\tfrac{2}{2})(\langle xyz\rangle, i\zeta{+}\overline{i\zeta})^G + \tfrac{1}{2}(\tfrac{2}{2})(\langle xyz\rangle, \text{-}i\zeta{+}\overline{\text{-}i\zeta})^G \\ & + \tfrac{1}{4}(\tfrac{4}{2} - (3\tfrac{2}{2} + 3\tfrac{2}{2}))(\langle \text{-}z\rangle, \text{-}\zeta{+}\overline{\text{-}\zeta})^G + \tfrac{1}{8}(\tfrac{4}{2} - (3\tfrac{2}{2} + 3\tfrac{2}{2} + \tfrac{4}{2}) \\ & + (3\tfrac{2}{2} + 3\tfrac{2}{2}))(\langle z\rangle, \zeta{+}\bar{\zeta})^G + \tfrac{1}{p}(\tfrac{2}{1})(Q_8, \chi)^G + \tfrac{1}{2p}(\tfrac{4}{2} - (\tfrac{2}{2} + \tfrac{2}{2} + \tfrac{2}{1}))(\langle x\rangle, i{+}\bar{i})^G \\ & + \tfrac{1}{2p}(\tfrac{4}{2} - (\tfrac{2}{2} + \tfrac{2}{2} + \tfrac{2}{1}))(\langle y\rangle, i{+}\bar{i})^G + \tfrac{1}{2p}(\tfrac{4}{2} - (\tfrac{2}{2} + \tfrac{2}{2} + \tfrac{2}{1}))(\langle xy\rangle, i{+}\bar{i})^G \\ & + \tfrac{1}{4p}(\tfrac{8}{4} - (3\tfrac{2}{2} + 3\tfrac{2}{2} + \tfrac{4}{2} + \tfrac{2}{1} + 3\tfrac{4}{2}) + (3 \cdot 2\tfrac{2}{2} + 3 \cdot 2\tfrac{2}{2} + 3\tfrac{2}{1}))(\langle \text{-}1\rangle, 2\varepsilon)^G \\ & + \tfrac{1}{8p}(\tfrac{8}{4} - (3\tfrac{2}{2} + 3\tfrac{2}{2} + \tfrac{4}{2} + \tfrac{4}{2} + \tfrac{2}{1} + 3\tfrac{4}{2} + \tfrac{8}{4}) + (3{\cdot}4\tfrac{2}{2} + 3{\cdot}4\tfrac{2}{2} + 2\tfrac{4}{2} + 4\tfrac{2}{1} + 3\tfrac{4}{2}) \\ & - (3^2\tfrac{2}{2} + 3^2\tfrac{2}{2} + 3\tfrac{2}{1}))(\langle 1\rangle, 2\mathbb{1})^G \\ = {} & (\langle xz\rangle, i\zeta{+}\overline{i\zeta})^G + (\langle yz\rangle, i\zeta{+}\overline{i\zeta})^G + (\langle xyz\rangle, i\zeta{+}\overline{i\zeta})^G - (\langle \text{-}z\rangle, \text{-}\zeta{+}\overline{\text{-}\zeta})^G \\ & + \tfrac{2}{p}(Q_8, \chi)^G - \tfrac{1}{p}(\langle x\rangle, i{+}\bar{i})^G - \tfrac{1}{p}(\langle y\rangle, i{+}\bar{i})^G - \tfrac{1}{p}(\langle xy\rangle, i{+}\bar{i})^G + \tfrac{1}{p}(\langle \text{-}1\rangle, 2\varepsilon)^G \end{aligned}$$

As this example shows, the symplectic induction formula may have non-integral coefficients.

Example 3.11. $G = C_{2^n} \times Q_8$

Let $n \in \mathbb{N}$, $Z =< z >$ the cyclic group of order 2^n and $Q_8 =< x, y >$ the quaternion group of order 8, as before. Let $G_n = Z \times Q_8$ be the direct product of these groups and $\rho_n : G \to \mathrm{Sp}(2)$ the symplectic representation given by sending x to $\begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}$, y to $\begin{pmatrix} j & 0 \\ 0 & j \end{pmatrix}$ and z to $\begin{pmatrix} \frac{\zeta+\zeta^{-1}}{2} & i\frac{\zeta-\zeta^{-1}}{2} \\ -i\frac{\zeta-\zeta^{-1}}{2} & \frac{\zeta+\zeta^{-1}}{2} \end{pmatrix}$, ζ a primitive 2^n-th root of unity. This is an irreducible representation for $n \geqslant 2$, and for $n = 1$ it is twice the one-dimensional symplectic representation Ψ defined by $x \mapsto i$, $y \mapsto j$ and $z \mapsto -1$.

Now let $n \geqslant 2$ and $G = G_n$, and let H_t be the subgroup generated by x,y and $z^{2^{n-t}}$, $0 \leqslant t \leqslant n$. Notice that $H_1 \cong G_1$ is normal in G_n with cyclic factor group, all the intermediate groups are the groups H_t $(1 \leqslant t \leqslant n-1)$, being isomorphic to G_t. Thus $\mathrm{Res}^{G_n}_{H_t}(\rho)$ stays irreducible for $t \geqslant 2$, while $\mathrm{Res}^G_{H_1}(\rho) = 2\Psi$. Hence the coefficient of $(H_1, \Psi)^G$ in $a_G^{Sp}(\rho)$ is $\frac{8\cdot 2}{8\cdot 2^n} \cdot \frac{2}{1} = \frac{1}{2^{n-2}}$.

As this example shows, the denominators of the coefficients in the symplectic induction formula may contain arbitrarily large 2-powers.

Proposition 3.12.
The homomorphism a_G^{Sp} is natural with respect to restriction: for $J \leqslant G$

$$a_J^{Sp} \circ \mathrm{Res}^G_J = \mathrm{Res}^G_J \circ a_G^{Sp} : \mathbb{Q}R^{Sp}(G) \to \mathbb{Q}R_+^{Sp}(J)\ .$$

Proof. The statement follows immediately from general results of Boltje ([**6**, Prop.1.4.1(ii)],[**7**, Prop.5.2(ii)] and [**6**, Prop.1.3.2],[**7**, Prop.2.4]) concerning an adjointness property of certain functors between conjugation functors and restriction functors on G, combined with certain isomorphisms. Let $(p_H)_{H\leqslant G}$ with $p_H : \mathbb{Q}R^{Sp}(H) \to \mathbb{Q}L^{Sp}(H)$ sending $\rho : H \to \mathrm{Sp}(n)$ to $\sum_{\Psi : H \to \mathrm{Sp}(1)} m(\rho, \Psi)\Psi$. Since those maps commute with conjugation maps, $(p_H)_{H\leqslant G}$ is a morphism between conjugation functors on G. According [**7**, Prop.5.2(ii)], this gives rise to a morphism $(r_H)_{H\leqslant G}$ of restriction functors on G with r_H mapping $\rho : H \to \mathrm{Sp}(n)$ to $(p_K(\mathrm{Res}^H_K(\rho)))_{K\leqslant H}$. Composed with the isomorphism

$$(a_K)_{K\leqslant H} \mapsto \sum_{L\leqslant K\leqslant H} \frac{|L|}{|H|}\mu(L:K)(L, \mathrm{Res}^K_L(a_K))^H$$

(see [**7**, 2.3a]) this turns out to be a morphism (denoted $(\widetilde{a}_H)_{H\leqslant G}$) of restriction functors on G with $\widetilde{a}_H$ sending $\rho : H \to \mathrm{Sp}(n)$ to

$$\begin{aligned}
\rho \quad &\mapsto \textstyle\sum_{L\leqslant K\leqslant H} \frac{|L|}{|H|}\mu(L:K)(L, \mathrm{Res}^K_L(\sum_{\Psi:K\to\mathrm{Sp}(1)} m(\mathrm{Res}^H_K(\rho),\Psi)\Psi))^H \\
&= \textstyle\sum_{L\leqslant K\leqslant H} \frac{|L|}{|H|}\mu(L:K)(L, \sum_{\Psi:K\to\mathrm{Sp}(1)} \; m(\mathrm{Res}^H_K(\rho),\Psi)\mathrm{Res}^K_L(\Psi))^H \\
&= \textstyle\sum_{L\leqslant K\leqslant H} \frac{|L|}{|H|}\mu(L:K)\sum_{\Psi:K\to\mathrm{Sp}(1)} \; m(\mathrm{Res}^H_K(\rho),\Psi)(L, \mathrm{Res}^K_L(\Psi))^H \\
&= \textstyle\sum_{(K_0,\Psi_0)\leqslant(K,\Psi)\leqslant H} \; \frac{|K_0|}{|H|}\mu(K_0:K)m(\mathrm{Res}^H_K(\rho),\Psi)(K_0,\Psi_0)^H \\
&= a^{Sp}_H(\rho)\ .
\end{aligned}$$

Thus $(a^{Sp}_H)_{H\leqslant G}$ is a morphism of restriction functors, in particular a^{Sp}_G natural with respect to restriction. □

Proposition 3.13.
Let $\rho : G \to \mathrm{Sp}(1)$ then $a^{Sp}_G(\rho) = (G,\rho)^G$.

Proof. Since (G,ρ) is the only element in $(G,\rho)^G$ and $m(\rho,\rho) = 1$, the coefficient of $(G,\rho)^G$ in $a^{Sp}_G(\rho)$ is 1. Now let $(H,\Psi) < (G,\rho)$. Only those elements may give other nontrivial contributions to $a^{Sp}_G(\rho)$. Since $\mathrm{Res}^G_{H_r}(\rho) = \Psi_r$ for $(H_r,\Psi_r) < (G,\rho)$, the multiplicities turn out to be 1. Thus we have to show that

$$\sum_{\substack{(H,\Psi)<(H_1,\Psi_1)<\ldots<(H_r,\Psi_r)\\(H_r,\Psi_r)\leqslant(G,\rho)}} (-1)^r = 0\ .$$

Consider the set, $\mathcal{R}$, of chains which the sum runs over. Let $\mathcal{P} < \mathcal{R}$ denote the subset of those chains which will not end in (G,ρ). Then

$$(\,(H,\Psi) < \ldots < (H_r,\Psi_r)\,) \;\mapsto\; (\,(H,\Psi) < \ldots < (H_r,\Psi_r) < (G,\rho)\,)$$

gives a bijection $\mathcal{P} \to \mathcal{R}\setminus\mathcal{P}$, where chains of length r are in correspondence to chains of length $r+1$. Since the terms cancel in pairs the sum above equals 0. □

Proposition 3.14.
The homomorphism $a^{Sp}_G : \mathbb{Q}R^{Sp}(G) \to \mathbb{Q}R^{Sp}_+(G)$ is the only homomorphism which is natural with respect to restriction and satisfies Prop. 3.13 when ρ is one-dimensional.

The proof is similar to the one of theorem 2.2.15 in [**41**].

Proposition 3.15.
The homomorphism a_G^{Sp} is natural with respect to inflation so that, for $N \lhd G$,

$$a_G^{Sp} \circ \mathrm{Infl}_{G/N}^G = \mathrm{Infl}_{G/N}^G \circ a_{G/N}^{Sp} : \mathbb{Q}R^{Sp}(G/N) \to \mathbb{Q}R_+^{Sp}(G)$$

Proof. We have to show that the coefficients of a basis element $(H, \Psi)^G$ in $a_G^{Sp}(\mathrm{Infl}_{G/N}^G(\overline{\rho}))$ and $\mathrm{Infl}_{G/N}^G(a_{G/N}^{Sp}(\overline{\rho}))$ for $\rho : G \to \mathrm{Sp}(n)$ with $N \leqslant \ker \rho$ coincide. Since all maps are morphisms, we can assume that ρ is irreducible. In case $\rho : G \to \mathrm{Sp}(1)$ the statement follows directly from 3.13.

Now we assume $n \geqslant 2$ and argue using induction on $|G|$. If $H = G$, the formula 3.9 tells that the coefficient of $(G, \Psi)^G$ in $a_G^{Sp}(\mathrm{Infl}_{G/N}^G(\overline{\rho}))$ is zero, as $\rho = \mathrm{Infl}_{G/N}^G(\overline{\rho})$ is irreducible, and on the other hand the coefficient of $(G, \Psi)^G$ in $\mathrm{Infl}_{G/N}^G(a_{G/N}^{Sp}(\overline{\rho}))$ also vanishes, because

$$\mathrm{Infl}_{G/N}^G\Big(\sum_{(HN/N,\overline{\Psi})^{G/N}} C_{(HN/N,\overline{\Psi})^{G/N}} (HN/N, \overline{\Psi})^{G/N}\Big)$$

contributes to this coefficient only from the base elements with $HN = G$, and for those the coefficient $C_{(G/N,\overline{\Psi})^{G/N}} = 0$ since $\overline{\Psi}$ is irreducible.

Now we suppose $H < G$ and use induction on $[G : H]$. The coefficient of $(H, \Psi)^H$ in $\mathrm{Res}_H^G(\sum_{(H_0,\Psi_0)^G} C_{(H_0,\Psi_0)^G} (H_0, \Psi_0)^G)$ is determined by $C_{(H_0,\Psi_0)^G}$ with $(H, \Psi)^G \leqslant (H_0, \Psi_0)^G$. Since by induction on $[G : H]$ we know that the coefficients coincide for $(H, \Psi)^G < (H_0, \Psi_0)^G$, it suffices to prove that the coefficient of $(H, \Psi)^H$ in $\mathrm{Res}_H^G(a_G^{Sp}(\mathrm{Infl}_{G/N}^G(\overline{\rho})))$ is the same as of $\mathrm{Res}_H^G(\mathrm{Infl}_{G/N}^G(a_{G/N}^{Sp}(\overline{\rho})))$. But in fact, with $\iota_{HN/N}^{H/(N\cap H)}$ denoting the isomorphism induced by the canonical group isomorphism $HN/N \cong H/(N \cap H)$,

$$\begin{aligned}
\mathrm{Res}_H^G(a_G^{Sp}(\mathrm{Infl}_{G/N}^G(\overline{\rho}))) &= a_H^{Sp}(\mathrm{Res}_H^G(\mathrm{Infl}_{G/N}^G(\overline{\rho}))) \\
&= a_H^{Sp}(\mathrm{Infl}_{H/(H\cap N)}^H(\iota_{HN/N}^{H/N\cap H}(\mathrm{Res}_{HN/N}^{G/N}(\overline{\rho})))) \\
&= \mathrm{Infl}_{H/(H\cap N)}^H(a_{H/(H\cap N)}^{Sp}(\iota_{HN/N}^{H/N\cap H}(\mathrm{Res}_{HN/N}^{G/N}(\overline{\rho})))) \\
&= \mathrm{Infl}_{H/(H\cap N)}^H(\iota_{HN/N}^{H/N\cap H}(a_{HN/N}^{Sp}(\mathrm{Res}_{HN/N}^{G/N}(\overline{\rho})))) \\
&= \mathrm{Infl}_{H/(H\cap N)}^H(\iota_{HN/N}^{H/N\cap H}(\mathrm{Res}_{HN/N}^{G/N}(a_{G/N}^{Sp}(\overline{\rho})))) \\
&= \mathrm{Res}_H^G(\mathrm{Infl}_{G/N}^G(a_{G/N}^{Sp}(\overline{\rho}))) \, .
\end{aligned}$$

We have used the induction hypothesis on G/N which is of smaller order than G, except in the trivial case $N = 1$. □

Definition 3.16. Let $\rho : G \longrightarrow \mathrm{Sp}(n)$ be a symplectic representation of a finite group, G. The *centre of* ρ, $Z(\rho)$, is the maximal subgroup H such that $\mathrm{Res}_H^G(\rho) = n\chi$ for some homomorphism of the form $\chi : H \longrightarrow \{\pm 1\} \subset \mathrm{Sp}(1)$. Since $\{\pm 1\}$ is central of Sp(1) it is easy to see that such a maximal $Z(\rho)$ exists and is unique (cf. [**41**] Corollary 2.2.40).

Proposition 3.17.
In the notation of Definition 3.16, the coefficient of $(H,\Psi)^G$ in $a_G^{Sp}(\rho)$ is zero unless $(Z(\rho),\chi) \leqslant (H,\Psi)$.

Proof. Since a_G^{Sp} commutes with inflation by 3.15 we may assume that ρ is injective. Recall the formula for $a_G^{Sp}(\rho)$

$$a_G^{Sp}(\rho) = \sum_{(H_0,\Psi_0)<..<(H_r,\Psi_r)} (-1)^r \frac{|H_0|}{|G|} \frac{\left\langle c(\mathrm{Res}^G_{H_r}(\rho)); c(\Psi_r)\right\rangle_{H_r}}{\langle c(\Psi_r); c(\Psi_r)\rangle_{H_r}} (H_0,\Psi_0)^G \in \mathbb{Q}R_+^{Sp}(G).$$

If H contains $Z(\rho)$ then $(Z(\rho),\chi) \leqslant (H,\Psi)$ because $\mathrm{Res}^G_{Z(\rho)}(\rho) = n\chi$. Therefore we must show that the coefficient of $(H,\Psi)^G$ is zero when H does not contain $Z(\rho)$. In this case $Z(\rho)$ is not trivial and we may choose $g \in Z(\rho)$ such that $\chi(g) = -1$, since ρ (and hence χ) is injective on $Z(\rho)$.

If $g \notin \widetilde{H}$ and $\widetilde{\Psi} : \widetilde{H} \longrightarrow \mathrm{Sp}(1)$ is a homomorphism then there exists a unique extension, $\widehat{\Psi}$, of $\widetilde{\Psi}$ to $\widehat{H} =< \widetilde{H}, g >$ such that $\widehat{\Psi}(g) = \chi(g) = -1$. Now consider the set, $\mathcal{R}$, of chains $(H,\Psi) < (H_1,\Psi_1) < .. < (H_r,\Psi_r)$ appearing in the formula with $\left\langle c(\mathrm{Res}^G_{H_r}(\rho)); c(\Psi_r)\right\rangle_{H_r}$ non-zero. Let $\mathcal{P} \subset \mathcal{R}$ denote the subset consisting of those chains for which no $H_i = \hat{H}_{i-1}$ and $H_1 \neq \hat{H}$. For each chain in $\mathcal{P}$ there exists a smallest integer, j, such that $g \notin H_{j-1}$ but $g \in H_j$. If there is no such H_j we set $j = r+1$. For each such chain we have $(H_{j-1},\Psi_{j-1}) < (\widehat{H}_{j-1},\widehat{\Psi}_{j-1}) < (H_j,\Psi_j)$ or $(H_r,\Psi_r) < (\hat{H}_r,\widehat{\Psi}_r)$ if $j = r+1$. Furthermore, when $j = r+1$ the multiplicities of $\widehat{\Psi}_r$ and Ψ_r in $\mathrm{Res}^G_{Z(\rho)}(\rho)$ are equal and hence non-zero.

Associating to each chain in $\mathcal{P}$ the unique chain obtained by interpolating $(\widehat{H}_{j-1},\widehat{\Psi}_{j-1})$ gives a multiplicity-preserving bijection between chains of length r in $\mathcal{P}$ and length $r+1$ in $\mathcal{R}\setminus\mathcal{P}$. This bijection shows that the terms in the coefficient of $(H,\psi)^G$ cancel in pairs, as required. ☐

Proposition 3.18.
The homomorphisms a_G and a_G^{Sp} are connected via complexification, that is

$$a_G \circ c_G = c_{+,G} \circ a_G^{Sp} : \mathbb{Q}R^{Sp}(G) \to \mathbb{Q}R_+(G)\ .$$

Proof. Let $\rho : G \to \mathrm{Sp}(n)$ and let $(J,\phi)^G$ be a base element of $\mathbb{Q}R_+(G)$. We have to show that the coefficients of $(J,\phi)^G$ in $a_G(c(\rho))$ and $c_+(a_G^{Sp}(\rho))$ coincide. The first one is easy to express, namely

$$\frac{|J|}{|G|} \sum_{(J_0,\phi_0)\in(J,\phi)^G} \sum_{(J_0,\phi_0)<..<(J_r,\phi_r)} (-1)^r \left\langle \mathrm{Res}^G_{J_r}(c(\rho)); \phi_r\right\rangle_{J_r}\ .$$

Now we calculate the coefficient, denoted C, of $(J,\phi)^G$ in

$$c_+(a_G^{Sp}(\rho)) = \sum_{(H_0,\Psi_0)<..<(H_t,\Psi_t)} (-1)^t \frac{|H_0|}{|G|} m(\mathrm{Res}^G_{H_t}(\rho),\Psi_t)\, c_+((H_0,\Psi_0)^G)\ .$$

Only those summands will have a contribution to C which have a non-zero term $(J,\phi)^G$ in $c_+((H_0,\Psi_0)^G)$. If $c(\Psi_0)=\phi_0+\overline{\phi_0}$, then $c_+((H_0,\Psi_0)^G)=(H_0,\phi_0)^G+(H_0,\overline{\phi_0})^G$, and this can only contributes to C, if $(H_0,\phi_0)^G=(J,\phi)^G$ or $(H_0,\overline{\phi_0})^G=(J,\phi)^G$. If $c(\Psi_0)=\psi_0$ is irreducible, then

$$c_+((H_0,\Psi_0)^G)=\mathrm{Ind}_{H_0}^G(a_{H_0}(\psi_0))=$$

$$\sum_{(J_0,\phi_0)<..<(J_t,\phi_t)}(-1)^t\frac{|J_0|}{|H_0|}\left\langle \mathrm{Res}_{J_t}^{H_0}(\psi_0);\phi_t\right\rangle_{J_t}(J_0,\phi_0)^G\,,$$

and this contributes

$$\frac{|J|}{|H_0|}\sum_{(J_0,\phi_0)\in(J,\phi)^G}\ \sum_{\substack{(J_0,\phi_0)<..<(J_t,\phi_t)\\ J_t\leqslant H_0}}(-1)^t\left\langle \mathrm{Res}_{J_t}^{H_0}(\psi_0);\phi_t\right\rangle_{J_t}.$$

Therefore, C can be expressed as

$$\begin{aligned}
C &= \sum_{\substack{(H_0,\Psi_0)<..<(H_t,\Psi_t)\\ c(\Psi_0)=\phi_0+\overline{\phi_0}\,,\,(H_0,\phi_0)\mathrm{xor}(H_0,\overline{\phi_0})\in(J,\phi)^G}}(-1)^t\tfrac{|H_0|}{|G|}m(\mathrm{Res}_{H_t}^G(\rho),\Psi_t)\\
&+\sum_{\substack{(H_0,\Psi_0)<..<(H_t,\Psi_t)\\ c(\Psi_0)=\phi_0+\overline{\phi_0}\,,\,(H_0,\phi_0)\mathrm{and}(H_0,\overline{\phi_0})\in(J,\phi)^G}}(-1)^t\tfrac{|H_0|}{|G|}2m(\mathrm{Res}_{H_t}^G(\rho),\Psi_t)\\
&+\sum_{\substack{(H_0,\Psi_0)<..<(H_t,\Psi_t)\\ c(\Psi_0)=\psi_0\ \mathrm{irred.}}}(-1)^t\tfrac{|H_0|}{|G|}m(\mathrm{Res}_{H_t}^G(\rho),\Psi_t)\times\\
&\qquad\left(\tfrac{|J|}{|H_0|}\sum_{\substack{(J_0,\phi_0)<..<(J_r,\phi_r)\\ (J_0,\phi_0)\in(J,\phi)^G\,,\,J_r<H_0}}(-1)^r\left\langle \mathrm{Res}_{J_r}^{H_0}(\psi_0);\phi_r\right\rangle_{J_r}\right)
\end{aligned}$$

Note that in each summand the factor $\frac{|J|}{|G|}$ appears and thus can be factored out. We expand the three subsums of C according to the type of decomposition of $c(\Psi_i)$, namely $c(\Psi_i)=\phi_i+\overline{\phi_i}$ or $c(\Psi_i)=\psi_i$ irreducible. The first subsum

$$\sum_{\substack{(H_0,\Psi_0)\,,\,c(\Psi_0)=\phi_0+\overline{\phi_0}\\ (H_0,\phi_0)\in(J,\phi)^G\,,\,(H_0,\overline{\phi_0})\notin(J,\phi)^G}}\ \sum_{(H_0,\Psi_0)<..<(H_t,\Psi_t)}(-1)^t m(\mathrm{Res}_{H_t}^G(\rho),\Psi_t)$$

splits into C_1+C_2 with

$$\begin{aligned}
C_1 &= \sum_{\substack{(H_0,\Psi_0)\,,\,c(\Psi_0)=\phi_0+\overline{\phi_0}\\ (H_0,\phi_0)\in(J,\phi)^G\,,\,(H_0,\overline{\phi_0})\notin(J,\phi)^G}}\ \sum_{\substack{(H_0,\Psi_0)<..<(H_t,\Psi_t)\\ c(\Psi_t)=\phi_t+\overline{\phi_t}}}(-1)^t m(\mathrm{Res}_{H_t}^G(\rho),\Psi_t)\\
&= \sum_{\substack{(H_0,\phi_0)\in(J,\phi)^G\\ (H_0,\overline{\phi_0})\notin(J,\phi)^G}}\ \sum_{(H_0,\phi_0)<..<(H_t,\phi_t)}(-1)^t\langle \mathrm{Res}_{H_t}^G(c(\rho));\phi_t\rangle_{H_t}
\end{aligned}$$

because $\langle c(\Psi_t);c(\Psi_t)\rangle_{H_t}=2$, $\langle \mathrm{Res}_{H_t}^G(c(\rho));\phi_r\rangle_{H_t}=\langle \mathrm{Res}_{H_t}^G(c(\rho));\overline{\phi_r}\rangle_{H_t}$ and

$$
\begin{aligned}
C_2 = & \sum_{\substack{(H_0,\Psi_0)\,,\,c(\Psi_0)=\phi_0+\overline{\phi_0}\\ (H_0,\phi_0)\in(J,\phi)^G\,,\,(H_0,\overline{\phi_0})\notin(J,\phi)^G}} \sum_{\substack{(H_0,\Psi_0)<..<(H_t,\Psi_t)\\ c(\Psi_t)=\psi_t}} (-1)^t m(\mathrm{Res}^G_{H_t}(\rho),\Psi_r) \\
= & \sum_{\substack{(H_0,\Psi_0)\,,\,c(\Psi_0)=\phi_0+\overline{\phi_0}\\ (H_0,\phi_0)\in(J,\phi)^G\,,\,(H_0,\overline{\phi_0})\notin(J,\phi)^G}} \sum_{\substack{(H_0,\Psi_0)<..<(H_r,\Psi_r)\\ c(\Psi_r)=\phi_r+\overline{\phi_r}}} \\
& \sum_{\substack{(H_{r+1},\Psi_{r+1})<..<(H_t,\Psi_t)\\ (H_r,\Psi_r)<(H_{r+1},\Psi_{r+1})\,,\,c(\Psi_r+1)=\psi_{r+1}}} (-1)^t m(\mathrm{Res}^G_{H_t}(\rho),\Psi_t) \\
= & \sum_{\substack{(J_0,\phi_0)<..<(J_r,\phi_r)\\ (J_0,\phi_0)\in(J,\phi)^G\,,\,(J_0,\overline{\phi_0})\notin(J,\phi)^G}} \\
& \sum_{\substack{(H_{r+1},\Psi_{r+1})<..<(H_t,\Psi_t)\\ J_r<H_{r+1}\,,\,c(\Psi_{r+1})=\psi_{r+1}}} (-1)^t \left\langle \mathrm{Res}^{H_{r+1}}_{J_r}(\psi_{r+1});\phi_r\right\rangle_{J_r} m(\mathrm{Res}^G_{H_t}(\rho),\Psi_t).
\end{aligned}
$$

The second subsum

$$
\sum_{\substack{(H_0,\Psi_0)\,,\,c(\Psi_0)=\phi_0+\overline{\phi_0}\\ (H_0,\phi_0)\in(J,\phi)^G \text{ and } (H_0,\overline{\phi_0})\in(J,\phi)^G}} \sum_{(H_0,\Psi_0)<..<(H_t,\Psi_t)} (-1)^t 2\cdot m(\mathrm{Res}^G_{H_t}(\rho),\Psi_t)
$$

splits into $C_3 + C_4$ with

$$
\begin{aligned}
C_3 &= \sum_{\substack{(H_0,\Psi_0)\,,\,c(\Psi_0)=\phi_0+\overline{\phi_0}\\ (H_0,\phi_0)\in(J,\phi)^G\,,\,(H_0,\overline{\phi_0})\in(J,\phi)^G}} \sum_{\substack{(H_0,\Psi_0)<..<(H_t,\Psi_t)\\ c(\Psi_t)=\phi_t+\overline{\phi_t}}} (-1)^t 2\cdot m(\mathrm{Res}^G_{H_t}(\rho),\Psi_t) \\
&= \sum_{\substack{(H_0,\phi_0)\in(J,\phi)^G\\ \text{with } (H_0,\overline{\phi_0})\in(J,\phi)^G}} \sum_{(H_0,\phi_0)<..<(H_t,\phi_t)} (-1)^t \left\langle \mathrm{Res}^G_{H_t}(c(\rho));\phi_t\right\rangle_{H_t},
\end{aligned}
$$

and

$$
\begin{aligned}
C_4 = & \sum_{\substack{(H_0,\Psi_0)\,,\,c(\Psi_0)=\phi_0+\overline{\phi_0}\\ (H_0,\phi_0)\in(J,\phi)^G\,,\,(H_0,\overline{\phi_0})\in(J,\phi)^G}} \sum_{\substack{(H_0,\Psi_0)<..<(H_t,\Psi_t)\\ c(\Psi_t)=\psi_t}} (-1)^t 2\cdot m(\mathrm{Res}^G_{H_t}(\rho),\Psi_r) \\
= & \sum_{\substack{(H_0,\Psi_0)\,,\,c(\Psi_0)=\phi_0+\overline{\phi_0}\\ (H_0,\phi_0)\in(J,\phi)^G\,,\,(H_0,\overline{\phi_0})\in(J,\phi)^G}} \sum_{\substack{(H_0,\Psi_0)<..<(H_r,\Psi_r)\\ c(\Psi_r)=\phi_r+\overline{\phi_r}}} \\
& \sum_{\substack{(H_{r+1},\Psi_{r+1})<..<(H_t,\Psi_t)\\ (H_r,\Psi_r)<(H_{r+1},\Psi_{r+1})\,,\,c(\Psi_r+1)=\psi_{r+1}}} (-1)^t 2\cdot m(\mathrm{Res}^G_{H_t}(\rho),\Psi_t) \\
= & \sum_{\substack{(J_0,\phi_0)<..<(J_r,\phi_r)\\ (J_0,\phi_0)\in(J,\phi)^G \text{ and } (J_0,\overline{\phi_0})\in(J,\phi)^G}} \\
& \sum_{\substack{(H_{r+1},\Psi_{r+1})<..<(H_t,\Psi_t)\\ J_r<H_{r+1}\,,\,c(\Psi_{r+1})=\psi_{r+1}}} (-1)^t \left\langle \mathrm{Res}^{H_{r+1}}_{J_r}(\psi_{r+1});\phi_r\right\rangle_{J_r} m(\mathrm{Res}^G_{H_t}(\rho),\Psi_t).
\end{aligned}
$$

In fact, for the calculation of C_3 we notice that if $\phi_0 = \overline{\phi_0}$ then either in case $\phi_t = \overline{\phi_t}$ the factor 2 cancels because of

$$2 \cdot m(\mathrm{Res}^G_{H_t}(\rho), \Psi_t) = 2\frac{\left\langle \mathrm{Res}^G_{H_t}(c(\rho)); 2\phi_t \right\rangle_{H_t}}{\langle 2\phi_t; 2\phi_t \rangle_{H_t}} = \left\langle \mathrm{Res}^G_{H_t}(c(\rho)); \phi_t \right\rangle_{H_t},$$

or in case $\phi_t \neq \overline{\phi_t}$ each chain $(H_0, \Psi_0) < .. < (H_t, \Psi_t)$ yields two chains $(H_0, \phi_0) < .. < (H_t, \phi_t)$ and

$$m(\mathrm{Res}^G_{H_t}(\rho), \Psi_t) = \frac{\left\langle \mathrm{Res}^G_{H_t}(c(\rho)); \phi_t + \overline{\phi_t} \right\rangle_{H_t}}{\left\langle \phi_t + \overline{\phi_t}; \phi_t + \overline{\phi_t} \right\rangle_{H_t}} = \left\langle \mathrm{Res}^G_{H_t}(c(\rho)); \phi_t \right\rangle_{H_t}.$$

Also if $\phi_0 \neq \overline{\phi_0}$ then the factor 2 vanishes because each chain

$$(H_0, \Psi_0) < .. < (H_t, \Psi_t)$$

yields two chains $(H_0, \phi_0) < .. < (H_t, \phi_t)$ starting in either (H_0, ϕ_0) or $(H_0, \overline{\phi_0})$, and again

$$m(\mathrm{Res}^G_{H_t}(\rho), \Psi_t) = \frac{\left\langle \mathrm{Res}^G_{H_t}(c(\rho)); \phi_t + \overline{\phi_t} \right\rangle_{H_t}}{\left\langle \phi_t + \overline{\phi_t}; \phi_t + \overline{\phi_t} \right\rangle_{H_t}} = \left\langle \mathrm{Res}^G_{H_t}(c(\rho)); \phi_t \right\rangle_{H_t}.$$

For the calculation of C_4 the same arguments hold for the subchains $(H_0, \Psi_0) < .. < (H_r, \Psi_r)$, where in case $\phi_r = \overline{\phi_r}$ we have $\left\langle \mathrm{Res}^{H_{r+1}}_{J_r}(\psi_{r+1}); \phi_r \right\rangle_{J_r} = 2$. Finally $C_2 + C_4$ add up to

$$\sum_{\substack{(J_0,\phi_0) < .. < (J_r,\phi_r) \\ (J_0,\phi_0) \in (J,\phi)^G}} \sum_{\substack{(H_0,\Psi_0) < .. < (H_t,\Psi_t) \\ J_r < H_0,\ c(\Psi_0) = \psi_{r+1}}} (-1)^{r+t-1} \left\langle \mathrm{Res}^{H_{r+1}}_{J_r}(\psi_{r+1}); \phi_r \right\rangle_{J_r} m(\mathrm{Res}^G_{H_t}(\rho), \Psi_t)$$

which now turns out to be the negative of the third subsum above, and $C_1 + C_3$ add up to

$$\sum_{(H_0,\phi_0) \in (J,\phi)^G} \sum_{(H_0,\phi_0) < .. < (H_t,\phi_t)} (-1)^t \left\langle \mathrm{Res}^G_{H_t}(c(\rho)); \phi_t \right\rangle_{H_t},$$

which coincides with the coefficient of $(J, \phi)^G$ in $a_G(c(\rho))$. $\square$

Example 3.19. $G = C_p \times Q_8$

We will apply the complexification map c_+ to formula calculated in

3.10:

$$
\begin{aligned}
c_+(a_G{}^{sp}(\rho)) &= (\langle xz\rangle, i\zeta)^G + (\langle xz\rangle, \overline{i\zeta})^G + (\langle yz\rangle, i\zeta)^G + (\langle yz\rangle, \overline{i\zeta})^G + (\langle xyz\rangle, i\zeta)^G \\
&+ (\langle xyz\rangle, \overline{i\zeta})^G - (\langle\text{-}z\rangle, \text{-}\zeta)^G - (\langle\text{-}z\rangle, \overline{\text{-}\zeta})^G + \frac{2}{p}((\langle x\rangle, i)^G + (\langle y\rangle, i)^G \\
&+ (\langle xy\rangle, i)^G - (\langle\text{-}1\rangle, \varepsilon)^G) - \frac{1}{p}(\langle x\rangle, i)^G - \frac{1}{p}(\langle x\rangle, \bar{i})^G \\
&- \frac{1}{p}(\langle y\rangle, i)^G - \frac{1}{p}(\langle y\rangle, \bar{i})^G - \frac{1}{p}(\langle xy\rangle, i)^G + \frac{1}{p}(\langle xy\rangle, \bar{i})^G + \frac{2}{p}(\langle\text{-}1\rangle, \varepsilon)^G \\
&= (\langle xz\rangle, i\zeta)^G + (\langle xz\rangle, \overline{i\zeta})^G + (\langle yz\rangle, i\zeta)^G + (\langle yz\rangle, \overline{i\zeta})^G + (\langle xyz\rangle, i\zeta)^G \\
&+ (\langle xyz\rangle, \overline{i\zeta})^G - (\langle\text{-}z\rangle, \text{-}\zeta)^G - (\langle\text{-}z\rangle, \overline{\text{-}\zeta})^G \\
&= a_G(\chi) + a_G(\chi') \\
&= a_G(c(\rho)).
\end{aligned}
$$

Theorem 3.20.

The map a_G^{Sp} yields an induction formula, that is

$$b_G^{Sp} a_G^{Sp} = \mathrm{id} : R^{Sp}(G) \to R^{Sp}(G)\ .$$

Proof. By 3.1 and 3.18 the diagram

$$
\begin{array}{ccccc}
\mathbb{Q}R^{Sp}(G) & \xrightarrow{a_G^{Sp}} & \mathbb{Q}R_+^{Sp}(G) & \xrightarrow{b_G^{sp}} & \mathbb{Q}R^{Sp}(G) \\
\downarrow{\scriptstyle c} & & \downarrow{\scriptstyle c_+} & & \downarrow{\scriptstyle c} \\
\mathbb{Q}R(G) & \xrightarrow{a_G} & \mathbb{Q}R_+(G) & \xrightarrow{b_G} & \mathbb{Q}R(G)
\end{array}
$$

is commutative. Hence, for $\rho : G \to \mathrm{Sp}(n)$, we use Theorem 2.4 to obtain

$$c(b_G^{sp}(a_G^{Sp}(\rho))) = b_G(a_G(c(\rho))) = c(\rho)\ ,$$

and since c is injective, we conclude $\rho = b_G^{sp}(a_G^{Sp}(\rho))$. □

Theorem 3.21.

Let $K/\mathbb{Q}_p$, (p odd), and let L/K be a finite, totally ramified Galois extension with group G. Then, for $\rho : G \to \mathrm{Sp}(n)$, $a_G^{Sp}(\rho) \in R_+^{Sp}(G)$.

Proof. First we observe that the structure of G is restricted, G is a semidirect product of a p-group by a cyclic group of order prime to p. Especially the 2-Sylow-subgroup of G has to be cyclic. Hence, if $\rho : G \to \mathrm{Sp}(n)$ is irreducible, then $c(\rho)$ will not be irreducible. In fact, if $c(\rho)$ were irreducible, its character would be real-valued and therefore of Schur index 2 (over $\mathbb{R}$). By the Brauer-Witt theorem ([**15**] §74.38) there would be a $(\mathbb{R}, 2)$-elementary subgroup H of G loaded with an irreducible character of Schur index 2. But H, a semidirect product of an odd cyclic normal subgroup with a 2-group, which is cyclic by itself, does not admit such a character.

Thus $c(\rho) = \theta + \overline{\theta}$ for some irreducible unitary representation $\theta : G \to \mathrm{U}(n)$. Since

$$a_G^{Sp}(\rho) = \sum_{(H,\Psi)^G} \alpha_{(H,\Psi)^G}(\rho)\ (H,\Psi)^G \text{ with}$$

$$\alpha_{(H,\Psi)^G}(\rho) = \frac{|H|}{|G|} \sum_{\substack{(H_0,\Psi_0)<..<(H_r,\Psi_r) \\ (H_0,\Psi_0)\in(H,\Psi)^G}} (-1)^r m(\mathrm{Res}^G_{H_r}\rho, \Psi_r) \ ,$$

we have to show that $\alpha_{(H,\Psi)^G}(\rho) \in \mathbb{Z}$. This we do by showing that these coefficients equal to coefficients of the canonical unitary induction formula of $c(\rho)$ or θ, which by Boltje's results are known to be integral.

Case 1: $c(\Psi) = \lambda + \overline{\lambda}$ with $\lambda \neq \overline{\lambda}$.

In this case the complexification of the one-dimensional symplectic representations Ψ_i split into $\lambda_i + \overline{\lambda_i}$ with $\lambda_i \neq \overline{\lambda_i}$. Hence $m(\mathrm{Res}^G_{H_r}(\rho), \Psi_r) = \langle \theta + \overline{\theta}; \lambda_r \rangle_{H_r}$. If $(H,\lambda)^G \neq (H,\overline{\lambda})^G$, we find that

$$\alpha_{(H,\Psi)^G}(\rho) = \frac{|H|}{|G|} \sum_{\substack{(H_0,\lambda_0)<..<(H_r,\lambda_r) \\ (H_0,\lambda_0)\in(H,\lambda)^G}} (-1)^r \langle \theta + \overline{\theta}; \lambda_r \rangle_{H_r} = \alpha_{(H,\lambda)^G}(\theta + \overline{\theta}) \ ,$$

the coefficient of $(H,\lambda)^G$ in $a_G(\theta + \overline{\theta})$. If $(H,\lambda)^G = (H,\overline{\lambda})^G$, each symplectic chain gives two unitary chains. Thus the same equation shows that $\alpha_{(H,\Psi)^G}(\rho) = \frac{1}{2}\alpha_{(H,\lambda)^G}(\theta + \overline{\theta})$. But as

$$\alpha_{(H,\lambda)^G}(\theta+\overline{\theta}) = \alpha_{(H,\lambda)^G}(\theta) + \alpha_{(H,\lambda)^G}(\overline{\theta}) = \alpha_{(H,\lambda)^G}(\theta) + \alpha_{(H,\overline{\lambda})^G}(\overline{\theta}) = 2\alpha_{(H,\lambda)^G}(\theta) \ ,$$

we conclude that $\alpha_{(H,\Psi)^G}(\rho) = \alpha_{(H,\lambda)^G}(\theta) \in \mathbb{Z}$.

Case 2: $c(\Psi) = 2\lambda$.

We show that $\alpha_{(H,\Psi)^G}(\rho) = \alpha_{(H,\lambda)^G}(\theta) \in \mathbb{Z}$. Therefore we take a chain

$$(*) \ \ (H_0,\Psi_0)<..<(H_r,\Psi_r)$$

and study the corresponding multiplicity. Let $c(\Psi_r) = \lambda_r + \overline{\lambda_r}$.
If $\lambda_r = \overline{\lambda_r}$ then $m(\mathrm{Res}^G_{H_r}(\rho), \Psi_r) = \frac{1}{2}\langle \mathrm{Res}^G_{H_r}(\theta + \overline{\theta}); \lambda_r \rangle_{H_r} = \langle \mathrm{Res}^G_{H_r}(\theta); \lambda_r \rangle_{H_r}$ and there is exactly one unitary chain derived from $(*)$, namely $(H_0,\lambda_0)<..<(H_r,\lambda_r)$. On the other side, if $\lambda_r \neq \overline{\lambda_r}$ then $m(\mathrm{Res}^G_{H_r}(\rho), \Psi_r) = \frac{1}{2}\langle \mathrm{Res}^G_{H_r}(\theta + \overline{\theta}); \lambda_r \rangle_{H_r}$ and $(*)$ affords the two unitary chains $(H_0,\lambda_0)<..<(H_r,\lambda_r)$ and $(H_0,\lambda_0)<..<(H_r,\overline{\lambda_r})$. Since $\langle \mathrm{Res}^G_{H_r}(\overline{\theta}); \lambda_r \rangle_{H_r} = \langle \mathrm{Res}^G_{H_r}(\theta); \overline{\lambda_r} \rangle_{H_r}$ we conclude that

$$\sum_{\substack{(H_0,\Psi_0)<..<(H_r,\Psi_r) \\ (H_0,\Psi_0)\in(H,\Psi)^G}} (-1)^r m(\mathrm{Res}^G_{H_r}(\rho), \Psi_r) \ = \sum_{\substack{(H_0,\lambda_0)<..<(H_r,\lambda_r) \\ (H_0,\lambda_0)\in(H,\lambda)^G}} (-1)^r m(\mathrm{Res}^G_{H_r}(\theta), \lambda_r) \ ,$$

which finishes the proof. $\square$

4. Symplectic Adams operations

In this section we are going to construct an additive homomorphism $\Psi^p : R_+^{Sp}(G) \longrightarrow R_+^{Sp}(G)$ which induces the Adams operation on $R^{Sp}(G)$. For $R_+(G)$ and complex representations this sort of construction first appeared in [**38**] but we shall explain in §5 precisely why the complex case will not suffice for the local root number applications we have in mind. Also it is with this type of application in mind that we content ourselves with the class of solvable G's which occur as local Galois groups in the odd residue characteristic case.

4.1. We may represent $\mathrm{Sp}(1) \cong \mathrm{SU}(2)$ by matrices in $\mathrm{U}(2)$ of the form

$$\begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix}$$

where a and b are complex numbers which satisfy the relation $|a|^2 + |b|^2 = 1$. The standard maximal torus in $\mathrm{Sp}(1)$ is the circle corresponding to $a = e^{i\theta}, b = 0$. Writing S^1 for this circle, the normaliser is given by

$$N_{\mathrm{Sp}(1)}S^1 = \{ \begin{pmatrix} a & 0 \\ 0 & \bar{a} \end{pmatrix}, \begin{pmatrix} 0 & b \\ -\bar{b} & 0 \end{pmatrix} \mid |a| = 1 = |b| \}.$$

Hence $N_{\mathrm{Sp}(1)}S^1 =< S^1, w >$ where

$$w = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Consider a finite subgroup, $H \subset N_{\mathrm{Sp}(1)}S^1$, then H may be conjugated by an element of $N_{\mathrm{Sp}(1)}S^1$ so that $H = H \cap S^1$ or $H =< H \cap S^1, w >$, such a subgroup H will temporarily be called *standard*. Write $C_m \subset S^1$ for the cyclic subgroup of order m and $Q_{4m} =< C_m, w >$. These are all the standard finite subgroups, $H \subset N_{\mathrm{Sp}(1)}S^1$. The inclusion, $i : H \subset \mathrm{Sp}(1)$, of a finite, standard subgroup yields a one-dimensional symplectic representation which is determined by the $\mathrm{Sp}(1)$-conjugacy class of i. However, if $H = H \bigcap S^1$ any automorphism of H induced by conjugation in $\mathrm{Sp}(1)$ may also be realised by conjugation in $N_{\mathrm{Sp}(1)}S^1$. The same is true for any standard subgroup of the form $H =< H \cap S^1, w >$ of order strictly larger than eight. The cyclic group C_4 of order four in S^1 may, however, be conjugated within $\mathrm{Sp}(1)$, onto $< w >= Q_4$. In fact, if $\Lambda(X)$ is given by

$$\Lambda(X) = \begin{pmatrix} 1/\sqrt{2} & i/\sqrt{2} \\ i/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} X \begin{pmatrix} 1/\sqrt{2} & -i/\sqrt{2} \\ -i/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}$$

then

$$\Lambda \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} = w, \qquad \Lambda(w) = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}$$

Also Λ induces an automorphism of the standard subgroup isomorphic to Q_8, the quaternion group of order eight.

Up to conjugation in $N_{\mathrm{Sp}(1)}S^1$ these are the only Sp(1)-conjugation automorphisms between standard subgroups which are not induced by conjugation in $N_{\mathrm{Sp}(1)}S^1$.

Let p be an odd prime. Then we may define a homomorphism

$$\Psi^p : N_{\mathrm{Sp}(1)}S^1 \longrightarrow N_{\mathrm{Sp}(1)}S^1$$

by the formulae

$$\Psi^p \begin{pmatrix} a & 0 \\ 0 & \overline{a} \end{pmatrix} = \begin{pmatrix} a^p & 0 \\ 0 & \overline{a}^p \end{pmatrix}, \quad \Psi^p \begin{pmatrix} 0 & b \\ -\overline{b} & 0 \end{pmatrix} = (-1)^{(p-1)/2} \begin{pmatrix} 0 & b^p \\ -\overline{b}^p & 0 \end{pmatrix}.$$

Proposition 4.2.

Let G be a finite solvable group which is isomorphic to the group of a Galois extension of local fields of odd residue characteristic. Let p be an odd prime. Then there is a natural homomorphism

$$\Psi^p : R_+^{Sp}(G) \longrightarrow R_+^{Sp}(G)$$

given by $\Psi^p((H,\psi)^G) = (H, \Psi^p \cdot \psi)^G$ when $\psi(H) \subset N_{\mathrm{Sp}(1)}S^1$ is standard in the sense of §4.1.

Proof. The hypothesis on G is inherited by H and by $\psi(H)$. This means that $\psi(H)$ is solvable and not isomorphic to the binary tetrahedral group of order twenty-four, which implies that we may conjugate in Sp(1) to get make $\psi(H)$ standard. The element, $(H,\psi)^G$, depends only on the $G - \mathrm{Sp}(1)$-conjugacy class of (H,ψ). Varying (H,ψ) by $G - N_{\mathrm{Sp}(1)}S^1$-conjugation does not alter $(H, \Psi^p \cdot \psi)^G$. By the discussion of §4.1, this means that Ψ^p is well-defined on $(H,\psi)^G$ except possibly if $\psi(H) = C_4, Q_4, Q_8$. However it is easily verified that the Sp(1)-conjugation, Λ, commutes with Ψ^p in these exceptional cases, which completes the proof. □

Proposition 4.3.

*As in ([**41**] p.109), define $\Psi^p : R_+(G) \longrightarrow R_+(G)$ by $\Psi^p((H,\phi)^G) = (H,\phi^p)^G$. Let G be a solvable group as in Proposition 4.2 and let p be an odd prime. Then*

$$\Psi^p \cdot c_+ = c_+ \cdot \Psi^p : R_+^{Sp}(G) \longrightarrow R_+(G).$$

Proof. If $\psi(H)$ is abelian and standard with $c(\psi) = \phi \oplus \overline{\phi}$ then $c(\Psi^p \cdot \psi) = \phi^p \oplus \overline{\phi}^p$ as complex representations of H so that

$$\Psi^p(c_+(H,\psi)^G) = (H,\phi^p)^G + (H,\overline{\phi}^p)^G = c_+(\Psi^p(H,\psi)^G).$$

Otherwise, being standard, $\psi(H)$ is Q_{4n} for some $n \geqslant 1$ and the result follows from the formulae of Proposition 2.6 for $a_{Q_{4n}}(Q_{4n},\psi)^{Q_{4n}}$. More precisely, from

Proposition 2.6 we have

$$a_{Q_{4n}}(c(\Psi)) =$$
$$\begin{cases} (< x >, \phi_x)^{Q_{4n}} + (< y >, \rho_y)^{Q_{4n}} + (< y >, \overline{\rho}_y)^{Q_{4n}} - (< y^2 >, \chi)^{Q_{4n}} \\ \qquad \qquad if\ n\ is\ odd, \\ (< x >, \phi_x)^{Q_{4n}} + (< y >, \rho_y)^{Q_{4n}} + (< xy >, \rho_{xy})^{Q_{4n}} - (< y^2 >, \chi)^{Q_{4n}} \\ \qquad \qquad if\ n\ is\ even \end{cases}$$

where $\rho_x(x) = \xi_{2n}$. Therefore, if n is odd, we have

$$\begin{aligned} &\Psi^p(c_+(H,\psi)^G) \\ &= \Psi^p((< x >, \phi_x)^G + (< y >, \rho_y)^G + (< y >, \overline{\rho}_y)^G - (< y^2 >, \chi)^G) \\ &= ((< x >, \phi_x^p)^G + (< y >, \rho_y^p)^G + (< y >, \overline{\rho}_y^p)^G - (< y^2 >, \chi^p)^G) \end{aligned}$$

On the other hand, if the symplectic representation $\psi : H \longrightarrow \mathrm{Sp}(1)$ satisfies $c(\psi) = \mathrm{Ind}_{C_{2n}}^{Q_{4n}}(\phi)$ then $c(\Psi^p(\psi)) = \mathrm{Ind}_{C_{2n}}^{Q_{4n}}(\phi^p)$ so that

$$\begin{aligned} &c_+(\Psi^p(H,\psi)^G)) \\ &= ((< x >, \phi_x^p)^G + (< y >, \rho_y^p)^G + (< y >, \overline{\rho}_y^p)^G - (< y^2 >, \chi^p)^G) \\ &= \Psi^p(c_+(H,\psi)^G). \end{aligned}$$

The case when n is even is similar. □

Corollary 4.4.
Let G be a solvable group as in Proposition 4.2 and let p be an odd prime. Then the composition

$$R^{Sp}(G) \xrightarrow{a_G^{sp}} R_+^{Sp}(G) \otimes \mathbb{Q} \xrightarrow{\Psi^p \otimes 1} R_+^{Sp}(G) \otimes \mathbb{Q} \xrightarrow{b_G \otimes 1} R^{Sp}(G) \otimes \mathbb{Q}$$

sends z to $\psi^p(z) \otimes 1$, where ψ^p is the usual Adams operation.

Proof. It suffices to show that $c(b_G \otimes 1(\Psi^p \otimes 1(a_G^{sp}(z))))$ is equal to $\psi^p(c(z)) \otimes 1$. However

$$\begin{aligned} c(b_G \otimes 1(\Psi^p \otimes 1(a_G^{sp}(z)))) &= b_G \otimes 1(c_+(\Psi^p \otimes 1(a_G^{sp}(z)))) \\ &= b_G \otimes 1(\Psi^p \otimes 1(c_+(a_G^{sp}(z)))) \\ &= b_G \otimes 1(\Psi^p \otimes 1(a_G(c(z)))) \\ &= \psi^p(c(z)) \otimes 1 \end{aligned}$$

by ([**41**] Theorem 4.1.6). □

Remark 4.5. *Orthogonal representations and* Ψ^2

We continue to assume that G is a finite solvable group which is isomorphic to the group of a Galois extension of local fields of odd residue characteristic.

Now we turn to the orthogonal group, O(2), whose maximal torus is the circle,

$$\mathrm{SO}(2) = \{ \begin{pmatrix} cos(\theta) & sin(\theta) \\ -sin(\theta) & cos(\theta) \end{pmatrix} \mid \theta \in \mathbb{R}\}.$$

This is normal is O(2) which may be written as a semi-direct product, $\mathbb{Z}/2 \ltimes S^1$, given in terms of generators and relations as

$$\mathrm{O}(2) = \{\tau, e^{i\theta}\ (\theta \in \mathbb{R}) \mid \tau^2 = 1,\ \tau e^{i\theta}\tau = e^{-i\theta}\}.$$

The formulae

$$\Psi^2(w) = \tau,\ \Psi^2(\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}) = e^{2i\theta}$$

define a homomorphism

$$\Psi^2 : N_{\mathrm{Sp}(1)}S^1 \longrightarrow \mathrm{O}(2)$$

since $\Psi^2(w^2) = \tau^2 = 1 = \Psi^2(-I)$ and $\tau e^{2i\theta}\tau = e^{-2i\theta}$.

However

$$\Psi^2(\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}) = -1,\ \Psi^2(w) = \tau$$

which are two elements of order two which are not conjugate in O(2). This means that we cannot define a homomorphism Ψ^2 by the formula of Proposition 4.2, in the light of the discussion of §4.1 of Sp(1)-conjugacy of standard subgroups. The difficulty occurs with the standard subgroups C_4, Q_4, Q_8. The following result is the best we can do.

Proposition 4.6.

Let G be a finite solvable group which is isomorphic to the group of a Galois extension of local fields of odd residue characteristic. Then there is a homomorphism

$$\Psi^2 : R_+^{Sp}(G) \longrightarrow R_+^{O}(G)$$

given by $\Psi^2((H,\psi)^G) = (H, \Psi^2 \cdot \psi)^G$ when $\psi(H) \subset N_{\mathrm{Sp}(1)}S^1$ is standard and different from C_4, Q_4, Q_8. When $\psi(H) \subset N_{\mathrm{Sp}(1)}S^1$ is standard and H is one of C_4, Q_4, Q_8 set

$$\Psi^2((H,\psi)^G) = (H, \Psi^2 \cdot \psi)^G + (H, \Psi^2 \cdot \psi')^G$$

where

$$\psi' = \begin{pmatrix} 1/\sqrt{2} & i/\sqrt{2} \\ i/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} \cdot \psi \cdot \begin{pmatrix} 1/\sqrt{2} & -i/\sqrt{2} \\ -i/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}.$$

5. Symplectic Local Root Numbers

5.1. *Local root numbers*

Now suppose that L/K is a Galois extension of p-adic local fields with group $G(L/K)$. An important invariant of a finite dimensional, complex representation ρ of $G(L/K)$ is the local root number $W_K(\rho)$, which is a complex number of unit norm ([**40**] §§1.4.10-1.4.14). When ρ is one-dimensional $W_K(\rho)$ is given by a Gauss sum/Artin conductor formula which extends uniquely to an exponential homomorphism on the representation ring $R(G(L/K))$ of the form

$$W_K : R(G(L/K)) \longrightarrow S^1 = \{z \in \mathbb{C}^* \mid |z| = 1\}$$

which satisfies the following properties:

(i) If $K \subset L \subset N$ is a chain of finite Galois extensions and $G(N/K) \longrightarrow G(L/K)$ is the canonical map then

$$W_K(\mathrm{Infl}_{G(L/K)}^{G(N/K)}(\rho)) = W_K(\rho).$$

(ii) If F is an intermediate field of L/K and $\rho : G(L/F) \longrightarrow \mathrm{GL}(V)$ is a representation then

$$W_K(\mathrm{Ind}_{G(L/F)}^{G(L/K)}(\rho - dim(\rho))) = W_F(\rho).$$

Note that $W_K(1) = 1$.

When ρ is the complexification of an orthogonal (i.e. real) representation, $\rho = c(\rho_1)$, then we have a formula of Deligne ([**16**]; see also [**36**], [**37**] Theorem 2.26 p.270)

$$W_K(\rho) = SW_2(\rho_1) \cdot W_K(det(\rho_1)).$$

Here $SW_2(\rho_1) \in H^2(K; \mathbb{Z}/2) \cong \{\pm 1\}$ is the second Stiefel-Whitney class of ρ_1 and $W_K(\det(\rho_1))$ is a fourth root of unity given by the quadratic Gauss sum/Artin conductor formula, since $\det(\rho_1)$ is a one-dimensional representation given by a quadratic character. In particular, this formula applies to the case of permutation representations $\rho = \mathrm{Ind}_{G(L/F)}^{G(L/K)}(1)$.

The case when ρ is the underlying complex representation of a symplectic (i.e. quaternionic) representation, $\rho = c(\rho_2)$, is particularly important in number theory (for example, see [**18**] and [**40**]). In this case $W_K(\rho) \in \{\pm 1\}$. On the other hand, the authors know of no formula for symplectic root numbers in general. When ρ_2 is one-dimensional of the form $\rho_2 : G(L/K) \longrightarrow \mathrm{Sp}(1)$ and K has odd residue characteristic the results of [**26**] and [**27**] amount to a semi-topological formula for $W_K(\rho)$ in the same spirit as the orthogonal local root number formula of (ii) above. The explicit construction and resulting formulae are derived in [**17**].

Let $G(L/K)$ denote the Galois group of a finite extension of local fields of odd residue characteristic. Suppose that $\rho : G(L/K) \longrightarrow \mathrm{Sp}(n)$ is a symplectic representation and that, in $R_+^{Sp}(G(L/K)) \otimes_{\mathbb{Z}} \mathbb{Q}$,

$$a_{G(L/K)}^{sp}(\rho) = \sum_{(G(L/F),\Psi)^{G(L/K)}} n_{(G(L/F),\Psi)^{G(L/K)}} \cdot (G(L/F), \Psi)^{G(L/K)}$$

is the symplectic Explicit Brauer Induction formula of 3.9. If each of the rational numbers $n_{(G(L/F),\Psi)^{G(L/K)}}$ actually lies in the 2-adic integers then $W_K(\rho)$ would be given by the formula

$$\begin{aligned} W_K(\rho) &= W_K(\rho - n) \\ &= \prod_{(G(L/F),\Psi)^{G(L/K)}} W_K(\mathrm{Ind}_{G(L/F)}^{G(L/K)}(\Psi - 1))^{n_{(G(L/F),\Psi)^{G(L/K)}}} \\ &= \prod_{(G(L/F),\Psi)^{G(L/K)}} W_F(\Psi)^{n_{(G(L/F),\Psi)^{G(L/K)}}}, \end{aligned}$$

which makes sense because $W_F(\Psi) \in \{\pm 1\}$ and $n_{(G(L/F),\Psi)^{G(L/K)}} \in \mathbb{Z}_2$.

The above formula for local symplectic roots numbers is the motivation for the following integrality conjecture.

Conjecture 5.2. Let G be a finite solvable group which is isomorphic to the group of a Galois extension of local fields of odd residue characteristic. Then, in §3.9,

$$a_{G(L/K)}^{sp}(\rho) \in R_+^{Sp}(G(L/K)) \otimes_{\mathbb{Z}} \mathbb{Z}_2.$$

Remark 5.3. *Evidence for Conjecture 5.2*

We have explained the motivation for Conjecture 5.2 in 5.1. Here are two pieces of evidence in its favour.

(i) In 3.11 we gave an example of a symplectic representation ρ_n of $G = Q_8 \times C_{2^n}$ for which $a_G^{sp}(\rho_n)$ was not 2-adically integral.

When can $G = Q_8 \times C_{2^n}$ occur as the Galois group of an extension of p-adic local fields? Never when p is odd!

If $p = 2$ one can take $L/\mathbb{Q}_2$ as in Case B or Case C of [**20**]. Namely either

$$L = \mathbb{Q}_2(\sqrt{2}, \sqrt{3})(\alpha_\pm)$$

where $\alpha_\pm^2 = \pm(\sqrt{6}/6)(1 + \sqrt{2})(\sqrt{2} + \sqrt{3})$ or

$$L = \mathbb{Q}_2(\sqrt{10}, \sqrt{3})(\alpha_\pm)$$

where $\alpha_\pm^2 = \pm(1+\sqrt{3}+\sqrt{10}/10+\sqrt{30}/10)$. In all these four cases $\mathbb{Q}_2^{nr}$, the maximal unramified extension of $\mathbb{Q}_2$, satisfies $\mathbb{Q}_2^{nr} \bigcap L = \mathbb{Q}_2$. Therefore if we take $K/\mathbb{Q}_2$ to be the unique unramified extension of degree 2^n then $LK/\mathbf{Q}_2$ is Galois with group $G(LK/\mathbb{Q}_2) \cong Q_8 \times C_{2^n}$.

On the other hand, if p is odd and F is a p-adic local field then $F^* \otimes \mathbb{Z}/2$ has four elements. Hence if L/F is Galois with $G(L/F) \cong Q_8$ then $N_{L/F}(L^*)$ is equal to the squares in F^*. If E/L is such that E/F is Galois with group $Q_8 \times C_{2^n}$ then there is an intermediate field M/F with $F^*/N_{M/F}(M^*) \cong C_2 \times C_2 \times C_2$ but then the surjection F^* onto $C_2 \times C_2 \times C_2$ must factor through $F^* \otimes \mathbf{Z}/2$ which has only four elements. Thus the expected counterexample to Conjecture 5.1 coming from 3.11 cannot exist when p is odd.

(ii) The prototypical integrality argument for an Explicit Brauer Induction formula is due to Boltje ([**41**] Theorem 2.3.43). The symplectic modification of that argument is rather more involved but can be used to establish 2-adic integrality in almost all cases.

In addition, 2-adic integrality holds under the conditions of Theorem 3.21.

Remark 5.4. *Conjecture 5.2 and* Ψ^p

In §5.1 we explained that the 2-adic conjecture would permit us to write the local root number of a symplectic representation ρ in a canonical formula in terms of the local root numbers of one-dimensional symplectic local Galois representations. However, treating ρ as a complex representation and using the Explicit Brauer Induction formula of [**35**] for the complexification $c(\rho)$ would accomplish more or less the same thing. This was first done in [**39**] and was the motivation for [**35**]. However, using Ψ^p in the case of local fields of odd residue characteristic p together with the formulae of [**17**] we can do much better. In [**17**] a topological formula in terms of Stiefel-Whitney classes is given for the ratio $W_K(\rho)/W_K(\Psi^p(\rho))$ when ρ is one-dimensional, symplectic. Since, in the notation of §5.1, $R_+^{Sp}(G(L/K))$ is the free abelian group on one-dimensional symplectic representations of subgroups of $G(L/K)$ the topological formula of [**17**] yields a homomorphism defined on $R_+^{Sp}(G(L/K))$ which, composed with $a_{G(L/K)}^{Sp}$, should yield a topological formula for $W_K(\rho)/W_K(\Psi^p(\rho))$ in general.

6. Induction formula for orthogonal representations

In Section 1 we described in the unitary case how canonical induction formulae may be used to derive the exponential property of the stable homotopy decomposition of BU and how we began this paper motivated by the possibility of similar behaviour in the symplectic case. However, in the orthogonal case, the transfer maps used in the stable decomposition of BO have long been known to have very complicated, non-exponential behaviour. This was first evaluated by Tornehave in [**45**]. The combinatorial results of this section are intended to describe, or at least to complement, Tornehave's results purely algebraically.

6.1. *Orthogonal representations*

Let G be a finite group. Let $R^O(G)$ denote the Grothendieck group of even-dimensional $\mathbb{R}G$-modules, $\mathbb{R}$ denoting the field of real numbers.

By definition, this is the quotient group of the free abelian group $\mathcal{F}^O(G)$ over the isomorphism classes of the category of even-dimensional $\mathbb{R}G$-modules, factored out the subgroup generated by expressions coming from short exact sequences. This gives a canonical surjective morphism

$$\kappa_G : \mathcal{F}^O(G) \to R^O(G) \; .$$

We identify $R^O(G)$ with the Grothendieck group of equivalence classes of even-dimensional orthogonal representations

$$\rho : G \longrightarrow \mathrm{O}(2n) := \mathrm{O}(2n, \mathbb{R})$$

for some $n \in \mathbb{N}$, and with the subgroup of $\mathbb{R}$-characters on G. Endowed with the standard maps

$$\mathrm{Res}_J^G : R^O(G) \to R^O(J) \quad \Big(\text{resp.} \quad \mathrm{Res}_J^G : \mathcal{F}^O(G) \to \mathcal{F}^O(J)\Big)$$

and

$$\mathrm{Ind}_J^G : R^O(J) \to R^O(G) \quad \Big(\text{resp.} \quad \mathrm{Ind}_J^G : \mathcal{F}^O(J) \to \mathcal{F}^O(G)\Big)$$

for $J \leqslant G$, this defines a G-Mackey functor structure on $H \mapsto R^O(H)$. But $H \mapsto \mathcal{F}^O(H)$ is not a G-Mackey functor because the Mackey formula does not hold.

Deligne [**22**] has shown that every representation $\rho : G \to \mathrm{O}(2n)$ is a $\mathbb{Z}$-linear combination of two-dimensional orthogonal representations on subgroups induced to G. So there exist $H_i \leqslant G$, $\Psi_i : H_i \to \mathrm{O}(2)$ and $n_i \in \mathbb{Z}$ such that

$$\rho = \sum_i n_i \mathrm{Ind}_{H_i}^G(\Psi_i) \ .$$

Thus let $L^O(G)$ (resp. $\mathcal{T}^O(G)$) denote the subgroup in $R^O(G)$ (resp. $\mathcal{F}^O(G)$) generated by the classes of two-dimensional orthogonal $\mathbb{R}H$-modules, that is the $\mathrm{O}(2)$-conjugacy classes of homomorphisms

$$\Psi : H \to \mathrm{O}(2) \ .$$

These two groups $L^O(G)$ and $\mathcal{T}^O(G)$ are canonically isomorphic via κ_G.

6.2. *The $_+$-construction*

Let $R_+^O(G)$ denote the G-Mackey functor obtained by $_+$-construction on $H \mapsto L^O(H)$. More precisely, let (H, Ψ) be a pair consisting of a subgroup $H \leqslant G$ and the equivalence class of an orthogonal representation $\Psi : H \to \mathrm{O}(2)$, and let $\mathcal{M}^O(G)$ be the set of all those pairs. There is an obvious action of G on $\mathcal{M}^O(G)$. Let $(H, \Psi)^G$ denote the G-orbit of (H, Ψ) in $\mathcal{M}^O(G)$, and let $\mathcal{M}^O(G)/_G$ denote the set of those orbits. Then $R_+^O(G)$ is defined as the free abelian group generated by the elements of $\mathcal{M}^O(G)/_G$. Indeed $H \mapsto R_+^O(H)$ is the G-Mackey functor induced by $H \mapsto L^O(H)$ and this comes with homomorphisms

$$\mathrm{Res}_J^G : R_+^O(G) \longrightarrow R_+^O(J)$$

and

$$\mathrm{Ind}_J^G : R_+^O(J) \longrightarrow R_+^O(G)$$

for $J \leqslant G$. For $N \triangleleft G$ we have the inflation map

$$\mathrm{Infl}_{G/N}^G : R_+^O(G/N) \to R_+^O(G)$$

defined by $\mathrm{Infl}_{G/N}^G((HN/N, \overline{\Psi})^G) = (HN, \Psi)^G$ for $\Psi : HN \to \mathrm{O}(2)$ with $N \leqslant \ker \Psi$.

Let $b_G^O : R_+^O(G) \to R^O(G)$ be the homomorphism defined by

$$b_G^O : (H, \Psi)^G \mapsto \mathrm{Ind}_H^G(\Psi) \ .$$

This map behaves naturally with respect to restriction, induction and inflation.

6.3. *Complexification*

Let c denote the natural homomorphism

$$c = c_G : R^O(G) \longrightarrow R(G)$$

given by embedding $\mathbb{R}$ into $\mathbb{C}$, which may also be regarded as tensoring with $\mathbb{C}$. So that for $\rho : G \to \mathrm{O}(2n)$,

$$c(\rho) : G \ni g \mapsto \rho(g) \in \mathrm{O}(2n) \subseteq \mathrm{U}(2n) \ .$$

For $\Psi : G \to \mathrm{O}(2)$ we will have to distinguish between the different behaviours of $c(\Psi)$. Either $c(\Psi)$ stays irreducible or it splits into a sum of two one-dimensional unitary representations. If $c(\Psi) = \psi$ is irreducible, we will indicate this by writing ψ instead of Ψ. If $c(\Psi) = \lambda + \overline{\lambda}$ with $\lambda : G \to \mathrm{U}(1)$ not real-valued and $\overline{\lambda}$ the complex conjugated character, we will write $\lambda + \overline{\lambda}$. If $c(\Psi) = 2\phi$ is twice a linear character, we use the notation 2ϕ, and finally in case $c(\Psi)$ is the sum of two different linear characters $\phi + \phi'$ taking values in ± 1 we will indicate this by writing $\phi + \phi'$. So ϕ will always denote a one-dimensional real-valued representation, λ a one-dimensional representation which differs from its complex conjugate denoted $\overline{\lambda}$, and ψ a two-dimensional irreducible representation.

Define the homomorphism

$$c_+ = c_{+,G} : R_+^O(G) \to R_+(G)$$

by the formula

$$c_+((H, \Psi)^G) = \mathrm{Ind}_H^G(a_H(c(\Psi)))\,.$$

This definition does not depend on the choice of (H, Ψ) in $(H, \Psi)^G$. Since we will have to apply this formula, we give the images in detail:

$$c_+((H, \Psi)^G) = \begin{cases} 2 \cdot (H, \phi)^G & \text{if } c(\Psi) = 2\phi \\ (H, \phi)^G + (H, \phi')^G & \text{if } c(\Psi) = \phi + \phi' \\ (H, \lambda)^G + (H, \overline{\lambda})^G & \text{if } c(\Psi) = \lambda + \overline{\lambda} \end{cases}$$

in case $c(\Psi)$ is reducible, and in case $c(\Psi) = \psi$ is irreducible we choose a representative (H, Ψ) of $(H, \Psi)^G$ and define

$$\begin{aligned} c_+((H, \Psi)^G) \quad &= \sum_{\substack{(K,\phi)<..<(K_s,\phi_s) \\ \text{in } \mathcal{M}(H)}} (-1)^s \tfrac{|K|}{|H|} \langle \mathrm{Res}_{K_s}^H(\psi); \phi_s \rangle (K, \phi)^G \\ &+ \sum_{\substack{(K,\phi)<..<(K_s,\lambda_s) \\ \text{in } \mathcal{M}(H)}} (-1)^s \tfrac{|K|}{|H|} \langle \mathrm{Res}_{K_s}^H(\psi); \lambda_s \rangle (K, \phi)^G \\ &+ \sum_{\substack{(K,\lambda)<..<(K_s,\lambda_s) \\ \text{in } \mathcal{M}(H)}} (-1)^s \tfrac{|K|}{|H|} \langle \mathrm{Res}_{K_s}^H(\psi); \lambda_s \rangle (K, \lambda)^G \end{aligned}$$

Obviously this definition does not depend on the choice of (H, Ψ).

Observe also that b_G and b_G^O are naturally connected via complexification, which means that $b_G \circ c_+ = c_+ \circ b_G^O : R_+^O(G) \to R(G)$.

6.4. *An orthogonal induction formula*

Let $\mathbb{Q}R^O(G)$ denote the $\mathbb{Q}$-vectorspace we receive from $R^O(G)$ by scalar extension. We identify $\mathbb{Q}R^O(G)$ with $R^O(G) \otimes_{\mathbb{Z}} \mathbb{Q}$, and similarly $\mathbb{Q}R_+^O(G)$ with $R_+^O(G) \otimes_{\mathbb{Z}} \mathbb{Q}$. All homomorphims on $R^O(G)$ and $R_+^O(G)$, especially Res_J^G and Ind_G^J, extend in a natural way to homomorphisms between those $\mathbb{Q}$-vectorspaces.

The homomorphism $a_G^O : \mathcal{F}^O(G) \to \mathbb{Q}R_+^O(G)$ is defined by mapping an orthogonal representation $\rho : G \to \mathrm{O}(2n)$ to

$$a^O_G(\rho) = \sum_{(H_0,\Psi_0)<..<(H_r,\Psi_r)} (-1)^r \frac{|H_0|}{|G|} m(\mathrm{Res}^G_{H_r}(\rho), \Psi_r)(H_0, \Psi_0)^G ,$$

where the sum runs over all chains $(H_0, \Psi_0)<..<(H_r, \Psi_r)$ in G. Here the multiplicity m given by the formula

$$m(\theta, \Psi) := \begin{cases} \langle c(\theta); \psi\rangle_H & \text{if } c(\Psi) = \psi \\ \langle c(\theta); \lambda\rangle_H & \text{if } c(\Psi) = \lambda + \overline{\lambda} \\ [\langle c(\theta); \phi\rangle_H/2] & \text{if } c(\Psi) = 2\phi \\ 1 & \text{if } c(\Psi) = \phi + \phi', \langle c(\theta); \phi\rangle_H \text{ odd and } \langle c(\theta); \phi'\rangle_H \text{ odd} \\ 0 & \text{otherwise} \end{cases} ,$$

for $\theta : H \to \mathrm{O}(2n)$, $\Psi : H \to \mathrm{O}(2)$, where $[x]$ denotes the integral part of a rational number x. Notice that $\langle c(\theta); \lambda\rangle_H = \left\langle c(\theta); \overline{\lambda}\right\rangle_H$ and that $m(\Psi, \Psi) = 1$ in all cases.

The following examples will show that in general this homomorphism does not factor through $R^O(G)$ and does not take values in $R^O_+(G)$. But first we give an analogue of 3.13.

Proposition 6.5.
Let $\rho : G \to \mathrm{O}(2)$ then

$$a^O_G(\rho) = (G, \rho)^G .$$

Proof. Since (G, ρ) is the only element in $(G, \rho)^G$ and $m(\rho, \rho) = 1$, the coefficient of $(G, \rho)^G$ in $a^O_G(\rho)$ is 1. Now let $(H, \Psi) < (G, \rho)$. Only those elements may give other nontrivial contributions to $a^O_G(\rho)$. Since $\mathrm{Res}^G_{H_r}(\rho) = \Psi_r$ for $(H_r, \Psi_r) < (G, \rho)$, the multiplicities turn out to be 1. Thus we have to show that

$$\sum_{\substack{(H,\Psi)<(H_1,\Psi_1)<..<(H_r,\Psi_r) \\ (H_r,\Psi_r)\leqslant(G,\rho)}} (-1)^r = 0 .$$

Consider the set, $\mathcal{R}$, of chains the sum runs over. Let $\mathcal{P} < \mathcal{R}$ denote the subset of those chains which will not end in (G, ρ). Then

$$(\,(H, \Psi) < \ldots < (H_r, \Psi_r)\,) \;\mapsto\; (\,(H, \Psi) < \ldots < (H_r, \Psi_r) < (G, \rho)\,)$$

gives a bijection $\mathcal{P} \to \mathcal{R} \setminus \mathcal{P}$, where chains of length r are in correspondence to chains of length $r+1$. So the terms cancel in pairs and indeed the sum above equals 0. □

Example 6.6. $G = C_2 \times C_2$

Let $G =< A, B|\ A^2 = B^2 = (AB)^2 = 1 >$ be the Klein 4-group. Let $\mathbb{1}$ denote the trivial character on G and ε_X the linear character on G with kernel $< X >$

for $X = A, B, AB$. On a cyclic group $< X >$ of order 2, ε denotes the non-trivial irreducible character. By 6.5

$$a_G^O(\mathbb{1} + \varepsilon_A) + a_G^O(\varepsilon_B + \varepsilon_{AB}) = (G, \mathbb{1} + \varepsilon_A)^G + (G, \varepsilon_B + \varepsilon_{AB})^G \ .$$

But for the regular $\rho = \mathbb{1} + \varepsilon_A + \varepsilon_B + \varepsilon_{AB}$ an easy but lengthy calculation gives

$$\begin{aligned} a_G^O(\rho) &= (G, \mathbb{1} + \varepsilon_A)^G + (G, \mathbb{1} + \varepsilon_B)^G + (G, \mathbb{1} + \varepsilon_{AB})^G \\ &+ (G, \varepsilon_A + \varepsilon_B)^G + (G, \varepsilon_A + \varepsilon_{AB})^G + (G, \varepsilon_B + \varepsilon_{AB})^G - 2(\langle A\rangle, \mathbb{1} + \varepsilon)^G \\ &- 2(\langle B\rangle, \mathbb{1} + \varepsilon)^G - 2(\langle AB\rangle, \mathbb{1} + \varepsilon)^G + 2(\langle 1\rangle, 2\mathbb{1})^G \end{aligned}$$

Since the element $(\mathbb{1} + \varepsilon_A) + (\varepsilon_B + \varepsilon_{AB}) - (\mathbb{1} + \varepsilon_A + \varepsilon_B + \varepsilon_{AB}) \in \mathcal{F}^O(G)$ will not be killed by a_G^O, a_G^O does not factor through $R^O(G)$. The situation is the same after applying c because

$$a_G(c(\rho)) = (G, \mathbb{1})^G + (G, \varepsilon_A)^G + (G, \varepsilon_B)^G + (G, \varepsilon_{AB})^G ,$$

while

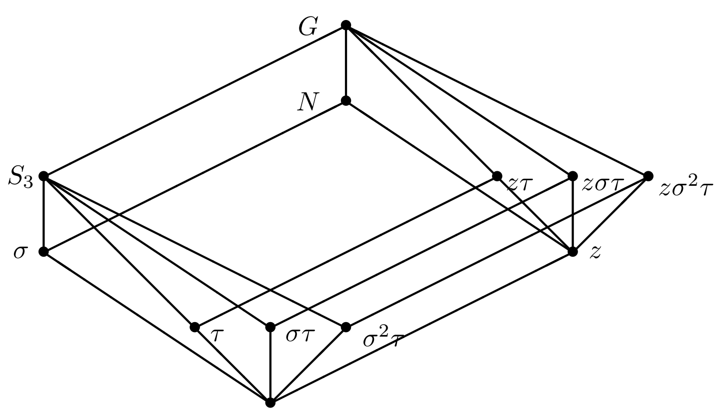
$$\begin{aligned} c_+(a_G^O(\rho)) &= 3(G, \mathbb{1})^G + 3(G, \varepsilon_A)^G + 3(G, \varepsilon_B)^G + 3(G, \varepsilon_{AB})^G \\ &+ \textstyle\sum_{X \neq 1} \left(-2(\langle X\rangle, \mathbb{1})^G - 2(\langle X\rangle, \varepsilon)^G\right) + 4(\langle 1\rangle, \mathbb{1})^G \ . \end{aligned}$$

Example 6.7. $G = C_p \times S_3$

Let $p \geqslant 5$ be an odd prime and let G be the direct product of a cyclic group of order p and the symmetric group on three letters, so

$$G = < z, \sigma, \tau | z^p = \sigma^3 = \tau^2 = 1, z\sigma = \sigma z, z\tau = \tau z, \tau\sigma = \sigma^2\tau > \ .$$

The lattice of subgroups of G is as pictured below.



Let ζ denote a primitive p-th root of unity. We calculate $a_G^O(\rho)$ for the orthogonal representation $\rho : G \to \mathrm{O}(4)$ defined by

$$\rho(\sigma) = \begin{pmatrix} \frac{-1}{2} & \frac{-\sqrt{3}}{2} & 0 & 0 \\ \frac{-\sqrt{3}}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{-1}{2} & \frac{-\sqrt{3}}{2} \\ 0 & 0 & \frac{-\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}, \qquad \rho(\tau) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix},$$

$$\rho(z) = \begin{pmatrix} \frac{\zeta+\zeta^{-1}}{2} & 0 & \frac{\zeta-\zeta^{-1}}{-2i} & 0 \\ 0 & \frac{\zeta+\zeta^{-1}}{2} & 0 & \frac{\zeta-\zeta^{-1}}{-2i} \\ \frac{\zeta-\zeta^{-1}}{2i} & 0 & \frac{\zeta+\zeta^{-1}}{2} & 0 \\ 0 & \frac{\zeta-\zeta^{-1}}{2i} & 0 & \frac{\zeta+\zeta^{-1}}{2} \end{pmatrix},$$

which is irreducible as an orthogonal representation, because $c(\rho)$ splits into the sum of two irreducible non-orthogonal representations $\chi\zeta$ and $\chi\zeta^{-1}$.

To calculate the formula one has to know the multiplicities $m = m(\mathrm{Res}^G_H(\rho), \Psi)$ for all subgroups $H \leqslant G$ and all $\Psi : H \to \mathrm{O}(2)$. This we can look up in the following table.

$H, c(\Psi)$	$N, \zeta\zeta_3{+}\overline{\zeta\zeta_3}$	$N, \zeta\zeta_3^2{+}\overline{\zeta\zeta_3^2}$	S_3, χ	$\langle\sigma\rangle, \zeta_3{+}\overline{\zeta_3}$	$\langle z\rangle, \zeta{+}\overline{\zeta}$
m	1	1	2	2	2

$H, c(\Psi)$	$\langle z\tau\rangle, \zeta{+}\overline{\zeta}$	$\langle z\tau\rangle, \varepsilon\zeta{+}\overline{\varepsilon\zeta}$	$\langle\tau\rangle, 2\mathbb{1}$	$\langle\tau\rangle, 2\varepsilon$	$1, 2\mathbb{1}$
m	1	1	1	1	2

Here ζ_3 denotes a fixed primitive third root of unity resp. the corresponding representation defined by $z \mapsto \zeta_3$, $\chi : S_3 \to \mathrm{O}(2)$ the faithful irreducible representation on S_3 and ε the character on $\langle\tau\rangle$ sending $\tau \mapsto -1$. Now

$$\begin{aligned}
a^O_G(\rho) &= \frac{1}{2}(1)(N, \zeta\zeta_3{+}\overline{\zeta\zeta_3})^G + \frac{1}{2}(1)(N, \zeta\zeta_3^2{+}\overline{\zeta\zeta_3^2})^G + \frac{1}{p}(2)(S_3, \chi)^G \\
&+ \frac{1}{2p}(2-(2+1+1))(\langle\sigma\rangle, \zeta_3{+}\overline{\zeta_3})^G + \frac{1}{3}(1)(\langle z\tau\rangle, \zeta{+}\overline{\zeta})^G + \frac{1}{3}(1)(\langle z\tau\rangle, \varepsilon\zeta{+}\overline{\varepsilon\zeta})^G \\
&+ \frac{1}{3}(1)(\langle z\sigma\tau\rangle, \zeta{+}\overline{\zeta})^G + \frac{1}{3}(1)(\langle z\sigma\tau\rangle, \varepsilon\zeta{+}\overline{\varepsilon\zeta})^G + \frac{1}{3}(1)(\langle z\sigma^2\tau\rangle, \zeta{+}\overline{\zeta})^G \\
&+ \frac{1}{3}(1)(\langle z\sigma^2\tau\rangle, \varepsilon\zeta{+}\overline{\varepsilon\zeta})^G + \frac{1}{6}(2-(3\cdot(1+1)+2))(\langle z\rangle, \zeta{+}\overline{\zeta})^G \\
&+ \frac{1}{3p}(1-1)(\langle\tau\rangle, 2\mathbb{1})^G + \frac{1}{3p}(1-1)(\langle\tau\rangle, 2\varepsilon)^G + \frac{1}{3p}(0-2)(\langle\tau\rangle, \mathbb{1}+\varepsilon)^G \\
&+ \frac{1}{3p}(1-1)(\langle\sigma\tau\rangle, 2\mathbb{1})^G + \frac{1}{3p}(1-1)(\langle\sigma\tau\rangle, 2\varepsilon)^G + \frac{1}{3p}(0-2)(\langle\sigma\tau\rangle, \mathbb{1}+\varepsilon)^G \\
&+ \frac{1}{3p}(1-1)(\langle\sigma^2\tau\rangle, 2\mathbb{1})^G + \frac{1}{3p}(1-1)(\langle\sigma^2\tau\rangle, 2\varepsilon)^G + \frac{1}{3p}(0-2)(\langle\sigma^2\tau\rangle, \mathbb{1}+\varepsilon)^G \\
&+ \frac{1}{6p}(2-(2+2+3(1+1)+(1+1)+3(1+1)+2)+(4\cdot 2+2(1+1)+3\cdot 2(1+1))) \\
&\quad (\langle 1\rangle, 2\mathbb{1})^G \\
&= (N, \zeta\zeta_3 + \overline{\zeta\zeta_3})^G + \frac{2}{p}(S_3, \chi)^G - \frac{1}{p}(\langle\sigma\rangle, \zeta_3 + \overline{\zeta_3})^G + (\langle z\tau\rangle, \zeta + \overline{\zeta})^G \\
&+ (\langle z\tau\rangle, \varepsilon\zeta + \overline{\varepsilon\zeta})^G - (\langle z\rangle, \zeta + \overline{\zeta})^G - \frac{2}{p}(\langle\tau\rangle, \mathbb{1}+\varepsilon)^G - \frac{1}{p}(\langle 1\rangle, 2\mathbb{1})^G
\end{aligned}$$

Proposition 6.8.
The homomorphism a^O_G is natural with respect to restriction so that, if $J \leqslant G$,

$$a^O_J \circ \mathrm{Res}^G_J = \mathrm{Res}^G_J \circ a^O_G : \mathcal{F}^O(G) \to \mathbb{Q}R^O_+(J) \ .$$

Proof. Similar to the proof of 3.12. □

Proposition 6.9.
Let G be a finite group. For $\rho : G \to \mathrm{O}(2n)$, the defect for commutativity in

$$\begin{tikzcd} \mathcal{F}^O(G) \arrow[r, "a^O_G"] \arrow[d, "c\circ\kappa_G"'] & \mathbb{Q}R^O_+(G) \arrow[d, "c_+"] \\ R(G) \arrow[r, "a_G"'] & \mathbb{Q}R_+(G) \end{tikzcd}$$

is given by

$$c_+(a^O_G(\rho)) \ - \ a_G(c(\rho)) = \sum_{\substack{(H_0,\phi_0)<..<(H_r,\phi_r) \\ \phi_r \text{ odd}}} (-1)^r \frac{|H_0|}{|G|}(n_{H_r}(\rho) - 2)(H_0, \phi_0)^G \ ,$$

where the sum runs over all chains ending in some pair (H_r, ϕ_r) with linear character ϕ_r (taking values in ± 1) such that $\left\langle \mathrm{Res}^G_{H_r}(c(\rho)); \phi_r \right\rangle_{H_r}$ is an odd number, and $n_{H_r}(\rho)$ denotes the number of such characters on H_r.

The proof of this fact follows directly from lemma 6.11 and lemma 6.12. It is straightforward, but one has to keep book on lots of cases. Therefore we have to introduce some more notation and will begin with two preparatory lemmas.

6.10. *Notation*

Recall the convention explained in 6.3 to denote a base element $(H, \Psi)^G$ of $\mathbb{Q}R^O_+(G)$ by $(H, c(\Psi))^G$ and to use ψ, λ and ϕ to indicate the type of splitting of $c(\Psi)$.
Since we will have to compare coefficients, we use for each base element $(H, \Psi)^G \in \mathcal{M}^O(G)/_G$ the homomorphism $\pi_{(H,\Psi)^G} : \mathbb{Q}R^O_+(G) \to \mathbb{Q}$ defined by

$$\pi_{(H,\Psi)^G}((H', \Psi')^G) = \begin{cases} 1 & \text{if } \ (H, \Psi)^G = (H', \Psi')^G \\ 0 & \text{else} \end{cases}$$

To abbreviate notation we denote for $(H, \Psi) \in \mathcal{M}^O(G)$

$$\mathcal{M}^O(H, \Psi) := \{(H', \Psi') \in \mathcal{M}(G) | (H', \Psi') \leqslant (H, \Psi)\}$$

and

$$\mathcal{M}(H, \Psi) := \{(H', \varepsilon') \in \mathcal{M}(G) | H' \leqslant H, \left\langle \mathrm{Res}^H_{H'}(c(\Psi)); \varepsilon' \right\rangle_{H'} > 0\} \ .$$

Furthermore, for $(H, \phi) \in \mathcal{M}(G)$ and $(K, \Psi) \in \mathcal{M}^O(G)$ we will write $(H, \phi) \preceq (K, \Psi)$, if $H \leqslant K$ and $\left\langle \mathrm{Res}^K_H(c(\Psi)); \phi \right\rangle_H > 0$. If additionally $H < K$ we write $(H, \phi) \prec (K, \Psi)$. For $(H, \phi)^G \in \mathcal{M}(G)/_G$ and $(K, \Psi)^G \in \mathcal{M}^O(G)/_G$ we use

$$(H, \phi)^G \preceq (K, \Psi)^G$$

(resp. $(H,\phi)^G \prec (K,\Psi)^G$) to express the fact that there exist $(H_0,\phi_0) \in (H,\phi)^G$ and $(K_0,\Psi_0) \in (K,\Psi)^G$ such that $(H_0,\phi_0) \preceq (K_0,\Psi_0)$ (resp. $(H_0,\phi_0) \prec (K_0,\Psi_0)$). Since ρ will not be changed through our calculations, we will write briefly $\langle\chi\rangle$ instead of $\left\langle \mathrm{Res}^G_H(c(\rho));\varepsilon\right\rangle_H$, where ε is an irreducible complex representation on a subgroup H of G. For $\phi : H \to \{\pm 1\}$ we will say "ϕ is odd" or briefly "ϕ odd" , if $\langle\phi\rangle = \left\langle \mathrm{Res}^G_H(\rho\otimes\mathbb{C});\phi\right\rangle_H$ is an odd number.

Lemma 6.11.
Let $(H,\lambda)^G \in \mathcal{M}(G)/_G$ for some $\lambda : H \to \mathrm{U}(1)$ with $\lambda \neq \overline{\lambda}$. Then

$$\pi_{(H,\lambda)^G}(a_G(c(\rho))) \;=\; \pi_{(H,\lambda)^G}(c_+(a^O_G(\rho)))\,.$$

Proof. Let $(H,\lambda)^G \in \mathcal{M}(G)/_G$. The coefficient B of $(H,\lambda)^G$ in $a_G(c(\rho))$ is, according to 2.2,

$$B = \frac{|H|}{|G|} \sum_{\substack{(H_0,\lambda_0)<..<(H_r,\lambda_r)\\ (H_0,\lambda_0)\in(H,\lambda)^G}} (-1)^r\langle\lambda_r\rangle\ .$$

The calculation of the coefficient C of $(H,\lambda)^G$ in $c_+(a^O_G(\rho))$ takes some more effort. Since c_+ is a homomorphism, it is the coefficient of $(H,\lambda)^G$ in

$$\sum_{(H_0,\Psi_0)<..<(H_r,\Psi_r)} (-1)^r\frac{|H_0|}{|G|} m(\mathrm{Res}^G_{H_r}(\rho),\Psi_r)\; c_+((H_0,\Psi_0)^G)\,.$$

But $(H,\lambda)^G$ can only have a nontrivial contributions from $c_+((\widetilde{H},\widetilde{\Psi})^G)$ if either $(\widetilde{H},\widetilde{\Psi})^G = (H,\lambda+\overline{\lambda})^G$ or $(\widetilde{H},\widetilde{\Psi})^G = (K,\psi)^G$ with $(H,\lambda+\overline{\lambda})^G{<}(K,\psi)^G$. So, if for $(\widetilde{H},\widetilde{\Psi}) \in \mathcal{M}^O(G)$

$$C_{\widetilde{H},\widetilde{\Psi}} := \pi_{(H,\lambda)^G}\left(\sum_{\substack{(H_0,\Psi_0)<..<(H_r,\Psi_r)\\ (H_0,\Psi_0)=(\widetilde{H},\widetilde{\Psi})}} (-1)^r\frac{|H_0|}{|G|} m(\mathrm{Res}^G_{H_r}(\rho),\Psi_r)\; c_+((\widetilde{H},\widetilde{\Psi})^G)\right)$$

then C can be expressed as

$$C = \sum_{\substack{(H_0,\lambda_0+\overline{\lambda}_0)\in\mathcal{M}^O(G)\\ (H_0,\lambda_0+\overline{\lambda}_0)^G=(H,\lambda+\overline{\lambda})^G}} C_{H_0,\lambda_0+\overline{\lambda}_0} \;+\; \sum_{\substack{(K,\psi)\in\mathcal{M}^O(G)\\ (K,\psi)^G>(H,\lambda+\overline{\lambda})^G}} C_{K,\psi}.$$

We shall calculate these coefficients in each case, starting with $C_{H_0,\lambda_0+\overline{\lambda}_0}$. Without loss of generality we may suppose that λ_0 is such that $(H_0,\lambda_0)^G = (H,\lambda)^G$. With this fixed λ_0 we change notation and write briefly H for H_0 and λ for λ_0. This will not cause any confusion. We decompose $C_{H,\lambda+\overline{\lambda}}$ into the sum of the coefficient of $(H,\lambda)^G$ in

$$\sum_{(H,\lambda+\overline{\lambda})<..<(H_r,\lambda_r+\overline{\lambda}_r)} (-1)^r\frac{|H|}{|G|}\langle\lambda_r\rangle((H,\lambda)^G + (H,\overline{\lambda})^G)$$

and the coefficient of $(H,\lambda)^G$ in

$$\sum_{\substack{(H,\lambda+\overline{\lambda})<..<(H_r,\lambda_r+\overline{\lambda}_r)<\\<(K_1,\psi_1)<..<(K_s,\psi_s)}} (-1)^{r+s}\frac{|H|}{|G|}\langle\psi_s\rangle((H,\lambda)^G+(H,\overline{\lambda})^G)\ .$$

Any long chain $((H_0,\lambda_0+\overline{\lambda}_0)<..<(H_r,\lambda_r+\overline{\lambda}_r)<(K_1,\psi_1)<..<(K_s,\psi_s))$ of length $r+s$ can be broken up uniquely into the *lower* chain $((H_0,\lambda_0+\overline{\lambda}_0)<..<(H_r,\lambda_r+\overline{\lambda}_r))$ in $\mathcal{M}^O(K_1,\psi_1)$ of length r and the *upper* chain $((K_1,\psi_1)<..<(K_s,\psi_s))$ of length $s-1$ and conversely any such lower and upper chain define a unique long chain. Hence the second sum turns out to be the coefficient of $(H,\lambda)^G$ in

$$\sum_{(K,\psi)>(H,\lambda+\overline{\lambda})}\ \sum_{\substack{(H,\lambda+\overline{\lambda})<..<(H_r,\lambda_r+\overline{\lambda}_r)\\ \text{in } \mathcal{M}(K,\psi)}}\ \sum_{(K,\psi)<(K_1,\psi_1)<..<(K_s,\psi_s)}(-1)^{r+s+1}\frac{|H|}{|G|}\langle\psi_s\rangle((H,\lambda)^G+(H,\overline{\lambda})^G)\ .$$

Now, for $(H_i,\lambda_i+\overline{\lambda}_i)>(H,\lambda+\overline{\lambda})$, we choose λ_i such that $\mathrm{Res}^{H_i}_H(\lambda_i)=\lambda$, and distinguish two cases.

If $(H,\lambda)^G\neq(H,\overline{\lambda})^G$, then every chain $((H,\lambda+\overline{\lambda})<..<(H_r,\lambda_r+\overline{\lambda}_r))$ starting in $(H,\lambda+\overline{\lambda})$ determines the unique chain $((H,\lambda)<..<(H_r,\lambda_r))$ starting in an element of $(H,\lambda)^G$, and the other way arround. Thus, in this case,

$$C_{H,\lambda+\overline{\lambda}}=\frac{|H|}{|G|}\left(\sum_{(H,\lambda)<..<(H_r,\lambda_r)}(-1)^r\langle\lambda_r\rangle\right.$$
$$\left.+\sum_{(K,\psi)>(H,\lambda+\overline{\lambda})}\ \sum_{\substack{(H,\lambda)<..<(H_r,\lambda_r)\\ \text{in } \mathcal{M}(K,\psi)}}\ \sum_{(K,\psi)<..<(K_s,\psi_s)}(-1)^{r+s+1}\langle\psi_s\rangle\right)$$

If $(H,\lambda)^G=(H,\overline{\lambda})^G$, then every chain $((H,\lambda+\overline{\lambda})<..<(H_r,\lambda_r+\overline{\lambda}_r))$ determines uniquely the two chains $((H,\lambda)<..<(H_r,\lambda_r))$ and $((H,\overline{\lambda})<..<(H_r,\overline{\lambda}_r))$ starting in elements of $(H,\lambda)^G$, and the other way arround. Therefore, in this case

$$C_{H,\lambda+\overline{\lambda}}=\frac{|H|}{|G|}\left(\sum_{(H,\lambda)<..<(H_r,\lambda_r)}(-1)^r\langle\lambda_r\rangle+\sum_{(H,\overline{\lambda})<..<(H_r,\overline{\lambda}_r)}(-1)^r\langle\lambda_r\rangle\right)$$

$$+\frac{|H|}{|G|}\sum_{(K,\psi)>(H,\lambda+\overline{\lambda})}\ \sum_{(K,\psi)<..<(K_s,\psi_s)}\left(\sum_{\substack{(H,\lambda)<..<(H_r,\lambda_r)\\ \text{in } \mathcal{M}(K,\psi)}}(-1)^{r+s+1}\langle\psi_s\rangle+\sum_{\substack{(H,\overline{\lambda})<..<(H_r,\overline{\lambda}_r)\\ \text{in } \mathcal{M}(K,\psi)}}(-1)^{r+s+1}\langle\psi_s\rangle\right)$$

Next we express $C_{K,\psi}$ with $(K,\psi)^G>(H,\lambda+\overline{\lambda})^G$ for some $(K,\psi)\in\mathcal{M}^O(G)$. By definition, the coefficient of $(H,\lambda)^G$ in $c_+((K,\psi)^G)$ is

$$\sum_{\substack{(H_0,\lambda_0)\in(H,\lambda)^G\\ (H_0,\lambda_0+\overline{\lambda}_0)<(K,\psi)}}\ \sum_{\substack{(H_0,\lambda_0)<..<(H_r,\lambda_r)\\ \text{in } \mathcal{M}(K,\psi)}}(-1)^r\frac{|H|}{|K|}\langle\mathrm{Res}^K_{H_r}(\psi);\lambda_r\rangle$$

Since $\langle \mathrm{Res}_{H_i}^K(\psi); \lambda_i \rangle = 1$, the coefficient $C_{K,\psi}$ of $(H,\lambda)^G$ in

$$\sum_{(K,\psi)<..<(K_s,\psi_s)} (-1)^s \frac{|K|}{|G|} \langle \mathrm{Res}_K^G(c(\rho)); \psi \rangle_K \; c_+((K,\psi)^G)$$

turns out to be

$$\begin{aligned} C_{K,\psi} &= \textstyle\sum_{(K,\psi)<..<(K_s,\psi_s)} (-1)^s \frac{|K|}{|G|} \langle \psi_s \rangle \sum_{\substack{(H_0,\lambda_0) \\ \in (H,\lambda)^G}} \sum_{\substack{(H_0,\lambda_0)<..<(H_r,\lambda_r) \\ \text{in } \mathcal{M}(K,\psi)}} \quad (-1)^r \frac{|H|}{|K|} \\ &= \textstyle\frac{|H|}{|G|} \sum_{(K,\psi)<..<(K_s,\psi_s)} \sum_{\substack{(H_0,\lambda_0) \\ \in (H,\lambda)^G}} \sum_{\substack{(H_0,\lambda_0)<..<(H_r,\lambda_r) \\ \text{in } \mathcal{M}(K,\psi)}} \quad (-1)^{s+r} \langle \psi_s \rangle. \end{aligned}$$

Using these three expressions for $C_{(\widetilde{H},\widetilde{\Psi})^G}$, the formula for C turns into

$$C = \textstyle\sum_{\substack{(H_0,\lambda_0+\overline{\lambda}_0) \\ \in (H,\lambda+\overline{\lambda})^G}} C_{H_0,\lambda_0+\overline{\lambda}_0} + \sum_{\substack{(K,\psi)\in \mathcal{M}^O(G) \\ (K,\psi)^G > (H,\lambda+\overline{\lambda})^G}} c_{K,\psi}$$

$$= \textstyle\frac{|H|}{|G|} \Big(\sum_{(H_0,\lambda_0)\in(H,\lambda)^G} \sum_{(H_0,\lambda_0)<..<(H_r,\lambda_r)} (-1)^r \langle \lambda_r \rangle$$

$$+ \textstyle\sum_{\substack{(K,\psi)\in \mathcal{M}^O(G) \\ (K,\psi)^G > (H,\lambda+\overline{\lambda})^G}} \sum_{(K,\psi)<..<(K_s,\psi_s)} \sum_{\substack{(H_0,\lambda_0)<..<(H_r,\lambda_r) \\ \text{in } \mathcal{M}(K,p), (H_0,\lambda_0)^G = (H,\lambda)^G}} (-1)^{r+s+1} \langle \psi_s \rangle$$

$$+ \textstyle\sum_{\substack{(K,\psi)\in \mathcal{M}^O(G) \\ (K,\psi)^G > (H,\lambda+\overline{\lambda})^G}} \sum_{(K,\psi)<..<(K_s,\psi_s)} \sum_{\substack{(H_0,\lambda_0) \\ \in (H,\lambda)^G}} \sum_{\substack{(H_0,\lambda_0)<..<(H_r,\lambda_r) \\ \text{in } \mathcal{M}(K,\psi)}} (-1)^{s+r} \langle \psi_s \rangle \Big)$$

The two last terms cancel out, and we finally get

$$C \quad = \textstyle\frac{|H|}{|G|} \sum_{\substack{(H_0,\lambda_0)\in \mathcal{M}(G) \\ (H_0,\lambda_0+\overline{\lambda}_0)^G = (H,\lambda+\overline{\lambda})^G}} \sum_{(H_0,\lambda_0)<..<(H_r,\lambda_r)} \quad (-1)^r \langle \lambda_r \rangle = B$$

as claimed in 6.11. □

Lemma 6.12.
Let $(H,\phi)^G \in \mathcal{M}(G)/_G$ with $\phi : H \to \{\pm 1\}$ an orthogonal representation. Then

$$\pi_{(H,\phi)^G}\big(c_+(a_G^O(\rho)) \; - \; a_G(c(\rho)) \big) = \sum_{\substack{(H_0,\phi_0)<..<(H_r,\phi_r) \\ (H_0,\phi_0)\in(H,\phi)^G \text{ , } \phi_r \text{ odd}}} \frac{|H_0|}{|G|} (-1)^r (n_{H_r} - 2) \; ,$$

where n_{H_r} denote the number of elements in the set $\{\phi_r : H_r \to \{\pm 1\} \mid \phi_r \,\mathrm{odd}\}$.

Proof. Let $(H,\phi)^G \in \mathcal{M}(G)/_G$. The coefficient B of $(H,\phi)^G$ in $a_G(c(\rho))$ is

$$B = \frac{|H|}{|G|} \left(\sum_{\substack{(H_0,\phi_0)<..<(H_r,\phi_r) \\ (H_0,\phi_0)\in(H,\phi)^G}} (-1)^r \langle \phi_r \rangle + \sum_{\substack{(H_0,\phi_0)<..<(H_r,\lambda_r) \\ (H_0,\phi_0)\in(H,\phi)^G}} (-1)^r \langle \lambda_r \rangle \right)$$

The coefficient C of $(H,\phi)^G$ in $c_+(a_G^O(\rho))$ turns out to be the sum of a lot of partial sums coming from different kinds of chains. Since c_+ is a homomorphism, C is the coefficient of $(H,\phi)^G$ in

$$\sum_{(H_0,\Psi_0)<\cdot\cdot<(H_r,\Psi_r)} (-1)^r \frac{|H_0|}{|G|} m(\mathrm{Res}^G_{H_r}(\rho),\Psi_r)\; c_+((H_0,\Psi_0)^G)\,.$$

Since $(H,\phi)^G$ can only have a nontrivial contribution from elements of the form $c_+((H_0,\Psi_0)^G)$ if $(H_0,\Psi_0)^G$ is either $(H,2\phi)^G$ or $(H,\phi+\phi')^G$ or $(K,\psi)^G$ with $(H,\phi)^G \prec (K,\psi)^G$, we can express C as

$$C = \sum_{\substack{(H_0,2\phi_0)\in\mathcal{M}^O(G)\\ (H_0,2\phi_0)^G=(H,2\phi)^G}} C_{H_0,2\phi_0} + \sum_{\substack{(H_0,\phi_0+\phi_0')\in\mathcal{M}^O(G)\\ (H_0,\phi_0+\phi_0')^G=(H,\phi+\phi')^G}} C_{H_0,\phi_0+\phi_0'} + \sum_{\substack{(H_0,\psi_0)\in\mathcal{M}^O(G)\\ (H_0,\psi_0)^G\prec(H,\phi)^G}} C_{H_0,\psi_0}$$

where the second subsum runs over all possible pairs $\phi+\phi'$, and, for $(\widetilde{H},\widetilde{\Psi})\in\mathcal{M}^O(G)$,

$$C_{\widetilde{H},\widetilde{\Psi}} := \pi_{(H,\phi)^G}\left(\sum_{\substack{(H_0,\Psi_0)<\cdot\cdot<(H_r,\Psi_r)\\ (H_0,\Psi_0)=(\widetilde{H},\widetilde{\Psi})}} (-1)^r \frac{|H_0|}{|G|} m(\mathrm{Res}^G_{H_r}(\rho),\Psi_r)\; c_+((\widetilde{H},\widetilde{\Psi})^G)\right)\,.$$

Now we calculate these coefficients $C_{\widetilde{H},\widetilde{\Psi}}$ in each of the three cases.

We begin with the case $(\widetilde{H},\widetilde{\Psi})^G=(\widetilde{H},2\widetilde{\phi})^G=(H,2\phi)^G$ and can, without causing confusion, change notation back to H instead of $\widetilde{H}$ and ϕ instead of $\widetilde{\phi}$. Using the explicit formula for the multiplicities, we split $C_{H,2\phi}$ into $\frac{|H|}{|G|}\sum_{i=1}^6 C_i(H,2\phi)$, where

$$\begin{aligned}
C_1(H,2\phi) &:= \textstyle\sum_{(H,2\phi)<\cdot\cdot<(H_r,2\phi_r)} \quad (-1)^r 2\,[\langle\phi_r\rangle/2]\\
C_2(H,2\phi) &:= \textstyle\sum_{\substack{(H,2\phi)<\cdot\cdot<(H_i,2\phi_i)<(H_{i+1},\phi_{i+1}+\phi'_{i+1})<\\ <\cdot\cdot<(H_r,\phi_r+\phi'_r)\,,\ \phi_r,\phi'_r\ \mathrm{odd}}} \quad (-1)^r 2\cdot 1\\
C_3(H,2\phi) &:= \textstyle\sum_{\substack{(H,2\phi)<\cdot\cdot<(H_i,2\phi_i)<\\ <(H_{i+1},\lambda_{i+1}+\overline{\lambda}_{i+1})<\cdot\cdot<(H_r,\lambda_r+\lambda'_r)}} (-1)^r 2\langle\lambda_r\rangle\\
C_4(H,2\phi) &:= \textstyle\sum_{\substack{(H,2\phi)<\cdot\cdot<(H_r,2\phi_r)<\\ <(K_1,\psi_1)<\cdot\cdot<(K_s,\psi_s)}} \quad (-1)^{r+s} 2\langle\psi_s\rangle\\
C_5(H,2\phi) &:= \textstyle\sum_{\substack{(H,2\phi)<\cdot\cdot<(H_i,2\phi_i)<(H_{i+1},\phi_{i+1}+\phi'_{i+1})<\cdot\cdot<\\ <(H_r,\phi_r+\phi'_r)<(K_1,\psi_1)<\cdot\cdot<(K_s,\psi_s)}} \quad (-1)^{r+s} 2\langle\psi_s\rangle\\
C_6(H,2\phi) &:= \textstyle\sum_{\substack{(H,2\phi)<\cdot\cdot<(H_i,2\phi_i)<(H_{i+1},\lambda_{i+1}+\overline{\lambda}_{i+1})<\cdot\cdot<\\ <(H_r,\lambda_r+\overline{\lambda}_r)<(K_1,\psi_1)<\cdot\cdot<(K_s,\psi_s)}} \quad (-1)^{r+s} 2\langle\psi_s\rangle
\end{aligned}$$

Now we will modify these terms using the following observations:

1) Of course we can identify any chain of the form $((H,2\phi)<\cdot\cdot<(H',2\phi'))$ in $\mathcal{M}^O(G)$ with the corresponding chain $((H,\phi)<\cdot\cdot<(H',\phi'))$ in $\mathcal{M}(G)$.

2) Clearly $2\,[\langle\phi_r\rangle/2]$ equals $\langle\phi_r\rangle-1$, if ϕ_r odd, and coincides with $\langle\phi_r\rangle$ otherwise.

3) Any chain $((H,2\phi)<\cdot\cdot<(H_r,\lambda_r+\overline{\lambda}_r))$ in $\mathcal{M}^O(G)$ corresponds uniquely to the pair of chains given by $((H,\phi)<\cdot\cdot<(H_r,\lambda_r))$ and $((H,\phi)<\cdot\cdot<(H_r,\overline{\lambda}_r))$. Thus, taking the sum over all chains $((H,\phi)<\cdot\cdot<(H_r,\lambda_r))$ instead of $((H,2\phi)<\cdot\cdot<(H_r,\lambda_r+\overline{\lambda}_r))$ gives twice as many summands, and this will take care of the factor 2.

4) A chain

$$((H,2\phi)\lessdot..\lessdot(H_r,2\phi_r)\lessdot(K_1,\psi_1)\lessdot..\lessdot(K_s,\psi_s))$$

of length $r+s$ breaks uniquely up into the *lower* chain $((H,2\phi)\lessdot..\lessdot(H_r,2\phi_r))$ in $\mathcal{M}^O(K_1,\psi_1)$ of length r and the *upper* chain $((K_1,\psi_1)\lessdot..\lessdot(K_s,\psi_s))$ of length $s-1$.

5) A chain

$$((H,2\phi)\lessdot..\lessdot(H_r,\phi_r+\phi'_r)\lessdot(K_1,\psi_1)\lessdot..\lessdot(K_s,\psi_s))$$

of length $r+s$ breaks up uniquely into the *lower* chain $((H,2\phi)\lessdot..\lessdot(H_r,\phi_r+\phi'_r))$ in $\mathcal{M}^O(K_1,\psi_1)$ of length r and the *upper* chain $((K_1,\psi_1)\lessdot..\lessdot(K_s,\psi_s))$ of length $s-1$.

6) A chain $((H,2\phi)\lessdot..\lessdot(H_r,\lambda_r+\overline{\lambda}_r)\lessdot(K_1,\psi_1)\lessdot..\lessdot(K_s,\psi_s))$ of length $r+s$ breaks up uniquely into the *lower* chain $((H,2\phi)\lessdot..\lessdot(H_r,\lambda_r+\overline{\lambda}_r))$ in $\mathcal{M}^O(K_1,\psi_1)$ of length r and the *upper* chain $((K_1,\psi_1)\lessdot..\lessdot(K_s,\psi_s))$ of length $s-1$.

Thus we have

$$\begin{aligned}
C_1(H,2\phi) &= \textstyle\sum_{(H,\phi)\lessdot..\lessdot(H_r,\phi_r)} (-1)^r\langle\phi_r\rangle - \sum_{\substack{(H,\phi)\lessdot..\lessdot(H_r,\phi_r)\\ \phi_r \text{ odd}}} (-1)^r\\
C_2(H,2\phi) &= 2\textstyle\sum_{\substack{(H,2\phi)\lessdot..\lessdot(H_r,\phi_r+\phi'_r)\\ \phi_r,\phi'_r \text{ odd}}} (-1)^r\cdot 1\\
C_3(H,2\phi) &= \textstyle\sum_{(H,\phi)\lessdot..\lessdot(H_r,\lambda_r)} (-1)^r\langle\lambda_r\rangle\\
C_4(H,2\phi) &= 2\textstyle\sum_{(K,\psi)>(H,2\phi)}\sum_{\substack{(H,\phi)\lessdot..\lessdot(H_r,\phi_r)\\ \text{in } \mathcal{M}(K,\psi)}}\sum_{(K,\psi)\lessdot..\lessdot(K_s,\psi_s)}(-1)^{r+s+1}\langle\psi_s\rangle\\
C_5(H,2\phi) &= 2\textstyle\sum_{(K,\psi)>(H,2\phi)}\sum_{\substack{(H,2\phi)\lessdot..\lessdot(H_r,\phi_r+\phi'_r)\\ \text{in } \mathcal{M}^O(K,\psi)}}\sum_{(K,\psi)\lessdot..\lessdot(K_s,\psi_s)}(-1)^{r+s+1}\langle\psi_s\rangle\\
C_6(H,2\phi) &= \textstyle\sum_{(K,\psi)>(H,2\phi)}\sum_{\substack{(H,\phi)\lessdot..\lessdot(H_r,\lambda_r)\\ \text{in } \mathcal{M}(K,\psi)}}\sum_{(K,\psi)\lessdot..\lessdot(K_s,\psi_s)}(-1)^{r+s+1}\langle\psi_s\rangle
\end{aligned}$$

Next we study the case

$$(\widetilde{H},\widetilde{\Psi})^G=(\widetilde{H},\widetilde{\phi}+\widetilde{\phi}')^G=(H,\phi+\phi')^G$$

for some $\phi'\neq\phi$. Again we change notation and write H instead of $\widetilde{H}$ and ϕ,ϕ' instead of $\widetilde{\phi},\widetilde{\phi}'$. The explicit formula for the multiplicities transform $C_{H,\phi+\phi'}$ into

$$C_{H,\phi+\phi'}=$$

$$\frac{|H|}{|G|}m_{H,\phi+\phi'}\left(\sum_{\substack{(H,\phi+\phi')\lessdot..\lessdot(H_r,\phi_r+\phi'_r)\lessdot\\ \lessdot(K_1,\psi_1)\lessdot..\lessdot(K_s,\psi_s)}}(-1)^{r+s+1}\langle\psi_s\rangle+\sum_{\substack{(H,\phi+\phi')\lessdot..\lessdot(H_r,\phi_r+\phi'_r)\\ \phi_r,\phi'_r \text{ odd}}}(-1)^r\right)$$

with $m_{H,\phi+\phi'}=2$, if $(H,\phi)^G=(H,\phi')^G$, and $m_{H,\phi+\phi'}=1$ otherwise.

We split any chain $((H,\phi+\phi')\lessdot..\lessdot(H_r,\phi_r+\phi'_r)\lessdot(K_1,\psi_1)\lessdot..\lessdot(K_s,\psi_s))$ of length $r+s$ into the *lower* chain $((H,\phi+\phi')\lessdot..\lessdot(H_r,\phi_r+\phi'_r))$ in $\mathcal{M}^O(K_1,\psi_1)$ of length

r and the *upper* chain $((K_1,\psi_1)\lessdot..\lessdot(K_s,\psi_s))$ of length $s-1$. So, if

$$C_7(H,\phi+\phi') \quad := m_{H,\phi+\phi'} \sum_{(K,\psi)>(H,\phi+\phi')} \sum_{\substack{(H,\phi+\phi')\lessdot..\lessdot(H_r,\phi_r+\phi'_r)\\ \text{in } \mathcal{M}^O(K,\psi)}} \sum_{(K,\psi)\lessdot..\lessdot(K_s,\psi_s)} (-1)^{r+s+1}\langle\psi_s\rangle$$

and

$$C_8(H,\phi+\phi') \quad := m_{H,\phi+\phi'} \sum_{\substack{(H,\phi+\phi')\lessdot..\lessdot(H_r,\phi_r+\phi'_r)\\ \phi_r,\phi'_r \text{ odd}}} (-1)^r\ ,$$

then $C_{H,\phi+\phi'} = \frac{|H|}{|G|}\left(\ C_7(H,\phi+\phi') + C_8(H,\phi+\phi')\ \right)$.

Finally we expand $C_{\widetilde{H},\widetilde{\Psi}}$ in the case $(\widetilde{H},\widetilde{\Psi})^G = (K,\psi)^G \succ (H,\phi)^G$ for some $(K,\psi)\in\mathcal{M}^O(G)$.

By definition of c_+, the coefficient of $(H,\phi)^G$ in $c_+((K,\psi)^G)$ is

$$\sum_{\substack{(H_0,\phi_0)\in(H,\phi)^G\\ (H_0,\phi_0)\prec(K,\psi)}} \frac{|H_0|}{|K|}\left(\sum_{\substack{(H_0,\phi_0)\lessdot..\lessdot(H_r,\phi_r)\\ \text{in } \mathcal{M}(K,\psi)}} (-1)^r\left\langle \mathrm{Res}^K_{H_r}(\psi);\phi_r\right\rangle + \sum_{\substack{(H_0,\phi_0)\lessdot..\lessdot(H_r,\lambda_r)\\ \text{in } \mathcal{M}(K,\psi)}} (-1)^r\left\langle \mathrm{Res}^K_{H_r}(\psi);\lambda_r\right\rangle\right).$$

Therefore, the coefficient $C_{K,\psi}$ of $(H,\phi)^G$ in

$$\sum_{(K,\psi)\lessdot..\lessdot(K_s,\psi_s)} (-1)^s\frac{|K|}{|G|} m(\mathrm{Res}^G_K(\rho),\psi)\ (\otimes\mathbb{C}_+)((K,\psi)^G)$$

is given by

$$C_{K,\psi} = \sum_{(K,\psi)\lessdot..\lessdot(K_s,\psi_s)} (-1)^s\tfrac{|K|}{|G|}\langle\psi_s\rangle \sum_{\substack{(H_0,\phi_0)\in(H,\phi)^G\\ (H_0,\phi_0)\prec(K,\psi)}} \tfrac{|H|}{|K|}\cdot$$

$$\left(\sum_{\substack{(H_0,\phi_0)\lessdot..\lessdot(H_r,\phi_r)\\ \text{in } \mathcal{M}(K,\psi)}} (-1)^r\left\langle \mathrm{Res}^K_{H_r}(\psi);\phi_r\right\rangle + \sum_{\substack{(H_0,\phi_0)\lessdot..\lessdot(H_r,\lambda_r)\\ \text{in } \mathcal{M}(K,\psi)}} (-1)^r\left\langle \mathrm{Res}^K_{H_r}(\psi);\lambda_r\right\rangle\right)$$

This we split into several subsums and obtain $C_{K,\psi} = \frac{|H|}{|G|}\sum_{i=4}^{7} C_i(K,\psi)$, where

$$C_4(K,\psi) \quad := \sum_{(K,\psi)\lessdot..\lessdot(K_s,\psi_s)} \sum_{\substack{(H_0,\phi_0)\in(H,\phi)^G\\ (H_0,\phi_0)\prec(K,\psi)}} \sum_{\substack{(H_0,\phi_0)\lessdot..\lessdot(H_r,\phi_r) \text{ in } \mathcal{M}(K,\psi)\\ \mathrm{Res}^K_{H_r}(\psi)=2\phi_r}} (-1)^{r+s}\langle\psi_s\rangle\cdot 2$$

$$C_5(K,\psi) \quad := \sum_{(K,\psi)\lessdot..\lessdot(K_s,\psi_s)} \sum_{\substack{(H_0,\phi_0)\in(H,\phi)^G\\ (H_0,\phi_0)\prec(K,\psi)}} \sum_{\substack{(H_0,\phi_0)\lessdot..\lessdot(H_r,\phi_r) \text{ in } \mathcal{M}(K,\psi)\\ \mathrm{Res}^K_{H_r}(\psi)=\phi_r+\phi'_r,\mathrm{Res}^K_{H_0}(\psi)=2\phi_0}} (-1)^{r+s}\langle\psi_s\rangle$$

$$C_6(K,\psi) \quad := \sum_{(K,\psi)\lessdot..\lessdot(K_s,\psi_s)} \sum_{\substack{(H_0,\phi_0)\in(H,\phi)^G\\ (H_0,\phi_0)\prec(K,\psi)}} \sum_{\substack{(H_0,\phi_0)\lessdot..\lessdot(H_r,\lambda_r) \text{in } \mathcal{M}(K,\psi)\\ \mathrm{Res}^K_{H_0}(\psi)=2\phi_0}} (-1)^{r+s}\langle\psi_s\rangle$$

$$C_7(K,\psi) \quad := \sum_{(K,\psi)\lessdot..\lessdot(K_s,\psi_s)} \sum_{\substack{(H_0,\phi_0)\in(H,\phi)^G\\ (H_0,\phi_0)\prec(K,\psi)}} \sum_{\substack{(H_0,\phi_0)\lessdot..\lessdot(H_r,\phi_r) \text{ in } \mathcal{M}(K,\psi)\\ \mathrm{Res}^K_{H_0}(\psi)=\phi_0+\phi'_0}} (-1)^{r+s}\langle\psi_s\rangle$$

Bringing all these expressions together, we conclude

$$\begin{aligned} C &= \sum_{\substack{(H_0,2\phi_0)\\ \in(H,2\phi)^G}} C_{H_0,2\phi_0} + \sum_{(H,\phi+\phi')^G} \sum_{\substack{(H_0,\phi_0+\phi_0')\\ \in(H,\phi+\phi')^G}} C_{H_0,\phi_0+\phi_0'} + \sum_{\substack{(K,\psi)\in\mathcal{M}^O(G)\\ (K,\psi)^G\succ(H,\phi)^G}} C_{K,\psi} \\ &= \frac{|H|}{|G|}\left(\sum_{i=1}^{6} \sum_{\substack{(H_0,2\phi_0)\\ \in(H,2\phi)^G}} C_i(H_0,2\phi_0) + \sum_{i=7}^{8} \sum_{(H,\phi+\phi')^G} \sum_{\substack{(H_0,\phi_0+\phi_0')\\ \in(H,\phi+\phi')^G}} C_i(H_0,\phi_0+\phi_0') \right. \\ &\qquad\qquad \left. + \sum_{i=4}^{7} \sum_{\substack{(K,\psi)\in\mathcal{M}^O(G)\\ (K,\psi)^G\succ(H,\phi)^G}} C_i(K,\psi)\right). \end{aligned}$$

Some of these terms cancel each other. The terms involving C_4 vanish, as

$$\begin{aligned} &\sum_{\substack{(H_0,2\phi_0)\\ \in(H,2\phi)^G}} C_4(H_0,2\phi_0) + \sum_{\substack{(K,\psi)\in\mathcal{M}^O(G)\\ (K,\psi)^G\succ(H,\phi)^G}} C_4(K,\psi) = \\ &\sum_{\substack{(H_0,2\phi_0)\\ \in(H,2\phi)^G}} 2 \sum_{(K,\psi)>(H_0,2\phi_0)} \sum_{\substack{(H_0,\phi_0)<..<(H_r,\phi_r)\\ \text{in } \mathcal{M}(K,\psi)}} \sum_{(K,\psi)<..<(K_s,\psi_s)} (-1)^{r+s+1}\langle\psi_s\rangle \\ &+ \sum_{\substack{(K,\psi)\in\mathcal{M}^O(G)\\ (K,\psi)^G\succ(H,\phi)^G}} \sum_{(K,\psi)<..<(K_s,\psi_s)} \sum_{\substack{(H_0,\phi_0)\in(H,\phi)^G\\ (H_0,\phi_0)\prec(K,\psi)}} \sum_{\substack{(H_0,\phi_0)<..<(H_r,\phi_r) \text{ in } \mathcal{M}(K,\psi)\\ \mathrm{Res}^K_{H_r}(\psi)=2\phi_r}} 2(-1)^{r+s}\langle\psi_s\rangle \\ &= 0. \end{aligned}$$

The C_5-terms add up to 0, because

$$\begin{aligned} &\sum_{\substack{(H_0,2\phi_0)\\ \in(H,2\phi)^G}} C_5(H_0,2\phi_0) + \sum_{\substack{(K,\psi)\in\mathcal{M}^O(G)\\ (K,\psi)^G\succ(H,\phi)^G}} C_5(K,\psi) \\ &= \sum_{\substack{(H_0,2\phi_0)\\ \in(H,2\phi)^G}} 2 \sum_{(K,\psi)>(H_0,2\phi_0)} \sum_{\substack{(H_0,2\phi_0)<..<(H_r,\phi_r+\phi_r')\\ \text{in } \mathcal{M}^O(K,\psi)}} \sum_{(K,\psi)<..<(K_s,\psi_s)} (-1)^{r+s+1}\langle\psi_s\rangle \\ &+ \sum_{\substack{(K,\psi)\in\mathcal{M}^O(G)\\ (K,\psi)^G\succ(H,\phi)^G}} \sum_{(K,\psi)<..<(K_s,\psi_s)} \sum_{\substack{(H_0,\phi_0)\in(H,\phi)^G\\ (H_0,\phi_0)\prec(K,\psi)}} \sum_{\substack{(H_0,\phi_0)<..<(H_r,\phi_r) \text{ in } \mathcal{M}(K,\psi)\\ \mathrm{Res}^K_{H_r}(\psi)=\phi_r+\phi_r',\mathrm{Res}^K_{H_0}(\psi)=2\phi_0}} (-1)^{r+s}\langle\psi_s\rangle \end{aligned}$$

Indeed, every chain $(H_0<..<H_r)$ of subgroups in K determine exactly two chains in $\mathcal{M}(K,\psi)$, namely $((H_0,\phi_0)<..<(H_r,\phi_r))$ and $((H_0,\phi_0)<..<(H_r,\phi_r'))$.

Also the C_6-terms cancel each other, as

$$\sum_{\substack{(H_0,2\phi_0)\\ \in(H,2\phi)^G}} C_6(H_0,2\phi_0)+\sum_{\substack{(K,\psi)\in\mathcal{M}^O(G)\\ (K,\psi)^G\succ(H,\phi)^G}} C_6(K,\psi)$$

$$=\sum_{\substack{(H_0,2\phi_0)\\ \in(H,2\phi)^G}}\sum_{(K,\psi)>(H_0,2\phi_0)}\sum_{\substack{(H_0,\phi_0)<..<(H_r,\lambda_r)\\ \text{in }\mathcal{M}^O(K,\psi)}}\sum_{(K,\psi)<..<(K_s,\psi_s)}(-1)^{r+s+1}\langle\psi_s\rangle$$

$$+\sum_{\substack{(K,\psi)\in\mathcal{M}^O(G)\\ (K,\psi)^G\succ(H,\phi)^G}}\sum_{(K,\psi)<..<(K_s,\psi_s)}\sum_{\substack{(H_0,\phi_0)\in(H,\phi)^G\\ (H_0,\phi_0)\prec(K,\psi)}}\sum_{\substack{(H_0,\phi_0)<..<(H_r,\lambda_r)\text{in }\mathcal{M}(K,\psi)\\ \mathrm{Res}^K_{H_0}(\psi)=2\phi_0}}(-1)^{r+s}\langle\psi_s\rangle$$

$$=0$$

Finally, the terms with C_7 cancel out, since

$$\sum_{(H,\phi+\phi')^G}\sum_{\substack{(H_0,\phi_0+\phi_0')\\ \in(H,\phi+\phi')^G}} C_7(H_0,\phi_0+\phi_0')+\sum_{\substack{(K,\psi)\in\mathcal{M}^O(G)\\ (K,\psi)^G\succ(H,\phi)^G}} C_7(K,\psi)$$

$$=\sum_{(H,\phi+\phi')^G}\sum_{\substack{(H_0,\phi_0+\phi_0')\\ \in(H,\phi+\phi')^G}} m_{H_0,\phi_0+\phi_0'}\sum_{(K,\psi)>(H_0,\phi_0+\phi_0')}$$

$$\sum_{\substack{(H_0,\phi_0+\phi_0')<..<(H_r,\phi_r+\phi_r')\\ \text{in }\mathcal{M}^O(K,\psi)}}\sum_{(K,\psi)<..<(K_s,\psi_s)}(-1)^{r+s+1}\langle\psi_s\rangle$$

$$+\sum_{\substack{(K,\psi)\in\mathcal{M}^O(G)\\ (K,\psi)^G\succ(H,\phi)^G}}\sum_{(K,\psi)<..<(K_s,\psi_s)}\sum_{\substack{(H_0,\phi_0)\in(H,\phi)^G\\ (H_0,\phi_0)\prec(K,\psi)}}\sum_{\substack{(H_0,\phi_0)<..<(H_r,\phi_r)\text{ in }\mathcal{M}(K,\psi)\\ \mathrm{Res}^K_{H_0}(\psi)=\phi_0+\phi_0'}}(-1)^{r+s}\langle\psi_s\rangle$$

In fact this is 0, because if we have a pair $(K,\psi)>(H_0,\phi_0+\phi_0')$ with $(H_0,\phi_0)^G=(H,\phi)^G\neq(H_0,\phi_0')^G$, the $m_{H_0,\phi_0+\phi_0'}=1$, and taking sums over chains in $\mathcal{M}^O(K,\psi)$ starting in $(H_0,\phi_0+\phi_0')$ is the same as taking sums over chains in $\mathcal{M}(K,\psi)$ starting in (H_0,ϕ_0). Otherwise, if a pair $(K,\psi)>(H_0,\phi_0+\phi_0')$ satisfies $(H_0,\phi_0)^G=(H,\phi)^G=(H_0,\phi_0')^G$, then $m_{H_0,\phi_0+\phi_0'}=2$, and taking sums over chains in $\mathcal{M}(K,\psi)$ starting in (H_0,ϕ_0) or (H_0,ϕ_0') is twice as much as taking sums over chains in $\mathcal{M}^O(K,\psi)$ starting in $(H_0,\phi_0+\phi_0')$.

So C reduces to

$$C=\frac{|H|}{|G|}\left(\sum_{i=1}^{3}\sum_{\substack{(H_0,2\phi_0)\\ \in(H,2\phi)^G}}C_i(H_0,2\phi_0)+\sum_{(H,\phi+\phi')^G}\sum_{\substack{(H_0,\phi_0+\phi_0')\\ \in(H,\phi+\phi')^G}}C_8(H_0,\phi_0+\phi_0')\right).$$

Now we can compare B and C, which is the coefficient of $(H,\phi)^G$ in $c_+(a_G^O(\rho))-a_G(c(\rho)$. We notice that, taking the expressions for B and C into account,

$$\frac{|H|}{|G|}\sum_{(H_0,\phi_0)\in(H,\phi)^G}\quad(C_1(H_0,2\phi_0)+C_3(H_0,2\phi_0))$$

$$=B+\frac{|H|}{|G|}\sum_{(H_0,\phi_0)\in(H,\phi)^G}\sum_{\substack{(H_0,\phi_0)<..<(H_r,\phi_r)\\ \phi_r\text{ odd}}}\quad(-1)^r\ .$$

Thus we have to compute

$$\sum_{(H_0,\phi_0)\in(H,\phi)^G} \left(\sum_{\substack{(H_0,2\phi_0)<..<(H_r,\phi_r+\phi'_r)\\ \phi_r,\phi'_r \text{ odd}}} 2(-1)^r - \sum_{\substack{(H_0,\phi_0)<..<(H_r,\phi_r)\\ \phi_r \text{ odd}}} (-1)^r \right)$$

and

$$\sum_{(H,\phi+\phi')^G} \sum_{(H_0,\phi_0+\phi'_0)\in(H,\phi+\phi')^G} C_8(H_0,\phi_0+\phi'_0).$$

The first term can be simplified by the following observation. Let $(H_0,\phi_0)\in(H,\phi)^G$ fixed and, for $H_0 \leqslant H_r \leqslant G$, let $X_{H_r,\phi_0} := \{\phi_r : H_r \to \pm 1 | \mathrm{Res}^{H_r}_{H_0}(\phi_r) = \phi_0, \phi_r \text{ odd}\}$ and $n = n_{H_r,\phi_0}$ denote the number of elements in X_{H_r,ϕ_0}. Then there are precisely n elements $(H_r,\phi_r)\in\mathcal{M}(G)$ with $(H_0,\phi_0)\leqslant(H_r,\phi_r)$ and ϕ_r odd, and precisely $\frac{n(n-1)}{2}$ elements $(H_r,\phi_r+\phi'_r)\in\mathcal{M}(G)$ with $(H_0,2\phi_0)\leqslant(H_r,\phi_r+\phi'_r)$ and ϕ_r,ϕ'_r odd. Thus

$$\begin{aligned}\sum_{\substack{(H_0,2\phi_0)<..<(H_r,\phi_r+\phi'_r)\\ \phi_r,\phi'_r \text{ odd}}} 2(-1)^r &= \sum_{H_0<..<H_r} 2\frac{n_{H_r,\phi_0}(n_{H_r,\phi_0}-1)}{2}(-1)^r \\ &= \sum_{\substack{(H_0,\phi_0)<..<(H_r,\phi_r)\\ \phi_r \text{ odd}}} (-1)^r(n_{H_r,\phi_0}-1)\end{aligned}$$

So the first term can be rewritten as

$$\sum_{(H_0,\phi_0)\in(H,\phi)^G} \sum_{\substack{(H_0,\phi_0)<..<(H_r,\phi_r)\\ \phi_r \text{ odd}}} (-1)^r(n_{H_r,\phi_0}-2)$$

Next we reduce the second expression. Let $(H_0,\phi_0)\in(H,\phi)^G$ fixed. For $H_0 \leqslant H_r \leqslant G$ let $X'_{H_r,\phi_0} := \{\phi_r : H_r \to \pm 1 | \mathrm{Res}^{H_r}_{H_0}(\phi_r) \neq \phi_0, \phi_r \text{ odd}\}$ and $n' = n'_{H_r,\phi_0}$ denote the number of elements in X'_{H_r,ϕ_0}. Now

$$\sum_{\phi'_0\neq\phi_0} \sum_{\substack{(H_0,\phi_0+\phi'_0)<..<(H_r,\phi_r+\phi'_r)\\ \phi_r,\phi'_r \text{ odd}}} (-1)^r = \sum_{\substack{(H_0,\phi_0)<..<(H_r,\phi_r)\\ \phi_r \text{ odd}}} (-1)^r n'_{H_r,\phi_0}\ ,$$

since, for any chain $(H_0,\phi_0)<..<(H_r,\phi_r)$ with ϕ_r odd, a chain $(H_0,\phi_0+\phi'_0)<..<(H_r,\phi_r+\phi'_r)$ with ϕ_r,ϕ'_r odd determines and is determined by $\phi'_r \in X'_{H_r,\phi_0}$. Furthermore, for (H_r,ϕ_r) fixed with $(H_r,\phi_r)^G \succeq (H,\phi)^G$,

$$\sum_{\substack{(H_0,\phi_0+\phi'_0)\\ (H_0,\phi_0+\phi'_0)^G\succeq(H,\phi_0)^G}} \sum_{\substack{(H_0,\phi_0+\phi'_0)<..<(H_r,\phi_r+\phi'_r)\\ \phi_r,\phi'_r \text{ odd}}} m_{H_0,\phi_0+\phi'_0}(-1)^r = \sum_{(H_0,\phi_0)\in(H,\phi)^G} \sum_{\substack{(H_0,\phi_0)<..<(H_r,\phi_r)\\ \phi_r \text{ odd}}} (-1)^r n'_{H_r,\phi_0},$$

as $m_{H_0,\phi_0+\phi'_0} = 1$ in the case $(H_0,\phi_0)^G \neq (H_0,\phi'_0)^G$ and $m_{H_0,\phi_0+\phi'_0} = 2$ in the case $(H_0,\phi_0)^G = (H_0,\phi'_0)^G$. Finally, we may take the sum over all (H_r,ϕ_r) to rewrite the second expression as

$$\sum_{(H_0,\phi_0)\in(H,\phi)^G} \sum_{\substack{(H_0,\phi_0)<..<(H_r,\phi_r)\\ \phi_r \text{ odd}}} (-1)^r n'_{H_r,\phi_0}\ .$$

Taking these simplifications into account, we conclude from $n_{H_r,\phi_0} + n'_{H_r,\phi_0} = n_{H_r}$, that

$$C - B = \frac{|H|}{|G|} \sum_{\substack{(H_0,\phi_0)<..<(H_r,\phi_r)\\ (H_0,\phi_0)\in(H,\phi)^G,\ \phi_r \text{ odd}}} (-1)^r(n_{H_r}-2)$$

which completes the proof of lemma 6.12. □

Theorem 6.13.
The map a_G^O induces an explicit induction formula, that is, for $\rho : G \to \mathrm{O}(2n)$ in $\mathcal{F}^O(G)$,

$$b_G^O(a_G^O(\rho)) = \kappa_G(\rho) \in R^O(G)\ ,$$

where $\kappa_G : \mathcal{F}^O(G) \to R^O(G)$ is the map of §6.1.

Proof. We study the diagram

$$\begin{tikzcd}
\mathcal{F}^O(G) \arrow[r, "a_G^O"] \arrow[d, "c\circ\kappa_G"'] & \mathbb{Q}R_+^O(G) \arrow[r, "b_G^O"] \arrow[d, "c_+"'] & \mathbb{Q}R^O(G) \arrow[d, "c"'] \\
\mathbb{Q}R(G) \arrow[r, "a_G"] & \mathbb{Q}R_+(G) \arrow[r, "b_G"] & \mathbb{Q}R(G)
\end{tikzcd}\ .$$

Note that the right square is a commutative diagram, since all maps involved are homomorphisms and a_G, used in the definition of c_+, is a section for b_G. Indeed

$$\begin{aligned}
b_G(c_+((H,\Psi)^G)) &= b_G(\mathrm{Ind}_H^G(a_H(c(\Psi)))\\
= \mathrm{Ind}_H^G(b_H a_H(c(\Psi))) &= \mathrm{Ind}_H^G(c(\Psi)) = c(\mathrm{Ind}_H^G(\Psi)).
\end{aligned}$$

Since c_+ is injective, it suffices to show

$$c_+((b_G^O \circ a_G^O)(\rho)) = c(\kappa_G(\rho))\ \ (=\ (b_G \circ a_G)(c\circ\kappa_G)(\rho)\,)\ .$$

Therefore it is enough to prove that the defect on the commutativity in the left square, given by (6.9), lies in the kernel of b_G. In fact, for $\rho : G \to \mathrm{O}(2n)$ and $n_H(\rho)$ as in 6.9, we calculate

$$\begin{aligned}
& b_G\left(c_+(a_G^O(\rho))\ -\ a_G(c(\rho))\right)\\
&= b_G\left(\sum_{\substack{(H,\phi)\\ \phi\text{ odd}}}(n_H(\rho)-2)\sum_{\substack{(H_0,\phi_0)<..<(H_r,\phi_r)\\ (H_r,\phi_r)=(H,\phi)}}(-1)^r\tfrac{|H_0|}{|G|}(H_0,\phi_0)^G\right)\\
&= \sum_{\substack{(H,\phi)\\ \phi\text{ odd}}}(n_H(\rho)-2)\sum_{\substack{(H_0,\phi_0)<..<(H_r,\phi_r)\\ (H_r,\phi_r)=(H,\phi)}}(-1)^r\tfrac{|H_0|}{|G|}\mathrm{Ind}_{H_0}^G(\phi_0)\\
&= \sum_{\substack{(H,\phi)\\ \phi\text{ odd}}}(n_H(\rho)-2)\tfrac{|H|}{|G|}\mathrm{Ind}_H^G(\sum_{\substack{(H_0,\phi_0)<..<(H_r,\phi_r)\\ (H_r,\phi_r)=(H,\phi)}}(-1)^r\tfrac{|H_0|}{|H|}\mathrm{Ind}_{H_0}^H(\phi_0))\\
&= \sum_{\substack{(H,\phi)\\ \phi\text{ odd}}}(n_H(\rho)-2)\tfrac{|H|}{|G|}\mathrm{Ind}_H^G(\phi\sum_{\substack{H_0<..<H_r\\ H_r=H}}(-1)^r\tfrac{|H_0|}{|H|}\mathrm{Ind}_{H_0}^H(\mathbb{1}))\\
&= 0\ ,
\end{aligned}$$

since firstly, for $H \leqslant G$ a fixed noncyclic group,

$$\sum_{H_0<..<H_r=H}(-1)^r\frac{|H_0|}{|H|}\mathrm{Ind}_{H_0}^H(\mathbb{1}) = 0\ . \tag{1}$$

(see for example [**6**, III.1.4]) and secondly, if $H \leqslant G$ has a nontrivial contribution to the sum above, then $n_H(\rho) > 0$ (so there exists an odd ϕ) and $n_H(\rho) - 2 \neq 0$ (so $n_H(\rho) \geqslant 4$), so that H has an elementary abelian 2 group of order at least 4 as a factor group and can not be cyclic. □

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Olaf Neiße `neisse@fh-furtwangen.de`

Department of Computer & Electrical Engineering
University of Furtwangen (FH)
78120 Furtwangen, Germany

Victor P. Snaith `V.Snaith@sheffield.ac.uk`

Department of Pure Mathematics
Hicks Building, University of Sheffield
Sheffield S3 7RH, England