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ERRATUM TO 'CATEGORY OF A_∞-CATEGORIES'

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(*communicated by Jim Stasheff*)

Abstract

The erroneous statement (HHA **5** (2003), no. 1, 1–48) that the collection of unital A_∞-categories, *all* A_∞-functors, and all A_∞-transformations (resp. equivalence classes of natural A_∞-transformations) form a $\mathcal{K}$-2-category $\mathcal{K}^u A_\infty$ (resp. ordinary 2-category ${}^u A_\infty$) is corrected as follows. All 2-category axioms are satisfied, except that $1_e \cdot f$ does not necessarily equal 1_{ef} for all composable 1-morphisms e, f. The axiom $e \cdot 1_f = 1_{ef}$ does hold. The mistake does not affect results on invertible 2-morphisms and quasi-invertible 1-morphisms in ${}^u A_\infty$.

Let $\mathcal{V} = (\mathcal{V}, \otimes, c, \mathbb{1})$ be a symmetric monoidal category. Besides the notions of a 1-unital 2-unital $\mathcal{V}$-2-category (Definition A.1) and a 1-unital non-2-unital $\mathcal{V}$-2-category (a 2-category enriched in $\mathcal{V}$ which has unit 1-morphisms, but does not have unit 2-morphisms) (Definition A.2) the article [**Lyu03**] should contain the following intermediate notion:

Definition A.3 (1-unital left-2-unital $\mathcal{V}$-2-category)**.** A 1-unital left-2-unital $\mathcal{V}$-2-category consists of a 1-unital non-2-unital $\mathcal{V}$-2-category $\mathfrak{A}$ plus a morphism $1_f : \mathbb{1} \to \mathfrak{A}(\mathcal{A}, \mathcal{B})(f, f)$ for any 1-morphism $f : \mathcal{A} \to \mathcal{B}$, which is a two-sided unit with respect to vertical composition of 2-morphisms m_2, such that

$$e \cdot 1_f \equiv \left(\mathcal{D} \xrightarrow{e} \mathcal{A} \underset{f}{\overset{f}{\underset{\Downarrow 1_f}{\rightrightarrows}}} \mathcal{B} \right) = 1_{ef} \tag{1}$$

for all composable 1-morphisms e, f. Moreover, if

$$1_f \cdot k \equiv \left(\mathcal{A} \underset{f}{\overset{f}{\underset{\Downarrow 1_f}{\rightrightarrows}}} \mathcal{B} \xrightarrow{k} \mathcal{C} \right) = 1_{fk} \tag{2}$$

for all composable 1-morphisms f, k, such $\mathfrak{A}$ is the same as a 1-unital 2-unital $\mathcal{V}$-2-category.

Let $\mathcal{K}$ denote the homotopy category of the differential graded category of complexes of $\Bbbk$-modules, $\Bbbk$ being a commutative ring with a unit. Morphisms of $\mathcal{K}$ are chain maps modulo homotopy. It is correctly stated in [**Lyu03**] that the collection

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of all A_∞-categories, all A_∞-functors and all A_∞-transformations (resp. equivalence classes of natural A_∞-transformations) is a 1-unital non-2-unital $\mathcal{K}$-2-category $\mathcal{K}A_\infty$ (resp. 1-unital non-2-unital 2-category A_∞). It is correctly stated there that the collection of unital A_∞-categories, unital A_∞-functors and all A_∞-transformations (resp. equivalence classes of natural A_∞-transformations) is a 1-unital 2-unital $\mathcal{K}$-2-category $\mathcal{K}A^u_\infty$ (resp. ordinary 2-category A^u_∞). However, it is claimed incorrectly in Corollaries 7.11, 7.12 [ibid.] that the latter property holds also for the collection of unital A_∞-categories, *all* A_∞-functors, and all A_∞-transformations (resp. equivalence classes of natural A_∞-transformations). The correct statement is that the stated collection constitutes a 1-unital left-2-unital $\mathcal{K}$-2-category $\mathcal{K}^u A_\infty$ (resp. 1-unital left-2-unital 2-category ${}^u A_\infty$). Fortunately, the notions of an invertible 2-morphism, of a 1-morphism which is an equivalence, etc. make sense in ${}^u A_\infty$. All other results of [**Lyu03**] which concern ${}^u A_\infty$ remain valid. For instance, if $\mathcal{B}, \mathcal{C}$ are unital A_∞-categories, $r\colon f \to g\colon \mathcal{B} \to \mathcal{C}$ is an isomorphism of A_∞-functors and f is unital, then g is unital as well.

The proof of property (1) for all A_∞-functors $e\colon \mathcal{D} \to \mathcal{A}$, $f\colon \mathcal{A} \to \mathcal{B}$ with unital A_∞-category $\mathcal{B}$ consists of the line $e \cdot 1_f = e \cdot (f\mathbf{i}^{\mathcal{B}})s^{-1} = (ef\mathbf{i}^{\mathcal{B}})s^{-1} = 1_{ef}$, where $\mathbf{i}^{\mathcal{B}}\colon \mathrm{id}_{\mathcal{B}} \to \mathrm{id}_{\mathcal{B}}\colon \mathcal{B} \to \mathcal{B}$ is the unit A_∞-transformation. For any A_∞-functor $f\colon \mathcal{A} \to \mathcal{B}$ and a unital A_∞-functor $k\colon \mathcal{B} \to \mathcal{C}$, property (2) follows from the chain maps

$$1_f \cdot k = (f\mathbf{i}^{\mathcal{B}}s^{-1}) \cdot k = (f\mathbf{i}^{\mathcal{B}}k)s^{-1}\colon \mathbb{k} \to (A_\infty(\mathcal{A}, \mathcal{C})(fk, fk), m_1),$$
$$1_{fk} = (fk\mathbf{i}^{\mathcal{C}})s^{-1}\colon \mathbb{k} \to (A_\infty(\mathcal{A}, \mathcal{C})(fk, fk), m_1)$$

being equal in $\mathcal{K}$. In fact, these cycles are homologous, since $\mathbf{i}^{\mathcal{B}}k \equiv k\mathbf{i}^{\mathcal{C}}$ implies $f\mathbf{i}^{\mathcal{B}}k \equiv fk\mathbf{i}^{\mathcal{C}}$.

The erroneous statement was also referred to (but not used in any reasoning) after Corollary 5.6 of [**LO06**]. Other articles on the subject are not influenced by the mistake described here.

References

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