Homology, Homotopy and Applications, vol. 9(1), 2007, pp.331–366

ON THE 2-ADIC K-LOCALIZATIONS OF H-SPACES

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(*communicated by Donald M. Davis*)

Abstract

We determine the 2-adic K-localizations for a large class of H-spaces and related spaces. As in the odd primary case, these localizations are expressed as fibers of maps between specified infinite loop spaces, allowing us to approach the 2-primary v_1-periodic homotopy groups of our spaces. The present v_1-periodic results have been applied very successfully to simply-connected compact Lie groups by Davis, using knowledge of the complex, real, and quaternionic representations of the groups. We also functorially determine the united 2-adic K-cohomology algebras (including the 2-adic KO-cohomology algebras) for all simply-connected compact Lie groups in terms of their representation theories, and we show the existence of spaces realizing a wide class of united 2-adic K-cohomology algebras with specified operations.

1. Introduction

In [**20**], Mahowald and Thompson determined the p-adic K-localizations of the odd spheres at an arbitrary prime p, expressing these localizations as homotopy fibers of maps between specified infinite loop spaces. Then, working at an odd prime p in [**8**], we generalized this result to give the p-adic K-localizations for a large class of H-spaces and related spaces. In the present paper, we obtain similar results for 2-adic K-localizations of such spaces, using our preparatory work in [**10**] and [**11**]. By a *2-adic K-localization*, we mean a $K/2_*$-localization (see [**2**], [**3**]), which is the same as a $K^*(-;\hat{\mathbb{Z}}_2)$-localization, since the $K/2_*$-equivalences of spaces or spectra are the same as the $K^*(-;\hat{\mathbb{Z}}_2)$-equivalences. Our localization results in this paper will apply to many (but not all) simply-connected finite H-spaces and to related spaces such as the spheres S^{4k-1} for $k \geqslant 1$. We show that these results allow computations of the v_1-periodic homotopy groups (see [**13**], [**15**]) of our spaces from their united 2-adic K-cohomologies, and thus allow computations of the v_1-periodic homotopy groups for a large class of simply-connected compact Lie groups from their complex, real, and quaternionic representation theories. The present results will be extended in a

Received December 18, 2006, revised January 20, 2007; published on March 6, 2007.
2000 Mathematics Subject Classification: 55N15, 55P60, 55Q51, 55S25.
Key words and phrases: K-localizations, v_1-periodic homotopy, 2-adic K-theory, united K-theory, compact Lie groups.

subsequent paper to cover the remaining simply-connected compact Lie groups and various spaces related to the remaining odd spheres. This work has been applied very successfully by Davis [**14**] to complete his 13-year program (with Bendersky) of calculating the v_1-periodic homotopy groups of all simply-connected compact Lie groups, and has also been applied by Bendersky, Davis, and Mahowald [**1**].

Throughout this paper, we work at the prime 2 and rely on the *united 2-adic K-cohomology*

$$K^*_{CR}(X;\hat{\mathbb{Z}}_2) \;=\; \{K^*(X;\hat{\mathbb{Z}}_2), KO^*(X;\hat{\mathbb{Z}}_2)\}$$

of a space or spectrum X as in [**10**]. This combines the usual periodic cohomologies with certain operations between them, such as complexification and realification. For our H-spaces and related spaces X, the cohomology $K^*_{CR}(X;\hat{\mathbb{Z}}_2)$ is essentially determined by the *2-adic Adams Δ-module*

$$\widetilde{K}^{-1}_{\Delta}(X;\hat{\mathbb{Z}}_2) \;=\; \{\widetilde{K}^{-1}(X;\hat{\mathbb{Z}}_2), \widetilde{KO}^{-1}(X;\hat{\mathbb{Z}}_2), \widetilde{KO}^{-5}(X;\hat{\mathbb{Z}}_2)\}$$

which combines the specified cohomologies with the additive operations among them (see Definition 6.1). In fact, for most simply-connected finite H-spaces X, we expect to have an isomorphism $K^*_{CR}(X;\hat{\mathbb{Z}}_2) \cong \hat{L}(M)$ where $M = \{M_C, M_R, M_H\}$ is the submodule of primitives in $\widetilde{K}^{-1}_{\Delta}(X;\hat{\mathbb{Z}}_2)$ and where $\hat{L}$ is a functor that we introduce in Lemma 4.5, extending the 2-adic exterior algebra functor on complex components. For a simply-connected compact Lie group G, the required 2-adic Adams Δ-module may be obtained as the indecomposables $\hat{Q}R_\Delta G = \{\hat{Q}RG, \hat{Q}R_R G, \hat{Q}R_H G\}$ of the complex, real, and quaternionic representation ring $R_\Delta G = \{RG, R_R G, R_H G\}$ (see Definition 10.1), and we have:

Theorem 1.1. *For a simply-connected compact Lie group G, there is a natural isomorphism $K^*_{CR}(G;\hat{\mathbb{Z}}_2) \cong \hat{L}(\hat{Q}R_\Delta G)$ of algebras.*

This will follow from Theorem 10.3. It extends results of Hodgkin [**17**], Seymour [**23**], Minami [**21**], and others on $K^*(G;\hat{\mathbb{Z}}_2)$ and $KO^*(G;\hat{\mathbb{Z}}_2)$. Our main result on $K/2_*$-localizations will apply to a space X with $K^*_{CR}(X;\hat{\mathbb{Z}}_2) \cong \hat{L}M$ for a 2-adic Adams Δ-module M that is *strong* (see Definition 7.11). This technical algebraic condition seems relatively mild and holds for $\hat{Q}R_\Delta G$ when G is a simply-connected compact simple Lie group *other than* E_6 or $Spin(4k+2)$ with k not a 2-power by work of Davis (see Lemma 10.5). For a strong 2-adic Adams Δ-module M, we obtain two stable 2-adic Adams Δ-modules $\bar{M} = \{\bar{M}_C, \bar{M}_R, \bar{M}_H\}$ and $\bar{\rho}\bar{M} = \{\bar{M}_C, \bar{M}_R + \bar{M}_H, \bar{M}_R \cap \bar{M}_H\}$ where $\bar{M}_C = M_C$, $\bar{M}_R = \mathrm{im}(M_R \to M_C)$, and $\bar{M}_H = \mathrm{im}(M_H \to M_C)$; and we obtain two corresponding $K/2_*$-local spectra $\mathcal{E}\bar{M}$ and $\mathcal{E}\bar{\rho}\bar{M}$ such that $K^{-1}_{\Delta}(\mathcal{E}\bar{M};\hat{\mathbb{Z}}_2) = \bar{M}$, $K^0(\mathcal{E}\bar{M};\hat{\mathbb{Z}}_2) = 0$, $K^{-1}_{\Delta}(\mathcal{E}\bar{\rho}\bar{M};\hat{\mathbb{Z}}_2) = \bar{\rho}\bar{M}$, and $K^0(\mathcal{E}\bar{\rho}\bar{M};\hat{\mathbb{Z}}_2) = 0$ (see Definition 8.1). Stated briefly, our main localization result is:

Theorem 1.2. *If X is a connected space with $K^*_{CR}(X;\hat{\mathbb{Z}}_2) \cong \hat{L}M$ for a strong 2-adic Adams Δ-module M, then its $K/2_*$-localization $X_{K/2}$ is the homotopy fiber of a map from $\Omega^\infty\mathcal{E}\bar{M}$ to $\Omega^\infty\mathcal{E}\bar{\rho}\bar{M}$ with low dimensional modifications.*

This will follow from Theorem 8.6. It will apply to simply-connected compact simple Lie groups with the above-mentioned exceptions, and it should apply to many other simply-connected finite H-spaces and related spaces; in fact, there must exist a great diversity of spaces with the required united 2-adic K-cohomology algebras by:

Theorem 1.3. *For each strong 2-adic Adams Δ-module M, there exists a simply-connected space X with $K^*_{CR}(X;\hat{\mathbb{Z}}_2) \cong \hat{L}M$.*

This will follow from Theorem 8.5. For our spaces X, we also obtain results on the 2-primary v_1-periodic homotopy groups $v_1^{-1}\pi_*X$, which are naturally isomorphic to stable homotopy groups $\pi_*\tau_2\Phi_1X$, where $\tau_2\Phi_1X$ is the 2-torsion part of the spectrum Φ_1X obtained using the v_1-*stabilization functor* Φ_1 constructed in [**4**], [**9**], [**16**], and [**18**]. From this standpoint, the homotopy $v_1^{-1}\pi_*X$ is essentially determined by the cohomology $KO^*(\Phi_1X;\hat{\mathbb{Z}}_2)$, since there is an exact sequence

$$\cdots \longrightarrow KO^{n-3}(\Phi_1X;\hat{\mathbb{Z}}_2) \xrightarrow{\psi^3-9} KO^{n-3}(\Phi_1X;\hat{\mathbb{Z}}_2) \longrightarrow (v_1^{-1}\pi_nX)^{\#}$$
$$\longrightarrow KO^{n-2}(\Phi_1X;\hat{\mathbb{Z}}_2) \xrightarrow{\psi^3-9} KO^{n-2}(\Phi_1X;\hat{\mathbb{Z}}_2) \longrightarrow \cdots$$

where $(-)^{\#}$ gives the Pontrjagin dual (see Theorem 9.2). A space X is called $K/2_*$-*durable* when the $K/2_*$-localization induces an isomorphism $v_1^{-1}\pi_*X \cong v_1^{-1}\pi_*X_{K/2}$ or equivalently $\Phi_1X \simeq \Phi_1X_{K/2}$. This condition holds for all connected H-spaces (and many other spaces), and our $K/2_*$-localization result implies:

Theorem 1.4. *If X is a connected $K/2_*$-durable space (e.g. H-space) with $K^*_{CR}(X;\hat{\mathbb{Z}}_2) \cong \hat{L}M$ for a strong 2-adic Adams Δ-module M, then there is a (co)fiber sequence of spectra $\Phi_1X \to \mathcal{E}\bar{M} \to \mathcal{E}\bar{\rho}\bar{M}$ with a $KO^*(-;\hat{\mathbb{Z}}_2)$ cohomology exact sequence*

$$0 \longrightarrow KO^{-8}(\Phi_1X;\hat{\mathbb{Z}}_2) \longrightarrow \bar{M}_C/(\bar{M}_R+\bar{M}_H) \xrightarrow{\lambda^2} \bar{M}_C/\bar{M}_R \longrightarrow KO^{-7}(\Phi_1X;\hat{\mathbb{Z}}_2)$$
$$\longrightarrow 0 \longrightarrow \bar{M}_H/(\bar{M}_R\cap\bar{M}_H) \longrightarrow KO^{-6}(\Phi_1X;\hat{\mathbb{Z}}_2) \longrightarrow \bar{M}_R\cap\bar{M}_H \xrightarrow{\lambda^2} \bar{M}_H \longrightarrow$$
$$KO^{-5}(\Phi_1X;\hat{\mathbb{Z}}_2) \longrightarrow 0 \longrightarrow 0 \longrightarrow KO^{-4}(\Phi_1X;\hat{\mathbb{Z}}_2) \longrightarrow \bar{M}_C/(\bar{M}_R\cap\bar{M}_H) \xrightarrow{\lambda^2}$$
$$\bar{M}_C/\bar{M}_H \longrightarrow KO^{-3}(\Phi_1X;\hat{\mathbb{Z}}_2) \longrightarrow (\bar{M}_R+\bar{M}_H)/(\bar{M}_R\cap\bar{M}_H) \xrightarrow{\lambda^2}\longrightarrow$$
$$KO^{-2}(\Phi_1X;\hat{\mathbb{Z}}_2) \longrightarrow \bar{M}_R+\bar{M}_H \xrightarrow{\lambda^2} \bar{M}_R \longrightarrow KO^{-1}(\Phi_1X;\hat{\mathbb{Z}}_2) \longrightarrow 0.$$

This will follow from Theorem 9.5. It allows effective computations of 2-primary v_1-periodic homotopy groups as shown by Davis [**14**], and its complex analogue implies that our spaces X are usually $\widehat{K}\Phi_1$-*good*, which means that $\hat{Q}K^n(X;\hat{\mathbb{Z}}_2)/\lambda^2 \cong K^n(\Phi_1X;\hat{\mathbb{Z}}_2)$ for $n=-1,0$.

Theorem 1.5. *If X is as in Theorem 1.4 with $\lambda^2\colon M_C \to M_C$ monic, then X is $\widehat{K}\Phi_1$-good.*

This will follow from Theorem 9.7. It will be used in a subsequent paper to show that all simply-connected compact Lie groups (and many other spaces) are $\widehat{K}\Phi_1$-good, which is useful because the v_1-periodic homotopy groups of $\widehat{K}\Phi_1$-good spaces are often accessible by [**10**], even when our $K/2_*$-localization theorems do not apply. From the perspective of [**10**], the present work verifies important examples of $\widehat{K}\Phi_1$-good spaces beyond the odd spheres.

Throughout the paper, spaces and spectra will belong to the usual pointed simplicial or CW homotopy categories. To provide a suitably precise setting for our main theorems and proofs, we must devote considerable attention to developing the algebraic infrastructure of united 2-adic K-cohomology theory. The paper is divided into the following sections:

1. **Introduction**
2. **The united 2-adic K-cohomologies of spectra and spaces**
3. **The 2-adic ϕCR-algebras**
4. **The universal 2-adic ϕCR-algebra functor $\hat{L}$**
5. **Stable 2-adic Adams operations and $K/2_*$-local spectra**
6. **On the united 2-adic K-cohomologies of infinite loop spaces**
7. **Strong 2-adic Adams Δ-modules**
8. **On the $K/2_*$-localizations of our spaces**
9. **On the v_1-periodic homotopy groups of our spaces**
10. **Applications to simply-connected compact Lie groups**
11. **Proofs of basic lemmas for $\hat{L}$**
12. **Proof of the Bott exactness lemma for $\hat{L}$**
13. **Proofs for regular modules**
14. **Proof of the realizability theorem for $\hat{L}M$**

Although we have long been interested in the K-localizations and v_1-periodic homotopy groups of spaces, we were prompted to develop the present results by Martin Bendersky and Don Davis. We thank them for their questions and comments.

2. The united 2-adic K-cohomologies of spectra and spaces

We now consider the united 2-adic K-cohomologies

$$K^*_{CR}(X;\hat{\mathbb{Z}}_2) \;=\; \{K^*(X;\hat{\mathbb{Z}}_2), KO^*(X;\hat{\mathbb{Z}}_2)\}$$

of spectra and spaces X, focusing on their basic structures as 2-adic CR-modules or CR-algebras. We first recall:

Definition 2.1 (The 2-adic CR-modules)**.** By a *2-adic CR-module*, we mean a CR-module over the category of 2-profinite abelian groups (see [**10**, 4.1]). Thus, a 2-adic CR-module $M = \{M_C, M_R\}$ consists of $\mathbb{Z}$-graded 2-profinite abelian groups M_C and M_R with continuous additive operations

$$B\colon M_C^* \cong M_C^{*-2}, \qquad t\colon M_C^* \cong M_C^*, \qquad B_R\colon M_R^* \cong M_R^{*-8},$$
$$\eta\colon M_R^* \to M_R^{*-1}, \qquad c\colon M_R^* \to M_C^*, \qquad r\colon M_C^* \to M_R^*,$$

satisfying the relations

$$\begin{array}{ccccc} 2\eta = 0, & \eta^3 = 0, & \eta B_R = B_R\eta, & \eta r = 0, & c\eta = 0, \\ t^2 = 1, & tB = -Bt, & rt = r, & tc = c, & cB_R = B^4c, \\ rB^4 = B_R r, & cr = 1 + t, & rc = 2, & rBc = \eta^2, & rB^{-1}c = 0. \end{array}$$

For $z \in M_C^*$ and $x \in M_R^*$, the elements $tz \in M_C^*$ and $rB^2cx \in M_R^*$ are sometimes written as z^* (or $\psi^{-1}z$) and ξx. For a spectrum or space X, the united 2-adic K-cohomology

$$K_{CR}^*(X;\hat{\mathbb{Z}}_2) \;=\; \{K^*(X;\hat{\mathbb{Z}}_2), KO^*(X;\hat{\mathbb{Z}}_2)\}$$

has a natural 2-adic CR-module structure with the usual periodicities $B\colon K^*(X;\hat{\mathbb{Z}}_2) \cong K^{*-2}(X;\hat{\mathbb{Z}}_2)$, and $B_R\colon KO^*(X;\hat{\mathbb{Z}}_2) \cong KO^{*-8}(X;\hat{\mathbb{Z}}_2)$, conjugation $t\colon K^*(X;\hat{\mathbb{Z}}_2) \cong K^*(X;\hat{\mathbb{Z}}_2)$, Hopf operation $\eta\colon KO^*(X;\hat{\mathbb{Z}}_2) \to KO^{*-1}(X;\hat{\mathbb{Z}}_2)$, complexification $c\colon KO^*(X;\hat{\mathbb{Z}}_2) \to K^*(X;\hat{\mathbb{Z}}_2)$, and realification $r\colon K^*(X;\hat{\mathbb{Z}}_2) \to KO^*(X;\hat{\mathbb{Z}}_2)$.

Definition 2.2 (Bott exactness)**.** As in [**10**, 4.1], we say that a 2-adic CR-module M is *Bott exact* when the Bott sequence

$$\cdots \longrightarrow M_R^{*+1} \xrightarrow{\eta} M_R^* \xrightarrow{c} M_C^* \xrightarrow{rB^{-1}} M_R^{*+2} \xrightarrow{\eta} \cdots$$

is exact, and we note that the 2-adic CR-module $K_{CR}^*(X;\hat{\mathbb{Z}}_2)$ is always Bott exact for a spectrum or space X. To compare CR-modules, we shall often use:

Lemma 2.3. *For Bott exact 2-adic CR-modules M and N, a map $f\colon M \to N$ is an isomorphism if and only if $f\colon M_C \to N_C$ is an isomorphism.*

Proof. For the "if" part, we treat the Bott sequences of M and N as exact couples, and we note that f induces an isomorphism of the associated spectral sequences since $f\colon M_C \cong N_C$. Using the map of second derived couples with $f\colon M_C^{(2)} \cong N_C^{(2)}$, we easily see that $f\colon \eta^2 M_R \cong \eta^2 N_R$; then using the map of first derived couples with $f\colon M_C^{(1)} \cong N_C^{(1)}$, we easily see that $f\colon \eta M_R \cong \eta N_R$; and finally using the original map of exact couples, we easily see that $f\colon M_R \cong N_R$. □

Definition 2.4 (The free 2-adic CR-modules)**.** For each integer n and $L = C, R$, there is a *monogenic free 2-adic CR-module* $F^L(g,n)$ on a generator $g \in F^L(g,n)_L^n$ having the universal property that, for each 2-adic CR-module M and $y \in M_L^n$, there is a unique map $f\colon F^L(g,n) \to M$ with $f(g) = y$. The 2-adic CR modules $F^C(g,n)$ and $F^R(g,n)$ are given more explicitly by

$$F^C(g,n)_C^{n-2i} = \hat{\mathbb{Z}}_2 \oplus \hat{\mathbb{Z}}_2 = \langle B^i g\rangle \oplus \langle B^i g^*\rangle, \quad F^C(g,n)_C^{n-2i-1} = 0,$$

$$F^C(g,n)_R^{n-2i} = \hat{\mathbb{Z}}_2 = \langle rB^i g\rangle, \quad F^C(g,n)_R^{n-2i-1} = 0,$$

$$F^R(g,n)_C^{n-2i} = \hat{\mathbb{Z}}_2 = \langle B^i cg\rangle, \quad F^R(g,n)_C^{n-2i-1} = 0,$$

$$F^R(g,n)_R^{n-8i} = \hat{\mathbb{Z}}_2 = \langle B_R^i g\rangle, \quad F^R(g,n)_R^{n-8i-1} = \mathbb{Z}/2 = \langle B_R^i \eta g\rangle,$$

$$F^R(g,n)_R^{n-8i-2} = \mathbb{Z}/2 = \langle B_R^i \eta^2 g\rangle, \quad F^R(g,n)_R^{n-8i-4} = \hat{\mathbb{Z}}_2 = \langle B_R^i \xi g\rangle,$$

$$F^R(g,n)_R^{n-8i-k} = 0 \;\text{ for }\; k = 3, 5, 6, 7.$$

We note that $F^C(g,n)$ and $F^R(g,n)$ are Bott exact for all n. In general, a *free*

2-adic CR-module on a finite set of generators may be constructed as a direct sum of the corresponding monogenic free 2-adic CR-modules. To test for this freeness, we may use:

Lemma 2.5. *For a Bott exact 2-adic CR-module M (e.g. for some* $M = K^*_{CR}(X;\hat{\mathbb{Z}}_2)$*), if* M^*_C *is a free module over* $\hat{K}^* = \hat{\mathbb{Z}}_2[B, B^{-1}]$ *on the generators* $\{ca_i\}_i \amalg \{b_j\}_j \amalg \{b^*_j\}_j$ *for finite sets of elements* $\{a_i\}_i$ *in* M^*_R *and* $\{b_j\}_j$ *in* M^*_C*, then M is a free 2-adic CR-module on the generators* $\{a_i\}_i$ *and* $\{b_j\}_j$*.*

Proof. The canonical map to M from the specified 2-adic CR-module is an isomorphism by Lemma 2.3. □

To describe the multiplicative structure of $K^*_{CR}(X;\hat{\mathbb{Z}}_2)$ for a space X, we introduce:

Definition 2.6 (The 2-adic CR-algebras)**.** By a *2-adic CR-algebra* $A = \{A_C, A_R\}$, we mean a 2-adic CR-module with continuous bilinear multiplications $A^m_L \times A^n_L \to A^{m+n}_L$ and elements $1 \in A^0_L$ for $m, n \in \mathbb{Z}$ and $L = C, R$ such that:

(i) the multiplication in A^*_C and A^*_R is graded commutative and associative with identity 1;

(ii) $B(zw) = (Bz)w = z(Bw)$ and $(zw)^* = z^*w^*$ for $z \in A^m_C$ and $w \in A^n_C$;

(iii) $B_R(xy) = (B_Rx)y = x(B_Ry)$, $\eta(xy) = (\eta x)y = x(\eta y)$, and $\xi(xy) = (\xi x)y = x(\xi y)$ for $x \in A^m_R$ and $y \in A^n_R$;

(iv) $c1 = 1$ and $c(xy) = (cx)(cy)$ for $x \in A^m_R$ and $y \in A^n_R$;

(v) $r((cx)z) = x(rz)$ and $r(z(cx)) = (rz)x$ for $x \in A^m_R$ and $z \in A^n_C$.

Equivalently, a 2-adic CR-algebra A consists of a 2-adic CR–module with a commutative associative multiplication $A\hat{\otimes}_{CR}A \to A$ with identity $\underline{e} \to A$ for $\underline{e} = F^R(1,0) \cong K^*_{CR}(\text{pt};\hat{\mathbb{Z}}_2)$, where $\hat{\otimes}_{CR}$ is the (symmetric monoidal) complete tensor product for 2-adic CR-modules [**11**, 2.6].

Definition 2.7 (Augmentations and nilpotency)**.** For a 2-adic CR-algebra A, an *augmentation* is a map $A \to \underline{e}$ of 2-adic CR-algebras which is left inverse to the identity $\underline{e} \to A$. When A is augmented, we let $\tilde{A} = \{\tilde{A}_C, \tilde{A}_R\}$ denote the augmentation ideal, and for $m \geqslant 1$ we let $\tilde{A}(m)$ denote the m-th power of $\tilde{A}$ given by the image of the m-fold product $\tilde{A}\hat{\otimes}_{CR} \cdots \hat{\otimes}_{CR}\tilde{A} \to \tilde{A}$. Thus, $\tilde{A}(m)_C$ is the image of the m-fold product $\tilde{A}^*_C\hat{\otimes} \cdots \hat{\otimes}\tilde{A}^*_C \to \tilde{A}^*_C$, while $\tilde{A}(m)_R$ is the image of the m-fold product $\tilde{A}^*_R\hat{\otimes} \cdots \hat{\otimes}\tilde{A}^*_R \to \tilde{A}^*_R$ plus the realification of $\tilde{A}(m)_C$. The *indecomposables* of A are given by the 2-adic CR-module $\hat{Q}A = \tilde{A}/\tilde{A}(2)$. We call A *nilpotent* when $\tilde{A}(m) = 0$ for sufficiently large m and call A *pro-nilpotent* when $\cap_m \tilde{A}(m) = 0$ or equivalently when $A \cong \lim_m A/\tilde{A}(m)$. For a space X, the cohomology $K^*_{CR}(X;\hat{\mathbb{Z}}_2)$ has a canonical augmentation $K^*_{CR}(X;\hat{\mathbb{Z}}_2) \to \underline{e}$ induced by the basepoint $\text{pt} \subset X$ with the usual augmentation ideal $\widetilde{K}^*_{CR}(X;\hat{\mathbb{Z}}_2) = \{\widetilde{K}^*(X;\hat{\mathbb{Z}}_2), \widetilde{KO}^*(X;\hat{\mathbb{Z}}_2)\}$. Moreover, when X is connected, the cohomology $K^*_{CR}(X;\hat{\mathbb{Z}}_2)$ is pro-nilpotent since it is the inverse limit of the cohomologies $K^*_{CR}(X_\alpha;\hat{\mathbb{Z}}_2)$ for the finite connected subspaces $X_\alpha \subset X$, where each $K^*_{CR}(X_\alpha;\hat{\mathbb{Z}}_2)$ is nilpotent.

3. The 2-adic ϕCR-algebras

To capture some additional features of the 2-adic CR-algebras $K^*_{CR}(X;\hat{\mathbb{Z}}_2)$ for spaces X, we now introduce the 2-adic ϕCR-algebras. These structures are often surprisingly rigid and will allow us to construct convenient bases for $K^*_{CR}(X;\hat{\mathbb{Z}}_2)$ in some important general cases, for instance, when X is a simply-connected compact Lie group.

Definition 3.1 (The 2-adic ϕCR-algebras)**.** By a *2-adic ϕCR-algebra A*, we mean a 2-adic CR-algebra with continuous functions $\phi\colon A^0_C \to A^0_R$ and $\phi\colon A^{-1}_C \to A^0_R$ such that:

(i) $c\phi a = a^*a$ and $c\phi x = B^{-1}x^*x$ for $a \in A^0_C$ and $x \in A^{-1}_C$;

(ii) $\phi(a+b) = \phi a + \phi b + r(a^*b)$ and $\phi(x+y) = \phi x + \phi y + rB^{-1}(x^*y)$ for $a, b \in A^0_C$ and $x, y \in A^{-1}_C$;

(iii) $\phi(ab) = (\phi a)(\phi b)$, $\phi(ax) = (\phi a)(\phi x)$, and $\phi B^{-1}(xy) = (\phi x)(\phi y)$ for $a, b \in A^0_C$ and $x, y \in A^{-1}_C$;

(iv) $\phi(1) = 1$, $\phi(ka) = k^2\phi a$, $\phi(a^*) = \phi a$, $\phi(kx) = k^2\phi x$, and $\phi(x^*) = -\phi x$ for $a \in A^0_C$, $x \in A^{-1}_C$, and $k \in \hat{\mathbb{Z}}_2$.

For convenience, we extend the operation ϕ periodically to give $\phi\colon A^{2i}_C \to A^0_R$ and $\phi\colon A^{2i-1}_C \to A^0_R$ with $\phi w = \phi B^i w$ for all i and elements w. For a space X, the cohomology $K^*_{CR}(X;\hat{\mathbb{Z}}_2)$ has a natural 2-adic ϕCR-algebra structure with $\phi\colon K^*(X;\hat{\mathbb{Z}}_2) \to KO^0(X;\hat{\mathbb{Z}}_2)$ as in [**11**, Section 3]. In particular, $\underline{e} \cong K^*_{CR}(\mathrm{pt};\hat{\mathbb{Z}}_2)$ is a 2-adic ϕCR-algebra with $\phi(k1) = k^2 1$ for $k \in \hat{\mathbb{Z}}_2$. For a 2-adic ϕCR-algebra A, an *augmentation* is a map $A \to \underline{e}$ of 2-adic ϕCR-algebras which is left inverse to the identity, and we retain the other notation and terminology of Definition 2.7. Thus, for a space X, the ϕCR-algebra $K^*_{CR}(X;\hat{\mathbb{Z}}_2)$ has a canonical augmentation and is pro-nilpotent whenever X is connected. To capture some other needed features, we introduce:

Definition 3.2 (The special 2-adic ϕCR-algebras)**.** A 2-adic ϕCR-algebra A is called *special* when:

(i) A is augmented and pro-nilpotent;

(ii) $z^2 = 0$ for $z \in A^n_C$ with n odd;

(iii) $y^2 = 0$ for $y \in A^n_R$ with $n \equiv 1, -3 \mod 8$;

(iv) $\phi c x = 0$ for $x \in A^n_R$ with $n \equiv -1, -5 \mod 8$.

For a connected space X, the cohomology $K^*_{CR}(X;\hat{\mathbb{Z}}_2)$ is a special 2-adic ϕCR-algebra by [**11**, Section 3].

Definition 3.3 (Simple systems of generators)**.** Let A be a special 2-adic ϕCR-algebra. By a *simple system of generators of odd degree* for A, we mean finite ordered sets of odd-degree elements $\{x_i\}_i$ in $\tilde{A}_R$ and $\{z_j\}_j$ in $\tilde{A}_C$ such that A_C is an exterior algebra over $\hat{K}^* = \hat{\mathbb{Z}}_2[B, B^{-1}]$ on the generators $\{cx_i\}_i \amalg \{z_j\}_j \amalg \{z^*_j\}_j$.

Such a simple system determines *associated products*

$$x_{i_1}\dots x_{i_m}(\phi z_{j_1})\dots(\phi z_{j_n})\in A_R,$$
$$(cx_{i_1})\dots(cx_{i_m})(c\phi z_{j_1})\dots(c\phi z_{j_n})w_{k_1}\dots w_{k_q}\in A_C$$

where: $i_1<\dots<i_m$ with $m\geqslant 0$; $j_1<\dots<j_n$ with $n\geqslant 0$; $k_1<\dots<k_q$ with $q\geqslant 1$; each w_{k_t} is z_{k_t} or $z^*_{k_t}$ with $w_{k_q}=z_{k_q}$; and $\{k_1,\dots,k_q\}$ is disjoint from $\{j_1,\dots,j_n\}$ in each complex product.

Proposition 3.4. *If A is a Bott exact special 2-adic ϕCR-algebra with a simple system of generators of odd degree, then A is a free 2-adic CR-module on the associated products.*

Proof. This follows by Lemma 2.5. □

When the cohomology $K^*_{CR}(X;\hat{\mathbb{Z}}_2)$ of a connected space X has a simple system of generators of odd degree, this result will determine the 2-adic CR-algebra structure of the cohomology, provided that we can compute the squares of the real simple generators of degree $\equiv -1,-5 \mod 8$, since the squares of the other simple generators and of their ϕ's must vanish. For a simply-connected compact Lie group G, we shall see that the cohomology $K^*_{CR}(G;\hat{\mathbb{Z}}_2)$ must always have a simple system of generators of odd degree by Theorem 10.3 below.

4. The universal 2-adic ϕCR-algebra functor $\hat{L}$

We must now go beyond simple systems of generators and develop functorial descriptions of cohomologies $K^*_{CR}(X;\hat{\mathbb{Z}}_2)$ using universal special 2-adic ϕCR-algebras. Our results will apply, for instance, when X is a suitable infinite loop space (Theorem 6.7) or a simply-connected compact Lie group (Theorem 10.3). We start by introducing the algebraic modules that will generate our universal algebras.

Definition 4.1 (The 2-adic Δ-modules)**.** By a *2-adic Δ-module* $N=\{N_C,N_R,N_H\}$, we mean a triad of 2-profinite abelian groups N_C, N_R, and N_H with continuous additive operations

$$t\colon N_C\cong N_C,\qquad c\colon N_R\to N_C,\qquad r\colon N_C\to N_R,$$
$$c'\colon N_H\to N_C,\qquad q\colon N_C\to N_H$$

satisfying the relations

$$t^2=1,\qquad cr=1+t,\qquad rc=2,\qquad tc=c,\qquad rt=r,$$
$$c'q=1+t,\qquad qc'=2,\qquad tc'=c',\qquad qt=q$$

as in [**10**, 4.5]. For $z\in N_C$, the element tz is sometimes written as z^* or $\psi^{-1}z$. For a 2-adic CR-module N and integer n, we obtain a 2-adic Δ-module $\Delta^nN=\{N^n_C,N^n_R,N^{n-4}_R\}$ with $c'=B^{-2}c\colon N^{n-4}_R\to N^n_C$ and $q=rB^2\colon N^n_C\to N^{n-4}_R$. In particular, we obtain a 2-adic Δ-module $K^n_\Delta(X;\hat{\mathbb{Z}}_2)=\Delta^nK^*_{CR}(X;\hat{\mathbb{Z}}_2)$ for a space X.

We say that a 2-adic Δ-module N is *torsion-free* when N_C, N_R, and N_H are torsion-free, and we say that N is *exact* when the sequence

$$\cdots \longrightarrow N_C \xrightarrow{(r,q)} N_R \oplus N_H \xrightarrow{c-c'} N_C \xrightarrow{1-t} N_C \xrightarrow{(r,q)} N_R \oplus N_H \longrightarrow \cdots$$

is exact (see [**10**, 4.5]). It is straightforward to show:

Lemma 4.2. *A 2-adic Δ-module $N = \{N_C, N_R, N_H\}$ is torsion-free and exact if and only if:*

(i) $c\colon N_R \to N_C$ *and* $c'\colon N_H \to N_C$ *are monic;*

(ii) N_C *is torsion-free with* $\ker(1+t) = \operatorname{im}(1-t)$ *for* $t\colon N_C \to N_C$*;*

(iii) $cN_R + c'N_H = \ker(1-t)$ *and* $cN_R \cap c'N_H = \operatorname{im}(1+t)$.

The 2-adic Δ-module

$$K_\Delta^{-1}(X;\hat{\mathbb{Z}}_2) \;=\; \{K^{-1}(X;\hat{\mathbb{Z}}_2), KO^{-1}(X;\hat{\mathbb{Z}}_2), KO^{-5}(X;\hat{\mathbb{Z}}_2)\}$$

of a space X has additional operations θ which we now include in:

Definition 4.3 (The 2-adic $\theta\Delta$-modules)**.** By a *2-adic $\theta\Delta$-module* $M = \{M_C, M_R, M_H\}$, we mean a 2-adic Δ-module with continuous additive operations $\theta\colon M_C \to M_C$, $\theta\colon M_R \to M_R$, and $\theta\colon M_H \to M_R$ satisfying the following relations for elements $z \in M_C$, $x \in M_R$, and $y \in M_H$:

$$\theta cx = c\theta x, \qquad \theta c'y = c\theta y, \qquad \theta tz = t\theta z, \qquad \theta qz = \theta rz, \qquad \theta\theta rz = \theta r\theta z.$$

In general, θrz may differ from $r\theta z$, and we let $\bar{\phi}\colon M_C \to M_R$ be the difference operation with $\bar{\phi}z = \theta rz - r\theta z$ for $z \in M_C$. Using the above relations, we easily deduce:

$$\begin{aligned} &\bar{\phi}cx = 0, && \bar{\phi}c'x = 0, && \bar{\phi}tz = \bar{\phi}z,\\ &2\bar{\phi}z = 0, && c\bar{\phi}z = 0, && \theta\bar{\phi}z = 0. \end{aligned}$$

For a space X, the cohomology $K_\Delta^{-1}(X;\hat{\mathbb{Z}}_2)$ has a natural 2-adic $\theta\Delta$-module structure by [**11**, Section 3] with the operations

$$\begin{aligned} \theta = -\lambda^2 &\colon K^{-1}(X;\hat{\mathbb{Z}}_2) \longrightarrow K^{-1}(X;\hat{\mathbb{Z}}_2),\\ \theta = -\lambda^2 &\colon KO^{-1}(X;\hat{\mathbb{Z}}_2) \longrightarrow KO^{-1}(X;\hat{\mathbb{Z}}_2),\\ \theta = -\lambda^2 &\colon KO^{-5}(X;\hat{\mathbb{Z}}_2) \longrightarrow KO^{-1}(X;\hat{\mathbb{Z}}_2). \end{aligned}$$

Moreover, this structure interacts with the 2-adic ϕCR-algebra structure of $K_{CR}^*(X;\hat{\mathbb{Z}}_2)$ in several ways.

Lemma 4.4. *For a space X, we have:*

(i) $\eta\phi z = \bar{\phi}z$ *for* $z \in K^{-1}(X;\hat{\mathbb{Z}}_2)$*;*

(ii) $x^2 = \eta\theta x$ *for* $x \in KO^{-1}(X;\hat{\mathbb{Z}}_2)$*;*

(iii) $y^2 = B_R\eta\theta y$ *for* $y \in KO^{-5}(X;\hat{\mathbb{Z}}_2)$.

Proof. This follows from [**11**, Section 3]. □

We shall take account of these relations in our universal algebras. For a 2-adic $\theta\Delta$-module M and a special 2-adic ϕCR-algebra A, an *admissible* map $\alpha\colon M \to A$ consists of a 2-adic Δ-module map $\alpha\colon M \to \Delta^{-1}\tilde{A}$ such that:

(i) $\eta\phi\alpha z = \alpha\bar{\phi}z$ in A_C^{-1} for each $z \in M_C$;

(ii) $(\alpha x)^2 = \eta\alpha\theta x$ in A_R^{-2} for each $x \in M_R$;

(iii) $(\alpha y)^2 = B_R\eta\alpha\theta y$ in A_R^{-10} for each $y \in M_H$.

We say that a special 2-adic ϕCR-algebra A with an admissible map $\alpha\colon M \to A$ is *universal* if, for each special 2-adic ϕCR-algebra B with admissible map $g\colon M \to B$, there exists a unique ϕCR-algebra map $\bar{g}\colon A \to B$ such that $\bar{g}\alpha = g$.

Lemma 4.5. *For each 2-adic $\theta\Delta$-module M, there exists a universal special 2-adic ϕCR-algebra $\hat{L}M$ with admissible map $\alpha\colon M \to \hat{L}M$.*

This will be proved later in Section 11. By universality, $\hat{L}M$ is unique up to isomorphism and is natural in M, so that we have a functor $\hat{L}$ from the category of 2-adic $\theta\Delta$-modules to the category of special 2-adic ϕCR-algebras. We believe that the ϕCR-algebra $\hat{L}M$ can be given canonical operations θ satisfying all the formulae of [**11**, Section 3] and that this provides a strengthened version of $\hat{L}$ that is right adjoint to $\Delta^{-1}\widetilde{()}$. However, for simplicity, we rely on the present basic functor $\hat{L}$. We can describe the algebra $(\hat{L}M)_C$ explicitly using the 2-adic exterior algebra $\hat{\Lambda}M_C$ with $\hat{\Lambda}M_C = \lim_\beta \hat{\Lambda}M_{C\beta}$ where $M_{C\beta}$ ranges over the finite 2-adic quotients of M_C (ignoring θ).

Lemma 4.6. *For a 2-adic $\theta\Delta$-module M, the canonical map $\hat{\Lambda}M_C \to (\hat{L}M)_C$ is an algebra isomorphism.*

This will be proved later in Section 11. We must impose extra conditions on M to ensure that $\hat{L}M$ is Bott exact and hence topologically relevant.

Definition 4.7 (The robust 2-adic $\theta\Delta$-modules)**.** We say that a 2-adic $\theta\Delta$-module M is *profinite* when it is the inverse limit of an inverse system of finite 2-adic $\theta\Delta$-modules, and we let $M/\bar{\phi}$ denote the 2-adic Δ-module $\{M_C, M_R/\bar{\phi}M_C, M_H\}$. We call M *robust* when:

(i) M is profinite;

(ii) $M/\bar{\phi}$ is torsion-free and exact;

(iii) $\ker\bar{\phi} = cM_R + c'M_H + 2M_C$.

When M is obtained from $K_\Delta^{-1}(X;\hat{\mathbb{Z}}_2)$ for a space X, the profiniteness condition will usually hold automatically since $K_\Delta^{-1}(X;\hat{\mathbb{Z}}_2) = \lim_{\alpha,i} K_\Delta^{-1}(X_\alpha;\hat{\mathbb{Z}}_2)/2^i$ for the system of finite subcomplexes $X_\alpha \subset X$ and $i \geqslant 1$. The following key lemma will be proved later in Section 12.

Lemma 4.8. *If M is a robust 2-adic $\theta\Delta$-module, then the special 2-adic ϕCR-algebra $\hat{L}M$ is Bott exact; in fact, $\hat{L}M$ is the inverse limit of an inverse system of finitely generated free 2-adic CR-modules.*

This leads to a crucial comparison theorem.

Theorem 4.9. *For a connected space X and a robust 2-adic $\theta\Delta$-module M, suppose that $g\colon M \to \widetilde{K}_{\Delta}^{-1}(X;\hat{\mathbb{Z}}_2)$ is a 2-adic $\theta\Delta$-module map that induces an isomorphism $\hat{\Lambda}M_C \cong K^*(X;\hat{\mathbb{Z}}_2)$. Then g induces an isomorphism $\hat{L}M \cong K^*_{CR}(X;\hat{\mathbb{Z}}_2)$ of special 2-adic ϕCR-algebras.*

Proof. Since g gives an admissible map $M \to K^*_{CR}(X;\hat{\mathbb{Z}}_2)$ by Lemma 4.4, the result follows by Lemmas 2.3, 4.6, and 4.8. □

When M is finitely generated in this theorem, we may easily choose a simple system of odd-degree generators (see Definition 3.3) for $K^*_{CR}(X;\hat{\mathbb{Z}}_2)$ from M_C, M_R, and M_H. However, the present description of $K^*_{CR}(X;\hat{\mathbb{Z}}_2)$ as $\hat{L}M$ is more natural and includes the full multiplicative structure. To check whether such a description is possible for a given space X, we may use:

Remark 4.10 (Determination of M from $K^*_{CR}(X;\hat{\mathbb{Z}}_2)$)**.** For a connected space X, we may take the indecomposables $\hat{Q}K^*_{CR}(X;\hat{\mathbb{Z}}_2)$ as in Definition 2.7 with the operations θ of Definition 4.3 to produce a 2-adic $\theta\Delta$-module

$$\hat{Q}K_{\Delta}^{-1}(X;\hat{\mathbb{Z}}_2) \;=\; \{\hat{Q}K^{-1}(X;\hat{\mathbb{Z}}_2), \hat{Q}KO^{-1}(X;\hat{\mathbb{Z}}_2), \hat{Q}KO^{-5}(X;\hat{\mathbb{Z}}_2)\}$$

together with a natural quotient map $\widetilde{K}_{\Delta}^{-1}(X;\hat{\mathbb{Z}}_2) \twoheadrightarrow \hat{Q}K_{\Delta}^{-1}(X;\hat{\mathbb{Z}}_2)$. Now by Lemma 4.11 below, whenever Theorem 4.9 applies to X, there is a canonical isomorphism $M \cong \hat{Q}K_{\Delta}^{-1}(X;\hat{\mathbb{Z}}_2)$ and the map $g\colon M \to \widetilde{K}_{\Delta}^{-1}(X;\hat{\mathbb{Z}}_2)$ in the theorem corresponds to a splitting of $\widetilde{K}_{\Delta}^{-1}(X;\hat{\mathbb{Z}}_2) \twoheadrightarrow \hat{Q}K_{\Delta}^{-1}(X;\hat{\mathbb{Z}}_2)$. When X is an H-space, we may often obtain the required splitting by mapping $\hat{Q}K_{\Delta}^{-1}(X;\hat{\mathbb{Z}}_2)$ to the primitives in $\widetilde{K}_{\Delta}^{-1}(X;\hat{\mathbb{Z}}_2)$. For instance, this applies when X is a suitable infinite loop space or simply-connected compact Lie group (see Theorems 6.7 and 10.3). Finally, we note that the 2-adic $\theta\Delta$-module $\hat{Q}K_{\Delta}^{-1}(X;\hat{\mathbb{Z}}_2)$ will automatically be robust by Proposition 3.4 whenever $K^*_{CR}(X;\hat{\mathbb{Z}}_2)$ has a simple system of odd-degree generators with no real generators of degree $\equiv 1, -3 \mod 8$. We have used:

Lemma 4.11. *For a $\theta\Delta$-module M, the canonical map $M \to \Delta^{-1}\hat{Q}\hat{L}M$ is an isomorphism.*

This will be proved later in Section 11.

5. Stable 2-adic Adams operations and $K/2_*$-local spectra

We now bring stable Adams operations into our united 2-adic K-cohomology theory and use this theory to classify the needed $K/2_*$-local spectra. We first recall some terminology from [**8**, 2.6].

Definition 5.1 (The stable 2-adic Adams modules)**.** By a *finite stable 2-adic Adams module A*, we mean a finite abelian 2-group with automorphisms $\psi^k\colon A \cong A$ for the odd $k \in \mathbb{Z}$ such that:

(i) $\psi^1 = 1$ and $\psi^j\psi^k = \psi^{jk}$ for the odd $j, k \in \mathbb{Z}$;

(ii) when n is sufficiently large, the condition $j \equiv k \mod 2^n$ implies $\psi^j = \psi^k$.

By a *stable 2-adic Adams module* A, we mean the topological inverse limit of an inverse system of finite stable 2-adic Adams modules. Such an A has an underlying 2-profinite abelian structure with continuous automorphisms $\psi^k\colon A \cong A$ for the odd $k \in \mathbb{Z}$ (and in fact for $k \in \hat{\mathbb{Z}}_2^\times$). We note that the operations ψ^{-1} and ψ^3 on A determine all of the other stable Adams operations ψ^k as in [**5**, 6.4]. Our main examples of stable 2-adic Adams modules are the cohomologies $K^n(X;\hat{\mathbb{Z}}_2)$ and $KO^n(X;\hat{\mathbb{Z}}_2)$ for a spectrum or space X and integer n with the usual stable Adams operations ψ^k. We let $\hat{\mathcal{A}}$ denote the abelian category of stable 2-adic Adams modules, and for $i \in \mathbb{Z}$ we let $\bar{S}^i\colon \hat{\mathcal{A}} \to \hat{\mathcal{A}}$ be the functor with $\bar{S}^i A$ equal to A as a group but with ψ^k on $\bar{S}^i A$ equal to $k^i\psi^k$ on A for the odd $k \in \mathbb{Z}$. We note that $\bar{S}^i A = A$ in $\hat{\mathcal{A}}$ for all i when $2A = 0$.

Definition 5.2 (The stable 2-adic Adams *CR*-modules)**.** By a *stable 2-adic Adams CR-module* M, we mean a 2-adic *CR*-module consisting of stable 2-adic Adams modules $\{M_C^*, M_R^*\}$ such that the operations $B\colon \bar{S}M_C^* \cong M_C^{*-2}$, $t\colon M_C^* \cong M_C^*$, $B_R\colon \bar{S}^4 M_R^* \cong M_R^{*-8}$, $\eta\colon M_R^* \to M_R^{*-1}$, $c\colon M_R^* \to M_C^*$, and $r\colon M_C^* \to M_R^*$ are all maps in $\hat{\mathcal{A}}$, where $\psi^{-1} = t$ in M_C^* and $\psi^{-1} = 1$ in M_R^*. For a spectrum or space X, the united 2-adic K-cohomology

$$K_{CR}^*(X;\hat{\mathbb{Z}}_2) \;=\; \{K^*(X;\hat{\mathbb{Z}}_2), KO^*(X;\hat{\mathbb{Z}}_2)\}$$

has a natural stable 2-adic Adams *CR*-module structure with the usual operations.

Definition 5.3 (The stable 2-adic Adams Δ-modules)**.** By a *stable 2-adic Adams* Δ*-module* N, we mean a 2-adic Δ-module consisting of stable 2-adic Adams modules $\{N_C, N_R, N_H\}$ such that the operations $t\colon N_C \cong N_C$, $c\colon N_R \to N_C$, $r\colon N_C \to N_R$, $c'\colon N_H \to N_C$, and $q\colon N_C \to N_H$ are all maps in $\hat{\mathcal{A}}$, where $\psi^{-1} = t$ in N_C and $\psi^{-1} = 1$ in both N_R and N_H. For a stable 2-adic Adams *CR*-module M and integer n, we obtain a stable 2-adic Adams Δ-module

$$\Delta^n M \;=\; \{M_C^n, M_R^n, \bar{S}^{-2} M_R^{n-4}\}$$

as in Definition 4.1. Thus, for a spectrum or space X and integer n, we now obtain a stable 2-adic Adams Δ-module

$$K_\Delta^n(X;\hat{\mathbb{Z}}_2) \;=\; \Delta^n K_{CR}^*(X;\hat{\mathbb{Z}}_2) \;=\; \{K^n(X;\hat{\mathbb{Z}}_2), KO^n(X;\hat{\mathbb{Z}}_2), \bar{S}^{-2} KO^{n-4}(X;\hat{\mathbb{Z}}_2)\}.$$

To give another example, we say that a 2-profinite abelian group G with involution $t\colon G \cong G$ is *positively torsion-free* when G is torsion-free with $\ker(1+t) = \operatorname{im}(1-t)$. By [**5**, Proposition 3.8], this is equivalent to saying that G factors as a (possibly infinite) product of $\hat{\mathbb{Z}}_2$'s with $t = 1$ and $\hat{\mathbb{Z}}_2 \oplus t\hat{\mathbb{Z}}_2$'s. For a positively torsion-free stable 2-adic Adams module A, we may use the operation $\psi^{-1}\colon A \cong A$ to construct a torsion-free exact stable 2-adic Adams Δ-module $\{A, A^+, A_+\}$ with $A^+ = \ker(1 - \psi^{-1})$, $A_+ = \operatorname{coker}(1 - \psi^{-1})$, $t = \psi^{-1}$, $c = 1$, $r = 1 + \psi^{-1}$, $c' = 1 + \psi^{-1}$, and $q = 1$.

We let $\hat{\mathcal{A}}\mathcal{CR}$ (resp. $\hat{\mathcal{A}}\Delta$) denote the abelian category of stable 2-adic Adams *CR*-modules (resp. Δ-modules), and we note that the functor $\Delta^n\colon \hat{\mathcal{A}}\mathcal{CR} \to \hat{\mathcal{A}}\Delta$ for $n \in \mathbb{Z}$ has a left adjoint $CR^n\colon \hat{\mathcal{A}}\Delta \to \hat{\mathcal{A}}\mathcal{CR}$ with $CR^n(N)_C^n = N_C$, with $CR^n(N)_C^{n-1} = 0$, and with

$$CR^n(N)_R^{n-i} = \begin{cases} N_R & \text{for } i=0 \\ N_R/r & \text{for } i=1 \\ \bar{S}N_C/c' & \text{for } i=2 \\ 0 & \text{for } i=3,7 \\ \bar{S}^2N_H & \text{for } i=4 \\ \bar{S}^2N_H/q & \text{for } i=5 \\ \bar{S}^3N_C/c & \text{for } i=6 \end{cases}$$

as in [**10**, 4.10]. We easily see that $CR^n(N)$ is Bott exact whenever N is torsion-free and exact. Our next lemma will often allow us to work in the simpler category $\hat{\mathcal{A}}\Delta$ instead of $\hat{\mathcal{A}}\mathcal{CR}$.

Lemma 5.4. *For $n \in \mathbb{Z}$, the adjoint functors $CR^n\colon \hat{\mathcal{A}}\Delta \to \hat{\mathcal{A}}\mathcal{CR}$ and $\Delta^n\colon \hat{\mathcal{A}}\mathcal{CR} \to \hat{\mathcal{A}}\Delta$ restrict to equivalences between the full subcategories of all torsion-free exact $N \in \hat{\mathcal{A}}\Delta$ and all Bott exact $M \in \hat{\mathcal{A}}\mathcal{CR}$ with M_C^n positively torsion-free and $M_C^{n-1} = 0$.*

Proof. For $M \in \hat{\mathcal{A}}\mathcal{CR}$ as above, we see that $\Delta^n M$ is a torsion-free exact Δ-module by [**10**, 4.4 and 4.7] with an adjunction isomorphism $CR^n\Delta^n M \to M$ by Lemma 2.3. The corresponding result for $N \in \hat{\mathcal{A}}\Delta$ is obvious. □

When E is a spectrum with $K^n(E;\hat{\mathbb{Z}}_2)$ positively torsion-free and $K^{n-1}(E;\hat{\mathbb{Z}}_2) = 0$ for some n, we now have $K^*_{CR}(E;\hat{\mathbb{Z}}_2) \cong CR^n(N)$ in $\hat{\mathcal{A}}\mathcal{CR}$ for the torsion-free exact module $N = \Delta^n K^*_{CR}(E;\hat{\mathbb{Z}}_2)$ in $\hat{\mathcal{A}}\Delta$, and we have the following existence theorem for such spectra in the stable homotopy category.

Theorem 5.5. *For each torsion-free exact $N \in \hat{\mathcal{A}}\Delta$ and $n \in \mathbb{Z}$, there exists a $K/2_*$-local spectrum $\mathcal{E}^n N$ with $K^*_{CR}(\mathcal{E}^n N;\hat{\mathbb{Z}}_2) \cong CR^n(N)$ in $\hat{\mathcal{A}}\mathcal{CR}$. Moreover, $\mathcal{E}^n N$ is unique up to (noncanonical) equivalence.*

Proof. This follows by Lemma 5.4 and [**10**, Theorem 5.3]. □

The spectrum $\mathcal{E}^n N$ in the theorem will be endowed with an isomorphism $K^*_{CR}(\mathcal{E}^n N;\hat{\mathbb{Z}}_2) \cong CR^n(N)$ in $\hat{\mathcal{A}}\mathcal{CR}$. Thus, for an arbitrary spectrum E, a map $g\colon E \to \mathcal{E}^n N$ induces a map $g^*\colon CR^n(N) \to K^*_{CR}(E;\hat{\mathbb{Z}}_2)$ in $\hat{\mathcal{A}}\mathcal{CR}$. Each algebraic map of this sort must come from a topological map by:

Theorem 5.6. *For a torsion-free exact $N \in \hat{\mathcal{A}}\Delta$, $n \in \mathbb{Z}$, and an arbitrary spectrum E, if $\gamma\colon CR^n(N) \to K^*_{CR}(E;\hat{\mathbb{Z}}_2)$ is a map in $\hat{\mathcal{A}}\mathcal{CR}$, then there exists a map of spectra $g\colon E \to \mathcal{E}^n N$ with $g^* = \gamma$.*

Proof. Let $\tau_2 E$ denote the 2-torsion part of E given by the homotopy fiber of its localization away from 2. By Pontrjagin duality [**10**, Theorem 3.1], the map γ corresponds to an ACR-module map $K_*^{CR}(\tau_2 E) \to K_*^{CR}(\tau_2\mathcal{E}^n N)$ in the sense of [**5**], where $K_*^{CR}(\tau_2\mathcal{E}^n N)$ is CR-exact with $K_*(\tau_2\mathcal{E}^n N)$ divisible. This ACR-module map prolongs canonically to an $ACRT$-module map $K_*^{CRT}(\tau_2 E) \to K_*^{CRT}(\tau_2\mathcal{E}^n N)$ by [**5**, Theorem 7.14], and the results of [**5**, 9.8 and 7.11] now show that this prolonged algebraic map must come from a topological map $\tau_2 E \to \tau_2 \mathcal{E}^n N$, which gives the desired $g\colon E \to \mathcal{E}^n N$. □

The map g in this theorem is generally not unique (see [**10**, 5.4]).

6. On the united 2-adic K-cohomologies of infinite loop spaces

In preparation for our work on $K/2_*$-localizations of spaces, we functorially determine the united 2-adic K-cohomologies of the needed infinite loop spaces (see Theorem 6.7). We must first introduce:

Definition 6.1 (The 2-adic Adams Δ-modules)**.** By a *2-adic Adams Δ-module M*, we mean a 2-adic $\theta\Delta$-module (see Definition 4.3) consisting of stable 2-adic Adams modules $\{M_C, M_R, M_H\}$ such that the operations $t\colon M_C \cong M_C$, $c\colon M_R \to M_C$, $r\colon M_C \to M_R$, $c'\colon M_H \to M_C$, $q\colon M_C \to M_H$, $\theta\colon M_C \to M_C$, $\theta\colon M_R \to M_R$, and $\theta\colon M_H \to M_R$ are all maps in $\hat{\mathcal{A}}$, where $\psi^{-1} = t$ in M_C and $\psi^{-1} = 1$ in both M_R and M_H. We let $\hat{\mathcal{M}}\Delta$ denote the abelian category of 2-adic Adams Δ-modules. We say that M is *θ-nilpotent* when it has $\theta^i = 0$ for sufficiently large i, and we say that M is *θ-pro-nilpotent* when it is the inverse limit of an inverse system of θ-nilpotent 2-adic Adams Δ-modules. Thus, M is θ-pro-nilpotent if and only if $M \cong \lim_i M/\theta^i$ where M/θ^i is the quotient module of M in $\hat{\mathcal{M}}\Delta$ with

$$\begin{aligned}(M/\theta^i)_C &= M_C/\theta^i M_C,\\ (M/\theta^i)_R &= M_R/(\theta^i M_R + \theta^i M_H + r\theta^i M_C),\\ (M/\theta^i)_H &= M_H/q\theta^i M_C\end{aligned}$$

for $i \geqslant 1$. More simply, M is θ-pro-nilpotent if and only if $\cap_i \theta^i M_C = 0$ and $\cap_i \theta^i M_R = 0$. It is not hard to show that whenever M is θ-pro-nilpotent, M must be profinite (i.e. M must be the inverse limit of an inverse system of finite 2-adic Adams Δ-modules). For a space X, the cohomology

$$\widetilde{K}_{\Delta}^{-1}(X;\hat{\mathbb{Z}}_2) \;=\; \{\widetilde{K}^{-1}(X;\hat{\mathbb{Z}}_2), \widetilde{KO}^{-1}(X;\hat{\mathbb{Z}}_2), \bar{S}^{-2}\widetilde{KO}^{-5}(X;\hat{\mathbb{Z}}_2)\}$$

has a natural 2-adic Adams Δ-module structure by [**11**, Section 3], and we find:

Lemma 6.2. *If X is a connected space with $H^1(X;\hat{\mathbb{Z}}_2) = 0$, then the 2-adic Adams Δ-module $\widetilde{K}_{\Delta}^{-1}(X;\hat{\mathbb{Z}}_2)$ is θ-pro-nilpotent.*

Proof. The condition $\cap_i \theta^i \widetilde{K}^0(\Sigma X;\hat{\mathbb{Z}}_2) = 0$ holds by [**6**, 5.4 and 5.5] since $H^2(\Sigma X;\hat{\mathbb{Z}}_2) = 0$, and a similar proof shows $\cap_i \theta^i \widetilde{KO}^0(\Sigma X;\hat{\mathbb{Z}}_2) = 0$ since $H^1(\Sigma X;\mathbb{Z}/2) = 0$. This proof uses the fact that the λ-ideal $\widetilde{KO}^0 Y$ is γ-nilpotent for a connected finite CW complex Y by [**10**, Theorem 6.7] and the fact that the real line bundles over Y are classified by $H^1(Y;\mathbb{Z}/2)$. □

Definition 6.3 (The functor $\tilde{F}$)**.** We shall construct a functor $\tilde{F}\colon \hat{\mathcal{A}}\Delta \to \hat{\mathcal{M}}\Delta$ where $\hat{\mathcal{A}}\Delta$ is the abelian category of stable 2-adic Adams Δ-modules and $\hat{\mathcal{M}}\Delta$

is that of 2-adic Adams Δ-modules (see Definitions 5.3 and 6.1). This functor will carry each $N \in \hat{\mathcal{A}}\Delta$ to a universal θ-pro-nilpotent target module $\tilde{F}N \in \hat{\mathcal{M}}\Delta$. For $N \in \hat{\mathcal{A}}\Delta$, we first let $N_{RH} \in \hat{\mathcal{A}}$ denote the pushout of $N_R \xleftarrow{r} N_C \xrightarrow{q} N_H$ with a map $\bar{c}\colon N_{RH} \to N_C$ induced by c and c', and with a map $\bar{r}\colon N_C \to N_{RH}$ induced by r or q. We also let $N_{C+} \in \hat{\mathcal{A}}$ denote $N_C/(1-t)N_C$ and let $N_{C\phi} \in \hat{\mathcal{A}}$ denote $N_C/(cN_R + c'N_H + 2N_C)$. We next let

$$\rho N \ = \ \{N_C, N_{RH} \oplus N_{C\phi}, N_{C+}\}$$

be the stable 2-adic Adams Δ-module with operations given by $tz = tz$, $c(x,w) = \bar{c}x$, $rz = (\bar{r}z, [z])$, $c'[z] = (1+t)z$, and $qz = [z]$. We then obtain a stable 2-adic Adams Δ-module

$$\tilde{F}N \ = \ N \times \rho N \times \rho N \times \cdots$$

with components

$$\begin{aligned} \tilde{F}_C N \ &= \ N_C \times N_C \times N_C \times \cdots\,, \\ \tilde{F}_R N \ &= \ N_R \times N_{RH} \times N_{C\phi} \times N_{RH} \times N_{C\phi} \times \cdots\,, \\ \tilde{F}_H N \ &= \ N_H \times N_{C+} \times N_{C+} \times \cdots\,. \end{aligned}$$

We finally define operations $\theta\colon \tilde{F}_C N \to \tilde{F}_C N$, $\theta\colon \tilde{F}_R N \to \tilde{F}_R N$, and $\theta\colon \tilde{F}_H N \to \tilde{F}_R N$ respectively by the formulae

$$\begin{aligned} \theta(z_1, z_2, z_3, \dots) &= (0, z_1, z_2, z_3, \dots), \\ \theta(x_1, x_2, z_2, x_3, z_3, \dots) &= (0, [x_1], 0, x_2, 0, x_3, 0, \dots), \\ \theta(y_1, z_2, z_3, \dots) &= (0, [y_1], 0, \bar{r}z_2, 0, \bar{r}z_3, 0, \dots). \end{aligned}$$

This gives a natural 2-adic Adams Δ-module $\tilde{F}N$ and hence a functor $\tilde{F}\colon \hat{\mathcal{A}}\Delta \to \hat{\mathcal{M}}\Delta$. We let $\iota\colon N \to \tilde{F}N$ be the map in $\hat{\mathcal{A}}\Delta$ with $\iota_C(z) = (z, 0, 0, \dots)$, $\iota_R(x) = (x, 0, 0, \dots)$, and $\iota_H(y) = (y, 0, 0, \dots)$, and we show:

Theorem 6.4. *For a stable 2-adic Adams Δ-module $N \in \hat{\mathcal{A}}\Delta$, the 2-adic Adams Δ-module $\tilde{F}N \in \hat{\mathcal{M}}\Delta$ is θ-pro-nilpotent and the map $\iota\colon N \to \tilde{F}N$ has the universal property that, for each θ-pro-nilpotent $M \in \hat{\mathcal{M}}\Delta$ and map $f\colon N \to M$ in $\hat{\mathcal{A}}\Delta$, there exists a unique map $\bar{f}\colon \tilde{F}N \to M$ in $\hat{\mathcal{M}}\Delta$ with $\bar{f}\iota = f$.*

Proof. $\tilde{F}N$ is θ-pro-nilpotent since it is the inverse limit of its quotient modules

$$\tilde{F}N/\theta^{i+1} \ \cong \ N \times \rho N \times \cdots \times \rho N.$$

For $i \geqslant 1$, we define a map $f^{(i)}\colon \rho N \to M$ in $\hat{\mathcal{A}}\Delta$ by

$$\begin{aligned} f_C^{(i)} \ &= \ \theta^i f_C\colon N_C \longrightarrow M_C, \\ f_R^{(i)} \ &= \ (\theta^i f_R, \theta^i f_H) + \bar{\phi}\theta^{i-1} f_C\colon N_{RH} \oplus N_{C\phi} \longrightarrow M_R, \\ f_H^{(i)} \ &= \ q\theta^i f_C\colon N_{C+} \longrightarrow M_H. \end{aligned}$$

We then define $\bar{f}\colon \tilde{F}N \to M$ as the inverse limit of the maps

$$f + f^{(1)} + \cdots + f^{(i)}\colon N \times \rho N \times \cdots \times \rho N \longrightarrow M/\theta^{i+1}$$

in $\hat{\mathcal{M}}\Delta$, and we check that $\bar{f}\iota = f$. The uniqueness condition for $\bar{f}$ follows since the

2-adic Adams Δ-modules $\tilde{F}N/\theta^{i+1} = N \times \rho N \times \cdots \times \rho N$ are generated by ιN. $\square$

To show the robustness (see Definition 4.7) of $\tilde{F}N$ for suitable N, we need:

Definition 6.5 (The functor $\bar{\rho}\colon \hat{\mathcal{A}}\Delta \to \hat{\mathcal{A}}\Delta$)**.** For $N \in \hat{\mathcal{A}}\Delta$, we let $\bar{\rho}N = \{N_C, N_{RH}, N_{C+}\}$ be the stable 2-adic Adams Δ-module with operations given by $tz = tz$, $cx = \bar{c}x$, $rz = \bar{r}z$, $c'[z] = (1+t)z$, and $qz = [z]$. Thus, $\bar{\rho}N$ is the quotient of $\rho N = \{N_C, N_{RH} \oplus N_{C\phi}, N_{C+}\}$ by $N_{C\phi}$. If N is torsion-free and exact, then $\bar{\rho}N$ is also torsion-free and exact by Lemma 4.2 since it is isomorphic to the module $\{N_C, N_R + N_H, N_R \cap N_H\}$ with c and c' treated as inclusions.

Lemma 6.6. *If $N \in \hat{\mathcal{A}}\Delta$ is torsion-free and exact, then $\tilde{F}N \in \hat{\mathcal{M}}\Delta$ is robust.*

Proof. We check that $\bar{\phi}\colon \tilde{F}_C N \to \tilde{F}_R N$ is given by

$$\bar{\phi}(z_1, z_2, z_3, \dots) \;=\; (0, 0, [z_1], 0, [z_2], 0, \dots)$$

for $z_i \in N_C$ and $[z_i] \in N_{C\phi}$. Thus, $\ker\bar{\phi} \;=\; c\tilde{F}_R N + c'\tilde{F}_H N + 2\tilde{F}_C N$ and $\tilde{F}N/\bar{\phi} \cong N \times \bar{\rho}N \times \bar{\rho}N \times \cdots$. Hence, $\tilde{F}N/\bar{\phi}$ is torsion-free and exact by Definition 6.5 as required. $\square$

Our main result in this section is:

Theorem 6.7. *If E is a 0-connected spectrum with $H^1(E;\hat{\mathbb{Z}}_2) = 0 = H^2(E;\hat{\mathbb{Z}}_2)$, with $K^0(E;\hat{\mathbb{Z}}_2) = 0$, and with $K^{-1}(E;\hat{\mathbb{Z}}_2)$ positively torsion-free (5.3), then there is a natural isomorphism $\hat{L}\tilde{F}K^{-1}_{\Delta}(E;\hat{\mathbb{Z}}_2) \cong K^*_{CR}(\Omega^\infty E;\hat{\mathbb{Z}}_2)$.*

Proof. Since $\widetilde{K}^{-1}_{\Delta}(\Omega^\infty E;\hat{\mathbb{Z}}_2)$ is θ-pro-nilpotent by Lemma 6.2, the infinite suspension map $\sigma\colon K^{-1}_{\Delta}(E;\hat{\mathbb{Z}}_2) \to \widetilde{K}^{-1}_{\Delta}(\Omega^\infty E;\hat{\mathbb{Z}}_2)$ induces a map $\bar{\sigma}\colon \tilde{F}K^{-1}_{\Delta}(E;\hat{\mathbb{Z}}_2) \to \widetilde{K}^{-1}_{\Delta}(\Omega^\infty E;\hat{\mathbb{Z}}_2)$ in $\hat{\mathcal{M}}\Delta$, where $\tilde{F}K^{-1}_{\Delta}(E;\hat{\mathbb{Z}}_2)$ is robust by Lemmas 5.4 and 6.6. Thus $\bar{\sigma}$ induces an isomorphism $\hat{L}\tilde{F}K^{-1}_{\Delta}(E;\hat{\mathbb{Z}}_2) \cong K^*_{CR}(\Omega^\infty E;\hat{\mathbb{Z}}_2)$ by Theorem 4.9, since it induces an isomorphism of the complex components by [**6**, Theorem 8.3]. $\square$

7. Strong 2-adic Adams Δ-modules

Our main results on $K/2_*$-localizations in Section 8 will involve a space X with $K^*_{CR}(X;\hat{\mathbb{Z}}_2) \cong \hat{L}M$ for a 2-adic Adams Δ-module M that is *strong* in the sense that it is *robust*, *ψ^3-splittable*, and *regular*. In this section, we provide the required algebraic definitions and explanations of these notions. We first recall:

Definition 7.1 (The robust modules)**.** We say that a 2-adic Adams Δ-module M is *robust* when it is robust in the sense of Definition 4.7, ignoring stable Adams operations. When M is robust, the underlying 2-adic Δ-module $M/\bar{\phi}$ satisfies the conditions of Lemma 4.2 and may be factored as a (possibly infinite) product of monogenic free 2-adic Δ-modules

$$\begin{aligned} F^C(z) &= \{\hat{\mathbb{Z}}_2 \oplus t\hat{\mathbb{Z}}_2, \hat{\mathbb{Z}}_2, \hat{\mathbb{Z}}_2\} = \{\langle z\rangle \oplus \langle tz\rangle, \langle rz\rangle, \langle qz\rangle\}, \\ F^R(x) &= \{\hat{\mathbb{Z}}_2, \hat{\mathbb{Z}}_2, \hat{\mathbb{Z}}_2\} = \{\langle cx\rangle, \langle x\rangle, \langle qcx\rangle\}, \\ F^H(y) &= \{\hat{\mathbb{Z}}_2, \hat{\mathbb{Z}}_2, \hat{\mathbb{Z}}_2\} = \{\langle c'y\rangle, \langle rc'y\rangle, \langle y\rangle\} \end{aligned}$$

by an argument using the factorization of positively torsion-free groups in Definition 5.3. We let $\operatorname{gen}_C M$, $\operatorname{gen}_R M$, and $\operatorname{gen}_H M$ respectively denote the number of complex, real, and quaternionic monogenic free factors of $M/\bar{\phi}$. These numbers do not depend on the factorization since they equal the dimensions of the respective $\mathbb{Z}/2$-vector spaces $(M_{C\phi})^{\#}$, $(M_R/(\bar{\phi}M_C + rM_C))^{\#}$, and $(M_H/qM_C)^{\#}$, where $(-)^{\#}$ is the Pontrjagin duality functor from 2-profinite abelian groups to discrete 2-torsion abelian groups. Using the factorization of $M/\bar{\phi}$, we find that

$$\operatorname{gen} M_C = 2\operatorname{gen}_C M + \operatorname{gen}_R M + \operatorname{gen}_H M$$

where $\operatorname{gen} M_C$ denotes the number of $\hat{\mathbb{Z}}_2$ factors in the 2-profinite abelian group M_C.

Definition 7.2 (The ψ^3-splittable modules)**.** For a 2-adic Adams Δ-module $M \in \hat{\mathcal{M}}\Delta$, we consider the stable 2-adic Adams Δ-module $\bar{M} = M/\bar{\phi} \in \hat{\mathcal{A}}\Delta$, and we say that M is *ψ^3-splittable* when the quotient map $M \twoheadrightarrow \bar{M}$ has a right inverse $s\colon \bar{M} \to M$ in $\hat{\mathcal{A}}\Delta$. We call such a map s a *ψ^3-splitting* of M, and we note that it corresponds to a left inverse $s'\colon M_R/rM_C \to \bar{\phi}M_C$ of the canonical map $\bar{\phi}M_C \to M_R/rM_C$ in the category $\hat{\mathcal{A}}$ of stable 2-adic Adams modules, or equivalently in the category of profinite $\mathbb{Z}/2$-modules with automorphisms ψ^3. We deduce that M is automatically ψ^3-splittable in some important cases:

Lemma 7.3. *If M is a robust 2-adic Adams Δ-module with $\operatorname{gen}_C M = 0$ or $\operatorname{gen}_R M = 0$, then M is ψ^3-splittable.*

Proof. Since M_C is positively torsion-free, the map $cr = 1 + t\colon M_{C+} \to M_C$ is monic, and hence $c\colon rM_C \to M_C$ is also monic. Thus, $\bar{\phi}M_C \cap rM_C = 0$ and there is a short exact sequence

$$0 \longrightarrow \bar{\phi}M_C \longrightarrow M_R/rM_C \longrightarrow M_R/(\bar{\phi}M_C + rM_C) \longrightarrow 0$$

in $\hat{\mathcal{A}}$. Since $\operatorname{gen}_C M = 0$ or $\operatorname{gen}_R M = 0$, this has $\bar{\phi}M_C = 0$ or $M_R/(\bar{\phi}M_C + rM_C) = 0$, and hence the map $\bar{\phi}M_C \to M_R/rM_C$ has an obvious left inverse in $\hat{\mathcal{A}}$. □

We shall use the ψ^3-splittability condition to give:

Definition 7.4 (The θ-resolutions of modules)**.** Let $M \in \hat{\mathcal{M}}\Delta$ be a 2-adic Adams Δ-module that is θ-pro-nilpotent, robust, and ψ^3-splittable. These conditions will hold when M is strong (see Definition 7.11). For a ψ^3-splitting $s\colon \bar{M} \to M$ in $\hat{\mathcal{A}}\Delta$, we shall construct an associated *θ-resolution*

$$0 \longrightarrow \tilde{F}\bar{\rho}\bar{M} \xrightarrow{\bar{d}} \tilde{F}\bar{M} \xrightarrow{\bar{s}} M \longrightarrow 0$$

of M in $\hat{\mathcal{M}}\Delta$, with $\bar{\rho}\bar{M} = \{\bar{M}_C, \bar{M}_{RH}, \bar{M}_{C+}\}$ as in Definition 6.5, where $\bar{s}\colon \tilde{F}\bar{M} \to$

M is induced by s via Theorem 6.4. To specify $\bar{d}$, we use the commutative square

$$\begin{array}{ccc} \bar{\rho}\bar{M} & \xrightarrow{\theta} & \bar{M} \\ \downarrow \sigma & & \downarrow s \\ \rho\bar{M} & \xrightarrow{s^{(1)}} & M \end{array}$$

in $\hat{\mathcal{A}}\Delta$ with $\rho\bar{M} = \{\bar{M}_C, \bar{M}_{RH} \oplus \bar{M}_{C\phi}, \bar{M}_{C+}\}$ as in Definition 6.3, where $s^{(1)}$ is given by the proof of Theorem 6.4, where $\theta = \{\theta, (\theta, \theta), q\theta\}$, and where $\sigma = \{1, (1, \theta_\phi), 1\}$, using the map $\theta_\phi \colon \bar{M}_{RH} \to \bar{M}_{C\phi} = M_{C\phi}$ given by the composition of the sequence

$$\bar{M}_{RH} \xrightarrow{s} M_{RH} \xrightarrow{(\theta,\theta)} M_R \;\cong\; \bar{M}_R \oplus M_{C\phi} \xrightarrow{\text{proj}} M_{C\phi}$$

in which the isomorphism is the inverse of $(s, \bar{\phi}) \colon \bar{M}_R \oplus M_{C\phi} \cong M_R$. The commutative square now gives a map

$$d \;=\; (\theta, -\sigma, 0, 0, \dots) \colon \bar{\rho}\bar{M} \to \tilde{F}\bar{M}$$

in $\hat{\mathcal{A}}\Delta$ with $\bar{s}d = 0$, and this induces the required map $\bar{d} \colon \tilde{F}\bar{\rho}\bar{M} \to \tilde{F}\bar{M}$ in $\hat{\mathcal{M}}\Delta$ with $\bar{s}\bar{d} = 0$.

Lemma 7.5. *If $M \in \hat{\mathcal{M}}\Delta$ is θ-pro-nilpotent and robust with a ψ^3-splitting $s \colon \bar{M} \to M$, then the θ-resolution $0 \to \tilde{F}\bar{\rho}\bar{M} \xrightarrow{\bar{d}} \tilde{F}\bar{M} \xrightarrow{\bar{s}} M \to 0$ is exact in $\hat{\mathcal{M}}\Delta$.*

Proof. We easily check that $0 \to \bar{\phi}(\tilde{F}\bar{\rho}\bar{M})_C \to \bar{\phi}(\tilde{F}\bar{M})_C \to \bar{\phi}M_C \to 0$ is exact and that $\bar{s}/\bar{\phi} \colon \tilde{F}\bar{M}/\bar{\phi} \to M/\bar{\phi}$ is onto. Hence, it suffices to show that the map $\tilde{F}\bar{\rho}\bar{M}/\bar{\phi} \to \ker(\bar{s}/\bar{\phi})$ is an isomorphism. This follows by [**10**, Lemma 4.8] since the map $(\tilde{F}\bar{\rho}\bar{M}/\bar{\phi})_C \to \ker(\bar{s}/\bar{\phi})_C$ is clearly an isomorphism and since the 2-adic Δ-modules $\tilde{F}\bar{\rho}\bar{M}/\bar{\phi}$ and $\ker(\bar{s}/\bar{\phi})$ are exact by Lemma 6.6 and by the short exact sequence rule of [**10**, 4.5]. □

To formulate our regularity condition for M, we use:

Definition 7.6 (The 2-adic Adams modules)**.** These are the unstable versions of the stable 2-adic Adams modules and were previously discussed in [**8**, 2.8]. By a *finite 2-adic Adams module* A, we mean a finite abelian 2-group with endomorphisms $\psi^k \colon A \to A$ for $k \in \mathbb{Z}$ such that:

(i) $\psi^1 = 1$ and $\psi^j\psi^k = \psi^{jk}$ for $j, k \in \mathbb{Z}$;

(ii) when n is sufficiently large, the condition $j \equiv k \mod 2^n$ implies $\psi^j = \psi^k$.

By a *2-adic Adams module A*, we mean the topological inverse limit of an inverse system of finite 2-adic Adams modules. Such an A has an underlying 2-profinite abelian group with continuous endomorphisms $\psi^k \colon A \to A$ for $k \in \mathbb{Z}$ (and in fact for $k \in \hat{\mathbb{Z}}_2$). For a space X, the cohomology $K^1(X; \hat{\mathbb{Z}}_2)$ is a 2-adic Adams module with the usual Adams operations ψ^k for $k \in \mathbb{Z}$ as in [**6**, Example 5.2]. We note that the operations ψ^2 and ψ^k, for k odd, in $K^1(X; \hat{\mathbb{Z}}_2)$ correspond via Bott periodicity to θ and to $k^{-1}\psi^k$ in $K^{-1}(X; \hat{\mathbb{Z}}_2)$. In general, for a θ-pro-nilpotent 2-adic Adams Δ-module M, we obtain a 2-adic Adams module M^C having the same group as M_C but having $\psi^0 = 0$ and having ψ^{k2^i} equal to $k^{-1}\psi^k\theta^i$ on M_C for k odd and $i \geqslant 0$.

Definition 7.7 (The linear and strictly nonlinear modules)**.** As in [**8**, Section 4] and [**7**, Section 2], a 2-adic Adams module H is called *linear* when it has $\psi^k = k$ for all $k \in \mathbb{Z}$, and H is called *quasilinear* when $2H \subset \psi^2 H$. Each 2-adic Adams module A has a largest linear quotient module

$$\operatorname{Lin} A \;=\; A/((\psi^2 - 2)A + (\psi^{-1} + 1)A + (\psi^3 - 3)A)$$

and also has a largest quasilinear submodule $A_{ql} \subset A$ by Lemma 13.1 below. A 2-adic Adams module A is called *strictly nonlinear* when $A_{ql} = 0$. This implies that A is torsion-free with $\cap_i (\psi^2)^i A = 0$, and A will be strictly nonlinear by Remark 13.2 and [**7**, 2.5] whenever it is torsion-free with $(\psi^2)^i A \subset 2^{i+1} A$ for some $i \geqslant 1$.

Definition 7.8 (The regular modules)**.** As in [**8**, 4.4], we say that a 2-adic Adams module A is *regular* when the kernel of $A \to \operatorname{Lin} A$ is strictly nonlinear. This implies that $\cap_i (\psi^2)^i A = 0$, and A will be regular whenever it is an extension of a strictly nonlinear submodule by a linear quotient module. We also say that a 2-adic Adams Δ-module M is *regular* when it is θ-pro-nilpotent with M^C regular as a 2-adic Adams module. For a connected space X with $H^1(X; \hat{\mathbb{Z}}_2) = 0$, the 2-adic Adams Δ-module $\widetilde{K}_\Delta^{-1}(X; \hat{\mathbb{Z}}_2)$ is always θ-pro-nilpotent by Lemma 6.2, and hence $\widetilde{K}_\Delta^{-1}(X; \hat{\mathbb{Z}}_2)$ is regular if and only if $\widetilde{K}^1(X; \hat{\mathbb{Z}}_2)$ is regular as a 2-adic Adams module. The following two lemmas will often guarantee regularity for our modules.

Lemma 7.9. *Let X be a connected space with $H^1(X; \hat{\mathbb{Z}}_2) = 0$, with $H^m(X; \hat{\mathbb{Z}}_2) = 0$ for sufficiently large m, and with $\widetilde{K}^1(X; \hat{\mathbb{Z}}_2)$ torsion-free. Then $\widetilde{K}^1(X; \hat{\mathbb{Z}}_2)$ is regular with $\psi^2 \colon \widetilde{K}^1(X; \hat{\mathbb{Z}}_2) \to \widetilde{K}^1(X; \hat{\mathbb{Z}}_2)$ monic, and hence $\widetilde{K}_\Delta^{-1}(X; \hat{\mathbb{Z}}_2)$ is regular with $\theta \colon \widetilde{K}^{-1}(X; \hat{\mathbb{Z}}_2) \to \widetilde{K}^{-1}(X; \hat{\mathbb{Z}}_2)$ monic.*

Lemma 7.10. *For a regular 2-adic Adams module A, each submodule is regular, and each torsion-free quotient module is regular when A is finitely generated over $\hat{\mathbb{Z}}_2$.*

The proofs are in Section 13. Combining the preceding definitions, we finally introduce:

Definition 7.11 (The strong modules)**.** We say that a 2-adic Adams Δ-module $M \in \hat{\mathcal{M}}\Delta$ is *strong* when:

(i) M is robust;
(ii) M is ψ^3-splittable;
(iii) M is regular.

Such an M is automatically θ-pro-nilpotent (and hence profinite) since it is regular.

8. On the $K/2_*$-localizations of our spaces

We recall that the $K/2_*$-localizations of spaces or spectra are the same as the $K^*(-; \hat{\mathbb{Z}}_2)$-localizations since the $K/2_*$-equivalences are the same as the $K^*(-; \hat{\mathbb{Z}}_2)$-equivalences. In this section, we give our main result (Theorem 8.6) on the $K/2_*$-localization of a connected space X with $K_{CR}^*(X; \hat{\mathbb{Z}}_2) \cong \hat{L} M$ for a strong 2-adic Adams Δ-module M. We first consider:

Definition 8.1 (Building blocks for $K/2_*$-localizations)**.** For a torsion-free exact stable 2-adic Adams Δ-module $N \in \hat{\mathcal{A}}\Delta$, we let $\mathcal{E}N$ denote the $K/2_*$-local spectrum $\mathcal{E}^{-1}N$ of Theorem 5.5 with an isomorphism $K^*_{CR}(\mathcal{E}N;\hat{\mathbb{Z}}_2) \cong CR^{-1}N$ in the category $\hat{\mathcal{A}}\mathcal{CR}$ of stable 2-adic Adams CR-modules. As in [**8**, 3.5], we let $\tilde{\mathcal{E}}N \to \mathcal{E}N \to \bar{P}^2\mathcal{E}N$ denote the Postnikov fiber sequence of spectra with $\pi_i\tilde{\mathcal{E}}N \cong \pi_i\mathcal{E}N$ for $i > 2$, with $\pi_i\tilde{\mathcal{E}}N = 0$ for $i < 2$, and with $\pi_2\tilde{\mathcal{E}}N \cong \hat{t}_2\pi_2\mathcal{E}N$, where $\hat{t}_2\pi_2\tilde{\mathcal{E}}N \subset \pi_2\mathcal{E}N$ denotes the Ext-2-completion of the torsion subgroup of $\pi_2\mathcal{E}N$. We now obtain a simply-connected infinite loop space $\Omega^\infty\tilde{\mathcal{E}}N$ which is $K/2_*$-local by [**8**, Theorem 3.8]. These $\Omega^\infty\tilde{\mathcal{E}}N$, with their companions $\Omega^\infty\tilde{\mathcal{E}}\bar{\rho}N$, will serve as our building blocks for $K/2_*$-localizations of spaces, where $\bar{\rho}N$ denotes the torsion-free exact stable 2-adic Adams Δ-module $\bar{\rho}N = \{N_C, N_R + N_H, N_R \cap N_H\}$ of Definition 6.5.

Definition 8.2 (Strict homomorphisms and isomorphisms)**.** For a 2-adic Adams Δ-module $M \in \hat{\mathcal{M}}\Delta$ and a connected space X, a *strict homomorphism* (resp. *strict isomorphism*) $\hat{L}M \to K^*_{CR}(X;\hat{\mathbb{Z}}_2)$ is a homomorphism (resp. isomorphism) of special 2-adic ϕCR-algebras induced by a map $M \to \widetilde{K}^{-1}_{\Delta}(X;\hat{\mathbb{Z}}_2)$ of 2-adic Adams Δ-modules. For instance, there is a strict isomorphism

$$\hat{L}\tilde{F}N \cong K^*_{CR}(\Omega^\infty\tilde{\mathcal{E}}N;\hat{\mathbb{Z}}_2)$$

for each torsion-free exact stable 2-adic Adams Δ-module $N \in \hat{\mathcal{A}}\Delta$ by Theorem 6.7, and we have:

Lemma 8.3. *For a torsion-free exact module $N \in \hat{\mathcal{A}}\Delta$ and a connected space X with $H^1(X;\hat{\mathbb{Z}}_2) = 0 = H^2(X;\hat{\mathbb{Z}}_2)$, each strict homomorphism $\hat{L}\tilde{F}N \to K^*_{CR}(X;\hat{\mathbb{Z}}_2)$ is induced by a (possibly non-unique) map $X \to \Omega^\infty\tilde{\mathcal{E}}N$.*

Proof. A strict homomorphism $\hat{L}\tilde{F}N \to K^*_{CR}(X;\hat{\mathbb{Z}}_2)$ corresponds successively to: a map $\tilde{F}N \to \widetilde{K}^{-1}_{\Delta}(X;\hat{\mathbb{Z}}_2)$ in $\hat{\mathcal{M}}\Delta$, a map $N \to \widetilde{K}^{-1}_{\Delta}(X;\hat{\mathbb{Z}}_2)$ in $\hat{\mathcal{A}}\Delta$, and a map $CR^{-1}N \to K^*_{CR}(\Sigma^\infty X;\hat{\mathbb{Z}}_2)$ in $\hat{\mathcal{A}}\mathcal{CR}$. By Theorem 5.6, this last map is induced by a map $\Sigma^\infty X \to \mathcal{E}N$, which lifts uniquely to a map $\Sigma^\infty X \to \tilde{\mathcal{E}}N$, and we can easily check that the adjoint map $X \to \Omega^\infty\tilde{\mathcal{E}}N$ induces the original strict homomorphism. □

Definition 8.4 (The key construction)**.** For a strong 2-adic Adams Δ-module $M \in \hat{\mathcal{M}}\Delta$, we may take a θ-resolution (see Definition 7.4)

$$0 \longrightarrow \tilde{F}\bar{\rho}\bar{M} \xrightarrow{\bar{d}} \tilde{F}\bar{M} \xrightarrow{\bar{s}} M \longrightarrow 0$$

using the torsion-free exact module $\bar{M} = M/\bar{\phi} \in \hat{\mathcal{A}}\Delta$. We may then apply Lemma 8.3 to give a map $f\colon \Omega^\infty\tilde{\mathcal{E}}\bar{M} \to \Omega^\infty\tilde{\mathcal{E}}\bar{\rho}\bar{M}$ inducing the $K^*_{CR}(-;\hat{\mathbb{Z}}_2)$-homomorphism $f^* = \hat{L}\bar{d}\colon \hat{L}\tilde{F}\bar{\rho}\bar{M} \to \hat{L}\tilde{F}M$. Any such f will be called a *companion map* of M, and its homotopy fiber $\operatorname{Fib} f$ will be $K/2_*$-local since $\Omega^\infty\tilde{\mathcal{E}}M$ and $\Omega^\infty\tilde{\mathcal{E}}\bar{\rho}M$ are. As in [**8**, 4.6] and Definition 8.1, we let

$$\widetilde{\operatorname{Fib}} f \longrightarrow \operatorname{Fib} f \longrightarrow \bar{P}^2 \operatorname{Fib} f$$

denote the Postnikov fiber sequence with $\pi_i\widetilde{\operatorname{Fib}} f \cong \pi_i \operatorname{Fib} f$ for $i > 2$, with $\pi_i\widetilde{\operatorname{Fib}} f = 0$ for $i < 2$, and with $\pi_i\widetilde{\operatorname{Fib}} f \cong \hat{t}_2\pi_2\widetilde{\operatorname{Fib}} f$. We note that $\bar{P}^2 \operatorname{Fib} f$ is an infinite loop space which is $K/2_*$-local by [**8**, Theorem 3.8], and we conclude that $\widetilde{\mathrm{Fib}} f$ is also $K/2_*$-local. Moreover, we have $K^*_{CR}(\widetilde{\mathrm{Fib}} f; \hat{\mathbb{Z}}_2) \cong \hat{L}M$ by:

Theorem 8.5. *For a strong 2-adic Adams Δ-module $M \in \hat{\mathcal{M}}\Delta$ and any companion map $f\colon \Omega^\infty \tilde{\mathcal{E}} \bar{M} \to \Omega^\infty \tilde{\mathcal{E}} \bar{\rho} \bar{M}$, there is a strict isomorphism $\hat{L}M \cong K^*_{CR}(\widetilde{\mathrm{Fib}} f; \hat{\mathbb{Z}}_2)$.*

Thus, $\hat{L}M$ is topologically realizable for each strong $M \in \hat{\mathcal{M}}\Delta$. This theorem will be proved in Section 14 and leads immediately to our main result on $K/2_*$-localizations of spaces.

Theorem 8.6. *If X is a connected space with a strict isomorphism $\hat{L}M \cong K^*_{CR}(X; \hat{\mathbb{Z}}_2)$ for a strong 2-adic Adams Δ-module $M \in \hat{\mathcal{M}}\Delta$, then there is an equivalence $X_{K/2} \simeq \widetilde{\mathrm{Fib}} f$ for some companion map $f\colon \Omega^\infty \tilde{\mathcal{E}} \bar{M} \to \Omega^\infty \tilde{\mathcal{E}} \bar{\rho} \bar{M}$ of M, where the equivalence induces the canonical isomorphism $K^*_{CR}(\widetilde{\mathrm{Fib}} f; \hat{\mathbb{Z}}_2) \cong \hat{L}M \cong K^*_{CR}(X; \hat{\mathbb{Z}}_2)$. Moreover, $H^1(X; \hat{\mathbb{Z}}_2) = 0 = H^2(X; \hat{\mathbb{Z}}_2)$.*

Proof. The last statement follows by [**6**, 5.4]. For the first, we take a θ-resolution $0 \to \tilde{F} \bar{\rho} \bar{M} \xrightarrow{\bar{d}} \tilde{F} \bar{M} \xrightarrow{\bar{s}} M \to 0$ of M and apply Lemma 8.3 to give a map $h\colon X \to \Omega^\infty \tilde{\mathcal{E}} \bar{M}$ with $h^* = \hat{L}\bar{s}\colon \hat{L}\tilde{F}\bar{M} \to \hat{L}M$. We then apply Lemma 8.3 again to give a map $k\colon \operatorname{Cof} h \to \Omega^\infty \tilde{\mathcal{E}} \bar{\rho} \bar{M}$ with

$$k^* = \hat{L}\bar{d}\colon \hat{L}\tilde{F}\bar{\rho}\bar{M} \longrightarrow K^*_{CR}(\operatorname{Cof} h; \hat{\mathbb{Z}}_2) \subset \hat{L}\tilde{F}\bar{M}.$$

Composing k with the cofiber map, we obtain a companion map $f\colon \Omega^\infty \tilde{\mathcal{E}} \bar{M} \to \Omega^\infty \tilde{\mathcal{E}} \bar{\rho} \bar{M}$ of M such that h lifts to a map $u\colon X \to \widetilde{\mathrm{Fib}} f$ which is a $K/2_*$-equivalence by Theorem 8.5. Since $\widetilde{\mathrm{Fib}} f$ is $K/2_*$-local, this gives the desired equivalence $X_{K/2} \simeq \widetilde{\mathrm{Fib}} f$. □

In this theorem, M is uniquely determined by the space X since there is a canonical isomorphism $M \cong \hat{Q} K^{-1}_{\Delta}(X; \hat{\mathbb{Z}}_2)$ in $\hat{\mathcal{M}}\Delta$ by Remark 4.10 and [**11**, Section 3].

9. On the v_1-periodic homotopy groups of our spaces

The p-primary v_1-periodic homotopy groups $v_1^{-1}\pi_* X$ of a space X at a prime p were defined by Davis and Mahowald [**15**] and have been studied extensively (see [**13**]). In this section, we apply the preceding result (Theorem 8.6) on the $K/2_*$-localizations of our spaces to approach v_1-periodic homotopy groups at $p = 2$ using:

Definition 9.1 (The functor Φ_1)**.** As in [**4**], [**9**], [**16**], and [**18**], there is a v_1-*stabilization functor* Φ_1 from the homotopy category of spaces to that of spectra such that:

(i) for a space X, there is a natural isomorphism $v_1^{-1}\pi_* X \cong \pi_* \tau_2 \Phi_1 X$ where $\tau_2 \Phi_1 X$ is the 2-torsion part of $\Phi_1 X$ (given by the fiber of its localization away from 2);

(ii) $\Phi_1 X$ is $K/2_*$-local for each space X;

(iii) for a spectrum E, there is a natural equivalence $\Phi_1(\Omega^\infty E) \simeq E_{K/2}$;

(iv) Φ_1 preserves fiber squares.

Various other properties of Φ_1 are described in [**10**, Section 2], and the isomorphism $v_1^{-1}\pi_* X \cong \pi_*\tau_2\Phi_1 X$ may be applied as in [**10**, Theorem 3.2] to show:

Theorem 9.2. *For a space X, there is a natural long exact sequence*

$$\cdots \longrightarrow KO^{n-3}(\Phi_1 X;\hat{\mathbb{Z}}_2) \xrightarrow{\psi^3-9} KO^{n-3}(\Phi_1 X;\hat{\mathbb{Z}}_2) \longrightarrow (v_1^{-1}\pi_n X)^{\#}$$
$$\longrightarrow KO^{n-2}(\Phi_1 X;\hat{\mathbb{Z}}_2) \xrightarrow{\psi^3-9} KO^{n-2}(\Phi_1 X;\hat{\mathbb{Z}}_2) \longrightarrow \cdots$$

where $(-)^{\#}$ is the Pontrjagin duality functor from discrete 2-torsion abelian groups to 2-profinite abelian groups.

This may be used to calculate $v_1^{-1}\pi_* X$ from $KO^*(\Phi_1 X;\hat{\mathbb{Z}}_2)$ up to extension. To approach $KO^*(\Phi_1 X;\hat{\mathbb{Z}}_2)$ or $K^*(\Phi_1 X;\hat{\mathbb{Z}}_2)$, we require:

Definition 9.3 (The $K/2_*$-durable spaces)**.** Following [**8**, 7.8], we say that a space X is *$K/2_*$-durable* when the $K/2_*$-localization $X \to X_{K/2}$ induces an equivalence $\Phi_1 X \simeq \Phi_1 X_{K/2}$ (or equivalently induces an isomorphism $v_1^{-1}\pi_* X \cong v_1^{-1}\pi_* X_{K/2}$), and we recall that each connected H-space is $K/2_*$-durable. For such X, we may apply our key result on $K/2_*$-localizations (Theorem 8.6) to deduce:

Theorem 9.4. *If X is a connected $K/2_*$-durable space (e.g. H-space) with a strict isomorphism $\hat{L}M \cong K^*_{CR}(X;\hat{\mathbb{Z}}_2)$ for a strong module $M \in \hat{\mathcal{M}}\Delta$, then there is a (co)fiber sequence of spectra $\Phi_1 X \to \mathcal{E}\bar{M} \xrightarrow{\epsilon} \mathcal{E}\bar{\rho}\bar{M}$ such that $\epsilon^*\colon K^*_{CR}(\mathcal{E}\bar{\rho}\bar{M};\hat{\mathbb{Z}}_2) \to K^*_{CR}(\mathcal{E}\bar{M};\hat{\mathbb{Z}}_2)$ is given by $CR^{-1}\theta\colon CR^{-1}\bar{\rho}\bar{M} \to CR^{-1}\bar{M}$.*

Here, the map $\theta\colon \bar{\rho}\bar{M} \to \bar{M}$ is given by

$$\theta \;=\; (\theta,\theta,\theta)\colon \{\bar{M}_C, \bar{M}_R + \bar{M}_H, \bar{M}_R \cap \bar{M}_H\} \longrightarrow \{\bar{M}_C, \bar{M}_R, \bar{M}_H\}$$

in $\hat{\mathcal{A}}\Delta$. This theorem will be proved below and may be used to calculate $K^*(\Phi_1 X;\hat{\mathbb{Z}}_2)$ and $KO^*(\Phi_1 X;\hat{\mathbb{Z}}_2)$ since it immediately implies:

Theorem 9.5. *For X as in Theorem 9.4, there is a $K^*(-;\hat{\mathbb{Z}}_2)$ cohomology exact sequence*

$$0 \longrightarrow K^{-2}(\Phi_1 X;\hat{\mathbb{Z}}_2) \longrightarrow \bar{M}_C \xrightarrow{\theta} \bar{M}_C \longrightarrow K^{-1}(\Phi_1 X;\hat{\mathbb{Z}}_2) \longrightarrow 0,$$

and there is a $KO^*(-;\hat{\mathbb{Z}}_2)$ cohomology exact sequence

$$0 \longrightarrow KO^{-8}(\Phi_1 X;\hat{\mathbb{Z}}_2) \longrightarrow \bar{M}_C/(\bar{M}_R + \bar{M}_H) \xrightarrow{\theta} \bar{M}_C/\bar{M}_R \longrightarrow$$
$$KO^{-7}(\Phi_1 X;\hat{\mathbb{Z}}_2) \longrightarrow 0 \longrightarrow \bar{M}_H/(\bar{M}_R \cap \bar{M}_H) \longrightarrow KO^{-6}(\Phi_1 X;\hat{\mathbb{Z}}_2) \longrightarrow$$
$$\bar{M}_R \cap \bar{M}_H \xrightarrow{\theta} \bar{M}_H \longrightarrow KO^{-5}(\Phi_1 X;\hat{\mathbb{Z}}_2) \longrightarrow 0 \longrightarrow 0 \longrightarrow KO^{-4}(\Phi_1 X;\hat{\mathbb{Z}}_2) \longrightarrow$$
$$\bar{M}_C/(\bar{M}_R \cap \bar{M}_H) \xrightarrow{\theta} \bar{M}_C/\bar{M}_H \longrightarrow KO^{-3}(\Phi_1 X;\hat{\mathbb{Z}}_2) \longrightarrow$$
$$(\bar{M}_R + \bar{M}_H)/(\bar{M}_R \cap \bar{M}_H) \xrightarrow{\theta} \bar{M}_R/(\bar{M}_R \cap \bar{M}_H) \longrightarrow$$
$$KO^{-2}(\Phi_1 X;\hat{\mathbb{Z}}_2) \longrightarrow \bar{M}_R + \bar{M}_H \xrightarrow{\theta} \bar{M}_R \longrightarrow KO^{-1}(\Phi_1 X;\hat{\mathbb{Z}}_2) \longrightarrow 0.$$

In these sequences, θ may be replaced by $\lambda^2 = -\theta$. Also, for $i, k \in \mathbb{Z}$ with k odd, the Adams operation ψ^k in $K^{2i-1}(\Phi_1 X; \hat{\mathbb{Z}}_2)$, $K^{2i-2}(\Phi_1 X; \hat{\mathbb{Z}}_2)$, $KO^{2i-1}(\Phi_1 X; \hat{\mathbb{Z}}_2)$, or $KO^{2i-2}(\Phi_1 X; \hat{\mathbb{Z}}_2)$ agrees with $k^{-i}\psi^k$ in the adjacent $\bar{M}$ terms.

Thus, for X as in Theorem 9.4, we may essentially calculate $v_1^{-1}\pi_* X$ from $\bar{M}$ (up to extension problems) using Theorems 9.2 and 9.5. By [**10**, 7.6], this approach to $v_1^{-1}\pi_* X$ may be extended to various other important spaces X using:

Definition 9.6 (The $\widehat{K}\Phi_1$-goodness condition)**.** For a space X, we let $\Phi_1 \colon \widetilde{K}^*_{CR}(X; \hat{\mathbb{Z}}_2) \longrightarrow K^*_{CR}(\Phi_1 X; \hat{\mathbb{Z}}_2)$ denote the v_1-*stabilization homomorphism* of [**10**, 7.1], and we recall that it induces a homomorphism $\Phi_1 \colon \hat{Q}K^n_\Delta(X; \hat{\mathbb{Z}}_2)/\theta \longrightarrow K^n_\Delta(\Phi_1 X; \hat{\mathbb{Z}}_2)$ in $\hat{\mathcal{A}}\Delta$ for $n = -1, 0$ by [**10**, 7.4], where $\hat{Q}K^n_\Delta(X; \hat{\mathbb{Z}}_2)/\theta$ is as in Remark 4.10 and Definition 6.1. Following [**10**, 7.5], we say that a space X is $\widehat{K}\Phi_1$-*good* when the complex v_1-stabilization homomorphism $\Phi_1 \colon \hat{Q}K^n(X; \hat{\mathbb{Z}}_2)/\theta \to K^n(\Phi_1 X; \hat{\mathbb{Z}}_2)$ is an isomorphism for $n = -1, 0$. Our next theorem will provide initial examples of $\widehat{K}\Phi_1$-good spaces from which other examples may be built.

Theorem 9.7. *If X is a connected $K/2_*$-durable space (e.g. H-space) with a strict isomorphism $\hat{L}M \cong K^*_{CR}(X; \hat{\mathbb{Z}}_2)$ for a strong module $M \in \hat{\mathcal{M}}\Delta$ such that $\theta \colon \bar{M}_C \to \bar{M}_C$ is monic, then X is $\widehat{K}\Phi_1$-good with $K^0(\Phi_1 X; \hat{\mathbb{Z}}_2) = 0$, with $K^{-1}(\Phi_1 X; \hat{\mathbb{Z}}_2) = \bar{M}_C/\theta$, and with $K^{-1}_\Delta(\Phi_1 X; \hat{\mathbb{Z}}_2) \cong \bar{M}/\theta$.*

To prove Theorems 9.4 and 9.7, we first consider the spectrum $\tilde{\mathcal{E}}N$ for a torsion-free exact module $N \in \hat{\mathcal{A}}\Delta$ and note that $\Phi_1 \Omega^\infty \tilde{\mathcal{E}}N \simeq (\tilde{\mathcal{E}}N)_{K/2} \simeq \mathcal{E}N$.

Lemma 9.8. *The space $\Omega^\infty \tilde{\mathcal{E}}N$ is $\widehat{K}\Phi_1$-good, and the v_1-stabilization gives a natural isomorphism*

$$\Phi_1 \colon \hat{Q}K^{-1}_\Delta(\Omega^\infty \tilde{\mathcal{E}}N; \hat{\mathbb{Z}}_2)/\theta \;\cong\; K^{-1}_\Delta(\mathcal{E}N; \hat{\mathbb{Z}}_2).$$

Proof. By [**10**, 7.1], the homomorphism $\Phi_1 \colon K^{-1}_\Delta(\Omega^\infty \tilde{\mathcal{E}}N; \hat{\mathbb{Z}}_2) \to K^{-1}_\Delta(\mathcal{E}N; \hat{\mathbb{Z}}_2)$ is left inverse to the infinite suspension homomorphism, and the lemma now follows by Theorem 6.7 together with Lemma 4.11, and Definition 6.3. □

Proof of Theorem 9.4. Applying the functor Φ_1 to the fiber sequence of Theorem 8.6, we obtain a (co)fiber sequence of spectra

$$\Phi_1 X_{K/2} \longrightarrow \Phi_1 \Omega^\infty \tilde{\mathcal{E}}\bar{M} \xrightarrow{\Phi_1 f} \Phi_1 \Omega^\infty \tilde{\mathcal{E}}\bar{\rho}\bar{M}$$

for some companion map f of M. We then deduce that $\Phi_1 f$ corresponds to a map $\mathcal{E}\bar{M} \to \mathcal{E}\bar{\rho}\bar{M}$ having the desired properties by Lemmas 9.8 and 5.4. □

Proof of Theorem 9.7. The results on $K^*(\Phi_1 X; \hat{\mathbb{Z}}_2)$ and $K^{-1}_\Delta(\Phi_1 X; \hat{\mathbb{Z}}_2)$ follow from Theorem 9.5. Since $K^*(X; \hat{\mathbb{Z}}_2) \cong \hat{\Lambda}M_C$ by Lemma 4.6, we obtain isomorphisms $\hat{Q}K^0(X; \hat{\mathbb{Z}}_2)/\theta = 0$ and $\hat{Q}K^{-1}(X; \hat{\mathbb{Z}}_2)/\theta \cong M_C/\theta$, and we deduce that $\Phi_1 \colon \hat{Q}K^n(X; \hat{\mathbb{Z}}_2)/\theta \cong K^n(\Phi_1 X; \hat{\mathbb{Z}}_2)$ for $n = -1, 0$ by Lemma 9.8 and naturality. □

10. Applications to simply-connected compact Lie groups

We now apply the preceding results to a simply-connected compact Lie group G. We first use the representation theory of G to functorially determine the united 2-adic K-cohomology ring $K^*_{CR}(G;\hat{\mathbb{Z}}_2) = \{K^*(G;\hat{\mathbb{Z}}_2), KO^*(G;\hat{\mathbb{Z}}_2)\}$ in Theorem 10.3. Then, with slight restrictions on the group, we use the representation theory of G to give expressions for the $K/2_*$-localization $G_{K/2}$, for the v_1-stabilization $\Phi_1 G$, and for the cohomology $KO^*(\Phi_1 G;\hat{\mathbb{Z}}_2)$, and we also show that G is $\widehat{K}\Phi_1$-good. Our results are summarized in Theorem 10.6 and permit calculations of the 2-primary v_1-periodic homotopy groups $v_1^{-1}\pi_* G$ using Theorem 9.2, as accomplished very successfully by Davis [**14**]. In this section, we assume some general familiarity with the representation rings of our Lie groups as described in [**12**, Sections II.6 and VI.4] and [**14**, Theorem 2.3].

Definition 10.1 (The representation ring $R_\Delta G$)**.** For a simply-connected compact Lie group G, we let RG be the complex representation ring and let $R_R G, R_H G \subset RG$ be the real and quaternionic parts of RG with the usual λ-ring structures on RG and $R_R G \oplus R_H G$. We also let $t = \psi^{-1}\colon RG \cong RG,\ c\colon R_R G \subset RG,\ r\colon RG \to R_R G,\ c'\colon R_H G \subset RG$, and $q\colon RG \to R_H G$ be the usual operations satisfying the Δ-module relations of Definition 4.1. These structures are compatible in the expected ways and combine to give a $\Delta\lambda$*-ring* $R_\Delta G = \{RG, R_R G, R_H G\}$ in the sense of [**10**, 6.2]. We let $\tilde{R}_\Delta G = \{\tilde{R}G, \tilde{R}_R G, \tilde{R}_H G\}$ be the *augmentation ideal* of $R_\Delta G$ given by the kernel $\tilde{R}G$ of the complex augmentation $\dim\colon RG \to \mathbb{Z}$, where $\tilde{R}_R G = R_R G \cap \tilde{R}G$ and $\tilde{R}_H G = R_H G \cap \tilde{R}G$. We also let $QR_\Delta G = \{QRG, QR_R G, QR_H G\}$ be the *indecomposables* of $R_\Delta G$ given by

$$QRG = \tilde{R}G/(\tilde{R}G)^2,$$
$$QR_R G = \tilde{R}_R G/((\tilde{R}_R G)^2 + (\tilde{R}_H G)^2 + r(\tilde{R}G)^2),$$
$$QR_H G = \tilde{R}_H G/((\tilde{R}_R G)(\tilde{R}_H G) + q(\tilde{R}G)^2).$$

It is straightforward to show that $\tilde{R}_\Delta G$ and $QR_\Delta G$ inherit $\Delta\lambda$-ring structures (without identities) from $R_\Delta G$. Since $QR_\Delta G$ is a $\Delta\lambda$-ring with trivial multiplication, it is equipped with additive operations $t\colon QRG \cong QRG,\ c\colon QR_R G \to QRG,\ r\colon QRG \to QR_R G,\ c'\colon QR_H G \to QRG,\ q\colon QRG \to QR_H G,\ \theta = -\lambda^2\colon QRG \to QRG,\ \theta = -\lambda^2\colon QR_R G \to QR_R G,\ \theta = -\lambda^2\colon QR_H G \to QR_R G,\ \psi^k\colon QRG \to QRG,\ \psi^k\colon QR_R G \to QR_R G$, and $\psi^k\colon QR_H G \to QR_H G$ for the odd $k \in \mathbb{Z}$. We now let $\hat{Q}R_\Delta G = \{\hat{Q}RG, \hat{Q}R_R G, \hat{Q}R_H G\}$ be the 2-adic completion of $QR_\Delta G$ with the induced additive operations on the components $\hat{Q}RG = \hat{\mathbb{Z}}_2 \otimes QRG$, $\hat{Q}R_R G = \hat{\mathbb{Z}}_2 \otimes QR_R G$, and $\hat{Q}R_H G = \hat{\mathbb{Z}}_2 \otimes QR_H G$.

Lemma 10.2. *For a simply-connected compact Lie group G, $\hat{Q}R_\Delta G$ is a robust 2-adic Adams Δ-module.*

This will be proved below. To determine the cohomology ring $K^*_{CR}(G;\hat{\mathbb{Z}}_2) = \{K^*(G;\hat{\mathbb{Z}}_2), KO^*(G;\hat{\mathbb{Z}}_2)\}$ from the representation theory of G, we now let $\beta\colon \hat{Q}R_\Delta G \to \widetilde{K}^{-1}_\Delta(G;\hat{\mathbb{Z}}_2)$ be the 2-adic Adams Δ-module homomorphism induced by the composition of the canonical homomorphisms $\tilde{R}_\Delta G \to \widetilde{K}^0_\Delta(BG;\hat{\mathbb{Z}}_2) \to \widetilde{K}^{-1}_\Delta(G;\hat{\mathbb{Z}}_2)$.

Theorem 10.3. *For a simply-connected compact Lie group G, there is a natural strict isomorphism $\bar{\beta}\colon \hat{L}(\hat{Q}R_\Delta G) \cong K^*_{CR}(G;\hat{\mathbb{Z}}_2)$.*

Proof. This follows by Lemma 10.2 and Theorem 4.9 since $\beta\colon \hat{Q}RG \to K^{-1}(G;\hat{\mathbb{Z}}_2)$ induces an isomorphism $\hat{\Lambda}(\hat{Q}RG) \cong K^*(G;\hat{\mathbb{Z}}_2)$ by [**17**]. □

We note that $K^*_{CR}(G;\hat{\mathbb{Z}}_2)$ has a simple system of generators (see Definition 3.3) consisting of the $\beta\tilde{z}_\gamma \in K^{-1}(G;\hat{\mathbb{Z}}_2)$, the $\beta\tilde{x}_\alpha \in KO^{-1}(G;\hat{\mathbb{Z}}_2)$, and the $\beta\tilde{y}_\beta \in KO^{-5}(G;\hat{\mathbb{Z}}_2)$ obtained from the analysis of $\hat{Q}R_\Delta G$ below in Remark 10.7. Thus, by Proposition 3.4, $K^*_{CR}(G;\hat{\mathbb{Z}}_2)$ is a free 2-adic CR-module on the associated products. However, our description of $K^*_{CR}(G;\hat{\mathbb{Z}}_2)$ as $\hat{L}(\hat{Q}R_\Delta G)$ is more natural and includes the full multiplicative structure. Moreover, it will let us apply our main results to G.

Lemma 10.4. *For a simply-connected compact Lie group G, the 2-adic Adams Δ-module $\hat{Q}R_\Delta G$ is regular with $\theta\colon \hat{Q}RG \to \hat{Q}RG$ monic.*

Proof. This follows by Lemmas 7.9 and 7.10 since $\beta\colon \hat{Q}RG \to \widetilde{K}^{-1}(G;\hat{\mathbb{Z}}_2)$ is monic by Theorem 10.3. □

Thus, $\hat{Q}R_\Delta G$ is strong (robust, ψ^3-splittable, and regular) if and only if it is ψ^3-splittable, and this is usually the case by:

Lemma 10.5. *For a simply-connected compact simple Lie group G, the 2-adic Adams Δ-module $\hat{Q}R_\Delta G$ is ψ^3-splittable (and hence strong) if and only if G is* not *E_6 or $Spin(4k+2)$ with k not a 2-power.*

This will be proved below using work of Davis [**14**]. For a simply-connected compact Lie group G, we now let $\underline{\hat{Q}_\Delta} = \{\hat{Q}, \hat{Q}_R, \hat{Q}_H\}$ briefly denote the associated stable 2-adic Adams Δ-module $\underline{\hat{Q}_\Delta RG} = (\hat{Q}_\Delta RG)/\bar{\phi}$. This agrees with the notation of [**10**, 9.2] and [**14**], since our $\hat{Q}_\Delta = \{\hat{Q}, \hat{Q}_R, \hat{Q}_H\}$ is the 2-adic completion of their $Q_\Delta = \{Q, Q_R, Q_H\}$. Our main results now give the following omnibus theorem, whose four parts may be expanded in the obvious ways to match the cited theorems.

Theorem 10.6. *Let G be a simply-connected compact Lie group such that the 2-adic Adams Δ-module $\hat{Q}_\Delta RG$ is ψ^3-splittable (see Lemma 10.5), and let $\hat{Q}_\Delta = \{\hat{Q}, \hat{Q}_R, \hat{Q}_H\}$ be the associated stable 2-adic Adams Δ-module. Then:*

(i) the $K/2_*$-localization $G_{K/2}$ is the homotopy fiber of a map $\Omega^\infty\tilde{\mathcal{E}}\hat{Q}_\Delta \to \Omega^\infty\tilde{\mathcal{E}}\bar{\rho}\hat{Q}_\Delta$ with low dimensional modifications as in Theorem 8.6;

(ii) the 2-adic v_1-stabilization $\Phi_1 G$ is the homotopy fiber of a map of spectra $\mathcal{E}\hat{Q}_\Delta \to \mathcal{E}\bar{\rho}\hat{Q}_\Delta$ as in Theorem 9.4;

(iii) there is an exact sequence

$$0 \longrightarrow KO^{-8}(\Phi_1 G;\hat{\mathbb{Z}}_2) \longrightarrow \hat{Q}/(\hat{Q}_R + \hat{Q}_H) \xrightarrow{\theta} \hat{Q}/\hat{Q}_R \longrightarrow \cdots$$

continuing as in Theorem 9.5;

(iv) G is $\widehat{K}\Phi_1$-good at the prime 2 as in Theorem 9.7.

The exact sequence in (iii) permits calculations of the 2-primary v_1-periodic homotopy groups $v_1^{-1}\pi_* G$ using Theorem 9.2 as accomplished by Davis [**14**]. This exact sequence was previously obtained in [**10**, Theorem 9.3] using indirect algebraic methods under the hard-to-verify condition that G was $\widehat{K}\Phi_1$-good. It is now obtained using the $KO^*(-;\hat{\mathbb{Z}}_2)$ cohomology exact sequence of the (co)fiber sequence in (ii) under an accessible algebraic condition that implies the $\widehat{K}\Phi_1$-goodness of G by (iv).

We devote the rest of the section to proving Lemmas 10.2 and 10.5 using:

Remark 10.7 (Generators for representation rings)**.** For a simply-connected compact Lie group G, standard results summarized in [**14**, Theorem 2.3] show that RG is a finitely generated polynomial ring $\mathbb{Z}[z_\gamma, z_\gamma^*, x_\alpha, y_\beta]_{\gamma,\alpha,\beta}$ on certain basic complex representations z_γ together with their conjugates $z_\gamma^* = tz_\gamma$, certain basic real representations x_α, and certain basic quaternionic representations y_β. Moreover, in terms of these generators, the $\mathbb{Z}/2$-graded ring $\{R_R G, R_H G\}$ is characterized by the fact that its quotient $\{R_R G/rRG, R_H G/qRG\}$ is a $\mathbb{Z}/2$-graded polynomial algebra $\mathbb{Z}/2[x_\alpha, \bar{\phi} z_\gamma, y_\beta]_{\alpha,\gamma,\beta}$ on the real generators x_α and $\bar{\phi} z_\gamma$ (with $c\bar{\phi} z_\gamma = z_\gamma^* z_\gamma$) and the quaternionic generators y_β. Consequently, the indecomposables $QR_\Delta G = \{QRG, QR_R G, QR_H G\}$ may be expressed as

$$\begin{aligned} QRG &= \mathbb{Z}\{\tilde{z}_\gamma, \tilde{z}_\gamma^*, c\tilde{x}_\alpha, c'\tilde{y}_\beta\}_{\gamma,\alpha,\beta}, \\ QR_R G &= \mathbb{Z}\{r\tilde{z}_\gamma, \tilde{x}_\alpha, rc'\tilde{y}_\beta\}_{\gamma,\alpha,\beta} \;\oplus\; \mathbb{Z}/2\{\bar{\phi}\tilde{z}_\gamma\}_\gamma, \\ QR_H G &= \mathbb{Z}\{q\tilde{z}_\gamma, qc\tilde{x}_\alpha, \tilde{y}_\beta\}_{\gamma,\alpha,\beta} \end{aligned}$$

where $\tilde{w}$ denotes $w - \dim w$ for $w \in RG$. Thus, the 2-adic indecomposables $\hat{Q}R_\Delta G = \{\hat{Q}RG, \hat{Q}R_R G, \hat{Q}R_H G\}$ may be expressed similarly using $\hat{\mathbb{Z}}_2$ in place of $\mathbb{Z}$, and the stable 2-adic indecomposables $\hat{Q}_\Delta = \{\hat{Q}, \hat{Q}_R, \hat{Q}_H\}$ may be expressed as

$$\begin{aligned} \hat{Q} &= \hat{\mathbb{Z}}_2\{\tilde{z}_\gamma, \tilde{z}_\gamma^*, c\tilde{x}_\alpha, c'\tilde{y}_\beta\}_{\gamma,\alpha,\beta}, \\ \hat{Q}_R &= \hat{\mathbb{Z}}_2\{r\tilde{z}_\gamma, \tilde{x}_\alpha, rc'\tilde{y}_\beta\}_{\gamma,\alpha,\beta}, \\ \hat{Q}_H &= \hat{\mathbb{Z}}_2\{q\tilde{z}_\gamma, qc\tilde{x}_\alpha, \tilde{y}_\beta\}_{\gamma,\alpha,\beta}. \end{aligned}$$

Proof of Lemma 10.2. Since $QR_\Delta G$ is a $\Delta\lambda$-ring with trivial multiplication, it is straightforward to check all of the required relations for operations (see Definitions 4.3 and 6.1) . In particular, we deduce $\theta\theta r = \theta r \theta$ from the relations $\lambda^4 r = r\lambda^4 + \bar{\phi}\lambda^2$, $\lambda^4 = -\lambda^2\lambda^2$, $\bar{\phi} = \lambda^2 r - r\lambda^2$, $2\bar{\phi} = 0$, and $\theta = -\lambda^2$, which hold generally in $\Delta\lambda$-rings with trivial multiplication [**10**, 6.2]. We next observe that $\hat{Q}RG$, $\hat{Q}R_R G$, and $\hat{Q}R_H G$ are stable 2-adic Adams modules by [**6**, 6.2], since QRG and $QR_R G \oplus QR_H G$ are γ-nilpotent and finitely generated abelian (because they have trivial multiplications and have finite generating sets of elements $\tilde{w}$ for representations w). Thus, $\hat{Q}R_\Delta G$ is a 2-adic Adams Δ-module, and it must be robust by the analysis of Remark 10.7. □

To check the ψ^3-splittability of $\hat{Q}R_\Delta G$, we let $hG = \ker(1-t)/\operatorname{im}(1+t)$ be the augmented algebra over $\mathbb{Z}/2$ obtained from RG using the involution $t = \psi^{-1}\colon RG \cong$

RG. This is a polynomial algebra $hG \cong \mathbb{Z}/2[c\tilde{x}_\alpha, \tilde{z}^*_\gamma\tilde{z}_\gamma, c'\tilde{y}_\beta]_{\alpha,\gamma,\beta}$ which is $\mathbb{Z}/2$-graded, since there is an isomorphism

$$c + c' \colon R_RG/rRG \oplus R_HG/qRG \ \cong\ hG,$$

and we let $Q_RhG \cong \mathbb{Z}/2\{c\tilde{x}_\alpha, \tilde{z}^*_\gamma\tilde{z}_\gamma\}_{\alpha,\gamma}$ denote the real (degree 0) indecomposables. We define a homomorphism $s\colon QRG \to Q_RhG$ by $s[u] = [u^*u]$ for $u \in \tilde{R}G$ and note that $sQRG = \mathbb{Z}/2\{\tilde{z}^*_\gamma\tilde{z}_\gamma\}_\gamma$. We view s as a homomorphism of ψ^3-modules (abelian groups with endomorphisms ψ^3) as in [**14**, 2.4].

Lemma 10.8. *For a simply-connected compact Lie group G, $\hat{Q}R_\Delta G$ is ψ^3-splittable if and only if the ψ^3-submodule $sQRG \subset Q_RhG$ is a direct summand.*

Proof. By Definition 7.2 and the proof of Lemma 7.3, $\hat{Q}R_\Delta G$ is ψ^3-splittable if and only if the ψ^3-submodule $\bar{\phi}\hat{Q}RG \subset \hat{Q}R_RG/r\hat{Q}RG$ (or equivalently $\bar{\phi}QRG \subset QR_RG/rQRG$) is a direct summand. The lemma now follows since $\bar{\phi}QRG$ corresponds to $sQRG$ under the isomorphism $c\colon QR_RG/rQRG \cong Q_RhG$. □

Proof of Lemma 10.5. By Lemma 10.8 and Davis [**14**, Theorem 1.3], the following conditions are successively equivalent: $\hat{Q}R_\Delta G$ is ψ^3-splittable; the ψ^3-submodule $sQRG \subset Q_RhG$ is a direct summand; G satisfies the Technical Condition of [**14**, Definition 2.4]; G is *not* E_6 or $Spin(4k+2)$ with k not a 2-power. □

11. Proofs of basic lemmas for $\hat{L}$

We shall prove Lemmas 4.5, 4.6, and 4.11 showing the basic properties of the functor $\hat{L}\colon \theta\Delta\hat{\mathcal{M}}od \to \phi\mathcal{CR}\hat{\mathcal{A}}lg$, where $\theta\Delta\hat{\mathcal{M}}od$ is the category of 2-adic $\theta\Delta$-modules and $\phi\mathcal{CR}\hat{\mathcal{A}}lg$ is that of special 2-adic ϕCR-algebras (see Definitions 4.3 and 3.2). We first introduce an intermediate category of modules.

Definition 11.1 (The 2-adic $\eta\Delta$-modules)**.** By a *2-adic $\eta\Delta$-module* $N = \{N_C, N_R, N_H, N_S\}$, we mean a 2-adic Δ-module $\{N_C, N_R, N_H\}$, with operations t, c, r, c', and q as in Definition 4.1, together with a 2-profinite abelian group N_S and continuous additive operations $\bar{\phi}\colon N_C \to N_R$, $\eta\colon N_R \to N_S$, $()^{[2]}\colon N_R \to N_S$, and $()^{[2]}\colon N_H \to N_S$ satisfying the following relations for elements $z \in N_C$, $x \in N_R$, and $y \in N_H$:

$$\bar{\phi}cx = 0, \qquad \bar{\phi}c'y = 0, \qquad \bar{\phi}tz = \bar{\phi}z, \qquad 2\bar{\phi}z = 0, \qquad c\bar{\phi}z = 0,$$
$$(\bar{\phi}z)^{[2]} = 0, \qquad 2\eta x = 0, \qquad \eta rz = 0, \qquad (qz)^{[2]} = (rz)^{[2]} = \eta\bar{\phi}z.$$

We let $\eta\Delta\hat{\mathcal{M}}od$ denote the category of 2-adic $\eta\Delta$-modules.

Remark 11.2 (A functorial interpretation of admissible maps)**.** Let $J\colon \theta\Delta\hat{\mathcal{M}}od \to \eta\Delta\hat{\mathcal{M}}od$ be the functor carrying a 2-adic $\theta\Delta$-module M to the 2-adic $\eta\Delta$-module $JM = \{M_C, M_R, M_H, M_R/rM_C\}$ having the original operations t, c, r, c', q, and $\bar{\phi}$ together with operations $\eta\colon M_R \to M_R/rM_C$, $()^{[2]}\colon M_R \to M_R/rM_C$, and $()^{[2]}\colon M_H \to M_R/rM_C$ given by $\eta x = [x]$, $x^{[2]} = [\theta x]$, and $y^{[2]} = [\theta y]$ for $x \in M_R$ and $y \in M_H$. Let $I\colon \phi\mathcal{CR}\hat{\mathcal{A}}lg \to \eta\Delta\hat{\mathcal{M}}od$ be the functor carrying a special 2-adic ϕCR-algebra A to the 2-adic $\eta\Delta$-module $IA = \{\tilde{A}^{-1}_C, \tilde{A}^{-1}_R, \tilde{A}^{-5}_R, \tilde{A}^{-2}_R\}$ having the operations t, c, r, c', and q of $\Delta^{-1}\tilde{A}$ (see Definition 4.1) together with operations $\bar{\phi}\colon \tilde{A}_C^{-1} \to \tilde{A}_R^{-1}$, $\eta\colon \tilde{A}_R^{-1} \to \tilde{A}_R^{-2}$, $()^{[2]}\colon \tilde{A}_R^{-1} \to \tilde{A}_R^{-2}$, and $()^{[2]}\colon \tilde{A}_R^{-5} \to \tilde{A}_R^{-2}$ given by $\bar{\phi}z = \eta\phi z$, $\eta x = \eta x$, $x^{[2]} = x^2$, and $y^{[2]} = B_R^{-1}y^2$ for $z \in A_C^{-1}$, $x \in A_R^{-1}$, and $y \in A_R^{-5}$. We now easily see:

Lemma 11.3. *For $M \in \theta\Delta\hat{\mathcal{M}}od$ and $A \in \phi CR\hat{A}lg$, an admissible map $f\colon M \to A$ is equivalent to a map $f\colon JM \to IA$ in $\eta\Delta\hat{\mathcal{M}}od$.*

To construct the functor $\hat{L}$, we need:

Lemma 11.4. *The functor $I\colon \phi\mathcal{CR}\hat{\mathcal{A}}lg \to \eta\Delta\hat{\mathcal{M}}od$ has a left adjoint $\hat{V}\colon \eta\Delta\hat{\mathcal{M}}od \to \phi\mathcal{CR}\hat{\mathcal{A}}lg$.*

Proof. This follows by the Special Adjoint Functor Theorem (see [**19**]) since I preserves small limits and since $\phi\mathcal{CR}\hat{\mathcal{A}}lg$ has a small cogenerating set by Lemma 11.5 below. □

A special 2-adic ϕCR-algebra A will be called *finite* when the groups $\tilde{A}_C^m$ and $\tilde{A}_R^m$ are finite for all m.

Lemma 11.5. *Each special 2-adic ϕCR-algebra A is the inverse limit of its finite quotients in $\phi\mathcal{CR}\hat{\mathcal{A}}lg$.*

Proof. This is similar to the corresponding result for topological rings in [**22**, 5.1.2]. For a 2-adic CR-submodule $G \subset \tilde{A}$ with $\tilde{A}/G$ finite, we must obtain a special 2-adic ϕCR-ideal H of A with $H \subset G$ and $\tilde{A}/H$ finite. We first obtain an ideal M of A_R (closed under B_R, B_R^{-1}, η, and ξ) with $M \subset G_R$ and $\tilde{A}_R/M$ finite as in [**22**]. We next obtain an ideal N of A_C (closed under B, B^{-1}, and t) with $N \subset G_C \cap r^{-1}M \cap \phi^{-1}M^0$ and $\tilde{A}_C/N$ finite as in [**22**]. The desired ideal H is now given by $H_C = N$ and $H_R = M \cap c^{-1}N$. □

Proof of Lemma 4.5. Using Lemmas 11.3 and 11.4, we obtain the desired universal algebra $\hat{L}M$ from the functor $\hat{L} = \hat{V}J\colon \theta\Delta\hat{\mathcal{M}}od \to \phi\mathcal{CR}\hat{\mathcal{M}}od$. □

A 2-adic $\eta\Delta$-module N is called *sharp* when $\eta\colon N_R/rN_C \to N_S$ is an isomorphism, and we may now derive the properties of $\hat{L}$ from the corresponding properties of $\hat{V}$ on such sharp modules.

Lemma 11.6. *For a sharp 2-adic $\eta\Delta$-module N, the canonical map $\hat{\Lambda}N_C \to (\hat{V}N)_C$ is an algebra isomorphism.*

Proof. Let $W\colon \phi\mathcal{CR}\hat{\mathcal{A}}lg \to \mathcal{C}\hat{\mathcal{A}}lg$ be the forgetful functor carrying each $A \in \phi\mathcal{CR}\hat{\mathcal{A}}lg$ to its complex part $A_C \in \mathcal{C}\hat{\mathcal{A}}lg$ where $\mathcal{C}\hat{\mathcal{A}}lg$ is the category of *special 2-adic C-algebras*, which are defined similarly to *special 2-adic ϕCR-algebras* (see Definition 3.2) but using only complex terms and their operations. The functor W has a right adjoint $H\colon \mathcal{C}\hat{\mathcal{A}}lg \to \phi\mathcal{CR}\hat{\mathcal{A}}lg$ where $(HX)_C = X$ and $(HX)_R = \{x \in X | tx = x\}$ with $c = 1$, $r = 1 + t$, $\eta = 0$, $\phi z = z^*z$ for $z \in X^0$, and $\phi w = B^{-1}w^*w$ for $w \in X^{-1}$. For each $N \in \eta\Delta\hat{\mathcal{M}}od$ and each $X \in \mathcal{C}\hat{\mathcal{A}}lg$, a map $N \to IHX$ in $\eta\Delta\hat{\mathcal{M}}od$ corresponds to a map $N_C \to \tilde{X}^{-1}$ respecting t, which in turn corresponds to a map $\hat{\Lambda}N_C \to X$ in $\mathcal{C}\hat{\mathcal{A}}lg$. Hence, since $W\hat{V}$ is left adjoint to IH, the canonical map $\hat{\Lambda}N_C \to W\hat{V}N$ is an isomorphism. □

Proof of Lemma 4.6. For a 2-adic $\theta\Delta$-module M, the canonical map $\hat{\Lambda}M_C \to (\hat{L}M)_C$ is an isomorphism by Lemma 11.6 and by the above proof of Lemma 4.5. $\square$

Let $\hat{Q}\colon \phi\mathcal{CR}\hat{\mathcal{A}}lg \to \phi\mathcal{CR}\hat{\mathcal{M}}od$ be the functor carrying each $A \in \phi\mathcal{CR}\hat{\mathcal{A}}lg$ to its indecomposables $\hat{Q}A \in \phi\mathcal{CR}\hat{\mathcal{M}}od$ where $\phi\mathcal{CR}\hat{\mathcal{M}}od$ is the category of *special 2-adic ϕCR-modules*, which may be defined as the augmentation ideals of the special 2-adic ϕCR-algebras having trivial multiplication.

Lemma 11.7. *For a sharp 2-adic $\eta\Delta$-module N, the canonical map $\{N_C, N_R, N_H\} \to \Delta^{-1}\hat{Q}\hat{V}N$ is an isomorphism.*

Proof. The functor $\hat{Q}$ has a right adjoint $E\colon \phi\mathcal{CR}\hat{\mathcal{M}}od \to \phi\mathcal{CR}\hat{\mathcal{A}}lg$ where $EX = \underline{e} \oplus X$. Since $\hat{Q}\hat{V}\colon \eta\Delta\hat{\mathcal{M}}od \to \phi\mathcal{CR}\hat{\mathcal{M}}od$ is left adjoint to IE, a detailed analysis shows that $\hat{Q}\hat{V}N$ is a special 2-adic ϕCR-module with $(\hat{Q}\hat{V}N)_C^{-1} = N_C$, $(\hat{Q}\hat{V}N)_R^{-1} = N_R$, and $(\hat{Q}\hat{V}N)_R^{-5} = N_H$. $\square$

Proof of Lemma 4.11. For a 2-adic $\theta\Delta$-module M, the canonical map $M \to \Delta^{-1}\hat{Q}\hat{L}M$ is an isomorphism by Lemma 11.7 and the above proof of Lemma 4.5. $\square$

12. Proof of the Bott exactness lemma for $\hat{L}$

We must now prove Lemma 4.8 showing the Bott exactness of $\hat{L}M$ for a robust 2-adic $\theta\Delta$-module M. This lemma will follow easily from the corresponding result for $\eta\Delta$-modules (Lemma 12.1), whose proof will extend through most of this section. We say that a 2-adic $\eta\Delta$-module N is *profinitely sharp* when it is the inverse limit of an inverse system of finite sharp 2-adic $\eta\Delta$-modules. This obviously implies that N is sharp. We call N *robust* when:

(i) N is profinitely sharp;

(ii) the 2-adic Δ-module $\{N_C, N_R/\bar{\phi}N_C, N_H\}$ is torsion-free and exact;

(iii) $\ker\bar{\phi} = cN_R + c'N_H + 2N_C$.

Lemma 12.1. *If N is a robust 2-adic $\eta\Delta$-module, then the special 2-adic ϕCR-algebra $\hat{V}N$ is Bott exact; in fact, $\hat{V}N$ is the inverse limit of an inverse system of finitely generated free 2-adic CR-modules.*

This will be proved at the end of the section.

Proof of Lemma 4.8. For a robust 2-adic $\theta\Delta$-module M, the 2-adic $\eta\Delta$-module JM is also robust, and hence $\hat{L}M$ has the required properties by Lemma 12.1 and the proof of Lemma 4.5 in Section 11. $\square$

Before proving Lemma 12.1, we must analyze the robust 2-adic $\eta\Delta$-modules, and we start with:

Definition 12.2 (The complex 2-adic $\eta\Delta$-modules)**.** The functor $(-)_C\colon \eta\Delta\hat{\mathcal{M}}od \to \hat{\mathcal{A}}b$ from the 2-adic $\eta\Delta$-modules to the 2-profinite abelian groups has a left adjoint $C\colon \hat{\mathcal{A}}b \to \eta\Delta\hat{\mathcal{M}}od$ with $C(G)_C = G \oplus G = G \oplus tG$, $C(G)_R = G \oplus G/2 = rG \oplus \bar{\phi}G$, $C(G)_H = G = qG$, and $C(G)_S = G/2 = (\bar{\phi}G)^{[2]}$ for $G \in \hat{\mathcal{A}}b$. A 2-adic $\eta\Delta$-module will be called *complex* when it is isomorphic to $C(G)$ for some G. If G is torsion-free, then $C(G)$ is obviously robust. For an arbitrary $N \in \eta\Delta\hat{\mathcal{M}}od$ and $G \in \hat{\mathcal{A}}b$, we may describe the possible maps $N \to C(G)$ as follows. Let $f\colon N_C \to G$ and $g\colon N_S \to G/2$ be maps such that the diagram

$$\begin{array}{ccc} N_R \oplus N_H & \xrightarrow{fc+fc'} & G \\ \big\downarrow ()^{[2]} & & \big\downarrow 1 \\ N_S & \xrightarrow{g} & G/2 \end{array}$$

commutes. Then there is a map $F(f,g)\colon N \to C(G)$ with components $(f,ft)\colon N_C \to G \oplus G$, $(fc, g\eta)\colon N_R \to G \oplus G/2$, $fc'\colon N_H \to G$, and $g\colon N_S \to G/2$. Moreover, each map $N \to C(G)$ is of the above form for some f and g. When N is robust, the compatibility condition on $f\colon N_C \to G$ and $g\colon N_S \to G/2$ may be expressed by the commutativity of the diagram

$$\begin{array}{ccc} N_C^+ & \xrightarrow{f} & G \\ \big\downarrow \pi & & \big\downarrow 1 \\ N_S & \xrightarrow{g} & G/2 \end{array}$$

where $N_C^+ = \{z \in N_C | tz = z\}$ and π is the composition of $(c,c')\colon N_R/\bar{\phi}N_C \coprod_{N_C} N_H \cong N_C^+$ and $()^{[2]}\colon N_R/\bar{\phi}N_C \coprod_{N_C} N_H \to N_S$. Letting $N_C^- = \{z \in N_C | tz = -z\}$, we now have:

Lemma 12.3. *If $\tilde{N} \subset N$ is an inclusion of robust 2-adic $\eta\Delta$-modules such that $N_C/\tilde{N}_C$ is torsion-free and $\tilde{N}_C^- = N_C^-$, then each map $\tilde{N} \to C(G)$ for $G \in \hat{\mathcal{A}}b$ may be extended to a map $N \to C(G)$ of 2-adic $\eta\Delta$-modules.*

Proof. For a given map $F(\tilde{f},\tilde{g})\colon \tilde{N} \to C(G)$, we first extend $\tilde{g}\colon \tilde{N}_S \to G/2$ to a map $g\colon N_S \to G/2$. Since $\tilde{N}_C/\tilde{N}_C^+ \cong \tilde{N}_C^-$, $N_C/N_C^+ \cong N_C^-$, and $\tilde{N}_C^- = N_C^-$, we see that N_C is the pushout of the inclusions $N_C^+ \leftarrow \tilde{N}_C^+ \to \tilde{N}_C$. Thus, the maps $g\pi\colon N_C^+ \to G/2$ and $[\tilde{f}]\colon \tilde{N}_C \to G/2$ induce a map $f'\colon N_C \to G/2$, and we obtain a commutative diagram

$$\begin{array}{ccc} \tilde{N}_C & \xrightarrow{\tilde{f}} & G \\ \big\downarrow \subset & & \big\downarrow 1 \\ N_C & \xrightarrow{f'} & G/2. \end{array}$$

Since $N_C/\tilde{N}_C$ is projective in $\hat{\mathcal{A}}b$, we may now choose a lifting $f\colon N_C \to G$ in the diagram, and this gives the desired extension $F(f,g)\colon N_C \to C(G)$ of $F(\tilde{f},\tilde{g})$. □

Lemma 12.4. *For a robust 2-adic $\eta\Delta$-module N, there exists a decomposition $N \cong C(G) \oplus P$ where G is torsion-free and P is robust with $t = 1$ on P_C.*

Proof. By the factorization of positively torsion-free groups in Definition 5.3, there exists a decomposition $N_C \cong (G \oplus tG) \oplus H$ with $t = 1$ on H, and we let $i\colon C(G) \to$

N be the induced map. Then i is monic since $i\colon \{G \oplus tG, G, G\} \to \{N_C, N_R/\bar{\phi}N_C, N_H\}$ is monic by [**10**, Lemma 4.8], and since $i\colon G/2 \to \bar{\phi}N_C$ and $\eta\colon \bar{\phi}N_C \to N_S$ are monic by the proof of Lemma 7.3. Thus, $i\colon C(G) \to N$ has a left inverse by Lemma 12.3, and the result follows. □

Definition 12.5 (The t-trivial 2-adic $\eta\Delta$-modules)**.** A 2-adic $\eta\Delta$-module N will be called *t-trivial* when $t = 1$ on N_C. When N is t-trivial and robust, it must have $\bar{\phi} = 0\colon N_C \to N_R$ since $N_C = cN_R + c'N_H$ by the exactness of $\{N_C, N_R/\bar{\phi}N_C, N_H\}$. Moreover, it must also have $(rN_C)^{[2]} = 0$, $(qN_C)^{[2]} = 0$, and $c + c'\colon N_R/rN_C \oplus N_H/qN_C \cong N_C/2$ by [**10**, Lemma 4.7]. Hence, the operations $()^{[2]}\colon N_R \to N_S$ and $()^{[2]}\colon N_H \to N_S$ induce operations $\bar{\theta}\colon N_R/rN_C \to N_R/rN_C$ and $\bar{\theta}\colon N_H/qN_C \to N_R/rN_C$, where the $\bar{\theta}$-module N_R/rN_C is profinite since N is profinitely sharp. In this way, a t-trivial robust 2-adic $\eta\Delta$-module N corresponds to a torsion-free group $G \in \hat{\mathcal{A}}b$ together with a decomposition $(G/2)_R \oplus (G/2)_H = G/2$ equipped with operations $\bar{\theta}\colon (G/2)_R \to (G/2)_R$ and $\bar{\theta}\colon (G/2)_H \to (G/2)_R$ such that the $\bar{\theta}$-module $(G/2)_R$ is profinite. We say that a 2-adic $\eta\Delta$-module N is of *finite type* when N_C, N_R, N_H, and N_S are finitely generated over $\hat{\mathbb{Z}}_2$, and we now easily deduce:

Lemma 12.6. *A t-trivial robust 2-adic $\eta\Delta$-module may be expressed as the inverse limit of an inverse system of t-trivial robust quotient modules of finite type.*

A similar result obviously holds for the robust 2-adic $\eta\Delta$-modules $C(G)$ with G torsion-free, and the following lemma will now let us restrict our study of $\hat{V}$ to the robust modules of finite type.

Lemma 12.7. *If a 2-adic $\eta\Delta$-module N is the inverse limit of an inverse system $\{N_\alpha\}_\alpha$ of quotient modules, then $\hat{V}N \cong \lim_\alpha \hat{V}N_\alpha$.*

Proof. For a finite special 2-adic ϕCR-algebra F, there is a canonical isomorphism $\mathrm{Hom}(\lim_\alpha \hat{V}N_\alpha, F) \cong \mathrm{Hom}(\hat{V}N, F)$. Hence the map $\hat{V}N \to \lim_\alpha \hat{V}N_\alpha$ is an isomorphism by Lemma 11.5. □

Proof of Lemma 12.1. It now suffices to show that $\hat{V}N$ is a free 2-adic CR-module when $N = C(G) \oplus P$ for a finitely generated free $\hat{\mathbb{Z}}_2$-module G and a t-trivial robust 2-adic $\eta\Delta$-module P of finite type. By Definition 7.1, we may choose finite ordered sets of elements $\{z_k\}_k$ in G, $\{x_i\}_i$ in P_R, and $\{y_j\}_j$ in P_H such that G is a free $\hat{\mathbb{Z}}_2$-module on $\{z_k\}_k$ and $\{P_C, P_R, P_H\}$ is a free 2-adic Δ-module on $\{x_i\}_i$ and $\{y_j\}_j$. Since P_S is a free $\mathbb{Z}/2$-module on the generators $\{\eta x_i\}_i$, there are expressions $x_i^{[2]} = r_i$ and $y_j^{[2]} = s_j$ for each i and j where the r_i and s_j are $\mathbb{Z}/2$-linear combinations of these generators. We may now obtain $\widetilde{\hat{V}N}$ as the free augmented 2-adic CR-algebra on the generators $x_i \in (\widetilde{\hat{V}N})^{-1}_R$, $y_j \in (\widetilde{\hat{V}N})^{-5}_R$, $z_k \in (\widetilde{\hat{V}N})^{-1}_C$, and $\phi z_k \in (\widetilde{\hat{V}N})^0_R$ subject to the relations $x_i^2 = r_i$, $y_j^2 = B_R s_j$, $z_k^2 = 0$, $z_k^* z_k = Bc\phi z_k$, and $(\phi z_k)^2 = 0$ for each i, j, and k. It follows by a straightforward analysis that $\hat{V}N$ is a free 2-adic CR-module on the associated products (see Definition 3.3) of $\{x_i\}_i$, $\{y_j\}_j$, and $\{z_k\}_k$. □

13. Proofs for regular modules

We first show that our strict nonlinearity condition (see Definition 7.7) for 2-adic Adams modules agrees with that of [**7**, 2.4], and we then prove Lemmas 7.9 and 7.10 for regular modules. For a 2-adic Adams module A, we let $TA \subset A$ be given by the pullback square

$$\begin{array}{ccc} TA & \longrightarrow & (A/\psi^2 A)\backslash 2 \\ \big\downarrow{\subset} & & \big\downarrow{\subset} \\ A & \longrightarrow & A/\psi^2 A \end{array}$$

where $(A/\psi^2 A)\backslash 2$ is the kernel of $2\colon A/\psi^2 A \to A/\psi^2 A$. Since the square is also a pushout, A is quasilinear if and only if $TA = A$. Now let $T^\infty A$ be the intersection of the submodules $T^i A \subset A$ for $i > 0$.

Lemma 13.1. *$T^\infty A$ is the largest quasilinear submodule of A, and hence $A_{ql} = T^\infty A$.*

Proof. Using the inverse limit of the pullback squares for $T^i A$ with $i > 0$, we find that $T^\infty A$ contains each quasilinear submodule of A and that $T(T^\infty A) = T^\infty A$. □

Remark 13.2 (Strict nonlinearity conditions)**.** Our definition of strict nonlinearity in Section 7 is equivalent to our earlier definition in [**7**, 2.3 and 2.4]. In fact, for a 2-adic Adams module A, the largest quasilinear submodule A_{ql} remains unchanged in the earlier category of 2-adic ψ^2-modules, since it is still given by $T^\infty A$. To prove Lemma 7.10, we need:

Lemma 13.3. *For a strictly nonlinear 2-adic Adams module A, each submodule is strictly nonlinear. Moreover, when A is finitely generated over $\hat{\mathbb{Z}}_2$, each torsion-free quotient module is strictly nonlinear.*

Proof. The first statement is clear, and we shall prove the second by working in the earlier category $\hat{\mathcal{N}}$ of 2-adic ψ^2-modules that are ψ^2-pro-nilpotent. Let $0 \to \tilde{A} \to A \to \bar{A} \to 0$ be a short exact sequence in $\hat{\mathcal{N}}$ with A strictly nonlinear and finitely generated over $\hat{\mathbb{Z}}_2$ and with $\bar{A}$ torsion-free. To show that $\bar{A}$ is strictly nonlinear, it suffices to show that $\mathrm{Hom}_{\hat{\mathcal{N}}}(H, \bar{A}) = 0$ for each torsion-free quasilinear $H \in \hat{\mathcal{N}}$ that is finitely generated over $\hat{\mathbb{Z}}_2$. Since $\bar{A}$ is torsion-free, it now suffices to show that $\mathrm{Hom}_{\hat{\mathcal{N}}}(H, \bar{A})$ is finite for such H. Hence, since $\mathrm{Hom}_{\hat{\mathcal{N}}}(H, A) = 0$ by strict nonlinearity, it suffices to show that $\mathrm{Ext}^1_{\hat{\mathcal{N}}}(H, \tilde{A})$ is finite for such H. This finiteness follows using the exact sequence

$$0 \longrightarrow \mathrm{Hom}_{\hat{\mathcal{N}}}(H, \tilde{A}) \longrightarrow \mathrm{Hom}_{\hat{\mathcal{A}b}}(H, \tilde{A}) \longrightarrow \mathrm{Hom}_{\hat{\mathcal{A}b}}(H, \tilde{A}) \longrightarrow \mathrm{Ext}^1_{\hat{\mathcal{N}}}(H, \tilde{A}) \longrightarrow 0$$

with $\mathrm{Hom}_{\hat{\mathcal{N}}}(H, \tilde{A}) = 0$ by strict nonlinearity, where $\hat{\mathcal{A}b}$ is the category of 2-profinite abelian groups. □

Proof of Lemma 7.10. This result follows easily from Definition 7.8 and Lemma 13.3. □

Proof of Lemma 7.9. By [**8**, Lemma 5.5], there is an exact sequence

$$0 \longrightarrow \widetilde{K}^1(X/X^3;\hat{\mathbb{Z}}_2) \longrightarrow \widetilde{K}^1(X;\hat{\mathbb{Z}}_2) \longrightarrow H^3(X;\hat{\mathbb{Z}}_2)$$

of 2-adic Adams modules with $H^3(X;\hat{\mathbb{Z}}_2)$ linear and $\widetilde{K}^1(X/X^3;\hat{\mathbb{Z}}_2)$ torsion-free, where X^3 is the 3-skeleton of X. Hence, it suffices to show that $\widetilde{K}^1(X/X^3;\hat{\mathbb{Z}}_2)$ is strictly nonlinear with monic ψ^2. Since $H^m(X;\hat{\mathbb{Z}}_2) = 0$ for sufficiently large m, the map $\widetilde{K}^1(X/X^3;\hat{\mathbb{Z}}_2) \to \widetilde{K}^1(X^m/X^3;\hat{\mathbb{Z}}_2)$ is monic for such m. Thus by skeletal induction, the operator ψ^2 on $Q \otimes \widetilde{K}^1(X/X^3;\hat{\mathbb{Z}}_2)$ is annihilated by the polynomial $f(x) = (x-2^2)(x-2^3)\dots(x-2^k)$ for sufficiently large k. It follows that $Q \otimes \widetilde{K}^1(X/X^3;\hat{\mathbb{Z}}_2)$ is the direct sum of the eigenspaces E_i of ψ^2 with eigenvalues 2^i for $2 \leqslant i \leqslant k$, and hence ψ^2 is monic on $\widetilde{K}(X/X^3;\hat{\mathbb{Z}}_2)$ as desired. Moreover, the projection to E_i is given by the operator $f_i(\psi^2)/f_i(2^i)$ on $Q \otimes \widetilde{K}^1(X/X^3;\hat{\mathbb{Z}}_2)$ where $f_i(x) = f(x)/(x-2^i)$. This implies that $2^v\widetilde{K}^1(X/X^3;\hat{\mathbb{Z}}_2)$ is contained in $\bigoplus_{i=2}^k E_i \cap \widetilde{K}^1(X/X^3;\hat{\mathbb{Z}}_2)$ where 2^v is the highest power of 2 dividing an integer $f_i(2^i)$ for some i. Since the above direct sum is strictly nonlinear, so is $2^v\widetilde{K}^1(X/X^3;\hat{\mathbb{Z}}_2)$ by Lemma 13.3, and hence so is $\widetilde{K}^1(X/X^3;\hat{\mathbb{Z}}_2)$. □

14. Proof of the realizability theorem for $\hat{L}M$

We shall prove Theorem 8.5, giving a strict isomorphism $\hat{L}M \cong K^*_{CR}(\widetilde{\mathrm{Fib}f};\hat{\mathbb{Z}}_2)$ for a companion map $f\colon \Omega^\infty\tilde{\mathcal{E}}\bar{M} \to \Omega^\infty\tilde{\mathcal{E}}\bar{\rho}\bar{M}$ of a strong 2-adic Adams Δ-module M. For this, it will suffice by Theorem 4.9 to obtain an isomorphism $\hat{\Lambda}M_C \cong K^*(\widetilde{\mathrm{Fib}f};\hat{\mathbb{Z}}_2)$ of the complex components. We do this by adapting our proof of the corresponding odd primary result (Theorem 4.7) in [**8**]. First, to determine the 2-adic K-cohomology of the loops on $\Omega^\infty\tilde{\mathcal{E}}\bar{M}$ or $\Omega^\infty\tilde{\mathcal{E}}\bar{\rho}\bar{M}$, we may replace Theorem 11.2 of [**8**] by the following two theorems.

Theorem 14.1. *If $X = \Omega^\infty E$ for a 1-connected spectrum E with $H^2(E;\hat{\mathbb{Z}}_2) = 0$, with $K^0(E;\hat{\mathbb{Z}}_2) = 0$, and with $K^1(E;\hat{\mathbb{Z}}_2)$ torsion-free, then $K^1(\Omega X;\hat{\mathbb{Z}}_2) = 0$ and $K^0(\Omega X;\hat{\mathbb{Z}}_2)$ is torsion-free.*

Proof. This follows from [**6**, Theorem 8.3]. □

Using notation and terminology of [**7**] for a 1-connected space X, we obtain an augmented 2-adic ψ^2-module $\hat{Q}K^1(X;\hat{\mathbb{Z}}_2) \downarrow H^3(X;\hat{\mathbb{Z}}_2)$ representing the Atiyah-Hirzebruch map $K^1(X;\hat{\mathbb{Z}}_2) \to H^3(X;\hat{\mathbb{Z}}_2)$, and we have:

Theorem 14.2. *If X is a 1-connected H-space with $K^1(\Omega X;\hat{\mathbb{Z}}_2) = 0$ and $K^0(\Omega X;\hat{\mathbb{Z}}_2)$ torsion-free, then $\sigma\colon U(\hat{Q}K^1(X;\hat{\mathbb{Z}}_2) \downarrow H^3(X;\hat{\mathbb{Z}}_2)) \cong K^0(\Omega X;\hat{\mathbb{Z}}_2)$.*

Proof. This follows from [**7**, Theorem 10.2]. □

When X is $\Omega^\infty\tilde{\mathcal{E}}\bar{M}$ or $\Omega^\infty\tilde{\mathcal{E}}\bar{\rho}\bar{M}$, we shall determine $H^3(X;\hat{\mathbb{Z}}_2)$ from the united 2-adic K-cohomology of X. For any 1-connected space X, we let $\alpha_R\colon \widetilde{KO}^{-1}(X;\hat{\mathbb{Z}}_2) \to H^3(X;\hat{\mathbb{Z}}_2)$ be the homomorphism induced by the Postnikov section $KO\hat{\mathbb{Z}}_2 \to$

$P^4KO\hat{\mathbb{Z}}_2$. Using the indecomposables $\hat{Q}KO^*(X;\hat{\mathbb{Z}}_2)$ of Definition 2.7 and Remark 4.10, we have:

Lemma 14.3. *If X is a 1-connected space with $H^2(X;\hat{\mathbb{Z}}_2)=0$, then $\alpha_R\colon \widetilde{KO}^{-1}(X;\hat{\mathbb{Z}}_2)\to H^3(X;\hat{\mathbb{Z}}_2)$ factors through $\hat{Q}KO^{-1}(X;\hat{\mathbb{Z}}_2)$ and vanishes on the following subgroups: $\bar{\phi}\widetilde{K}^{-1}(X;\hat{\mathbb{Z}}_2)$, $(\theta-2)\widetilde{KO}^{-1}(X;\hat{\mathbb{Z}}_2)$, $(\theta-rB^{-2}c)\widetilde{KO}^{-5}(X;\hat{\mathbb{Z}}_2)$, and $(\psi^3-9)\widetilde{KO}^{-1}(X;\hat{\mathbb{Z}}_2)$.*

Proof. The map α_R factors through $\hat{Q}KO^{-1}(X;\hat{\mathbb{Z}}_2)$ by a suspension argument using the isomorphism $H^3(X;\hat{\mathbb{Z}}_2)\cong H^2(\Omega X;\hat{\mathbb{Z}}_2)$. Since X is 1-connected with $H^2(X;\hat{\mathbb{Z}}_2) = 0$, there is a natural isomorphism $H^3(X;\hat{\mathbb{Z}}_2)\cong(\pi_2(\tau_2X))^{\#}$ by [**8**, Lemma 11.4]. Thus, it suffices by naturality to prove the desired vanishing results when X is $S^2\cup_{2^k}e^3$ for $k\geqslant 1$, and these results now follow from the elementary case $X=S^3$ since the collapsing map $S^2\cup_{2^k}e^3\to S^3$ induces epimorphisms of the cohomologies $\widetilde{K}^{-1}(-;\hat{\mathbb{Z}}_2)$, $\widetilde{KO}^{-1}(-;\hat{\mathbb{Z}}_2)$, and $\widetilde{KO}^{-5}(-;\hat{\mathbb{Z}}_2)$. □

For a 1-connected space X with $H^2(X;\hat{\mathbb{Z}}_2)=0$, the above α_R now induces a homomorphism $\bar{\alpha}_R\colon \operatorname{Lin}^{\Delta}\hat{Q}K_{\Delta}^{-1}(X;\hat{\mathbb{Z}}_2)\to H^3(X;\hat{\mathbb{Z}}_2)$ where $\hat{Q}K_{\Delta}^{-1}(X;\hat{\mathbb{Z}}_2)$ is the 2-adic Adams Δ-module of indecomposables given by Remark 4.10 and Definition 6.1, and where $\operatorname{Lin}^{\Delta}$ carries a 2-adic Adams Δ-module M to the group

$$\operatorname{Lin}^{\Delta}M \;=\; M_R/(\bar{\phi}M_C+(\theta-2)M_R+(\theta-rc')M_H+(\psi^3-9)M_R).$$

To determine $H^3(X;\hat{\mathbb{Z}}_2)$ when X is $\Omega^{\infty}\tilde{\mathcal{E}}\bar{M}$ or $\Omega^{\infty}\tilde{\mathcal{E}}\bar{\rho}\bar{M}$, we may replace Proposition 11.3 of [**8**] by:

Proposition 14.4. *If N is a torsion-free exact stable 2-adic Adams Δ-module, then*

$$\bar{\alpha}_R\colon \operatorname{Lin}^{\Delta}\hat{Q}K_{\Delta}^{-1}(\Omega^{\infty}\tilde{\mathcal{E}}N;\hat{\mathbb{Z}}_2)\cong H^3(\Omega^{\infty}\tilde{\mathcal{E}}N;\hat{\mathbb{Z}}_2).$$

Proof. Since there is a stable isomorphism $\bar{\alpha}_R\colon KO^{-1}(\tilde{\mathcal{E}}N;\hat{\mathbb{Z}}_2)/(\psi^3-9)\cong H^3(\tilde{\mathcal{E}}N;\hat{\mathbb{Z}}_2)$ by [**10**, Theorem 3.2] and [**8**, Lemma 11.4], the proposition follows using Theorem 6.7 and Lemma 4.11. □

For any θ-pro-nilpotent 2-adic Adams Δ-module M, we obtain a homomorphism $r\colon M^C\to\operatorname{Lin}^{\Delta}M$ of 2-adic Adams modules with M^C as in Definition 7.6 and $\operatorname{Lin}^{\Delta}M$ linear. Such a homomorphism is called *properly torsion-free* [**7**, 4.5] when its source is torsion-free and its kernel is strictly nonlinear (see Definition 7.7). We shall need:

Lemma 14.5. *If M is a strong 2-adic Adams Δ-module, then $r\colon M^C\to\operatorname{Lin}^{\Delta}M$ is properly torsion-free.*

Proof. Since M is strong, M^C is torsion-free and $\ker(M^C\to\operatorname{Lin}M^C)$ is strictly nonlinear. Using the maps $r\colon \operatorname{Lin}M^C\to\operatorname{Lin}^{\Delta}M$ and $c\colon \operatorname{Lin}^{\Delta}M\to\operatorname{Lin}M^C$ with $cr=2$, we see that $2\ker(M^C\to\operatorname{Lin}^{\Delta}M)$ is contained in $\ker(M^C\to\operatorname{Lin}M)$. Thus $\ker(M^C\to\operatorname{Lin}^{\Delta}M)$ is strictly nonlinear by Lemma 13.3. □

As in [**8**, Section 11], for a strong 2-adic Adams Δ-module M and a companion map f, we obtain a ladder of p-complete fiber sequences

$$\begin{array}{ccccc} \widetilde{\mathrm{Fib}}f & \longrightarrow & X & \xrightarrow{\tilde{f}} & Y \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{Fib}\, f & \longrightarrow & \Omega^\infty\tilde{\mathcal{E}}\bar{M} & \xrightarrow{f} & \Omega^\infty\tilde{\mathcal{E}}\bar{\rho}\bar{M} \end{array}$$

such that:

(i) X and Y satisfy the hypotheses of Theorems 14.1 and 14.2;

(ii) the vertical maps from X and Y are $K^*(-;\hat{\mathbb{Z}}_2)$-equivalences;

(iii) $H^3(Y;\hat{\mathbb{Z}}_2) = 0$ and the sequence $H^3(\Omega^\infty\tilde{\mathcal{E}}\bar{\rho}\bar{M};\hat{\mathbb{Z}}_2) \to H^3(\Omega^\infty\tilde{\mathcal{E}}\bar{M};\hat{\mathbb{Z}}_2) \to H^3(X;\hat{\mathbb{Z}}_2) \to 0$ is exact.

Lemma 14.6. *There is a canonical isomorphism* $H^3(X;\hat{\mathbb{Z}}_2) \cong \mathrm{Lin}^\Delta M$.

Proof. Since $f^*\colon K^*_{CR}(\Omega^\infty\tilde{\mathcal{E}}\bar{\rho}\bar{M};\hat{\mathbb{Z}}_2) \to K^*_{CR}(\Omega^\infty\tilde{\mathcal{E}}\bar{M};\hat{\mathbb{Z}}_2)$ is equivalent to $\hat{L}\bar{d}\colon \hat{L}\tilde{F}\bar{\rho}\bar{M} \to \hat{L}\tilde{F}\bar{M}$ for the θ-resolution map $\bar{d}$, the homomorphism $f^*\colon H^3(\Omega^\infty\tilde{\mathcal{E}}\bar{\rho}\bar{M};\hat{\mathbb{Z}}_2) \to H^3(\Omega^\infty\tilde{\mathcal{E}}\bar{M};\hat{\mathbb{Z}}_2)$ is equivalent to $\mathrm{Lin}^\Delta\,\bar{d}\colon \mathrm{Lin}^\Delta\,\tilde{F}\bar{\rho}\bar{M} \to \mathrm{Lin}^\Delta\,\tilde{F}\bar{M}$ by Proposition 14.4. Hence, there is an isomorphism of cokernels $H^3(X;\hat{\mathbb{Z}}_2) \cong \mathrm{Lin}^\Delta M$. □

Proof of Theorem 8.5. The proof of Theorem 4.7 in [**8**] is now easily adapted to give Theorem 8.5. In more detail, Propositions 11.5 and 11.6 of [**8**] remain valid in our setting using Lemmas 14.5 and 14.6 together with the short exact sequence

$$0 \longrightarrow (\tilde{F}M^C \downarrow 0) \longrightarrow (\tilde{F}M^C \downarrow \mathrm{Lin}^\Delta M) \longrightarrow (M^C \downarrow \mathrm{Lin}^\Delta M) \longrightarrow 0$$

induced by the θ-resolution. Propositions 11.7 and 11.8 likewise remain valid, and thus $\hat{\Lambda}M^C \cong K^*(\widetilde{\mathrm{Fib}}f;\hat{\mathbb{Z}}_2)$, so that Theorem 8.5 follows by Theorem 4.9. □

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