Homology, Homotopy and Applications, vol. 8(2), 2006, pp.115–131

DEFORMATIONS OF ASSOCIATIVE ALGEBRAS WITH INNER PRODUCTS

JOHN TERILLA AND THOMAS TRADLER

(*communicated by Jim Stasheff*)

Abstract

We develop the deformation theory of A_∞ algebras together with ∞-inner products and identify a differential graded Lie algebra that controls the theory. This generalizes the deformation theories of associative algebras, A_∞ algebras, associative algebras with inner products, and A_∞ algebras with inner products.

1. Introduction

A natural consideration for an algebraic structure in topology is whether it is a homotopy invariant. The C_∞ structure on the cochains of a space is a classic example. While manifolds are distinguished by the inner product afforded by Poincaré duality, an inner product is not a homotopy invariant concept. The right—meaning the homotopy robust—concept is an ∞-inner product as introduced in [**12**]. In algebraic generality, an ∞-inner product is defined in the setting of an A_∞ algebra. In this paper, we describe the deformation theory of A_∞ algebras together with ∞-inner products by giving a controlling differential graded Lie algebra.

An application that we have in mind involves string topology. It is known that if X and Y have the same homotopy type, then they have the same string topology operations [**1**]. One may assign an A_∞ algebra A_X with an ∞-inner product I_X to a Poincare duality space X. Based on results in [**11, 13**], it is reasonable to think that if the two differential graded Lie algebras controlling the deformations of (A_X, I_X) and (A_Y, I_Y) are quasi-isomorphic, then X and Y have the same string topology operations. One may speculate that the quasi-isomorphism class of the differential graded Lie algebra controlling the deformations (A_X, I_X) determines the "string topology type" of the space X (much the same way that the C_∞ structure on the cochains on a space determines the rational homotopy type of a space; see [**10**]). In any event, it would be interesting to probe this controlling differential graded Lie algebra for its invariants.

The authors would like to thank Jim Stasheff, Martin Markl, and Dennis Sullivan for many helpful discussions.
Received March 12, 2006, revised April 7, 2006, July 20, 2006; published on August 16, 2006.
2000 Mathematics Subject Classification: 55P10, 16S10, 16S80.
Key words and phrases: homotopy, inner product, deformation theory.
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Let us review the basic idea of a deformation theory governed by a differential graded Lie algebra [**9, 3, 5, 6**]. Fix a ground field k of characteristic 0. For any differential graded Lie algebra $(\mathfrak{g} = \oplus_i \mathfrak{g}^i, d, [\,,\,])$ over k, one can consider deforming the differential d in the direction of an inner derivation. Informally, such a deformation is given by an (equivalence classes of) α making

$$d_\alpha := d + \mathrm{ad}(\alpha)$$

into a differential. The map d_α is always a derivation and the condition that $d_\alpha^2 = 0$ translates into the Mauer–Cartan equation:

$$d\alpha + \frac{1}{2}[\alpha, \alpha] = 0.$$

The deformed differential d_α may involve parameters from the maximal ideal m of a $\mathbb{Z}$ graded Artin local ring: $\alpha \in (\mathfrak{g} \otimes_k m)^1$. If m is the maximal ideal of a local Artin ring R and $\alpha \in (\mathfrak{g} \otimes_k m)^1$ is a solution to the Mauer–Cartan equation, then one may call d_α a *deformation of d over R*. A ring map $R \to S$ will transport a deformation of d over R to a deformation of d over S.

More formally, one has a functor $Def_{\mathfrak{g}}$ from the category of $\mathbb{Z}$ graded Artin local rings with residue field k to the category of sets, assigning to such a ring R with maximal ideal m the set

$$Def_{\mathfrak{g}}(R) = \{\alpha \in (\mathfrak{g} \otimes_k m)^1 : d\alpha + \frac{1}{2}[\alpha, \alpha] = 0\}/\sim .$$

Here, $\sim$ is the equivalence relation determined by the action of the gauge group, which we now recall. Since R is an Artin ring, m is a nilpotent algebra, and $(\mathfrak{g} \otimes_k m)^0 \subseteq \mathfrak{g} \otimes_k m$ is a nilpotent Lie algebra. Therefore, there exists a group $G = \{\exp \beta : \beta \in (\mathfrak{g} \otimes_k m)^0\}$, called the gauge group, with multiplication defined by the Baker–Campbell–Hausdorff formula. The action of $e^\beta \in G$ on an element $\alpha \in (g \otimes_k m)^1$ is determined by the infinitesimal action:

$$\alpha \mapsto \beta \cdot \alpha = [\beta, \alpha] - d\beta, \quad \alpha \in (\mathfrak{g} \otimes m)^1,\ \beta \in (\mathfrak{g} \otimes m)^0.$$

This action satisfies

$$e^{\mathrm{ad}\,\beta} d_\alpha e^{-\mathrm{ad}\,\beta} = d_{e^\beta \cdot \alpha},$$

and preserves the set of solutions to the Maurer–Cartan equation.

In this paper, we work with A_∞ algebras equipped with ∞ inner products. One has the notion of a deformation of an A_∞ algebra with an ∞ inner product over a ring R, and there is a natural equivalence on the set of deformations. A ring map $R \to S$ transports deformations over R to deformations over S. The association

$$R \mapsto \left\{ \begin{array}{c} \text{deformations of the } A_\infty \text{ algebra with} \\ \text{the } \infty \text{ inner product over } R \end{array} \right\} \Big/ \left\{ \begin{array}{c} \text{equivalent} \\ \text{deformations} \end{array} \right\}$$

defines a covariant deformation functor.

We construct a differential graded Lie algebra $(\mathfrak{h} = \oplus_i \mathfrak{h}^i, d, [\,,\,])$ associated to an A_∞ algebra with an ∞ inner product, and prove that the functor described above is isomorphic to $Def_{\mathfrak{h}}$. This is the precise mathematical content of the statement *the differential graded Lie algebra* $(\mathfrak{h}, [\,,\,], d)$ *controls the deformations of the* A_∞ *algebra with an* ∞ *inner product.*

2. Definitions of A_∞ algebras and ∞ inner products

We now review the concept of an ∞ inner product on an A_∞ algebra [**12**], [**11**]. The concepts of A_∞ algebras, A_∞ bimodules, A_∞ bimodule maps, and A_∞ inner products are generalizations of the usual concepts of associative algebras, bimodules, bimodule maps, and invariant inner products.

2.1. A_∞ algebras

Let $V = \bigoplus_{j\in\mathbb{Z}} V^j$ be a graded module over a ring S. Recall that the suspension $V[1]$ of V is defined to be $V[1] = \bigoplus_{j\in\mathbb{Z}} (V[1])^j$ with $(V[1])^j := V^{j-1}$. For a graded S-module A, we denote by TA the tensor algebra of the suspended space $A[1]$, $TA = S \oplus A[1] \oplus A[1]^{\otimes 2} \oplus \dots$. An A_∞ *algebra over* S is defined to be a pair (A, D) where A is a graded S module and $D \in \mathrm{Coder}(TA)$ of degree -1 with $D^2 = 0$. In addition, we require the "no homotopy unit" convention—that D has no component $S \to TA$.

Suppose that (A, D) and (A', D') are A_∞ algebras over S. Then, an A_∞ map from (A', D') to (A, D) is a map $\lambda : TA' \to TA$ satisfying $\lambda \circ D' = D \circ \lambda$.

2.2. A_∞ bimodules

Let (A, D) be an A_∞ algebra over S, and let M be a graded S module. Let $T^M A$ denote the tensor bicomodule $T^M A := \bigoplus_{k,l\geqslant 0} A[1]^{\otimes k} \otimes M[1] \otimes A[1]^{\otimes l}$ of $M[1]$ over TA. An A_∞ *bimodule structure on* M *over* A is defined to be a coderivation $D^M \in \mathrm{Coder}_D(T^M A, T^M A)$ over D of degree -1 with $(D^M)^2 = 0$.

Let (M, D^M) and (N, D^N) be A_∞ bimodules over A. Let $\mathrm{Comap}(T^M A, T^N A)$ denote the maps $F : T^M A \to T^N A$ satisfying

$$\begin{array}{ccc} T^M A & \xrightarrow{\Delta^M} & (TA \otimes T^M A) \oplus (T^M A \otimes TA) \\ \downarrow{\scriptstyle F} & & \downarrow{\scriptstyle (\mathrm{Id}\otimes F)\oplus(F\otimes \mathrm{Id})} \\ T^N A & \xrightarrow[\Delta^N]{} & (TA \otimes T^M A) \oplus (T^M A \otimes TA) \end{array}$$

The space $\mathrm{Comap}(T^M A, T^N A)$ carries a differential defined by

$$\delta^{M,N}(F) := D^N \circ F - (-1)^{|F|} F \circ D^M.$$

In this case, an A_∞ *bimodule map from* M *to* N is defined to be an element $F \in \mathrm{Comap}(T^M A, T^N A)$ of degree 0 with $\delta^{M,N}(F) = 0$, i.e.

$$D^N \circ F = F \circ D^M.$$

2.3. ∞ inner products

For any $f \in \mathrm{Coder}(TA)$, there are induced coderivations $f^A \in \mathrm{Coder}_f(T^A A, T^A A)$ and $f^{A^*} \in \mathrm{Coder}_f(T^{A^*} A, T^{A^*} A)$, where $A^* = \hom_S(A, S)$ denotes the dual of A.

One also has an induced map

$$\delta_f : \mathrm{Comap}(T^M A, T^N A) \to \mathrm{Comap}(T^M A, T^N A)$$

given by $\delta_f(F) = f^{A^*} \circ F - (-1)^{|f||F|} \cdot F \circ f^A$. Note that, in particular, if (A, D) is an A_∞ algebra, then A and A^* have A_∞ bimodule structures given by D^A and D^{A^*}.

Definition 2.1. Let (A, D) be an A_∞ algebra over S. We define an ∞ *inner product on* A *over* S to be an A_∞ bimodule map I from A to A^*. Equivalently, an ∞ inner product is an element $I \in \mathrm{Comap}(T^A A, T^{A^*} A)$ satisfying

$$\delta_D(I) = D^{A^*} \circ I - I \circ D^A = 0.$$

Every inner product $\langle\, , \rangle : A \otimes A \to S$ defines an element $I \in \mathrm{Comap}(T^A A, T^{A^*} A)$. In this case, the condition $D^{A^*} \circ I - I \circ D^A = 0$ is equivalent to $\langle D(a_1, \ldots, a_n), a_{n+1}\rangle = \pm\langle a_1, D(a_2, \ldots, a_{n+1})\rangle$. See the appendix for additional illustrations.

2.4. Induced maps

Recall that if $\lambda : A' \to A$ is an algebra map between two associative algebras, then every module over A is also a module over A', and similarly for module maps. Also, $\lambda : A' \to A$ and $\lambda^* : A^* \to (A')^*$ will be module maps over A'. Here we give the corresponding homotopy generalizations.

Suppose that λ is an A_∞ map from (A', D') to (A, D). First, every A_∞ bimodule (M, D^M) over A is also an A_∞ bimodule over A', whose structure map is determined by the lowest components (which are maps $T^M A \to M$)

$$\begin{aligned}&(D^M)^\lambda(a'_1, \ldots, a'_k, m, a'_{k+1}, \ldots, a'_{k+l})\\ &\quad= \sum \pm pr_M \circ D^M(\lambda(a'_1, \ldots), \ldots, \lambda(\ldots, a'_k), m, \lambda(a'_{k+1}, \ldots), \ldots, \lambda(\ldots, a'_{k+l})).\end{aligned}$$

Here, pr_M denotes the projection onto M. The signs are given by the usual sign rule, namely introducing a sign $(-1)^{|\alpha|\cdot|\beta|}$, whenever α jumps over β. The relevant degrees are the degrees given in $T^M A$.

Also, any A_∞ bimodule map $F : T^M A \to T^N A$ over A induces an A_∞ bimodule map $F^\lambda : T^M A' \to T^N A'$ over A' given by

$$\begin{aligned}&F^\lambda(a'_1, \ldots, a'_k, m, a'_{k+1}, \ldots, a'_{k+l})\\ &\quad= \sum \pm pr_N \circ F(\lambda(a'_1, \ldots), \ldots, \lambda(\ldots, a'_k), m, \lambda(a'_{k+1}, \ldots), \ldots, \lambda(\ldots, a'_{k+l})).\end{aligned}$$

Furthermore, λ induces the two A_∞ bimodule maps over A'

$$\overline{\lambda} : T^{A'} A' \to T^A A' \text{ and } \widetilde{\lambda} : T^{A^*} A' \to T^{(A')^*} A'$$

defined by the components

$$\overline{\lambda}(a'_1, \ldots, a'_{k+l+1}) = pr_A \circ \lambda(a'_1, \ldots, a'_{k+l+1})$$

and

$$(\widetilde{\lambda}(a'_1, \ldots, a^*, \ldots, a'_{k+l}))(a') = \pm a^*(pr_A \circ \lambda(a'_{k+1}, \ldots, a'_{k+l}, a', a'_1, \ldots, a'_k)).$$

3. Deformations of A_∞ algebras and ∞ inner products

Before we define the specific differential graded Lie algebra $(\mathfrak{h}, d, [\ ,\ ])$ that controls the deformations of A_∞ structures and ∞ inner products, we discuss a simple example, which is relevant to our setting, and make a remark.

Example 3.1. Any graded associative algebra $\mathfrak{g}$ becomes a Lie algebra by defining the bracket to be the usual commutator. An element $\alpha \in \mathfrak{g}^1$ satisfying $\alpha^2 = 0$ is sometimes called a *polarization*. With a polarization $\alpha \in \mathfrak{g}^1$, $\mathfrak{g}$ becomes a differential graded Lie algebra by setting the differential to be $\delta = \mathrm{ad}(\alpha)$. With δ so defined, the Mauer–Cartan equation becomes

$$0 = \delta(\gamma) + \frac{1}{2}[\gamma, \gamma] = \frac{1}{2}[\alpha + \gamma, \alpha + \gamma].$$

In other words, $\gamma \in \mathfrak{g}^1$ satisfies the Mauer–Cartan equation if and only if $\alpha + \gamma$ is another polarization.

Now let S be a graded ring and consider $\mathfrak{g}$ defined by

$$\mathfrak{g} = \left\{ \begin{pmatrix} a & 0 \\ b & a \end{pmatrix} \middle| \, a, b \in S \right\}$$

with the bracket defined as the usual graded commutator of matrix multiplication:

$$\left[\begin{pmatrix} a & 0 \\ b & a \end{pmatrix}, \begin{pmatrix} c & 0 \\ d & c \end{pmatrix} \right] = \begin{pmatrix} [a,c] & 0 \\ [b,c] + [a,d] & [a,c] \end{pmatrix}.$$

Then, $\begin{pmatrix} D & 0 \\ I & D \end{pmatrix} \in \mathfrak{g}^1$ is a polarization if and only if

$$0 = [D, D] = 2 \cdot D^2 \text{ and } 0 = [D, I] + [I, D] = 2 \cdot [D, I].$$

Having chosen a polarization $P = \begin{pmatrix} D & 0 \\ I & D \end{pmatrix}$, the formula for $\delta = \mathrm{ad}(P)$ is given by

$$\delta \begin{pmatrix} f & 0 \\ i & f \end{pmatrix} = \left[\begin{pmatrix} D & 0 \\ I & D \end{pmatrix}, \begin{pmatrix} f & 0 \\ i & f \end{pmatrix} \right] = \begin{pmatrix} [D,f] & 0 \\ [D,i] + [f,I] & [D,f] \end{pmatrix}.$$

Now we look at the gauge equivalence. First of all, the gauge group $G = \exp(\mathfrak{g}^0)$ is the Lie group consisting of matrices of the form e^A, for any $A \in \mathfrak{g}^0$. The gauge action of G on $\mathfrak{g}$ is then determined by $e^{\mathrm{ad}(A)} \cdot B = \mathrm{Ad}(e^A)(B) = e^A B e^{-A}$. A computation shows that

$$\exp \begin{pmatrix} f & 0 \\ i & f \end{pmatrix} = \begin{pmatrix} e^f & 0 \\ x & e^f \end{pmatrix}, \text{ where } x = \sum_{n \geqslant 1} \frac{1}{n!} \sum_{k+l=n-1} f^k \cdot i \cdot f^l.$$

Then the gauge equivalence summarizes as

$$e^A \begin{pmatrix} D & 0 \\ I & D \end{pmatrix} e^{-A} = \begin{pmatrix} e^f D e^{-f} & 0 \\ e^f I e^{-f} + [e^f D e^{-f}, x e^{-f}] & e^f D e^{-f} \end{pmatrix}.$$

This concludes the example.

Remark 3.2. Let N be a graded coalgebra over S. Then $\mathrm{hom}(N, N)$ will be a graded associative algebra by composition of linear maps and a Lie algebra with the bracket defined by the graded commutator of composition. The space $\mathrm{Coder}(N) \subseteq \hom(N, N)$ is not an associative subalgebra, but it is a Lie subalgebra. In particular, for any vector space A, $\mathrm{Coder}(TA)$ is a graded Lie algebra. An A_∞ structure on A consists of an element $D \in \mathrm{Coder}(TA)$ satisfying $D^2 = 0$. Thus, one can say that an A_∞ structure on A is a choice of polarization $D \in \mathrm{Coder}(TA)$. Hence, if (A, D) is an A_∞ algebra, $\mathrm{Coder}(TA)$ carries a differential $\delta : \mathrm{Coder}(TA) \to \mathrm{Coder}(TA)$ defined by

$$\delta(f) := [D, f] = D \circ f - (-1)^{|f|} f \circ D.$$

The complex $(\mathrm{Coder}(TA), \delta)$ is called the Hochschild cochain complex of A. Together with the bracket from $\hom(TA, TA)$, it is a differential graded Lie algebra that controls the deformations of the A_∞ algebra (A, D). In order to make this statement precise, we recall the deformation theory of A_∞ algebras (see for example [**2**]). As a first observation, one may note that γ is a solution to the Mauer–Cartan equation in the Hochschild differential graded Lie algebra if and only if $D + \gamma$ is another polarization in $\mathrm{Coder}(TA)$; i.e., another A_∞ structure on A.

3.1. Deformations of A_∞ algebras

Let A be a graded vector space over a field k of characteristic zero and let R be a graded Artin local algebra with residue field k. Let m denote the maximal ideal of R. We have the decomposition $R \simeq R/m \oplus m \simeq k \oplus m$ and the projection $pr_k : R \to k$, hence the decomposition $A \otimes R \simeq A \oplus (A \otimes m)$ and the projection $pr_A : A \otimes R \to A$. For definiteness, the reader may have the concrete example $R = k[t]/t^{l+1}$ in mind. In this example, the maximal ideal is $m = tk[t]/t^{l+1}$, $A \otimes R \simeq A + At + At^2 + \cdots + At^l$ (with the tensor signs suppressed) and the natural projection pr_A maps $a_0 + a_1 t + a_2 t^2 + \cdots + a_l t^l \mapsto a_0$.

Let (A, D) be an A_∞ algebra over k. A *deformation of* (A, D) *over* R is an A_∞ algebra $(A \otimes R, D')$ over R with the property that the projection

$$pr : T(A \otimes R) \simeq TA \otimes R \to TA$$

is a morphism of A_∞ algebras over k. This means that $pr \circ D' = D \circ pr$.

Suppose, that D' is a deformation of (A, D) over R. Via any map $R \to S$, one can view $A \otimes R$ as an S module and $(A \otimes R, D')$ as a deformation of (A, D) over S.

Let $\pi \in \hom(R \otimes R, R)$ denote the multiplication in R. Let D_R denote the A_∞ structure $D \otimes \pi$ on $A \otimes R$. The A_∞ algebra $(A \otimes R, D_R)$ is the model for a trivial deformation of (A, D). That is, $(A \otimes R, D')$ is a *trivial deformation* if it is isomorphic to $(A \otimes R, D_R)$ as an A_∞ algebra. This means that there is an automorphism

$$\lambda : T(A \otimes R) \to T(A \otimes R)$$

satisfying $\lambda \circ D' = D_R \circ \lambda$. Two deformations are equivalent if and only if they differ by a trivial one.

3.2. Deformations of A_∞ algebras with ∞ inner products

Definition 3.3. Let A be a graded vector space over a field k. We define the graded Lie algebra $(\mathfrak{h} = \oplus_i \mathfrak{h}^i, [\,,\,])$ by

$$\mathfrak{h}^i = \mathrm{Coder}(TA)^{-i} \oplus \mathrm{Comap}(T^A A, T^{A^*} A)^{1-i} \tag{1}$$

and

$$[(f,i),(g,j)] = ([f,g], \delta_f(j) - (-1)^{|f||g|}\delta_g(i))$$
$$= (fg - (-1)^{|f||g|}gf, f^{A^*}j - (-1)^{|f||j|}jf^A - (-1)^{|f||g|}g^{A^*}i + (-1)^{|g|\cdot(|f|+|i|)}ig^A). \tag{2}$$

The skew-symmetry and Jacobi identity of $[\,,\,]$ are straightforward to check after one notices that $\delta_f \circ \delta_g - (-1)^{|f||g|}\delta_g \circ \delta_f = \delta_{f\circ g - (-1)^{|f||g|}g\circ f}$.

Proposition 3.4. *A pair $(D,I) \in \mathfrak{h}$ is an A_∞ structure with ∞ inner product on A if and only if $[(D,I),(D,I)] = 0$.*

Proof. This is immediate:

$$0 = [(D,I),(D,I)] \Leftrightarrow 0 = [D,D] = 2\cdot D^2 \text{ and } 0 = 2\cdot\delta_D(I) = 2(D^{A^*}\circ I - I\circ D^A).$$

The condition $D^2 = 0$ means that D defines an A_∞ structure on A and the condition $D^{A^*}\circ I - I \circ D^A = 0$ means that I defines a compatible ∞-inner product. □

Now fix an A_∞ structure together with an ∞ inner product, which is to say, fix a pair $(D,I) \in \mathfrak{h}$ with $[(D,I),(D,I)] = 0$. Then, define $d : \mathfrak{h} \to \mathfrak{h}$ by

$$d(f,i) = [(D,I),(f,i)]. \tag{3}$$

The triple $(\mathfrak{h}, d, [\,,\,])$ is a differential graded Lie algebra.

Definition 3.5. A *deformation of an A_∞ algebra with ∞ inner product (A,D,I) over R* is an A_∞ algebra over R with ∞ inner product $(A\otimes R, D', I')$, such that the projection

$$pr : T(A\otimes R) \to TA$$

is a morphism of A_∞ algebras over k compatible with the ∞-inner products. Compatibility with the ∞ inner product means that the following diagram of A_∞-bimodule maps over k is commutative:

$$\begin{array}{ccc} T^{A\otimes R}(A\otimes R) & \xrightarrow{\overline{pr}} & T^{A}(A\otimes R) \\ \downarrow{\scriptstyle pr_k\circ I'} & & \downarrow{\scriptstyle I^{pr}} \\ T^{(A\otimes R)^*}(A\otimes R) & \xleftarrow[\widetilde{pr}]{} & T^{A^*}(A\otimes R) \end{array}$$

Here, the ∞-inner product I' on $A\otimes R$ over R induces an ∞-inner product on $A\otimes R$ over k by composing with the map induced by the projection $\hom_R(A\otimes R, R) \to \hom_k(A\otimes R, k)$, $f \mapsto pr_k \circ f$.

There is a natural extension of I to an ∞-inner product $I_R = I\otimes\pi$ on $(A\otimes R, D_R)$.

Definition 3.6. We say that (D', I') is a *trivial deformation* of (D, I) provided the triple $(A \otimes R, D', I')$ is isomorphic to $(A \otimes R, D_R, I_R)$ as A_∞ algebras with ∞ inner products. That is, if there exists an automorphism

$$\lambda : T(A \otimes R) \to T(A \otimes R)$$

and a comap

$$\rho : T^{A\otimes R}(A \otimes R) \to T^{(A\otimes R)^*}(A \otimes R)$$

satisfying

(i) $\lambda \circ D' = D_R \circ \lambda$,

(ii) $I' - \widetilde{\lambda} \circ (I_R)^\lambda \circ \overline{\lambda} = D'^{(A\otimes R)^*} \circ \rho + \rho \circ D'^{A\otimes R}$.

It may be helpful to think of the second condition in Definition 3.6 as saying I' equals I_R under a change of coordinates (given by λ) up to a homotopy (given by ρ). That is, the following diagram commutes, up to a homotopy defined by $\rho \in \mathrm{Comap}(T^{A\otimes R}(A \otimes R))$.

$$\begin{array}{ccc} T^{A\otimes R}(A\otimes R) & \xrightarrow{\overline{\lambda}} & T^{A\otimes R}(A\otimes R) \\ \Big\downarrow{\scriptstyle I'} & & \Big\downarrow{\scriptstyle (I_R)^\lambda} \\ T^{(A\otimes R)^*}(A\otimes R) & \xleftarrow[\widetilde{\lambda}]{} & T^{(A\otimes R)^*}(A\otimes R) \end{array}$$

Two deformations are equivalent if and only if they differ by a trivial one.

Now, the conclusion:

Theorem 3.7. *Let (A, D) be an A_∞ algebra and let I be an ∞-inner product. Then the differential graded Lie algebra $(\mathfrak{h}, d, [\,,\,])$ defined by equations* (1), (2) *and* (3) *controls the deformations of the A_∞ algebra with ∞ inner product (A, D, I).*

Proof. The content of this theorem is summarized in the following two statements.

- Deformations, over R, of the (A, D, I) correspond to solutions to the Mauer–Cartan equation in $\mathfrak{h} \otimes m$,
- and equivalent deformations correspond to gauge equivalent solutions.

First we prove the first statement. Let $\alpha = (f, i) \in (\mathfrak{h} \otimes m)^1$. Observe that

$$\begin{aligned} d\alpha + \frac{1}{2}[\alpha, \alpha] &= [(D_R, I_R), (f, i)] + \frac{1}{2} \cdot [(f, i), (f, i)] \\ &= \frac{1}{2} \cdot [(D_R + f, I_R + i), (D_R + f, I_R + i)]. \end{aligned}$$

Then, Proposition 3.4 proves that $d\alpha + \frac{1}{2}[\alpha, \alpha] = 0$ if and only if $(A \otimes R, D_R + f, I_R + i)$ is a deformation of (A, D, I). It is immediate that any $(A \otimes R, D', I')$ that is a deformation of (A, D, I) must satisfy $[(D', I'), (D', I')] = 0 \in \mathfrak{h} \otimes R$. The fact that $pr : T(A \otimes R) \to TA$ is a map of A_∞ algebras with ∞ inner products implies that $D' = D_R + f$ and $I' = I_R + i$ for some $(f, i) \in \mathfrak{h} \otimes m$.

Now we prove the second statement. Let $\alpha = (f, i) \in (\mathfrak{h} \otimes m)^0$. The gauge action for $\mathfrak{h}$ becomes

$$e^{\mathrm{ad}(f,i)} \cdot (D_R, I_R) = \sum_{n \geqslant 0} \frac{\mathrm{ad}(f,i)^n}{n!}(D_R, I_R).$$

It follows from

$$\delta_f\left(\delta_{\mathrm{ad}(f)^r(D_R)}((\delta_f)^s(i))\right) = \delta_{\mathrm{ad}(f)^{r+1}(D_R)}((\delta_f)^s(i)) + \delta_{\mathrm{ad}(f)^r(D_R)}((\delta_f)^{s+1}(i)),$$

that $\mathrm{ad}(f,i)^n(D_R, I_R)$ is given by

$$\left(\mathrm{ad}(f)^n(D_R), (\delta_f)^n(I_R) - \sum_{k+l=n-1} \frac{n!}{k!(l+1)!} \cdot \delta_{\mathrm{ad}(f)^k(D_R)} \circ (\delta_f)^l(i)\right).$$

Now define $\lambda^{-1} = e^f = \sum_{k \geqslant 0} \frac{1}{k!} f^k$ and $\rho = \sum_{l \geqslant 0} \frac{-1}{(l+1)!} \cdot (\delta_f)^l(i)$. Then for the automorphism λ and the homotopy ρ, we have

$$\begin{aligned}
&\sum_{n \geqslant 0} \frac{\mathrm{ad}(f,i)^n}{n!}(D_R, I_R) \\
&\quad = \left(\sum_{n \geqslant 0} \frac{\mathrm{ad}(f)^n}{n!}(D_R), \sum_{n \geqslant 0} \frac{(\delta_f)^n}{n!}(I_R) + \delta_{\sum_{k \geqslant 0} \frac{\mathrm{ad}(f)^k}{k!}(D_R)}\left(\sum_{l \geqslant 0} \frac{-1}{(l+1)!} \cdot (\delta_f)^l(i)\right)\right) \\
&\quad = \left(\lambda^{-1} D_R \lambda,\, \widetilde{\lambda}(I_R)^{\lambda}\overline{\lambda} + \delta_{\lambda^{-1} D_R \lambda}(\rho)\right).
\end{aligned}$$

This proves that $e^{\mathrm{ad}(f,i)} \cdot (D_R, I_R)$ is a trivial deformation of (D, I).

It is not hard to see that every trivial deformation of (D, I) arises from an element gauge equivalent to the identity. The condition that the A_∞ algebra map $\lambda : T(A \otimes R) \to T(A \otimes R)$ is an automorphism implies that $\lambda = e^f$ for some $f \in (\mathrm{Coder}(TA) \otimes m)^0$. Also, since $\rho = -i - \frac{1}{2}\delta_f(i) - \cdots$, the map

$$i \mapsto \rho(i) = \sum_{l \geqslant 0} \frac{-1}{(l+1)!} \cdot (\delta_f)^l(i)$$

is invertible. So one can obtain any homotopy ρ, by choosing a suitable element $i = \sum_{m \geqslant 0} c_m \cdot (\delta_f)^m(\rho) \in (\mathfrak{h} \otimes m)^0$ with $\rho(i) = \rho$. □

4. Moduli, infinitesimal deformations, and relationship to cyclic cohomology

Let us return briefly to general deformation theory in order to review the notions of infinitesimal deformations and moduli space. Let $(\mathfrak{g}, d, [\,,\,])$ be a differential graded Lie algebra and assume that $\mathrm{Ker}(d)/\mathrm{Im}(d) =: H(\mathfrak{g}) = \oplus_{i=-m}^{m} H^i(\mathfrak{g})$ is finite dimensional. Consider the (graded version of the) ring of dual numbers $R = k[t_{-m}, \ldots, t_m]/t_i t_j$. Here $\deg(t_i) = i - 1$ and the maximal ideal of R is $m = \oplus_i t_i R$.

From a solution $\sum(\gamma_j \otimes t_j) \in (\mathfrak{g} \otimes m)^1$ to the Mauer–Cartan equation, one may produce the map $d + \sum t_j\, \mathrm{ad}(\gamma_j) : \mathfrak{g} \otimes k[t_{-m}, \ldots, t_m] \to \mathfrak{g} \otimes k[t_{-m}, \ldots, t_m]$ which satisfies

$$\Big(d + \sum t_j \operatorname{ad}(\gamma_j)\Big)^2 = 0 \text{ modulo } t_i t_j.$$

One refers to $\gamma = \sum \gamma_j$ as an infinitesimal deformation. One can readily check that

$$Def_{\mathfrak{g}}(R) = \operatorname{Ker}(d)/\operatorname{Im}(d) = H(\mathfrak{g}).$$

Suppose $Def_{\mathfrak{g}}$ is prorepresentable. That is, there exists a projective limit of (graded) local Artin rings $\mathcal{O}$ and an equivalence of the functors

$$Def_{\mathfrak{g}}(\cdot) \simeq \hom(\mathcal{O}, \cdot).$$

In the case that $\mathcal{O} = \mathcal{O}_{\mathcal{M}}$ is the ring of local functions at the base point of a pointed $\mathbb{Z}$ graded space $\mathcal{M}$, then $\mathcal{M}$ is the local moduli space for $Def_{\mathfrak{g}}$. Denote the base point of $\mathcal{M}$ by p. One can check that

$$T_p(\mathcal{M}) \simeq \hom(\mathcal{O}_{\mathcal{M}}, R).$$

It follows that the graded tangent space to the moduli space at the base point is isomorphic to the cohomology of $(\mathfrak{g}, d)$:

$$T_p(\mathcal{M}) \simeq H(\mathfrak{g}).$$

Now, let (A, D) be an A_∞ algebra and let I be an ∞ inner product on (A, D). Theorem 3.7 says that the differential graded Lie algebra controlling deformations of (A, D, I) is

$$\mathfrak{h} = \operatorname{Coder}(TA) \oplus \operatorname{Comap}(T^A A, T^{A^*} A)$$

with bracket

$$[(f,i),(g,j)] = ([f,g], \delta_f(j) - (-1)^{|f||g|}\delta_g(i))$$

and a differential

$$d(f,i) = [(D,I),(f,i)].$$

Thus follows the expected infinitesimal statement:

Corollary 4.1. *The graded tangent space to the moduli space of A_∞ structures with ∞ inner products is isomorphic to $H(\mathfrak{h})$.*

As a final remark, we mention some connections between the cohomology $H(\mathfrak{h})$ and a couple of its cousins. If (A, D, I) is an A_∞ algebra with ∞-inner product, we have the Hochschild differential graded Lie algebra $(\operatorname{Coder}(TA), \delta, [\,,\,])$ and the sub differential graded Lie algebra of cyclic Hochschild cochains $\operatorname{Coder}(TA)_{\mathrm{Cyclic}}$, defined by

$$\operatorname{Coder}(TA)_{\mathrm{Cyclic}} = \{f \in \operatorname{Coder}(TA) : \delta_f(I) = 0\}.$$

If I consists of an ordinary symmetric inner product $I = \langle\,,\,\rangle$, then the condition $\delta_f(I) = f^{A^*} \circ I - I \circ f^A = 0$ is equivalent to

$$\langle f(a_1, ..., a_n), a_{n+1}\rangle = \pm\langle a_1, f(a_2, ..., a_{n+1})\rangle.$$

We have the following maps of differential graded Lie algebras:

$$(\mathrm{Coder}(TA)_{\mathrm{Cyclic}}, \delta, [\,,\,]) \longrightarrow (\mathfrak{h}, d, [\,,\,]) \text{ and } (\mathfrak{h}, d, [\,,\,]) \longrightarrow (\mathrm{Coder}(TA), \delta, [\,,\,]). \quad (4)$$

The first map is the injection $f \mapsto (f, 0) \in \mathfrak{h}$, which is a cochain map

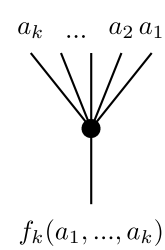
$$d(f, 0) = ([D, f], \pm(f^{A^*} \circ I - I \circ f^A)) = (\delta f, 0),$$

because elements of the domain are cyclic. The induced map in cohomology describes a statement from [**7**], namely that the first order deformations of D compatible with the inner product are classified by cyclic cohomology. We do not know under what conditions the map $f \mapsto (f, 0) \in \mathfrak{h}$ induces an isomorphism in cohomology. The second map in (4) is simply the projection $\mathrm{Coder}(TA) \oplus \mathrm{Comap}(T^A A, T^{A^*} A) \to \mathrm{Coder}(TA)$ and the induced map in cohomology describes the simple statement that any infinitesimal deformation of the pair (D, I) gives an infinitesimal deformation of D.

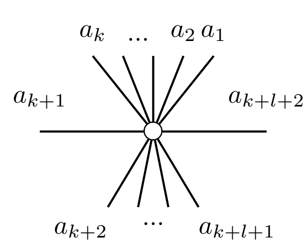
Appendix A. Explicit formulas of $\delta_f(i)$

Let $f \in \mathrm{Coder}(TA)$ and $i \in \mathrm{Comap}(T^A A, T^{A^*} A)$. We want to describe the term $\delta_f(i) = f^{A^*} \circ i - (-1)^{|f||i|} \cdot i \circ f^A \in \mathrm{Comap}(T^A A, T^{A^*} A)$ more explicitly. Here, $f : \oplus_{k \geqslant 1} A^{\otimes k} \to A$ and $i : \oplus_{k,l \geqslant 0} A^{\otimes k} \otimes A \otimes A^{\otimes l} \otimes A \to S$ have the components

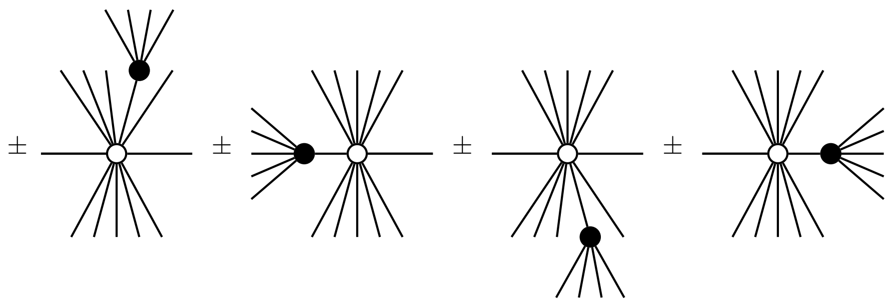
$$f_k : A^{\otimes k} \to A$$



$$i_{k,l} = \langle \cdots \rangle_{k,l} : A^{\otimes k} \otimes A \otimes A^{\otimes l} \otimes A \to S$$

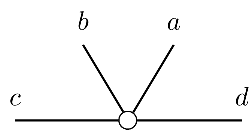


By convention, the inputs are always inserted using the counterclockwise direction. Then $f^{A^*} \circ i - (-1)^{|f||i|} \cdot i \circ f^A$ is given by inserting f into i in all possible combinations.

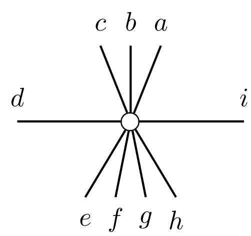


First, here are some examples of how these diagrams are to be read.

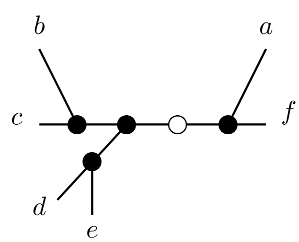
$\langle a, b, c, d\rangle_{2,0}$



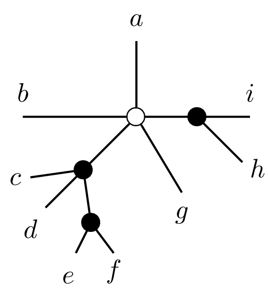
$\langle a, b, c, d, e, f, g, h, i\rangle_{3,4}$



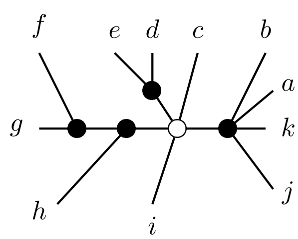
$\langle f_2(f_2(b, c), f_2(d, e)), f_2(f, a)\rangle_{0,0}$



$\langle a, b, f_3(c, d, f_2(e, f)), g, f_2(h, i)\rangle_{1,2}$



$\langle c, f_2(d, e), f_2(f_2(f, g), h), i, f_4(j, k, a, b)\rangle_{2,1}$



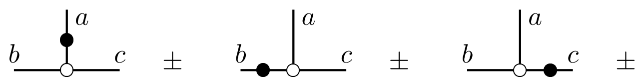
Here are the terms of $\delta_f(i) = f^{A^*} \circ i - (-1)^{|f||i|} \cdot i \circ f^A$ up to sign, when they are being applied to elements from $A^{\otimes k} \otimes A \otimes A^{\otimes l} \otimes A$:

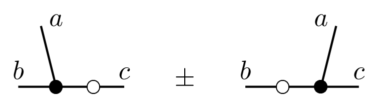
<u>$k = 0$, $l = 0$:</u>

$$\langle f_1(a), b\rangle_{0,0} \pm \langle a, f_1(b)\rangle_{0,0}$$

<u>$k = 1$, $l = 0$:</u>

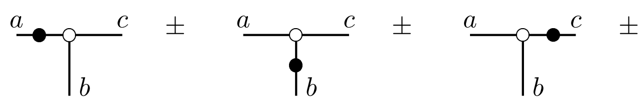
$$\langle f_1(a), b, c\rangle_{1,0} \pm \langle a, f_1(b), c\rangle_{1,0} \pm \langle a, b, f_1(c)\rangle_{1,0} \pm \langle f_2(a, b), c\rangle_{0,0} \pm \langle b, f_2(c, a)\rangle_{0,0}$$

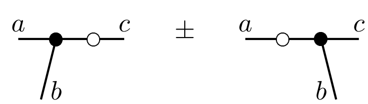




<u>$k = 0$, $l = 1$:</u>

$$\langle f_1(a), b, c\rangle_{0,1} \pm \langle a, f_1(b), c\rangle_{0,1} \pm \langle a, b, f_1(c)\rangle_{0,1} \pm \\ \langle f_2(a,b), c\rangle_{0,0} \pm \langle a, f_2(b,c)\rangle_{0,0}$$

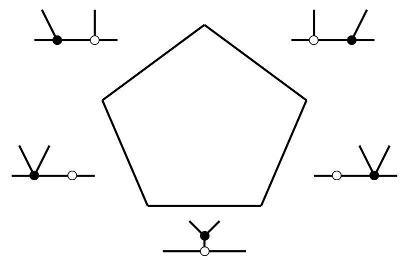




<u>$k = 2$, $l = 0$:</u>

$$\langle f_1(a), b, c, d\rangle_{2,0} \pm \langle a, f_1(b), c, d\rangle_{2,0} \pm \\ \langle a, b, f_1(c), d\rangle_{2,0} \pm \langle a, b, c, f_1(d)\rangle_{2,0} \pm \\ \langle f_2(a,b), c, d\rangle_{1,0} \pm \langle a, f_2(b,c), d\rangle_{1,0} \pm \langle b, c, f_2(d,a)\rangle_{1,0} \pm \\ \langle f_3(a,b,c), d\rangle_{0,0} \pm \langle c, f_3(d,a,b)\rangle_{0,0}$$

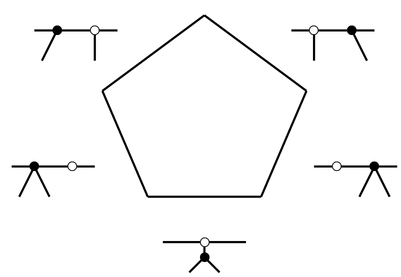
Note that for example the term $\langle a, b, f_2(c,d)\rangle_{2,0}$ does not appear, because c and d are the two special elements of $a \otimes b \otimes c \otimes d \in A^{\otimes 2} \otimes A \otimes A^{\otimes 0} \otimes A$, which are put on the horizontal line of the diagram. The two special elements from $A^{\otimes k} \otimes A \otimes A^{\otimes l} \otimes A$ can never be inside any f_n.



<u>$k = 0$, $l = 2$:</u>

$$\begin{aligned} \langle f_1(a), b, c, d\rangle_{0,2} \pm \langle a, f_1(b), c, d\rangle_{0,2} \pm \\ \langle a, b, f_1(c), d\rangle_{0,2} \pm \langle a, b, c, f_1(d)\rangle_{0,2} \pm \\ \langle f_2(a,b), c, d\rangle_{0,1} \pm \langle a, f_2(b,c), d\rangle_{0,1} \pm \langle a, b, f_2(c,d)\rangle_{0,1} \pm \\ \langle f_3(a,b,c), d\rangle_{0,0} \pm \langle a, f_3(b,c,d)\rangle_{0,0} \end{aligned}$$

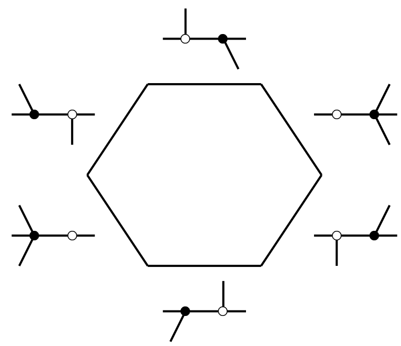
The special elements are a and d from $a \otimes b \otimes c \otimes d \in A^{\otimes 0} \otimes A \otimes A^{\otimes 2} \otimes A$.



<u>$k = 1$, $l = 1$:</u>

$$\begin{aligned} \langle f_1(a), b, c, d\rangle_{1,1} \pm \langle a, f_1(b), c, d\rangle_{1,1} \pm \\ \langle a, b, f_1(c), d\rangle_{1,1} \pm \langle a, b, c, f_1(d)\rangle_{1,1} \pm \\ \langle f_2(a,b), c, d\rangle_{0,1} \pm \langle b, c, f_2(d,a)\rangle_{0,1} \pm \\ \langle a, f_2(b,c), d\rangle_{1,0} \pm \langle a, b, f_2(c,d)\rangle_{1,0} \pm \\ \langle f_3(a,b,c), d\rangle_{0,0} \pm \langle b, f_3(c,d,a)\rangle_{0,0} \end{aligned}$$

The special elements are b and d from $a \otimes b \otimes c \otimes d \in A^{\otimes 1} \otimes A \otimes A^{\otimes 1} \otimes A$.

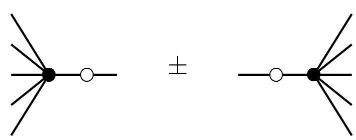


$i = \langle \, , \, \rangle_{0,0}$ for any k, l: Assume that $i = \langle \, , \, \rangle_{0,0}$ has only lowest component, but f has all higher components. We apply $f^{A^*} \circ i - (-1)^{|f||i|} \cdot i \circ f^A$ to the element

$$a_1 \otimes ... \otimes a_k \otimes a_{k+1} \otimes a_{k+2} \otimes ... \otimes a_{k+l+1} \otimes a_{k+l+2} \in A^{\otimes k} \otimes A \otimes A^{\otimes l} \otimes A$$

to get

$$\langle f(a_1, ..., a_{k+l+1}), a_{k+l+2} \rangle_{0,0} \pm \langle a_{k+1}, f(a_{k+2}, ..., a_{k+l+2}, a_1, ..., a_k) \rangle_{0,0}$$



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John Terilla `john@qc.edu`

Department of Mathematics
Queens College of the City University of New York
65-30 Kissena Blvd.
Flushing, NY 11367

Thomas Tradler `ttradler@citytech.cuny.edu`

Department of Mathematics
College of Technology of the City University of New York
300 Jay Street
Brooklyn, NY 11201

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