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DG-MODELS OF PROJECTIVE MODULES AND NAKAJIMA QUIVER VARIETIES

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(*communicated by Johannes Huebschmann*)

Abstract

Associated to each finite subgroup Γ of $\mathtt{SL}_2(\mathbb{C})$ there is a family of noncommutative algebras $O^\tau(\Gamma)$, which is a deformation of the coordinate ring of the Kleinian singularity $\mathbb{C}^2/\Gamma$. We study finitely generated projective modules over these algebras. Our main result is a bijective correspondence between the set of isomorphism classes of rank one projective modules over O^τ and a certain class of quiver varieties associated to Γ. We show that this bijection is naturally equivariant under the action of a "large" Dixmier-type automorphism group G. Our construction leads to a completely explicit description of ideals of the algebras O^τ.

1. Introduction

This paper is inspired by the recent work of Berest and Chalykh [**5**] on the right ideals of the first Weyl algebra $A_1(\mathbb{C})$ and Calogero-Moser spaces. The main result of [**5**] is an explicit construction of the Calogero-Moser correspondence refining the earlier work of Berest-Wilson [**6, 7**].

Our purpose is to extend the ideas and techniques of [**5**] to a broader class of algebras of geometric origin. More specifically, we will study right ideals in *quantized* coordinate rings of Kleinian singularities $\mathbb{C}^2/\Gamma$, where Γ is a finite cyclic subgroup of $\mathtt{SL}_2(\mathbb{C})$. Such rings form a family of noncommutative algebras O^τ (parametrized by the elements τ of the group algebra $\mathbb{C}\Gamma$), whose properties are similar to the properties of the Weyl algebra. For generic parameter values, O^τ, like $A_1(\mathbb{C})$, are simple, hereditary, Noetherian domains, having no nontrivial finite-dimensional representations. However, unlike A_1, they have a nontrivial K-group. A conjectural description of stably free ideals of O^τ, generalizing the work of Berest and Wilson, was suggested by Crawley-Boevey and Holland (see [**9**]). Recently, Baranovsky, Ginzburg and Kuznetsov [**2**] have refined and proved this conjecture using the methods of noncommutative projective geometry. The main idea of [**2**] (exploited earlier in [**16**] and [**7**]) consists of replacing O^τ by a graded algebra $\mathbb{B}^\tau$, which, by analogy with the geometric case, can be treated as the homogeneous coordinate ring of a noncommutative projective variety (see [**1**]). Projective modules over O^τ can then be extended

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to "vector bundles" on such a "variety" and the latter can be classified using the standard tools from algebraic geometry (the Beilinson spectral sequence and Barth's monads). Despite its naturality, this geometric approach has some disadvantages. First, it is fairly complicated and far from being explicit. Second, it involves a lot of choices (most notably the choice of filtration on the algebra O^τ), which are not intrinsic to the original problem. Third, it hides some interesting "affine" features of the problem, present in the case of the Weyl algebra: namely, the action of the Dixmier automorphism group on the ideal classes and the equivariance of the corresponding classifying map.

In the present paper we will give a new proof of the Crawley-Boevey-Holland conjecture, which is free from the above disadvantages. As in [**5**], our construction is elementary and independent of the choice of filtration on O^τ. It leads to a completely explicit description of ideals of O^τ, and more importantly, it is G-equivariant with respect to a certain "large" automorphism group G, which acts naturally on both the space of ideal classes and the associated quiver varieties $\mathfrak{M}^\tau$. This brings the picture with Kleinian singularities closer to the original example of the Weyl algebra and raises many interesting questions regarding the action of the group G on $\mathfrak{M}^\tau$ (see e.g. [**6**]).

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2. Background and statement of results

2.1. The algebras B^τ and O^τ

Let (L, ω) be a two-dimensional complex symplectic vector space with symplectic form ω, and let Γ be a finite subgroup of $\mathtt{Sp}(L, \omega)$. We can extend the natural (contragradient) action of Γ on L^* diagonally to TL^*, the tensor algebra of L^*, and define R to be the crossed product of TL^* with Γ. The form ω is a skew symmetric element of $L^* \otimes L^* \subset TL^* \subset R$; thus for each $\tau \in Z(\mathbb{C}\Gamma)$ we can define

$$B^\tau = R/R(\omega - \tau)R,$$
$$O^\tau = eB^\tau e,$$

where e is the symmetrizing idempotent $\sum_{g\in\Gamma} g/|\Gamma|$ in $\mathbb{C}\Gamma \subset B^\tau$. The algebras B^τ and O^τ were introduced and studied by W. Crawley-Boevey and M. Holland in [**9**].

It is convenient to choose a symplectic basis $\{e_x, e_y\}$ in L and identify L with $\mathbb{C}^2$, and $\mathtt{Sp}(L, \omega)$ with $\mathtt{SL}_2(\mathbb{C})$. If $\{x, y\}$ is the dual basis in L^* then we have an algebra isomorphism:

$$B^\tau \cong R/R(xy - yx - \tau)R, \tag{1}$$

where $R \cong \mathbb{C}\langle x, y\rangle * \Gamma$ is a crossed product of the free algebra on two generators with the group Γ.

In this paper we will be concerned with the case when Γ is a cyclic group $\mathbb{Z}_m$. One can give a more elementary description of B^τ in this case. We fix an embedding $\Gamma \hookrightarrow \mathtt{SL}_2(\mathbb{C})$ so that L decomposes as $\epsilon \oplus \epsilon^{-1}$, where ϵ is a primitive character of Γ. Now we choose a basis $\{x, y\}$ in L^* so that Γ acts on x by ϵ and on y by ϵ^{-1}. Then as an algebra, B^τ is generated by the elements x, y and $g \in \Gamma$ satisfying the relation:

$$g \cdot x = \epsilon(g) x \cdot g, \quad g \cdot y = \epsilon^{-1}(g)\, y \cdot g, \quad \forall g \in \Gamma, \tag{2}$$

$$x \cdot y - y \cdot x = \tau. \tag{3}$$

The corresponding algebras O^τ are called, in this case, *the type-A deformations of Kleinian singularities*. They were studied earlier by Hodges [**12**], Smith [**21**] and Bavula [**4**].

Homological and ring-theoretical properties of O^τ depend drastically on the values of the parameter τ. Through the McKay correspondence we can associate to the group Γ the affine Dynkin graph of type-A. The group algebra $\mathbb{C}\Gamma$ is then identified with the dual of the space spanned by the simple roots of the corresponding affine root system and, following [**9**], we say that an element $\tau \in \mathbb{C}\Gamma$ is *generic* if it does not belong to any root hyperplane in $\mathbb{C}\Gamma$.

From now on we will assume τ to be generic. In this case, B^τ and O^τ are Morita equivalent (see [**9**, Theorem 0.4]); the equivalence $F\colon \mathtt{Mod}(B^\tau) \to \mathtt{Mod}(O^\tau)$ between the categories of right modules is given by

$$M \mapsto M \otimes_{B^\tau} B^\tau e. \tag{4}$$

Hence these rings share the following properties: noetherianness, simplicity, having global dimension one.

2.2. Nakajima varieties

Given a pair (U, W) of finite-dimensional Γ-modules, consider the space of Γ-equivariant linear maps

$$\mathbb{M}_\Gamma(U, W) = \mathtt{Hom}_\Gamma(U, U \otimes L) \oplus \mathtt{Hom}_\Gamma(W, U) \oplus \mathtt{Hom}_\Gamma(U, W). \tag{5}$$

The group $G_\Gamma(U)$ of Γ-equivariant automorphisms of U acts on $\mathbb{M}_\Gamma(U, W)$ in the natural way:

$$g(B, \bar{i}, \bar{j}) = (gBg^{-1}, g\bar{i}, \bar{j}g^{-1}).$$

This action is free on the subvariety $\tilde{\mathfrak{M}}^\tau_\Gamma(U, W) \subseteq \mathbb{M}_\Gamma(U, W)$ defined by the conditions:

(i) $[B, B] + \tau|_U = \bar{i}\bar{j}$.

(ii) There is no proper submodule $U' \subset U$ such that $B(U') \subset U' \otimes L$ and $\bar{i}(W) \subset U'$. (6)

Here $[B, B]$ stands for the composition of the following maps

$$U \xrightarrow{B} BU \otimes L \xrightarrow{B \otimes id_L} U \otimes L \otimes L \xrightarrow{id_U \otimes \omega} U \otimes \mathbb{C} \cong U.$$

Definition 2.1. The *Nakajima variety* associated to the pair (U, W) is defined by

$$\mathfrak{M}^\tau_\Gamma(U, W) := \tilde{\mathfrak{M}}^\tau_\Gamma(U, W) /\!\!/ G_\Gamma(U).$$

The relation of this variety to the original definition of Nakajima [**19**] can be obtained via the McKay correspondence (see e.g. [**2**]).

As $L \cong \epsilon \oplus \epsilon^{-1}$, we can write down the map $B\colon U \to U \otimes L$ in the following form

$$B(v) = \bar{X}(v) \otimes \epsilon + \bar{Y}(v) \otimes \epsilon^{-1}, \tag{7}$$

with $\bar{X}, \bar{Y} \in \mathtt{End}_{\mathbb{C}}(U)$. The points of the Nakajima variety $\mathfrak{M}^{\tau}_{\Gamma}(U,W)$ can be represented then by quadruples $(\bar{X}, \bar{Y}, \bar{i}, \bar{j}) \in \mathtt{End}(U)^{\oplus 2} \oplus \mathtt{Hom}_{\Gamma}(W,U) \oplus \mathtt{Hom}_{\Gamma}(U,W)$ satisfying

$$\bar{X}\,\bar{Y} - \bar{Y}\,\bar{X} + \bar{T} = \bar{i}\,\bar{j}, \tag{8}$$

$$\bar{X}\,\bar{G} = \epsilon(g)\,\bar{G}\,\bar{X}, \qquad \bar{Y}\,\bar{G} = \epsilon^{-1}(g)\,\bar{G}\,\bar{Y}, \tag{9}$$

where $\bar{T}$ and $\bar{G}$ are endomorphisms corresponding to the action of τ and $g \in \Gamma$ in U respectively.

In the case when W is a one-dimensional Γ-module with character χ_W we can think of $\bar{i}$ and $\bar{j}$ just as linear maps $\bar{i} \in \mathtt{Hom}(W,U)$ and $\bar{j} \in \mathtt{Hom}(U,W)$ satisfying the conditions:

$$\bar{i}(w.g) = \chi_W(g)\,\bar{i}(w)\,,\ \bar{j}(v.g) = \chi_W(g)\,\bar{j}(v) \text{ for all } g \in \Gamma \text{ and } w \in W, v \in U. \tag{10}$$

2.3. Statement of results

We start this section by reminding the reader of the following result due to Berest and Wilson [**6, 7**].

Theorem 2.2.

(a) *There is a natural bijection between the space $\mathcal{R}$ of isomorphism classes of right ideals of the Weyl algebra $A_1(\mathbb{C})$ and the union $C = \bigsqcup C_n$ of Calogero-Moser algebraic varieties:*

$$C_n = \{\bar{X}, \bar{Y} \in M_n(\mathbb{C}) \,|\, rk(\bar{X}\bar{Y} - \bar{Y}\bar{X} - Id) = 1\}/GL_n(\mathbb{C}),$$

where $GL_n(\mathbb{C})$ acts on $(\bar{X}, \bar{Y})$ by simultaneous conjugation.

(b) *The automorphism group $G = Aut_{\mathbb{C}}(A_1)$ acts naturally on the varieties C_n, and this action is transitive for each $n = 0, 1, 2, \ldots$.*

(c) *The bijection $\mathcal{R} \longleftrightarrow C$ is equivariant under G, and thus the varieties C_n can be identified with the orbits of the natural action of G on the set $\mathcal{R}$ of ideal classes.*

The algebras O^{τ} are obvious generalizations of A_1 and one might expect that a result similar to Theorem 2.2 holds for the ideals of O^{τ}. In fact, Crawley-Boevey and Holland have conjectured that there is a bijection between the space of isomorphism classes of ideals of O^{τ} and certain Nakajima varieties related to Γ. Such a classification of ideals in terms of “Nakajima data” suggests the existence of some finite-dimensional modules associated to ideals. The natural candidates for such modules would be finite-dimensional representations of the algebra O^{τ}, but, since O^{τ} is simple, such representations do not exist. Nevertheless, stretching the notion of a module may overcome this problem. To be precise, we would like to extend the category of modules over O^{τ} to the category of DG-modules over a certain DG-algebra closely related to O^{τ}. In this extended category we will construct objects whose isomorphism classes are in a natural bijection with isomorphism classes of ideals in O^τ, and from which we can determine the Nakajima data corresponding to a given ideal. The idea of this approach goes back to [**5**], where the ideals of $A_1(\mathbb{C})$ are "modelled" by certain A_∞-modules. Relative to the ideals of $A_1(\mathbb{C})$ and O^τ the A_∞- and DG-modules play a role similar to "small" minimal models in the theory of differential graded algebras (see e.g. [**13, 15**]).

The basic property of O^τ (see Section 2.1) is that it is a *hereditary* Noetherian ring, which means that its ideals are finitely generated projective modules. So it is natural to classify first ideals up to stable isomorphism in the category of projective modules.

Let $\mathcal{R}'$ be the set of isomorphism classes of ideals of O^τ, and let $\mathcal{R}$ be the set of isomorphism classes of B^τ-submodules of eB^τ. Then the Morita equivalence (4) induces the bijection

$$\mathcal{R} \simeq \mathcal{R}', \qquad \mathbf{cl}(M) \mapsto \mathbf{cl}(Me), \tag{11}$$

where $\mathbf{cl}(M)$ stands for the isomorphism class of M. Thus, the problem of classifying the ideals of O^τ is equivalent to classifying projective B^τ-modules in $\mathcal{R}$.

Now let $K_0(\Gamma)$, $K_0(B^\tau)$ and $K_0(O^\tau)$ be the Grothendieck groups of the algebras $\mathbb{C}\Gamma$, B^τ and O^τ respectively. By a well-known theorem of Quillen the induction functor $P \mapsto P \otimes_{\mathbb{C}\Gamma} B^\tau$ gives an isomorphism $K_0(\Gamma) \cong K_0(B^\tau)$. Further, since B^τ and O^τ are Morita equivalent algebras, the corresponding equivalence functor induces another isomorphism $K_0(B^\tau) \cong K_0(O^\tau)$. We will use these isomorphisms to identify $K_0(B^\tau)$ and $K_0(O^\tau)$ with $K_0(\Gamma)$. By Theorem 3.4 below, there is a map:

$$\gamma\colon \mathcal{R} \to K_0(\Gamma) \times \hat{\Gamma}, \qquad \mathbf{cl}(M) \mapsto ([V], W),$$

where $\hat{\Gamma}$ is the group of characters of Γ, such that M_1 is stably isomorphic to M_2 if and only if $\gamma(\mathbf{cl}(M_1)) = \gamma(\mathbf{cl}(M_2))$. Thus, we can write $\mathcal{R}$ as a disjoint union of stable isomorphism classes

$$\mathcal{R} = \bigsqcup_{V,W} \mathcal{R}(V, W), \tag{12}$$

where $\mathcal{R}(V, W) = \gamma^{-1}([V], W)$. The advantage of working with B^τ(rather than O^τ) is that B^τ is a "one-relator" algebra: it has a presentation as a quotient of the quasi-free algebra R by a two-sided ideal generated by a single element (see (1)). Following [**5**], we can think of this presentation as a differential graded resolution of B^τ. To be precise, let $\boldsymbol{B}$ denote the graded associative algebra $I \oplus R$ having two nonzero components: the algebra $R = \mathbb{C}\langle x, y\rangle * \Gamma$ in degree zero and its (two-sided) ideal $I := R\nu R$ in degree -1. The differential on $\boldsymbol{B}$ is defined by the natural inclusion $d\colon I \hookrightarrow R$ (so that $d\nu = xy - yx - \tau \in R$, and $da \equiv 0$ for all $a \in R$). Now there exists a canonical quasi-isomorphism of DG-algebras given by the projection $\eta\colon \boldsymbol{B} \to B^\tau$. This map yields the restriction functor $\eta_*\colon \mathtt{Mod}(B^\tau) \to \mathtt{DGMod}(\boldsymbol{B})$, which is an exact embedding. It is well-known [**14**] that at the level of derived categories this functor induces an equivalence of triangulated categories $D(\mathtt{Mod}(B^\tau)) \to D(\mathtt{DGMod}(\boldsymbol{B}))$.

Now, let M be a projective B^τ-module representing a class in $\mathcal{R}$. We will associate to M an object $\boldsymbol{L}$ in $\mathtt{DGMod}(\boldsymbol{B})$ together with a quasi-isomorphism $M \to \boldsymbol{L}$, which we will call a *DG-model* of M. The DG-models are characterized by simple axioms (see Definition 2 in Section 4.1), which determine $\boldsymbol{L}$ for each M uniquely up to isomorphism (see Theorem 6.4 in Section 6.2). Thus our first result is

Theorem 2.3. *Let $\mathcal{M}$ be the set of isomorphism classes of DG-models. Then taking cohomology $\boldsymbol{L} \to H^{\bullet}(\boldsymbol{L})$ induces a bijection $\omega_1 \colon \mathcal{M} \tilde{\to} \mathcal{R}$.*

Next, in Section 4.2, we will show that each DG-model determines a point in the union of Nakajima varieties:

$$\mathfrak{M}^{\tau} = \bigsqcup_{V,W} \mathfrak{M}^{\tau}(V, W), \tag{13}$$

where V runs over the set of isomorphism classes of all finite-dimensional Γ-modules and W runs over the set of one-dimensional ones. Conversely, there is an explicit construction assigning to each point in $\mathfrak{M}^{\tau}$ a DG-model in $\mathtt{DGMod}(\boldsymbol{B})$ (see Section 4.3). In this way we will establish

Theorem 2.4. *There is a natural bijection $\omega_2 \colon \mathfrak{M}^{\tau} \tilde{\to} \mathcal{M}$.*

Combining Theorems 2.3 and 2.4 together we arrive at the following result (originally due to Baranovsky, Ginzburg and Kuznetsov [**2**]):

Theorem 2.5. *The isomorphism classes of projective B^{τ}-modules of rank one are in one-to-one correspondence with points of the Nakajima varieties $\mathfrak{M}^{\tau}$.*

If compared with [**2**], our proof of Theorem 2.4 has two main advantages. First, we construct the bijection $\Omega \colon \mathfrak{M}^{\tau} \longrightarrow \mathcal{M}$ as the composition of two maps ω_1 and ω_2, each of which is easy to describe. As a result, we give a completely explicit description of rank one, projective B^{τ}-modules (and thence, the right ideals of O^{τ}).

To be precise, let $\{W_0, W_1, \dots, W_{m-1}\}$ be the complete set of irreducible representations of $\Gamma = \mathbb{Z}_m$ such that $W_n \cong \epsilon^n$, and let $\{e_0, e_1, \dots, e_{m-1}\}$ be the corresponding idempotents in $\mathbb{C}\Gamma \subset B^{\tau}$. Writing $\mathfrak{M}^{\tau} = \bigsqcup_{n=0}^{m-1} \mathfrak{M}^{\tau}(W_n)$, we denote by Ω_n the restriction of $\Omega \colon \mathfrak{M}^{\tau} \to \mathcal{R}$ to the n-th stratum $\mathfrak{M}^{\tau}(W_n)$. Then we have the following theorem which extends the main result of [**5**].

Theorem 2.6. *The map $\Omega_n \colon \mathfrak{M}^{\tau}(W_n) \to \mathcal{R}$ sends a point of $\mathfrak{M}^{\tau}(W_n)$ represented by a quadruple $(\bar{X}, \bar{Y}, \bar{i}, \bar{j})$ to the class of the fractional ideal of B^{τ}:*

$$M = e_n \det(\bar{Y} - yI) B^{\tau} + e_n \kappa \det(\bar{X} - xI) B^{\tau},$$

where κ is the following element

$$\kappa = 1 - \bar{j}(\bar{Y} - yI)^{-1}(\bar{X} - xI)^{-1}\bar{i}(e_n)$$

in the classical ring of quotients of B^{τ}.

One of the interesting features of the Calogero-Moser correspondence in the case of the Weyl algebra is its equivariance with respect to the action of the automorphism group $Aut(A_1)$. Our approach allows us to extend this result to the case of noncommutative Kleinian singularities as follows. Let G be the group of Γ-equivariant automorphisms of the algebra $R = \mathbb{C}\langle x, y\rangle * \Gamma$, preserving the element $xy - yx \in R$. For each $\tau \in \mathbb{C}\Gamma$, the canonical projection $R \to B^{\tau}$ yields a group homomorphism $G \to Aut_\Gamma(B^\tau)$, and thus we have an action of G on the space $\mathcal{R}$ (induced by twisting the right B^τ-module structure by automorphisms of B^τ). On the other hand, there is a natural action of G on the Nakajima varieties $\mathfrak{M}^\tau(V, W)$. Finally, we observe that each $\sigma \in G$ extends naturally to an automorphism of the DG-algebra $\boldsymbol{B}$ and thus defines an auto-equivalence σ_* on the category $\mathtt{DGMod}(\boldsymbol{B})$. It is easy to see that our axiomatic of DG-models is invariant under such auto-equivalences; hence we have the induced action of G on $\mathcal{M}$. Now, the two bijections $\omega_1 \colon \mathcal{M} \to \mathcal{R}$ and $\omega_2 \colon \mathfrak{M}^\tau \to \mathcal{M}$ obviously commute with the actions of G defined above. Thus, we have the following

Theorem 2.7. *The map $\Omega \colon \mathfrak{M}^\tau \to \mathcal{R}$ is G-equivariant.*

We remark that if $\Gamma = \{e\}$, then the group G is isomorphic to $Aut_{\mathbb{C}}(A_1)$ (by a result of Makar-Limanov [**17**]) and in this case our Theorem 2.7 becomes one of the main results of Berest-Wilson. In general, comparing our results with [**6**] suggests that the (nonempty) subvarieties $\mathfrak{M}^\tau(V, W)$ in (13) are precisely the orbits of the given action of G on $\mathfrak{M}^\tau$. We will verify this conjecture in our subsequent paper.

3. K-theory

The purpose of this section is to give a K-theoretical classification of ideals of O^τ, that is a classification of ideals up to stable isomorphism in the category of finitely generated projective modules.

Let $\mathcal{R}'$ be the set of isomorphism classes of ideals of O^τ, and let $\mathcal{R}$ be the set of isomorphism classes of B^τ-submodules of eB^τ. Then, by (11), these sets are in natural bijection to each other. We will construct a map $\gamma \colon \mathcal{R} \to K_0(\Gamma) \times \hat{\Gamma}$ such that for two isomorphism classes, $\mathbf{cl}(M_1)$ and $\mathbf{cl}(M_2)$ in $\mathcal{R}$, the modules M_1 and M_2 are stably isomorphic if and only if $\gamma(\mathbf{cl}(M_1)) = \gamma(\mathbf{cl}(M_2))$.

First, we would like to make some remarks about the Grothendieck groups $K_0(\Gamma)$, $K_0(B^\tau)$ and $K_0(O^\tau)$. We write $[\,\cdot\,]$ for a stable isomorphism class in the respective K-group. By a well-known theorem of Quillen, the functor $P \mapsto P \otimes_{\mathbb{C}\Gamma} B^\tau$ gives an isomorphism of groups $K_0(\Gamma) \cong K_0(B^\tau)$, and since the set $\{[W_n]\}_{n=0}^{m-1}$ generates $K_0(\Gamma)$, the class $[e_n B^\tau]\}$ gives a set of generators of $K_0(B^\tau)$. Furthermore, since B^τ and O^τ are Morita equivalent algebras, the corresponding equivalence functor (4) induces an isomorphism $K_0(B^\tau) \cong K_0(O^\tau)$. We will use these isomorphisms to identify $K_0(B^\tau)$ and $K_0(O^\tau)$ with $K_0(\Gamma)$. Now, the map assigning to a finite-dimensional module of Γ its dimension extends to a group homomorphism $\dim \colon K_0(\Gamma) \to \mathbb{Z}$ and we have the following result:

Proposition 3.1. *Under the above identification of $K_0(\Gamma)$ and $K_0(O^\tau)$, the dimension function coincides with the rank function on projective modules $\mathtt{rk} \colon K_0(O^\tau) \to \mathbb{Z}$.*

Proof. Let $\widehat{\dim} \colon K_0(O^\tau) \to \mathbb{Z}$ be the composition of the isomorphism $K_0(O^\tau) \cong K_0(\Gamma)$ with $\dim \colon K_0(\Gamma) \to \mathbb{Z}$. We need to show that $\widehat{\dim} = \mathtt{rk}$. It suffices to check this on generators of $K_0(O^\tau)$, say $\{[e_n B^\tau e]\}$. By definition of the dimension function we have $\widehat{\dim}([e_n B^\tau e]) = 1$. On the other hand, each of the modules $e_n B^\tau e$ can be embedded into O^τ as an ideal and therefore $\mathtt{rk}(e_n B^\tau e) = 1$. □

Let us mention the following important result due to Baranovsky, Ginzburg and Kuznetsov (see [**2**, Proposition 1.3.11]).

Lemma 3.2. *Let $P \in K_0(\Gamma)$ be such that $\dim(P) = 1$. Then there exist unique Γ-modules W and V, such that $P = [W] + [V] \cdot ([L] - 2[W_0])$ in $K_0(\Gamma)$. Moreover, $\dim(W) = 1$ and V does not contain the regular representation as a submodule.*

Remark 3.3. If $U = \mathbb{C}\Gamma^{\oplus k}$ is a multiple of the regular representation of Γ then we have an isomorphism of Γ-modules $U \otimes L \cong U \oplus U$; hence

$$[U] \cdot ([L] - 2[W_0]) = [U \otimes L] - 2[U] = 0.$$

Thus, with the above identification of $K_0(B^\tau)$ and $K_0(\Gamma)$, we can define a map

$$\gamma\colon \mathcal{R} \to K_0(\Gamma) \times \hat{\Gamma}, \qquad \mathbf{cl}(M) \mapsto ([V], W),$$

where (V, W) is the pair of Γ-modules from Lemma 3.2. Now, restating this lemma in terms of B^τ-modules gives a classification of modules in $\mathcal{R}$ (or equivalently, in $\mathcal{R}'$) up to stable isomorphism.

Theorem 3.4. *For two isomorphism classes $\mathbf{cl}(M_1), \mathbf{cl}(M_2) \in \mathcal{R}$, we have $[M_1] = [M_2]$ in $K_0(B^\tau)$ if and only if $\gamma(\mathbf{cl}(M_1)) = \gamma(\mathbf{cl}(M_2))$.*

We will now give a construction of the map γ by showing how to explicitly determine the Γ-modules V and W for a given class $\mathbf{cl}(M) \in \mathcal{R}$.

Filter B^τ by assigning degree 1 to the generators x and y and degree 0 to all elements of Γ. Let us denote by $\bar{B^\tau} \cong \mathbb{C}[x, y] * \Gamma$ the associated graded algebra and let $\bar{M}$ be the associated graded module of a module $M \in \mathtt{Mod}(B^\tau)$ equipped with a good filtration. Each ideal M of B^τ can be equipped with the induced filtration (which is good as $\bar{B}^\tau$ is Noetherian).

Proposition 3.5. *For any isomorphism class $\mathbf{cl}(M) \in \mathcal{R}$, there exists a unique $n \in \{0, 1, \ldots, m-1\}$ such that $\bar{M} \hookrightarrow e_n\bar{B}^\tau$ and $\dim_{\mathbb{C}}(e_n\bar{B}^\tau/\bar{M}) < \infty$.*

Proof. This follows from Lemma 6.1 (see Section 6 below). □

The quotient $e_n\bar{B}^\tau/\bar{M}$ can be viewed as a (finite-dimensional) Γ-module via the canonical inclusion $\mathbb{C}\Gamma \to \bar{B}^\tau$.

Lemma 3.6. *Let $\mathbf{cl}(M_1), \mathbf{cl}(M_2) \in \mathcal{R}$ be such that $\bar{M}_1 \hookrightarrow e_n\bar{B}^\tau$ and $\bar{M}_2 \hookrightarrow e_k\bar{B}^\tau$ with finite-dimensional quotients, for some $n, k \in \{0, 1, \ldots, m-1\}$. Then $[M_1] = [M_2]$ in $K_0(B^\tau)$ if and only if $n = k$ and $e_n\bar{B}^\tau/\bar{M}_1 \cong e_k\bar{B}^\tau/\bar{M}_2 \oplus \mathbb{C}\Gamma^{\oplus N}$ as Γ-modules for some $N \in \mathbb{Z}_{\geqslant 0}$.*

This lemma allows us to give an explicit construction of the map γ. Specifically, let $\mathbf{cl}(M) \in \mathcal{R}$ be such that $\bar{M} \hookrightarrow e_n\bar{B}^\tau$ and $\dim(e_n\bar{B}^\tau/\bar{M}) < \infty$; then we can assign to $\mathbf{cl}(M)$ the pair $([e_n\bar{B}^\tau/\bar{M}], W_n) \in K_0(\Gamma) \times \hat{\Gamma}$. We will show that this map coincides with γ.

Let $G_0(\bar{B}^\tau)$ be the Grothendieck group of finitely generated modules over $\bar{B}^\tau$. Then it is well-known (see e.g. [**11**, Corollary 1.3]) that the class of $\bar{M}$ in $G_0(\bar{B}^\tau)$

does not depend on a choice of good filtration on M, thus defining a map

$$\psi\colon K_0(B^\tau) \to G_0(\bar{B}^\tau), [M] \mapsto [\bar{M}].$$

Now, since both $\mathbb{C}\Gamma$ and $\bar{B}^\tau \cong \mathbb{C}[x,y] * \Gamma$ are Noetherian rings of finite global dimension, ψ is an isomorphism of groups. Moreover, we have the following commutative diagram:

$$\begin{array}{ccc} & K_0(\Gamma) & \\ {}^{\phi_1}\swarrow & & \searrow^{\phi_2} \\ K_0(B^\tau) & \xrightarrow{\ \psi\ } & G_0(\bar{B}^\tau)\,, \end{array} \tag{14}$$

where ϕ_1 and ϕ_2 are group isomorphisms induced via canonical embeddings of $\mathbb{C}\Gamma$ in the algebras B^τ and $\bar{B}^\tau$ respectively.

Lemma 3.7. *Let L be the natural two-dimensional representation of Γ and let V be a finite-dimensional module over $\bar{B}^\tau$. Then in $K_0(\Gamma)$, we have*

$$\phi_2^{-1}([V]) = [V](2[W_0] - [L]). \tag{15}$$

Proof. We consider the following sequence of $\bar{B}^\tau$-modules

$$0 \to V \otimes_{\mathbb{C}\Gamma} \bar{B}^\tau \xrightarrow{d_2} \begin{array}{c} (V \otimes \epsilon) \otimes_{\mathbb{C}\Gamma} \bar{B}^\tau \\ \oplus \\ (V \otimes \epsilon^{-1}) \otimes_{\mathbb{C}\Gamma} \bar{B}^\tau, \end{array} \xrightarrow{d_1} V \otimes_{\mathbb{C}\Gamma} \bar{B}^\tau \xrightarrow{d_0} V \to 0 \tag{16}$$

where the maps are given by

$$d_0(v_1 \otimes b_1) = v_1.b_1\,,$$

$$d_1(v_1 \otimes \epsilon \otimes b_1,\, v_2 \otimes \epsilon^{-1} \otimes b_2) = (v_1.y \otimes b_1 - v_1 \otimes y \cdot b_1) - (v_2.x \otimes b_2 - v_2 \otimes x \cdot b_2),$$

$$d_2(v_1 \otimes b_1) = (v_1.x \otimes \epsilon \otimes b_1 - v_1 \otimes \epsilon \otimes x \cdot b_1\,,\, v_1.y \otimes \epsilon^{-1} \otimes b_1 - v_1 \otimes \epsilon^{-1} \otimes y \cdot b_1),$$

for $v_i \in V$ and $b_i \in \bar{B}^\tau$ $(i = 1, 2)$. We claim that this sequence is exact. First, it is easy to see that $d_2 \circ d_1 = d_1 \circ d_0 = 0$. Second, it is clear that d_0 is surjective and that $\mathtt{Ker}(d_0) = \mathtt{Im}(d_1)$. So we only need to prove that d_2 is injective and that $\mathtt{Ker}(d_1) = \mathtt{Im}(d_2)$.

Let $\{v_1, \ldots, v_n\}$ be a basis of the finite-dimensional space V and let $\bar{X} = (X_{ij})$ and $\bar{Y} = (Y_{ij})$ be matrices corresponding to the actions of x and y in this basis. Now if $u = \sum_{i=1}^n v_i \otimes b_i \in \mathtt{Ker}(d_2)$, then $xb_i = \sum_{j=1}^n X_{ij} b_j$. Assuming that $u \neq 0$ we let b_{i_0} be the element of largest degree among $\{b_1, \ldots, b_n\} \subset \bar{B}^\tau$ with respect to the above filtration. Then, $\mathtt{deg}(xb_{i_0}) > \mathtt{deg}(\sum_{j=1}^n X_{i_0 j} b_j)$ which contradicts the above equality and therefore $u = 0$, so d_2 is injective.

Now if $(u, u') = (\sum_{i=1}^n v_i \otimes \epsilon \otimes b_i,\, \sum_{i=1}^n v_i \otimes \epsilon^{-1} \otimes c_i) \in \mathtt{Ker}(\mathtt{d_1})$, then

$$\sum_{i=1}^n v_i \otimes (\sum_{j=1}^n Y_{ij} b_j - y b_i) = \sum_{i=1}^n v_i \otimes (\sum_{j=1}^n X_{ij} c_j - x, c_i) \tag{17}$$

and

$$\sum_{j=1}^{n} Y_{ij}b_j - yb_i = \sum_{j=1}^{n} X_{ij}c_j - xc_i, \quad i = 1, \ldots, n, \tag{18}$$

which we can simply write as follows: $(\bar{Y} - yI)\boldsymbol{b} = (\bar{X} - xI)\boldsymbol{c}$, where $\boldsymbol{b}$ and $\boldsymbol{c}$ are column vectors consisting of b_i and c_i, $i = 1, \ldots, n$, respectively.

To prove that $\mathtt{Ker}(d_1) = \mathtt{Im}(d_2)$ we must show that there exists $u'' = \sum_{i=1}^{n} v_i \otimes d_i$ such that $u = (\bar{X} - xI)u''$ and $u' = (\bar{Y} - yI)u''$. This is equivalent to finding a column vector $\boldsymbol{d}$ consisting of d_i, $i = 1, \ldots, n$ such that $\boldsymbol{b} = (\bar{X} - xI)\boldsymbol{d}$ and $\boldsymbol{c} = (\bar{Y} - yI)\boldsymbol{d}$. From the matrix equation (18) we can derive that each b_i is divisible by $\mathtt{det}(\bar{X} - xI)$ and each c_i by $\mathtt{det}(\bar{Y} - yI)$. Now, if we choose $\boldsymbol{d} := (\bar{X} - xI)^{-1}\boldsymbol{b} = (\bar{Y} - yI)^{-1}\boldsymbol{c}$, then it satisfies the required property. This proves the exactness of the sequence (16).

Thus, from (16) we obtain the following class equation in $G_0(\bar{B}^\tau)$:

$$\begin{aligned}[V] = [V \otimes_{\mathbb{C}\Gamma} \bar{B}^\tau] - [(V \otimes \epsilon) \otimes_{\mathbb{C}\Gamma} \bar{B}^\tau] \\ - [(V \otimes \epsilon^{-1}) \otimes_{\mathbb{C}\Gamma} \bar{B}^\tau] + [V \otimes_{\mathbb{C}\Gamma} \bar{B}^\tau].\end{aligned} \tag{19}$$

Now applying ϕ_2^{-1} to (19) we get the desired identity. □

Proof of Lemma 3.6. We recall that $\psi\colon K_0(B^\tau) \to G_0(\bar{B}^\tau)$ is a group isomorphism and therefore $[M_1] = [M_2]$ in $K_0(B^\tau)$ if and only if $[\bar{M}_1] = [\bar{M}_2]$ in $G_0(\bar{B}^\tau)$. The inclusions $\bar{M}_1 \hookrightarrow e_n\bar{B}^\tau$ and $\bar{M}_2 \hookrightarrow e_k\bar{B}^\tau$ yield the following identities in $G_0(\bar{B}^\tau)$:

$$[\bar{M}_1] = [e_n\bar{B}^\tau] - [e_n\bar{B}^\tau/\bar{M}_1] \quad \text{and} \quad [\bar{M}_2] = [e_k\bar{B}^\tau] - [e_k\bar{B}^\tau/\bar{M}_k].$$

Applying to these identities the group isomorphism $\phi_2^{-1}\colon G_0(\bar{B}^\tau) \to K_0(\Gamma)$ and using (15) we obtain

$$\phi_2^{-1}([\bar{M}_1]) = [W_n] + [e_n\bar{B}^\tau/\bar{M}_1]([L] - 2[W_0]),$$
$$\phi_2^{-1}([\bar{M}_2]) = [W_k] + [e_k\bar{B}^\tau/\bar{M}_2]([L] - 2[W_0]),$$

in $K_0(\Gamma)$. By Lemma 3.2 we have $W_n = W_k$ and then by the remark following Lemma 3.2 we obtain $e_n\bar{B}^\tau/\bar{M}_1 \cong e_k\bar{B}^\tau/\bar{M}_2 \oplus \mathbb{C}\Gamma^{\oplus N}$ as Γ-modules for some $N \in \mathbb{Z}_{\geqslant 0}$. □

4. DG-models

4.1. Axioms

Let us recall that we denote by $\boldsymbol{B}$ the graded associative algebra $I \oplus R$ having two nonzero components: the quasi-free algebra $R = \mathbb{C}\langle x, y\rangle * \Gamma$ in degree zero and its (two-sided) ideal $:= R\nu R$ in degree -1. The differential on $\boldsymbol{B}$ is defined by the natural inclusion $d\colon I \hookrightarrow R$ (so that $d\nu = xy - yx - \tau \in R$ and $da \equiv 0$ for all $a \in R$). The canonical map $f\colon R \to \boldsymbol{B}$ yields the restriction functor $f_*\colon \mathtt{DGMod}(\boldsymbol{B}) \to \mathtt{Com}(R)$. Thus any DG-module may be viewed as a complex of R-modules and, in particular, as a complex of $\mathbb{C}\Gamma$-modules (via the inclusion of $\mathbb{C}\Gamma$ into R).

We also recall that $\mathcal{R}(V, W)$ is the set of isomorphism classes of finitely generated, projective (right) modules M over B^τ such that $[M] = [W] + [V]([L] - 2[W_0])$ in

$K_0(B^\tau) = K_0(\Gamma)$. So if $\mathbf{cl}(M) \in \mathcal{R}(V, W)$ for some finite-dimensional Γ-module V and $W = W_n \in \hat{\Gamma}$, then we introduce the following definition (see [**5**]).

Definition 4.1. A *DG-model* of M is a quasi-isomorphism $q\colon M \to \boldsymbol{L}$ in the category $\mathtt{DGMod}(\boldsymbol{B})$, where $\boldsymbol{L} = L^0 \oplus L^1$ is a *DG*-module with two nonzero components (in degrees 0 and 1) satisfying the conditions:

- *Finiteness:*
$$\dim L^1 < \infty. \tag{20}$$
- *Existence of a cyclic vector:*
$$\text{There exists a } \Gamma \text{ linear map } i\colon W \to L^0 \text{ such that } i(W).R = L^0. \tag{21}$$
- *'Rank one' condition:*
$$\boldsymbol{L}.\nu \subseteq \mathtt{Im}(i), \tag{22}$$
where $\boldsymbol{L}.\nu$ denotes the action of ν on $\boldsymbol{L}$ and $\mathtt{Im}(i)$ denotes the image of i in L^0.

The following properties are almost immediate from the above definition.

1. Since W is a one-dimensional Γ-module there is a canonical inclusion $W \hookrightarrow \mathbb{C}\Gamma$ under which $W = \mathbb{C}e_n$; hence condition (21) implies that L^0 is a cyclic R-module with cyclic vector $i(e_n)$ which we denote by i_n.

2. The differential on $\boldsymbol{L}$ is given by a surjective R-linear map: $\boldsymbol{L}\colon L^0 \to L^1$. This follows from (20) and the fact that B^τ does not have finite-dimensional modules ([**9**, Theorem 0.4]). Composing $d_{\boldsymbol{L}}$ with i, one obtains the map $\bar{i}\colon W \to L^1$ and arguing as in 1, we can conclude that L^1 is a cyclic R module with cyclic vector $\bar{i}_n := \bar{i}(e_n)$.

3. Since ν is a degree -1 element in $\boldsymbol{B}$ we have $L^0.\nu = 0$. Thus condition (22) is equivalent to $L^1.\nu \subseteq \mathtt{Im}(i)$. By Schur's lemma the map i is injective; therefore we can define a map
$$\bar{j}\colon L^1 \to W, \qquad v.\nu = i(\bar{j}(v)).$$
Since $\Gamma \subset \mathtt{SL}_2(\mathbb{C})$ and d is an inclusion we have $g\nu = \nu g$ for all $g \in \Gamma$ which implies that $\bar{j}$ is a Γ-linear map. Composing $\bar{j}$ with $d_{\boldsymbol{L}}$ we obtain another Γ-linear map $j\colon L^0 \to W$.

The following results give a useful characterization of *DG*-models in the case of $\tau = 0$.

Proposition 4.2. *Suppose that $\boldsymbol{B}_0 = I_0 \oplus R$, where $I_0 = R\nu_0 R$, is a DG-algebra such that $d\nu_0 = xy - yx$. If $\boldsymbol{L} = L^0 \oplus L^1 \in \mathtt{DGMod}(\boldsymbol{B}_0)$ satisfies* (20)–(22) *then $L^1.\nu_0 = 0$ on $\boldsymbol{L}$.*

Proof. If $L^1 = 0$, then there is nothing to prove; thus we may assume that $L^1 \neq 0$. Then $d_{\boldsymbol{L}}(i_n) \neq 0$ for the map $f\colon R \to L^1$, $a \mapsto d_{\boldsymbol{L}}(i_n).a$, is surjective by (21). By using the notation (23)–(26) and arguing as in Lemma 4.4 below, we can compute $[\bar{X}, \bar{Y}] = \bar{i}\,\bar{j}$. On the other hand, since $\bar{i}_n.g = \epsilon^n(g)\,\bar{i}_n$ the set of vectors $\{\bar{Y}^m \bar{X}^k(\bar{i}_n)\}$ spans L^1 and $\dim L^1 < \infty$. An elementary lemma from linear algebra (see, e.g., [**19**, Lemma 2.9]) forces then $\bar{j} = 0$. □

Corollary 4.3. *Let $\boldsymbol{B}_0$ and $\boldsymbol{L}$ be as in Proposition 4.2; then $\boldsymbol{L}$ can be identified with a complex of $\bar{B}^\tau$-modules.*

4.2. The Nakajima data

Let $\boldsymbol{L}$ be a DG-module satisfying the axioms (20)–(22). Denote by X, Y, G (resp., $\bar{X}, \bar{Y}, \bar{G}$) the action of the canonical generators of R on L^0 (resp., L^1); i.e.,

$$X(u) := u.\, x \in \mathtt{End}_{\mathbb{C}}(L^0), \quad \bar{X}(v) := v.\, x \in \mathtt{End}_{\mathbb{C}}(L^1), \tag{23}$$

$$Y(u) := u.\, y \in \mathtt{End}_{\mathbb{C}}(L^0), \quad \bar{Y}(v) := v.\, y \in \mathtt{End}_{\mathbb{C}}(L^1), \tag{24}$$

$$G(u) := u.\, g \in \mathtt{End}_{\mathbb{C}}(L^0), \quad \bar{G}(v) := v.\, g \in \mathtt{End}_{\mathbb{C}}(L^1). \tag{25}$$

One can easily check that these maps satisfy the following conditions

$$\bar{X}\, d_{\boldsymbol{L}} = d_{\boldsymbol{L}}\, X, \quad \bar{Y}, d_{\boldsymbol{L}} = d_{\boldsymbol{L}}\, Y, \quad \bar{G}\, d_{\boldsymbol{L}} = d_{\boldsymbol{L}}\, G. \tag{26}$$

The next lemma shows that the linear data $(\bar{X}, \bar{Y}, \bar{i}, \bar{j})$ determined by a DG-model satisfy conditions (8) and (9) and hence correspond to a point of the Nakajima variety $\mathfrak{M}^{\tau}$.

Lemma 4.4. *The data introduced above satisfy the equations:*

$$X\, Y - Y\, X + T = i\, j, \quad \bar{X}\; \bar{Y} - \bar{Y}\, \bar{X} + \bar{T} = \bar{i}\, \bar{j}, \tag{27}$$

$$X\, G = \epsilon(g) G\, X, \quad Y\, G = \epsilon(g) G\, Y, \quad \bar{X}\, \bar{G} = \epsilon(g) \bar{G}\, \bar{X}, \quad \bar{Y}\, \bar{G} = \epsilon(g) \bar{G}\, \bar{Y}. \tag{28}$$

Proof. In view of (26) and surjectivity of $d_{\boldsymbol{L}}$, the second parts of equations (27) and (28) can be derived from the first ones, and the first of (27) follows easily from the Leibnitz rule:

$$\begin{aligned} T(u) &= u.\, \tau = u.(xy - yx - d\nu) = u.xy - u.yx - u.d\nu \\ &= (u.x).y - (u.y).x + d_{\boldsymbol{L}}(u).\nu = Y\, X(u) - X\, Y(u) + i(\bar{j}\, d_{\boldsymbol{L}}(u)) \\ &= Y\, X(u) - X\, Y(u) + i(j(u)) = (Y\, X - X\, Y + i\, j)u, \end{aligned}$$

for all $u \in L^0$.

To prove (28) we notice that

$$u.gx = (u.g).x = X\, G(u).$$

On the other hand, we see that

$$u.\epsilon(g)xg = \epsilon(g)(u.x).g = \epsilon(g) G\, X(u). \qquad \square$$

4.3. From the Nakajima data to DG-models

Let $(\bar{X}, \bar{Y}, \bar{i}, \bar{j}) \in \mathtt{End}(U, U)^{\oplus 2} \oplus \mathtt{Hom}_{\Gamma}(W, U) \oplus \mathtt{Hom}_{\Gamma}(U, W)$, where $W \cong W_n$, is a quadruple representing a point in the Nakajima variety. As a Γ-module, U can be uniquely written as $U \cong V \oplus \mathbb{C}\Gamma^{\oplus k}$ for some nonnegative integer k and a module V which does not contain the regular representation.

If we let $L^1 := U$, then via the endomorphisms $\bar{X}$ and $\bar{Y}$ we can make L^1 an R-module. Due to the stability condition (ii) in (6), it is clear that L^1 is a cyclic module over R with a cyclic vector $\bar{i}(e_n) = \bar{i}_n$.

To define an L^0-component of a DG-model we introduce a functional $\lambda : R \to \mathbb{C}$ so that $\bar{j}(\bar{i}_n . a) = \lambda(a)\, \bar{i}_n$. In view of (27), one can immediately see that

$$\lambda(g) = \epsilon^n(g)\, \mathtt{tr}(\bar{T}), \qquad \forall g \in \Gamma, \tag{29}$$

where $\mathtt{tr}$ stands for the trace of a matrix.

Proposition 4.5. *The functional λ is defined by its values on the elements of the form $g\, x^k y^l$, where $k, l \in \mathbb{N}$ and $g \in \Gamma$. Moreover, $\lambda(g\, x^k y^l) = 0$ for all k and l such that $k \not\equiv l (\mathrm{mod}\ m)$.*

Proof. The first part of the proposition easily follows from (8).

For the second part, since $\lambda(g\, x^k y^l) = \lambda(g)\, \lambda(x^k y^l)$, it suffices to show that λ vanishes on the elements $x^k y^l$. By the definition of λ we have

$$\lambda(x^k y^l)\bar{i}_n = \bar{j}(\bar{Y}^l \bar{X}^k(\bar{i}_n)). \tag{30}$$

Applying $g \in \Gamma$ to (30) and then using (10) and (9) sufficiently many times, we get

$$\begin{aligned} \epsilon^n(g)\, \lambda(x^k y^l)\bar{i}_n &= \bar{j}(\bar{G}\, \bar{Y}^l \bar{X}^k(\bar{i}_n)) \\ &= \epsilon^{n+l-k}(g)\, \bar{j}(\bar{Y}^l \bar{X}^k(\bar{i}_n)) \\ &= \epsilon^{n+l-k}(g)\, \lambda(x^k y^l)\bar{i}_n a. \end{aligned} \tag{31}$$

Finally, comparing the first and the last expressions of (31) we obtain the desired identity. □

Now we form the following *right* ideal in R:

$$J := \sum_{a \in R} (a\,(xy - yx - \tau) + \lambda(a))\, R, \tag{32}$$

and define $L^0 := W \otimes_{\mathbb{C}\Gamma} R/J$. Since $W \cong \mathbb{C}e_n$ we have that L^0 is a cyclic module over R with the generator $e_n \otimes [\,1\,]_J$ and we can define a map $i\colon W \to L^0$ by $w \mapsto w \otimes [\,1\,]_J$.

If we consider the map $W \otimes_{\mathbb{C}} R \to L^1$, $w \otimes a \mapsto \bar{i}(w).\, a$, then elements of the form $w.g \otimes a - w \otimes ga$ are annihilated by this map for any $w \in W$ and $a \in R$. Therefore this map factors through the canonical projection $W \otimes_{\mathbb{C}} R \to W \otimes_{\mathbb{C}\Gamma} R$ inducing a map $f\colon W \otimes_{\mathbb{C}\Gamma} R \to L^1$.

Further, it is easy to see that

$$\bar{i}_n . \,[a(xy - yx - \tau) + \lambda(a)] = 0, \quad \forall a \in R, \tag{33}$$

which allows us to factor f through yet another canonical projection $W \otimes_{\mathbb{C}\Gamma} R \to W \otimes_{\mathbb{C}\Gamma} R/J$ by producing a map from L^0 to L^1. We denote this map by $d_{\boldsymbol{L}}$. Being a composition of Γ-linear maps, $d_{\boldsymbol{L}}$ is also Γ-linear.

Thus, we have constructed a complex of cyclic R-modules

$$\boldsymbol{L} := [\,0 \to L^0 \xrightarrow{d_{\boldsymbol{L}}} L^1 \to 0\,],$$

with differential $d_{\boldsymbol{L}}$. We want to endow this complex with a DG-module structure over $\boldsymbol{B}$. For this it is sufficient to define the action of ν on L^1 and we define it as follows: $(\bar{i}_n.\, a).\, \nu = -e_n \otimes [\, \lambda(a)\,]_J$. Due to (33) this action is well-defined and it is also clear that $L^1.\nu \subseteq \mathtt{Im}(i)$.

Summing up, starting with Nakajima data $(\bar{X}, \bar{Y}, \bar{i}, \bar{j})$ we have constructed a DG-module $\boldsymbol{L}$ that satisfies all the axioms of Definition 1.

Finally we have to show that $\boldsymbol{L}$ represents a rank one projective module over B^τ of an appropriate class in K_0.

Lemma 4.6. *Let $\boldsymbol{L}$ be a DG-module over $\boldsymbol{B}$ constructed above. Then its cohomology $H^0(\boldsymbol{L})$ is a rank one finitely generated, projective module over B^τ such that $[H^0(\boldsymbol{L})] = [W_n] + [V]([L] - 2[W_0])$ in $K_0(B^\tau) \cong K(\Gamma)$, and consequently $H^0(\boldsymbol{L})$ is a representative of some class in $\mathcal{R}(V, W_n)$.*

Proof. Let us fix some standard filtration on R: say $R_k = \operatorname{span}\{x^p y^q g \colon p + q \leqslant k, g \in \Gamma\}$, and put the induced filtration on I, so that $\mathtt{gr}(R) \cong R$ and $\mathtt{gr}(I) \cong I_0$. We can then filter the complex $\boldsymbol{L}$ as follows: $L^0_k := i_n.R_k$ and $L^1_k := \bar{i}_n.R_k$. Using (27) it is easy to see that the given DG structure on $\boldsymbol{L}$ descends to the associated graded complex $\mathtt{gr}(\boldsymbol{L}) := \oplus_{n \geqslant 0} \boldsymbol{L}_k / \boldsymbol{L}_{k-1}$ making it into a DG-module over $I_0 \oplus R$. This module satisfies the same axioms (20)–(22) as $\boldsymbol{L}$; hence by Corollary 4.3, we have the following short exact sequence of $\bar{B}^\tau$-modules:

$$0 \to \overline{H^0(\boldsymbol{L})} \to \bar{L}^0 \to \bar{L}^1 \to 0, \tag{34}$$

where $\bar{L}^0 := \mathtt{gr}(L^0)$ and $\bar{L}^1 := \mathtt{gr}(L^1)$. In particular, we have an isomorphism of $\bar{B}^\tau$-modules $\bar{L}^0 \cong W_n \otimes_{\mathbb{C}\Gamma} \bar{B}^\tau$. Passing from $\mathtt{Mod}(\bar{B}^\tau)$ to $\mathtt{Mod}(e\bar{B}^\tau e)$ we see that $\overline{H^0(\boldsymbol{L})}e$ is a submodule of $W_n \otimes_{\mathbb{C}\Gamma} \bar{B}^\tau e$. The module $W_n \otimes_{\mathbb{C}\Gamma} \bar{B}^\tau e \cong e_n \bar{B}^\tau e$ can be identified with an ideal of $e\bar{B}^\tau e$ and so can $\overline{H^0(\boldsymbol{L})}e$. Thus $\overline{H^0(\boldsymbol{L})}e$ is a finitely generated, rank one, torsion-free module over $e\bar{B}^\tau e$. By standard filtration arguments all of the above properties lift to $H^0(\boldsymbol{L})e$, which is viewed as a module over the algebra $O^\tau = eB^\tau e$. Now by [**9**, Theorem 0.4], the global dimension $\mathtt{gldim}(O^\tau) = 1$ and therefore $H^0(\boldsymbol{L})e$ is projective. Hence, using Morita equivalence between O^τ and B^τ, we conclude that $H^0(\boldsymbol{L})$ is a projective B^τ-module.

Now we need to show that $\phi_1^{-1}([H^0(\boldsymbol{L})]) = [W_n] + [V]([L] - 2[W_0])$ which is equivalent, by (14), to showing that $\phi_2^{-1}([\overline{H^0(\boldsymbol{L})}]) = [W_n] + [V]([L] - 2[W_0])$. From (34) we have $[\overline{H^0(\boldsymbol{L})}] = [\bar{L}^0] - [\bar{L}^1]$ in $K_0(\bar{B}^\tau)$. Since $\bar{L}^0 \cong W_n \otimes_{\mathbb{C}\Gamma} \bar{B}^\tau$, we get that $\phi_2^{-1}([\bar{L}^0]) = [W_n]$. Next we know that $\bar{L}^1$ is a finite-dimensional module over $\bar{B}^\tau$ isomorphic to $V \oplus \mathbb{C}\Gamma^{\oplus k}$, and therefore by Lemma 3.7 and the remark after Lemma 3.2 we obtain $\phi_2^{-1}([\bar{L}^1]) = [V](2[W_0] - [L])$. □

5. *DG*-models and injective resolutions

In this section we show how to construct some explicit representatives of (the isomorphism class of) a module M, such that $\mathbf{cl}(M) \in \mathcal{R}(V, W_n)$, from its DG-model $M \xrightarrow{r} \boldsymbol{L}$. The key idea is to relate $\boldsymbol{L}$ to a minimal injective resolution of M (see [**5**]).

Let $\varepsilon \colon M \to \boldsymbol{E}$ be a minimal injective resolution of M in $\mathtt{Mod}(B^\tau)$. Since the global dimension of B^τ is one, the resolution $\boldsymbol{E}$ has length one; i.e. $\boldsymbol{E} = [0 \to E^0 \xrightarrow{\mu_1} E^1 \to 0]$, and is determined (by M) uniquely up to isomorphism in $\mathtt{Com}(B^\tau)$. Recall that $\mathtt{DGMod}_\infty(\boldsymbol{B})$ denotes the category of DG-modules over $\boldsymbol{B}$ with morphisms given by A_∞-homomorphisms. Then, when regarded as an object in $\mathtt{DGMod}_\infty(\boldsymbol{B})$, $\boldsymbol{E}$ is in the same quasi-isomorphism class as $\boldsymbol{L}$. It is natural to find a quasi-isomorphism that 'embeds' $\boldsymbol{L}$ into $\boldsymbol{E}$. By Lemma A.2 (see Appendix A below) any A_∞-quasi-isomorphism between such modules is determined by two components $f = (f_1, f_2)$:

$$f_1 \colon \boldsymbol{L} \to \boldsymbol{E}, \qquad (u, v) \mapsto (f_1(u), \bar{f}_1(v)),$$
$$f_2 \colon \boldsymbol{L} \to \boldsymbol{E}, \qquad (u, v) \otimes a \mapsto (f_2(v, a), 0),$$

which are subject to relations (75)–(79).

Theorem 5.1. *Let $r \colon M \to \boldsymbol{L}$ be a DG-model of M, and let $\varepsilon \colon M \to \boldsymbol{E}$ be a minimal injective resolution. Then there is a unique A_∞-quasi-isomorphism $f_x \colon \boldsymbol{L} \to \boldsymbol{E}$ such that $(f_x)_1 \circ r = \varepsilon$, and*

$$(f_x)_2\,(v, x) = 0 \quad \textit{and} \quad (f_x)_2\,(v, g) = 0 \quad \forall\, v \in L^1, \quad \forall\, g \in \Gamma. \tag{35}$$

Remark 5.2. First, a similar result can be stated if we replace x by y. We will denote the corresponding quasi-isomorphism by $f_y \colon \boldsymbol{L} \to \boldsymbol{E}$. Second, the last equation of (35) implies that f_2 induces (and is determined by) the map $L^1 \otimes_{\mathbb{C}\Gamma} R \to E^0$, which we also denote by f_2.

The following lemma is essential for the proof of Theorem 5.1.

Lemma 5.3. *E^0 is a torsion-free module over $\mathbb{C}[x]$.*

Proof. We notice that since M is an ideal of B^τ it is a torsion-free module over $\mathbb{C}[x]^\Gamma$. Let $n \in E^0$ be a torsion element; then there is $q \in \mathbb{C}[x]^\Gamma$ such that $q \neq 0$ and $nq = 0$. Since E^0 is the injective envelope of M we can find nonzero $b \in B^\tau$ and $m \in M$ such that $m = nb$. Now the elements of $S = \mathbb{C}[x]^\Gamma \setminus \{0\}$ act ad-nilpotently on B^τ which implies that S is an Ore set. Hence there are elements $t \in S$ and $c \in B^\tau$ such that $bt = qc$. By multiplying the expression $m = nb$ by t, we get

$$mt = nbt = nqc = 0,$$

which contradicts that M is $\mathbb{C}[x]^\Gamma$ torsion-free. This proves that E^0 is a torsion-free module over $\mathbb{C}[x]^\Gamma$. Now $\mathbb{C}[x]$ is a finite integral extension of $\mathbb{C}[x]^\Gamma$. Hence, for any nonzero $u \in \mathbb{C}[x]$, there exists a minimal monic polynomial

$$f(v) = v^l + a_{l-1}(x)v^{l-1} + \ldots + a_1(x)v + a_0(x)$$

with coefficients in $\mathbb{C}[x]^\Gamma$ such that $f(u) = 0$; therefore we have

$$u(u^{l-1} + a_{l-1}(x)u^{l-2} + \ldots + a_1(x)) = -a_0(x).$$

If we had nonzero $n \in E^0$ such that $nu = 0$ this would imply that $na_0(x) = 0$ which contradicts that E^0 is torsion-free over $\mathbb{C}[x]^\Gamma$. $\square$

Proof of Theorem 5.1. We observe that, since there is a canonical inclusion of $\mathbb{C}[x] * \Gamma$ into R, the complex $\boldsymbol{L}$ can be viewed as a complex over $\mathbb{C}[x] * \Gamma$. Now since B^τ is projective over $\mathbb{C}[x] * \Gamma$ (in fact, it is a free module $B^\tau = \oplus_{k=0}^\infty y^k \mathbb{C}[x] * \Gamma$), the complex $\boldsymbol{E}$ consists of $\mathbb{C}[x] * \Gamma$ injective modules. Hence, $\varepsilon \colon M \to \boldsymbol{E}$ extends to a $\mathbb{C}[x] * \Gamma$-linear morphism $f_1 \colon \boldsymbol{L} \to \boldsymbol{E}$ such that the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M & \xrightarrow{r} & L^0 & \xrightarrow{d_{\boldsymbol{L}}} & L^1 & \longrightarrow & 0 \\ & & \Vert & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle \bar{f}_1} & & \\ 0 & \longrightarrow & M & \xrightarrow{\varepsilon} & E^0 & \xrightarrow{\mu_1} & E^1 & \longrightarrow & 0 \end{array} \tag{36}$$

commutes in $\mathtt{Com}(\mathbb{C}[x] * \Gamma)$. We claim that such an extension is unique. Indeed, if $f_1' \colon L^0 \to E^0$ is another map in $\mathtt{Mod}(\mathbb{C}[x] * \Gamma)$ which satisfies $f_1 \circ r = f_1' \circ r = \varepsilon$, then $f_1' - f_1 \equiv 0$ on $\mathtt{Ker}(d_{\boldsymbol{L}})$ by exactness of the first row of (36). Thus the difference $\Delta := f_1' - f_1$ induces a $\mathbb{C}[x] * \Gamma$-linear and hence $\mathbb{C}[x]$-linear map $\bar{\Delta} \colon L^1 \to E^0$. Since $\dim_{\mathbb{C}} L^1 < \infty$, L^1 is torsion over $\mathbb{C}[x]$, while E^0 is torsion-free by the lemma above. Hence, $\bar{\Delta} = 0$ and therefore $f_1' = f_1$. This implies, of course, that $f_1' = f_1$ as morphisms in $\mathtt{Com}(\mathbb{C}[x] * \Gamma)$.

We now show how to define the second morphism f_2. Since $d_{\boldsymbol{L}}$ is surjective by part (b) of Lemma A.2 morphisms f_1, f_2 need only satisfy (75) and (76). To construct f_2 for which these relations hold it suffices to show

$$f_1(u.a) = f_1(u).a, \qquad \forall u \in \mathtt{Ker}(d_{\boldsymbol{L}}),\ \forall a \in R. \tag{37}$$

This can be easily shown from diagram (36). Now, we can define f_2 as follows:

$$f_2(v, a) := f_1(d_{\boldsymbol{L}}^{-1}(v).a) - f_1(d_{\boldsymbol{L}}^{-1}(a)).a, \tag{38}$$

which, in view of (37), is a well-defined map.

Finally, the morphism f_1 being $\mathbb{C}[x] * \Gamma$-linear is equivalent to

$$f_2(v, x) = 0 \quad \text{and} \quad f_2(v, g) = 0\ , \quad \forall\, v \in L^1, \quad \forall g \in \Gamma,$$

and these are the exact relations as in (35). □

To find the image of M in E^0 we need to give an explicit construction of f_1. Formula (78) in Appendix A which relates f_1 and f_2 is useful for this purpose. Indeed, by substituting ν, the generator of two-sided ideal I, for c in this formula we get

$$f_1(v.\nu) - f_1(v).\nu = -f_2(v, d\nu).$$

Then as $\boldsymbol{E}$ is a complex over B^τ the second term on the left-hand side vanishes. Now since $v.\nu = \bar{j}(v) i_n$ and $d\nu = xy - yx - \tau$, we obtain the following equation on f_2:

$$i_x \bar{j}(v) = -f_2(v, xy) + f_2(v, yx) - f_2(v, \tau), \tag{39}$$

where $i_x := f_1(i_n) \in E^0$. Using (79) and the fact that f_1 is Γ-linear (35), we can rewrite this equation in the following form:

$$f_2(v, y) \cdot x - f_2(\bar{X}(v), y) = \bar{j}(v)\, i_x. \tag{40}$$

Once this functional equation is solved one can recover f_1 from (76). The solution of (40) is given in Theorem 5.5 below. To state this theorem we need the following important result.

Lemma 5.4. *The ring B^τ has the classical (right) ring of quotients $Q(B^\tau)$.*

Proof. By [**9**, Theorem 0.4] the ring O^τ is simple. Being a simple ring is a Morita invariant property so the ring B^τ is also simple. Now, as B^τ is a Noetherian, the existence of $Q(B^\tau)$ is a consequence of Goldie's theorem (see e.g. [**23**] pp. 54–56). □

Theorem 5.5. *Let f_x be a A_∞-quasi-isomorphism defined in Theorem 5.1, and f_y be its counterpart obtained by interchanging x and y (see the remark following Theorem 5.1). Then f_x and f_y are given explicitly by*

$$\begin{aligned}(f_x)_1\,(i_n.\,x^k y^m) &= i_x \cdot \left(x^k y^m + \Delta_x^{km}(\bar{i}_n)\right), \\ (f_x)_2\,(v,\,x^k y^m) &= i_x \cdot \Delta_x^{km}(v),\end{aligned} \tag{41}$$

$$\begin{aligned}(f_y)_1\,(i_n.\,x^k y^m) &= i_y \cdot \left(x^k y^m + \Delta_y^{km}(\bar{i}_n)\right), \\ (f_y)_2\,(v,\,x^k y^m) &= i_y \cdot \Delta_y^{km}(v),\end{aligned} \tag{42}$$

where $i_x := (f_x)_1(i_n)$ and $i_y := (f_y)_1(i_n)$ in E^0, and

$$\Delta_x^{km}(v) := -\bar{j}(\bar{X} - xI)^{-1}(\bar{Y} - yI)^{-1}(\bar{Y}^m - y^m I)\,\bar{X}^k v, \tag{43}$$

$$\Delta_y^{km}(v) := \bar{j}(\bar{Y} - yI)^{-1}(\bar{X} - xI)^{-1}(\bar{X}^k - x^k I)\,y^m v, \tag{44}$$

where $I := Id_{L^1}$. Moreover,

$$i_x.g = \epsilon^n(g) i_x, \quad i_y.g = \epsilon^n(g) i_y, \quad \forall\, g \in \Gamma, \tag{45}$$

$$i_x = i_y \cdot \kappa, \tag{46}$$

where $\kappa \in Q$ is given by the formula $\kappa = 1 - \bar{j}(\bar{Y} - yI)^{-1}(\bar{X} - xI)^{-1}\bar{i}_n$ and satisfies the equation

$$e_n \kappa (1 - e_n) = 0 \quad \text{in } Q. \tag{47}$$

Let us give some comments on the theorem.

1. Since i_n is a cyclic vector of a one-dimensional Γ-module W, the elements $\{i_n.\,x^k y^m\}$ form a basis of L^0. Thus it suffices to define the maps f_x and f_y only on these elements.

2. Formulas (43) and (44) define the maps $\Delta_{x,y}^{km}\colon L^1 \to Q(B^\tau)$ for $m, k \geqslant 0$, which could be written more accurately as follows:

$$\Delta_x^{km}(v) := -J[(\bar{X} - xI)^* \sum_{l=1}^{m} \bar{Y}^{m-l} \bar{X}^k(v) \otimes y^{l-1}] \cdot \det(\bar{X} - xI)^{-1},$$

where $(\bar{X} - x\,I)^* \in \mathtt{End}_{\mathbb{C}}(L^1) \otimes_{\mathbb{C}} R$ denotes the classical adjoint of the matrix $\bar{X} - xI$ and the map J stands for the composition of the following maps:

$$L^1 \otimes_{\mathbb{C}} R \xrightarrow{\bar{j}\otimes\pi} W_n \otimes_{\mathbb{C}} B^\tau \longrightarrow W_n \otimes_{\mathbb{C}\Gamma} B^\tau \cong e_n B^\tau \hookrightarrow Q(B^\tau).$$

3. The dot in the right-hand sides of (41) and (42) denotes the (right) action of B^τ on $\boldsymbol{E}$. Even though $\Delta_{x,y}^{km}(v) \in Q(B^\tau)$, these formulas make sense because both E^0 and E^1 are injective, and hence *divisible* modules over B^τ.

Proof. We will give proofs of statements only for the map f_x and leave the calculations for f_y to the reader. Let us start with the formula for $(f_x)_2$. By the uniqueness result from Theorem 5.1 it suffices to check that $(f_x)_2$ satisfies the defining relation (40). This can be done in two steps. First, we verify this relation for $k = 0$ and $m = 1$ by simply substituting the corresponding expression for f_2 into (40). Second, using (79) and the fact that f_x is linear with respect to x, we can check (40) for all k, $m \in \mathbb{N}$. Now the expression for $(f_x)_1$ in (41) can be easily derived from formula (76) which relates f_1 and f_2.

The formulas in (45) can be derived from the fact that both f_x and f_y are Γ-linear maps.

Let $p(x) := \det(\bar{X} - xI)$. Then, by the Hamilton-Cayley theorem, $\bar{i}_n.\, p(x) = 0$ which implies that $i_n.\, p(x)$ is in the image of r. Since $f_x \circ r = \varepsilon = f_y \circ r$, we have

$$(f_x)_1(i_n.\, p(x)) = (f_y)_1(i_n.\, p(x)).$$

Using (42) and (44) we obtain

$$i_x \cdot p(x) = i_y \cdot (1 - \bar{j}\,(\bar{Y} - yI)^{-1}(\bar{X} - xI)^{-1}\bar{i}_n)\, p(x)$$

and, since E^0 is a divisible module over $\mathbb{C}[x]$, we derive formula (46) by simply dividing the last identity by $p(x)$.

In order to prove (47) it suffices to show that $e_n\kappa \cdot g = \epsilon^n(g)\, e_n\kappa$ for all $g \in \Gamma$. For this we expand $e_n\kappa$ into the formal series:

$$\begin{aligned} e_n\kappa &= e_n - e_n \sum_{l,k \geqslant 0} \bar{j}\left(\bar{Y}^l \bar{X}^k\, \bar{i}_n\right) y^{-l-1}x^{-k-1} \\ &= e_n - e_n \sum_{l \equiv k (\operatorname{mod} m)} \lambda_{kl}\, y^{-l-1}x^{-k-1}, \end{aligned} \tag{48}$$

where $\lambda_{kl} = \lambda(x^k y^l)$ and the last equality follows from Proposition 4.5. Now multiplying this series by g we obtain

$$e_n\kappa \cdot g = \epsilon^n(g)\, e_n \left(1 - \sum_{l \equiv k (\operatorname{mod} m)} \epsilon^{k-l}(g)\lambda_{kl}\, y^{-l-1}x^{-k-1}\right) = \epsilon^n(g) e_n\kappa, \tag{49}$$

where the last equality follows from (48) and the fact that $\epsilon^{k-l}(g) = 1$ for $l \equiv k (\operatorname{mod} m)$. □

Corollary 5.6. *Let* $\boldsymbol{L}$ *be an DG-envelope of* M*, a representative of some class in* $\mathcal{R}(V, W_n)$*, and let the quadruple* $(\bar{X}, \bar{Y}, \bar{i}, \bar{j})$ *be the Nakajima data associated with* $\boldsymbol{L}$*. Then,* M *is isomorphic to each of the following (fractional) ideals*

$$M_x := e_n \det(\bar{X} - xI)\, B^\tau + e_n\mu\, \det(\bar{Y} - yI)\, B^\tau, \tag{50}$$

$$M_y := e_n \det(\bar{Y} - yI)\, B^\tau + e_n\kappa\, \det(\bar{X} - xI)\, B^\tau, \tag{51}$$

where $\mu := 1 + \bar{j}\,(\bar{X} - xI)^{-1}(\bar{Y} - yI)^{-1}\bar{i}_n$ *is such that*

$$e_n\kappa \cdot e_n\mu = e_n\mu \cdot e_n\kappa = e_n;$$

hence $M_y = e_n\kappa M_x$.

Proof. Since M is an ideal of B^τ and $E^0(M)$ is a divisible module there is an inclusion $Q(B^\tau) \hookrightarrow E^0(M)$. The key idea of the proof is to realize M in its injective envelope $E^0(M)$ by investigating the image of $r(M)$ under the maps f_x and f_y.

Let $p(x) := \det(\bar{X} - xI)$ and $q(y) := \det(\bar{Y} - yI)$. Then by arguing as above, we can conclude from the Hamilton-Cayley theorem that $D = i_n.\, p(x)R + i_n.\, q(y)R$ is an R-submodule of $\mathtt{Ker}(d_{\boldsymbol{L}}) = \mathtt{Im}(r)$. It is easy to see that D is a submodule of finite codimension in L^0 and hence D is of finite codimension in $\mathtt{Im}(r)$. Now, since f is an injective map, $f_x(D)$ is a subspace of finite codimension in $f_x(\mathtt{Im}(r))$. Further, it is clear that $f_x(\mathtt{Im}(r)) = \varepsilon(M)$ is a B^τ-module.

If we show that $f_x(D)$ is also a B^τ-module, then since the algebra B^τ does not have finite-dimensional modules, we will obtain $f_x(D) = f_x(\mathtt{Im}(r))$.

By (41) and (43) we have $f_x(i_n.\, p(x)R) = i_x \cdot p(x)B^\tau$. Further, since $f_x \circ r = \varepsilon = f_y \circ r$ we obtain $f_x(i_n.\, q(y)R) = f_y(i_n.\, q(y)R)$; therefore by (42) and (44) we have $f_y(i_n.\, q(y)R) = i_y \cdot q(y)B^\tau$.

Now, since

$$[\bar{X} - xI, \bar{Y} - yI] = [\bar{X}, \bar{Y}] + [x, y]I = \bar{i} \circ \bar{j} - \bar{T} + \tau I,$$

it is easy to check that $e_n\mu \cdot e_n\kappa = e_n\kappa \cdot e_n\mu = e_n$ and hence $i_y = i_x \cdot \mu$. Thus, we get

$$f_x(D) = f_x(i_n.\, p(x)R) + f_y(i_n.\, q(y)R) = i_x \det(\bar{X} - xI)\, B^\tau + i_x\mu\, \det(\bar{Y} - yI)\, B^\tau.$$

By the above arguments we obtain $M \cong \varepsilon(M) = f_x(\mathtt{Im}(r)) = f_x(D)$. To finish the proof we notice that there is a B^τ-linear automorphism of $E^0(M)$, sending i_x to e_n. □

6. Existence and uniqueness

6.1. Distinguished representatives

In the previous section, in Corollary 5.6, for every DG-model $\boldsymbol{L} \in \mathcal{M}$ we have constructed two different realizations of $H^0(\boldsymbol{L})$ as fractional ideals of B^τ. Our main goal in this section is to present an analogous result for any $\mathbf{cl}(M) \in \mathcal{R}$. This result will be essential for proving the existence and uniqueness of DG-models.

First, we notice that $S_1 = \mathbb{C}[x] \setminus \{0\}$ is an Ore set in B^τ. Indeed we have already shown in Lemma 5.3 that the set $S = \mathbb{C}[x]^\Gamma \setminus \{0\}$ is an Ore set. Since S is an integral extension of S_1 then for any $u \in S_1$,

$$u(u^{k-1} + a_{k-1}(x)u^{k-2} + \ldots + a_1(x)) = -a_0(x),$$

where $a_1(x), \ldots, a_{k-1}(x) \in \mathbb{C}[x]^\Gamma$ and $a_0(x) \in S$. Thus, for any $b \in B^\tau$ there exist $c \in B^\tau$ and $a \in \mathbb{C}[x]^\Gamma$ such that

$$ab = ca_0(x) = [-c(u^{k-1} + a_{k-1}(x)u^{k-2} + \ldots + a_1(x))]u,$$

which proves that S_1 is an Ore set. Now let $B^\tau[S_1^{-1}]$ be the ring of fractions of B^τ with respect to S_1. Then M_x of Corollary 5.6 has the following properties:

$$\begin{aligned}
&(1)\ M_x \subset e_n B^\tau[S_1^{-1}] \text{ and } M_x \cap e_n\mathbb{C}[x] \neq \{0\}\ ,\\
&(2)\ \text{if } e_n(a_k(x)y^k + a_{k-1}(x)y^{k-1} + \ldots) \in M_x \text{ then } a_k(x) \in \mathbb{C}[x],\\
&(3)\ M_x \text{ contains elements of the form } e_n(y^k + a_{k-1}(x)y^{k-1} + \ldots).
\end{aligned} \tag{52}$$

We can also introduce the set $S_2 = \mathbb{C}[y] \setminus \{0\}$ and show that M_y satisfies similar properties.

Lemma 6.1. *Let* $\mathbf{cl}(M) \in \mathcal{R}$*. Then there exists a fractional ideal* M_x *of* B^τ *isomorphic to* M *and satisfying conditions (1)–(3) for some* $n \in \{0, 1, \ldots, m-1\}$*.*

Proof. First of all, without loss of generality, we may assume that M is a submodule of eB^τ such that $M \cap e\mathbb{C}[x] \neq \{0\}$ (see [**3**, Lemma 6.4]). Let $\boldsymbol{w} = (w_1, w_2)$ be a pair of nonnegative real numbers such that $\boldsymbol{w} = w_1 + w_2 > 0$i. Then we introduce a natural increasing filtration on

$$B^\tau\colon F^0_{\boldsymbol{w}} B^\tau = \mathbb{C}\Gamma, F^i_{\boldsymbol{w}} B^\tau = \{x^k y^m g | \mathtt{deg}_{\boldsymbol{w}}(x^k y^m g) := kw_1 + mw_2 \leqslant i, g \in \Gamma\}.$$

We can extend this filtration on Q by the following formula:

$$\mathtt{deg}_{\boldsymbol{w}}(ab^{-1}) := \mathtt{deg}_{\boldsymbol{w}}(a) - \mathtt{deg}_{\boldsymbol{w}}(b)$$

and $F^i_{\boldsymbol{w}} Q = \{q \in Q | \, \mathtt{deg}_w(q) \leqslant i\}$. With respect to this filtration we define the associated graded algebra $\mathtt{gr}_{\boldsymbol{w}} B^\tau = \oplus_{k=0}^\infty F^k_w B^\tau / F^{k-1}_w B^\tau \cong \mathbb{C}[\bar{x}, \bar{y}] * \Gamma$, where $\bar{x} := \mathtt{gr}(x)$ and $\bar{y} := \mathtt{gr}(y)$. If we now choose $\boldsymbol{w} = (0, 1)$ and denote the associated graded module of M with respect to this filtration by $\mathtt{gr}_y(M)$, then

$$\mathtt{gr}_y(M) = \bigoplus_{k \geqslant 0} e\, I_k(\bar{x})\, \bar{y}^k,$$

where $I_0 \subseteq I_1 \subseteq I_2 \subseteq \ldots$ is an ascending chain in $\mathbb{C}[\bar{x}]$ with $I_0(\bar{x}) \neq 0$. Since $\mathbb{C}[\bar{x}]$ is a principal ideal domain, each I_k is cyclic and the sequence of ideals $\{I_k\}$ stabilizes:

$$I_{n_0} = I_{n_0+1} = I_{n_0+2} = \ldots$$

starting from some $n = n_0 \geqslant 0$. We write $p = p(\bar{x})$ for the principal generator of I_{n_0} in $\mathbb{C}[\bar{x}]$. Now we claim that $p(\bar{x}) = \bar{x}^j \tilde{p}(\bar{x})$ for some $j \in \{0, 1, \ldots, m-1\}$ and $\tilde{p}(\bar{x}) \in \mathbb{C}[\bar{x}]^\Gamma$. It is clear that we can write $p(x)$ as follows: $p(\bar{x}) = \sum_{k=0}^{m-1} \bar{x}^k p_k(\bar{x})$, where each $p_j(\bar{x})$ is a Γ-invariant polynomial. Then $\mathtt{gcd}[p_0(\bar{x}), \bar{x}p_1(\bar{x}), \ldots, \bar{x}^{m-1} p_{m-1}(\bar{x})]$, the greatest common divisor of these polynomials, can be expressed as $\bar{x}^j \tilde{p}(\bar{x})$ where $\tilde{p}$ is Γ-invariant. Hence, there exists an element in M of the following form

$$b = e\, \bar{x}^j \tilde{p}(\bar{x}) \bar{y}^{n_0} + \sum_{k=0}^{n_0-1} \tilde{p}_k(\bar{x}) \bar{y}^k.$$

Therefore $\mathtt{gr}_y(b) = e\, \bar{x}^j \tilde{p}(\bar{x}) \bar{y}^{n_0}$ which implies our claim. Finally, let $M_x = p^{-1}(x) M$. Then since $ex^j = x^j e_{m-j}$, we obtain a fractional ideal M_x satisfying conditions (1)–(3). □

Corollary 6.2. *Let* M_x *and* M'_x *be two fractional ideals isomorphic to* M *and satisfying (1)–(3) above. Let* q *be an element of* Q *such that* $M'_x = qM_x$*. Then* $q \in \mathbb{C}e_n$ *(and hence* $M'_x = M_x$*).*

Proof. It is clear from (2) of (52) that $\mathrm{gr}_y(M_x) \subset \mathtt{gr}_y B^\tau \cong e_n \mathbb{C}[\bar{x}, \bar{y}]$. Moreover, due to conditions (1) and (3) this embedding is of finite codimension. This in turn implies that $\mathrm{gr}_y(q) \in \mathbb{C}e_n$. Now since $M_x \cap e_n \mathbb{C}[x] \neq \{0\}$ we have $q \in e_n \mathbb{C}(x)[y]$. Combining these last two facts, we conclude that $q \in \mathbb{C}e_n$. □

Reversing the roles of x and y, we obtain another distinguished representative M_y. The statement similar to Lemma 6.2 will also be true for M_y. In Corollary 5.6

we have seen that there is an element $\kappa \in Q$ such that $M_y = e_n \kappa M_x$. The following corollary claims such an element is unique.

Corollary 6.3. *Let M_x and M_y be fractional ideals isomorphic to M and defined as above, and q be an element of Q such that $M_y = qM_x$. Then q is uniquely determined up to a constant factor of e_n.*

Proof. Suppose we have $q_1, q_2 \in Q$ such that $M_y = q_i M_x$ $(i = 1, 2)$. Since both M_x and M_y are submodules of $e_n Q(B^\tau)$ we obtain $q_1, q_2 \in e_n Q(B^\tau) e_n$. Hence $M_x = qM_x$ where $q = q_2^{-1} q_1 \in Q$. Now by the above lemma $q \in \mathbb{C} e_n$. □

6.2. Uniqueness

In this section we will establish uniqueness of DG-models up to isomorphism of DG-modules. First we remind the reader of the definition of a linear functional, which was introduced earlier:

$$\lambda \colon R \to \mathbb{C}, \quad a \mapsto \lambda(a), \text{ where } \lambda(a)\, \bar{i}_n = \bar{j}(\bar{i}_n . a).$$

From Section 4.3 we recall that λ is completely determined by its special values:

$$\lambda_{kl} := \lambda(x^k y^l)\ ,\ k, l \in \mathbb{N} \text{ and } \ k \equiv l (\operatorname{mod} m). \tag{53}$$

We will prove

Theorem 6.4. *Let $\boldsymbol{L}$ and $\tilde{\boldsymbol{L}}$ be two DG-models of $\mathbf{cl}(M) \in \mathcal{R}(V, W_n)$. Then the following are equivalent:*

(*a*) *$\boldsymbol{L}$ and $\tilde{\boldsymbol{L}}$ are isomorphic as DG-modules over $\boldsymbol{B}$,*
(*b*) *$\boldsymbol{L}$ and $\tilde{\boldsymbol{L}}$ are A_∞-isomorphic,*
(*c*) *$\boldsymbol{L}$ and $\tilde{\boldsymbol{L}}$ are A_∞ quasi-isomorphic,*
(*d*) *$\boldsymbol{L}$ and $\tilde{\boldsymbol{L}}$ determine the same functional $\lambda \colon R \to \mathbb{C}$ (i.e. $\lambda = \tilde{\lambda}$).*

Proof. The implications $(a) \Rightarrow (b) \Rightarrow (c)$ are obvious. It suffices only to show that $(c) \Rightarrow (d)$ and $(d) \Rightarrow (a)$.

If $\boldsymbol{L}$ satisfies (20)–(22) then, by Lemma 4.6, the cohomology $H^0(\boldsymbol{L})$ represents a class in $\mathcal{R}(V, W_n)$. By Corollary 5.6, $H^0(\boldsymbol{L})$ is then isomorphic to the fractional ideals M_x and M_y related by $M_y = e_n \kappa M_x$ (see (50), (51)). Expanding $e_n \kappa$ into the formal series as in (48), we have

$$e_n \kappa = e_n - e_n \sum_{l \equiv k (\operatorname{mod} m)} \lambda_{kl}\, y^{-l-1} x^{-k-1}]. \tag{54}$$

Now, $e_n \kappa$ is determined uniquely, up to a constant factor of e_n, by the isomorphism class of $H^0(\boldsymbol{L})$ (see Corollary 6.3). Hence, if $\boldsymbol{L}$ are $\tilde{\boldsymbol{L}}$ are quasi-isomorphic A_∞-modules, we have $H^0(\tilde{\boldsymbol{L}}) \cong H^0(\boldsymbol{L})$ and therefore $e_n \tilde{\kappa} = c \cdot e_i \kappa$ for some $c \in \mathbb{C}\, e_n$. Comparing the coefficients of (54) yields at once $c = e_n$ and $\tilde{\lambda}_{lk} = \lambda_{lk}$ for all $l, k \geqslant 0$. Thus, we conclude $(c) \Rightarrow (d)$.

Now if we let $\lambda = \tilde{\lambda}$, then for the ideals J and $\tilde{J}$ defined in (32) we have $J = \tilde{J}$. Hence the map $f_1^0 \colon L^0 \to \tilde{L}^0$ defined as $i_n . a \mapsto \tilde{i}_n . a$ is an isomorphism of R-modules. Since $H^0(\tilde{\boldsymbol{L}}) \cong H^0(\boldsymbol{L})$, the induced map $f_1^1 \colon L^1 \to \tilde{L}^1$ is also an isomorphism. Finally since $d_{\tilde{L}} \circ f_1^0 = f_1^1 \circ d_L$ the pair (f_1^0, f_1^1) produces the necessary DG-isomorphism which proves implication $(d) \Rightarrow (a)$. □

6.3. Existence

Let us start by stating the main result of the section.

Theorem 6.5. *For each class of projective module* $\mathbf{cl}(M) \in \mathcal{R}(V, W_n)$ *there is a DG-model satisfying axioms of Definition* 1.

We need to produce for each class $\mathbf{cl}(M) \in \mathcal{R}$ a right DG-module which is a two-complex of vector spaces, quasi-isomorphic to M and satisfying conditions (20)–(22). Constructing such a DG-module $\boldsymbol{L}$ is equivalent to constructing a DG-algebra homomorphism Ψ from $\boldsymbol{B}$ to $\boldsymbol{C} := \underline{\mathtt{Hom}}_{\mathbb{C}}(\boldsymbol{L}, \boldsymbol{L})^{\mathtt{opp}}$, the opposite of the DG-algebra $\underline{\mathtt{Hom}}_{\mathbb{C}}(\boldsymbol{L}, \boldsymbol{L})$. The algebra $\boldsymbol{C}$ is concentrated in degrees $-1, 0$ and 1:

$$\boldsymbol{C} = C^{-1} \oplus C^0 \oplus C^1,$$

$$C^{-1} = \mathtt{Hom}_{\mathbb{C}}(L^1, L^0),\ C^0 = \mathtt{End}_{\mathbb{C}}(L^0) \oplus \mathtt{End}_{\mathbb{C}}(L^1),\ C^1 = \mathtt{Hom}_{\mathbb{C}}(L^0, L^1).$$

The multiplication is defined by $f * g := g \circ f$ where $\circ$ is just the usual composition of homomorphisms, and the differential is defined as:

$$d_{\boldsymbol{C}}(f) := d_{\boldsymbol{L}} \circ f - (-1)^j f \circ d_{\boldsymbol{L}}, \ \text{ for } f \in C^j.$$

Now the algebra $\boldsymbol{B}$ has the generators $x, y, g (\in \Gamma)$ in degree zero and one generator ν in degree minus one such that $d_{\boldsymbol{B}}(\nu) = xy - yx - \tau$. Thus the map Ψ is given in the following form

$$x \mapsto (X, \bar{X}),\ y \mapsto (Y, \bar{Y}),\ g \mapsto (G, \bar{G}),\ \nu \mapsto f,$$

where $X, Y, G \in \mathtt{End}_{\mathbb{C}}(L^0)$, $\bar{X}, \bar{Y}, \bar{G} \in \mathtt{End}_{\mathbb{C}}(L^1)$ and $f \in \mathtt{Hom}_{\mathbb{C}}(L^1, L^0)$. Moreover, Ψ must satisfy the conditions $d_{\boldsymbol{C}}(\Psi(x)) = d_{\boldsymbol{C}}(\Psi(y)) = 0$ and $\Psi(d_{\boldsymbol{B}}\nu) = d_{\boldsymbol{C}}(\Psi(\nu))$, which are equivalent to:

$$d_{\boldsymbol{L}} \circ X = \bar{X} \circ d_{\boldsymbol{L}}, \quad d_{\boldsymbol{L}} \circ Y = \bar{Y} \circ d_{\boldsymbol{L}}, \tag{55}$$

$$X\,Y - Y\,X + T = f_0, \quad \bar{X}\,\bar{Y} - \bar{Y}\,\bar{X} + \bar{T} = f_1, \tag{56}$$

where $(f_0, f_1) = d_{\boldsymbol{C}}(f)$. The rest of this section focuses on the construction of such a complex $\boldsymbol{L}$.

In Section 6.1 we have shown that the ideals M_x and M_y defined in Corollary 5.6 are uniquely characterized by properties (1)–(3) of (52). Moreover, by Corollary 6.3, there is $e_n\kappa \in Q$ such that $M_y = e_n\kappa M_x$ and $e_n\kappa$ is uniquely defined up to a constant factor of e_n. We choose $e_n\kappa$ such that $\mathtt{gr}_y(e_n\kappa) = e_n$. Now even though M_x and M_y are fractional ideals we can embed them into e_nB^τ by means of the following maps

$$\rho_x \colon e_nB^\tau[S_1^{-1}] \to e_nB^\tau, \quad e_nb(x)y^m \mapsto e_nb(x)_+y^m, \tag{57}$$

$$\rho_y \colon e_nB^\tau[S_2^{-1}] \to e_nB^\tau, \quad e_nb(y)x^m \mapsto e_nb(y)_+x^m, \tag{58}$$

where "$+$" stands for taking the polynomial part of the corresponding rational function. Let $r_x \colon M_x \to e_nB^\tau$ and $r_y \colon M_y \to e_nB^\tau$ be restrictions of the above maps to M_x and M_y correspondingly and let $V_x = e_nB^\tau/r_x(M_x)$ and $V_y = e_nB^\tau/r_y(M_y)$. It is not difficult to see first that r_x is $\mathbb{C}[y] * \Gamma$-linear and r_y is $\mathbb{C}[x] * \Gamma$-linear maps and second that both V_x and V_y are finite-dimensional Γ-modules. Now let us consider the following complexes of Γ-modules:

$$\boldsymbol{L}_x := [0 \to e_n B^\tau \to V_x \to 0] \quad \text{and} \quad \boldsymbol{L}_y := [0 \to e_n B^\tau \to V_y \to 0]. \tag{59}$$

We can extend the isomorphism $M_x \xrightarrow{e_n\kappa\cdot} M_y$ to an isomorphism of the above complexes:

$$\begin{array}{ccc} M_x & \xrightarrow{r_x} & \boldsymbol{L}_x \\ {\scriptstyle e_n\kappa\cdot}\downarrow & & \downarrow{\scriptstyle \Phi} \\ M_y & \xrightarrow{r_y} & \boldsymbol{L}_y. \end{array}$$

First let us introduce some notation. Let $B^\tau[S_1^{-1}][S_2^{-1}]$ be a Γ-module where B^τ first localized by the set S_1 and next by S_2. Then it is easy to see that $e_n B^\tau[S_1^{-1}][S_2^{-1}] \cong e_n\mathbb{C}(x)(y)$ and $e_n B^\tau[S_2^{-1}][S_1^{-1}] \cong e_n\mathbb{C}(y)(x)$. We now introduce four linear maps:

$$e_n\mathbb{C}[x](y) \xleftarrow{\grave{\rho}_x} e_n B^\tau[S_1^{-1}][S_2^{-1}] \xrightarrow{\acute{\rho}_y} e_n\mathbb{C}(x)[y], \qquad e_n\mathbb{C}(y)[x] \xleftarrow{\acute{\rho}_x} e_n B^\tau[S_2^{-1}][S_1^{-1}] \xrightarrow{\grave{\rho}_y} e_n\mathbb{C}[y](x), \tag{60}$$

which are defined as follows: $\grave{\rho}_x\colon e_n f(x)g(y) \mapsto e_n f(x)_+ g(y)$, $\acute{\rho}_y\colon e_n f(x)g(y) \mapsto e_n f(x)g(y)_+$, $\acute{\rho}_x\colon e_n g(y)f(x) \mapsto e_n g(y)f(x)_+$, and $\grave{\rho}_y\colon e_n g(y)f(x) \mapsto e_n g(y)_+ f(x)$. It is clear that all of these maps are Γ-equivariant. We then define a Γ-equivariant map $\phi\colon e_n B^\tau \to e_n B^\tau$ by

$$\phi(e_n b) := \grave{\rho}_y\, \acute{\rho}_x(e_n\kappa \cdot e_n b) = \grave{\rho}_y\, \acute{\rho}_x(e_n\kappa b), \quad b \in B^\tau. \tag{61}$$

Now one can argue as in Lemma 7 of [**5**] to prove the following result.

Proposition 6.6. *Let $\phi\colon e_n B^\tau \to e_n B^\tau$ be a map as in* (61)*. Then:*

(1) ϕ extends κ through r_x, i.e. $\phi \circ r_x = r_y \circ e_n\kappa$.

(2) ϕ is invertible with $\phi^{-1}\colon e_n B^\tau \to e_n B^\tau$ given by $\phi^{-1}(a) = \grave{\rho}_x\, \acute{\rho}_y(e_n\mu b)$.

(3) We have $\phi(e_n b) = e_n b$ whenever $b \in \mathbb{C}[x]$ or $b \in \mathbb{C}[y]$.

Proof. Denote by $\mathbb{C}(x)_-$ the subspace of $\mathbb{C}(x)$ consisting of functions vanishing at infinity. Then we can extend our earlier notation writing, for example, $\mathbb{C}(x)_-(y)$ for the subspace of $\mathbb{C}(x)(y)$ spanned by all elements $f(x)g(y)$ with $f(x) \in k(x)_-$ and $g(y) \in k(y)$.

(1) Since $M_x \subset e_n\mathbb{C}(x)[y]$ we have $r_x(m) - m \in e_n\mathbb{C}(x)_-[y] = \mathbb{C}[y](x)_-$ for any $m \in M_x$. Hence, $e_n\kappa \cdot (r_x(m) - m) \in e_n\mathbb{C}(y)(x)_-$ and therefore $\acute{\rho}_x(e_n\kappa \cdot r_x(m)) = \acute{\rho}_x(e_n\kappa \cdot m)$. On the other hand, if $m \in M_x$, then $e_n\kappa \cdot m \in M_y \subset \mathbb{C}(y)[x]$ and therefore $\acute{\rho}_x(e_n\kappa \cdot m) = e_n\kappa \cdot m$. Thus,

$$\phi(r_x(m)) = \grave{\rho}_y\acute{\rho}_x(e_n\kappa \cdot m) = \grave{\rho}_y(e_n\kappa \cdot m) = r_y(e_n\kappa \cdot m).$$

(2) From the definition of ϕ it follows that $\grave{\rho}_y\,\acute{\rho}_x(\phi(e_nb)-\kappa\cdot e_nb)=0$; therefore

$$\begin{aligned}\phi(e_nb)-e_n\kappa\cdot b &\in e_n\mathbb{C}(y)(x)_- + e_n\mathbb{C}(y)_-(x)\\ &= e_n\mathbb{C}(x)_-[y]+e_n\mathbb{C}[x](y)_- + e_n\mathbb{C}(y)_-(x)_-.\end{aligned}$$

Now multiplying the last expression by $e_n\mu$ and using the fact that $e_n\mu-1\in e_n\mathbb{C}(x)_-(y)_-$, we obtain

$$\begin{aligned}e_n(\mu\cdot\phi(e_nb)-b) &\in e_n\mathbb{C}(x)_-(y)+e_n\mathbb{C}(x)(y)_-\\ &\quad + e_n\mathbb{C}(y)_-(x)_- + e_n\mathbb{C}(x)_-(y)_-(x)_-.\end{aligned}$$

On the other hand, since $\phi(e_nb)\in e_nB^\tau$, we have $e_n\mu\cdot\phi(e_nb)-e_nb\in e_n\mathbb{C}(x)(y)$. By comparing the last two inclusions we obtain

$$e_n\mu\cdot\phi(e_nb)-e_nb \;\in\; e_n\mathbb{C}(x)_-(y)+e_n\mathbb{C}(x)(y)_-.$$

Hence $\grave{\rho}_x\,\acute{\rho}_y(e_n\mu\cdot\phi(e_nb)-e_nb)=0$ and therefore $\grave{\rho}_x\,\acute{\rho}_y(e_n\mu\cdot\phi(e_nb))=e_nb$ for all $b\in B^\tau$. Defining now $\phi^{-1}\colon e_nB^\tau\to e_nB^\tau$ by the formula $\phi^{-1}(e_nb):=\grave{\rho}_x\,\acute{\rho}_y(e_n\mu\cdot b)$ we see that $\phi^{-1}\circ\phi=\mathtt{Id}_{e_nB^\tau}$. On the other hand, reversing the roles of ϕ and ϕ^{-1} in the above argument would give obviously $\phi\circ\phi^{-1}=\mathtt{Id}_{e_nB^\tau}$. Thus, ϕ is an isomorphism of a vector space, and ϕ^{-1} is indeed its inverse.

(3) is immediate from the definition of ϕ. For example, if $b\in\mathbb{C}[x]$ then $e_n\kappa\cdot b-e_nb\in\mathbb{C}(y)_-(x)$ and therefore

$$\phi(e_nb):=\grave{\rho}_y\,\acute{\rho}_x(e_n\kappa\cdot b)=\grave{\rho}_y\,\acute{\rho}_x(e_nb)=b.$$

This finishes the proof of the proposition. □

Remark 6.7. Once the isomorphism ϕ satisfying condition (1) of Proposition 6.6 is established one can easily determine the isomorphism of quotient spaces $\bar{\phi}\colon V_x\to V_y$ and hence the isomorphism of complexes $\Phi=(\phi,\bar{\phi})\colon \boldsymbol{L}_x\to\boldsymbol{L}_y$.

We will now define our *DG*-module. Let $\boldsymbol{L}:=\boldsymbol{L}_x$ and endomorphisms $X, Y\in\mathtt{End}_{\mathbb{C}}(L^0)$ and $\bar{X}, \bar{Y}\in\mathtt{End}_{\mathbb{C}}(L^1)$ are given by

$$X(e_nb):=\phi^{-1}(\phi(e_nb)\cdot x),\qquad Y(e_nb)=e_nb\cdot y, \tag{62}$$

$$\bar{X}(e_nb):=\bar{\phi}^{-1}(\bar{\phi}(\overline{e_nb})\cdot x),\qquad \bar{Y}(e_nb)=e_nb\cdot y, \tag{63}$$

where " $\cdot$ " stands for the usual multiplication in B^τ. It is clear from the construction that these endomorphisms satisfy (55). We next define the 'cyclic' vectors:

$$i\colon W_n\to L^0,\ e_n\mapsto e_n,\ \text{ and }\ \bar{i}\colon W_n\to L^1,\ e_n\mapsto d_{\boldsymbol{L}}(e_n). \tag{64}$$

Now condition (56) is a consequence of the following proposition.

Proposition 6.8. *The endomorphisms* (62) *and* (63) *satisfy the equations*

$$X\,Y-Y\,X+T=i\,j,\qquad \bar{X}\,\bar{Y}-\bar{Y}\,\bar{X}+\bar{T}=\bar{i}\,\bar{j} \tag{65}$$

for some $j\colon L^0\to W_n$ *and* $\bar{j}\colon L^1\to W_n$ *related by* $j=\bar{j}d_{\boldsymbol{L}}$.

Proof. It suffices to show that

$$X\,Y(e_nb)-Y\,X(e_nb)+T(e_nb)\in\mathbb{C}e_n \text{ for any } b\in B^\tau.$$

Indeed, if it holds, we can define $j(e_nb)=X\,Y(e_nb)-Y\,X(e_nb)+T(e_nb)$. By the previous proposition it is then easy to see that $j(e_n b) = 0$ on $\mathtt{Im}(r_x)$, and since $\mathtt{Im}(r_x) = \mathtt{Ker}(d_{\boldsymbol{L}})$ the second equation follows from the first.

Let $\tilde{b} := X(e_n b) - e_n b \cdot x$; then using (62) we have

$$\phi(\tilde{b}) = \phi(e_n b) \cdot x - \phi(e_n b \cdot x) = \grave{\rho}_y\,(\acute{\rho}_x(e_n \kappa b) \cdot x - \acute{\rho}_x(e_n \kappa b \cdot x)).$$

It is clear that the last expression lies in $e_n\mathbb{C}[y]$ and therefore, by Proposition 6.6(3), we get $\tilde{b} \in e_n\mathbb{C}[y]$ for all $b \in B^\tau$. Now we have

$$\begin{aligned}(XY - YX)(e_n b) + T(e_n b) &= \phi^{-1}(\phi(e_n b y)x) - \phi^{-1}(\phi(e_n b)x)y + T(e_n b)\\ &= \left(\phi^{-1}(\phi(e_n b y)x) - e_n b y x\right)\\ &\quad - \left(\phi^{-1}(\phi(e_n b)x) - e_n b x\right) y \in e_n\mathbb{C}[y].\end{aligned}$$

On the other hand, if besides (62) we define X', $Y' \in \mathtt{End}_{\mathbb{C}}(e_n B^\tau)$ by

$$X'(e_n b) := e_n b \cdot x\ , \quad Y'(e_n b) := \phi(\phi^{-1}(e_n b) \cdot y),$$

then by symmetry $(X'Y' - Y'X')e_n b + e_n b \in e_n\mathbb{C}[x]$ for all $b \in B^\tau$. But $\phi\, X = X'\, \phi$ and $\phi\, Y = Y'\, \phi$. Hence

$$\phi\,([X,\, Y]e_n b + e_n b) = [X',\, Y']\phi(e_n b) + \phi(e_n b) \in e_n\mathbb{C}[x];$$

therefore

$$[X,\, Y]e_n b + e_n b \ \in\ e_n\mathbb{C}[y] \,\cap\, \phi^{-1}(e_n\mathbb{C}[x]) = e_n\mathbb{C}[y] \,\cap\, e_n\mathbb{C}[x] = \mathbb{C}e_n,$$

where $\phi^{-1}(e_n\mathbb{C}[x]) = e_n\mathbb{C}[x]$ is due to Proposition 6.6(3). □

Finally, if we choose $f = i\,\bar{j}$ then $d_{\boldsymbol{C}}(f) = (f_1,\, f_2) = (i\,j,\, \bar{i}\,\bar{j})$ and therefore, by Proposition 6.8, condition (56) holds.

7. Bijective correspondences

Let us remind the reader that $\mathcal{R}(V,W)$ is the set of isomorphism classes of projective modules M over B^τ such that $[M] = [W] + [V]([L] - 2[W_0])$ under $K_0(B^\tau) \cong K_0(\Gamma)$. Further let $\mathcal{M}(V,W)$ be the set of strict isomorphism classes of DG-models as defined in Definition 1. Finally, let $\tilde{\mathfrak{M}}^\tau_\Gamma(V,W) = \bigsqcup_{k=0}^{\infty} \mathfrak{M}^\tau_\Gamma(V \oplus \mathbb{C}\Gamma^{\oplus k}, W)$ be a disjoint union of Nakajima spaces. We then establish the following bijective correspondences.

Theorem 7.1. *There are four maps*

$$\mathcal{R}(V,W) \underset{\omega_1}{\overset{\theta_1}{\rightleftarrows}} \mathcal{M}(V,W) \underset{\omega_2}{\overset{\theta_2}{\rightleftarrows}} \tilde{\mathfrak{M}}^\tau_\Gamma(V,W), \qquad (66)$$

such that (θ_1, ω_1) and (θ_2, ω_2) are pairs of mutually inverse bijections.

Proof. The map θ_1 is given by the construction in Section 6.3 which assigns to an ideal M its DG-model $M \xrightarrow{r} \boldsymbol{L}$ (Theorem 6.5). Passing from M to an isomorphic module produces a DG-model quasi-isomorphic to $\boldsymbol{L}$, which by the uniqueness theorem implies that they are DG isomorphic. Therefore this map is well-defined.

The map ω_1 is defined simply by taking the cohomology of a DG-model which is by definition a projective module of B^τ such that $\phi_1^{-1}([M]) = [W] + [V]([L] - 2[W_0])$. Now it is clear that $\omega_1 \circ \theta_1 = Id_{\mathcal{R}}$, while $\theta_1 \circ \omega_1 = Id_{\mathcal{M}}$ follows again from the uniqueness theorem.

In Section 2.2 we have constructed Nakajima data from a DG-model. Since the action of $\boldsymbol{B}$ commutes with DG-module isomorphism we get a well-defined map θ_2 from $\mathcal{M}$ to $\tilde{\mathfrak{M}}_\Gamma^\tau(V,W)$.

In Section 2.3 we have shown how to get a DG-model from a point in $\tilde{\mathfrak{M}}_\Gamma^\tau(V,W)$. Now if we replace $(\bar{X}, \bar{Y}, \bar{i}, \bar{j})$ by equivalent data $(g\bar{X}g^{-1}, g\bar{Y}g^{-1}, g(\bar{i}), \bar{j}g^{-1})$, where $g \in \mathtt{GL}(V \oplus \mathbb{C}\Gamma^{\oplus k})$, then the functional λ remains the same, and hence so do the ideal J and the R-module L^0. On the other hand, the differential $d_{\boldsymbol{L}}$ gets changed to $gd_{\boldsymbol{L}}$. As a result, we obtain a DG-module $\tilde{\boldsymbol{L}}$ strictly isomorphic to $\boldsymbol{L}$, the isomorphism $\boldsymbol{L} \to \tilde{\boldsymbol{L}}$ being given by $(\mathtt{Id}_{L^0}, g)$. Thus, the construction of Section 2.3 yields a well-defined map $\omega_2 \colon \tilde{\mathfrak{M}}_\Gamma^\tau(V,W) \to \mathcal{M}$.

Now we have to show that $\theta_2 \circ \omega_2 = Id$ and $\omega_2 \circ \theta_2 = \mathtt{Id}_{\mathcal{M}}$. The first equality follows immediately from the constructions in Sections 2.2 and 2.3. The second equality follows from Theorem 6.4 since both $\boldsymbol{L}$ and $\omega_2 \circ \theta_2(\boldsymbol{L})$ have the same linear data $(\bar{X}, \bar{Y}, \bar{i}, \bar{j})$ and hence produce the same λ. □

8. G-equivariance

Let $G = Aut_\Gamma(R)$ be the group of Γ-equivariant automorphisms of the algebra $R = \mathbb{C}\langle x, y\rangle * \Gamma$ preserving the form $\omega = xy - yx \in R$. In this section we show that G acts naturally on each of the spaces $\mathcal{R}(V,W)$, $\mathfrak{M}_\Gamma^\tau(V,W)$ and $\mathcal{M}(V,W)$ and the bijections of Theorem (7.1) are equivariant with respect to these actions.

We start by describing the action of G on the space of ideals $\mathcal{R}(V,W)$. First, we observe that G maps to the group $Aut_\Gamma(B^\tau)$ of Γ-equivariant automorphisms of the algebra B^τ as B^τ is, by definition, a quotient of the algebra R. Now, $Aut_\Gamma(B^\tau)$ acts naturally on the category $\mathtt{Mod}(B^\tau)$ by twisting the structure of B^τ-modules by automorphisms: to be precise, for each $\sigma \in Aut_\Gamma(B^\tau)$ we have an auto-equivalence $\sigma_* \colon \mathtt{Mod}(B^\tau) \to \mathtt{Mod}(B^\tau)$, given by $\sigma_*(M) = M_{\sigma^{-1}}$. Clearly, the functors σ_* restrict to the subcategory $\mathtt{PMod}(B^\tau)$ of finitely generated projective B^τ-modules and their action preserves the rank of projective modules. Thus, for each $\sigma \in Aut_\Gamma(B^\tau)$ we have a bijection $\mathcal{R} \to \mathcal{R}$ induced by σ_*, and this defines an action of G on $\mathcal{R}$ via the group homomorphism $G \to Aut_\Gamma(B^\tau)$. We claim

Lemma 8.1. *The action of G on $\mathcal{R}$ defined above respects the stratification* (12).

Proof. The action of the group G on the category $\mathtt{PMod}(B^\tau)$ by exact additive functors yields a well-defined group homomorphism $G \to Aut_\Gamma(B^\tau) \to Aut(K_0(B^\tau))$; thus for each $\sigma \in G$, we have an abelian group automorphism

$$\sigma_* \colon K_0(B^\tau) \to K_0(B^\tau), \qquad [M] \mapsto [M_{\sigma^{-1}}].$$

Now, in the view of Lemma 3.2, if $M \in \mathcal{R}(V,W)$, its stable isomorphism class $[M]$

can be decomposed $K_0(B^\tau)$ as

$$[M] = [W \otimes_{\mathbb{C}\Gamma} B^\tau] + [(V \otimes L) \otimes_{\mathbb{C}\Gamma} B^\tau] - 2[V \otimes_{\mathbb{C}\Gamma} B^\tau]. \tag{67}$$

Since $\sigma \in G$ is Γ-equivariant, the corresponding algebra automorphism $\sigma\colon B^\tau \to B^\tau$ an isomorphism $B^\tau \cong (B^\tau)_{\sigma^{-1}}$ of $\mathbb{C}\Gamma$-B^τ-bimodules. Hence, with decomposition (67), we set at once that $[M_{\sigma^{-1}}] = [M]$ for every $M \in \mathcal{R}$ and $\sigma \in G$. This finishes the proof of the lemma. □

Thus, with Lemma 8.1, we can define an action of the group G on $\mathcal{R}(V,W)$ simply by restricting its natural action on $\mathcal{R}$.

Next, we define an action of G on $\mathcal{M}(V,W)$. Again, we start by observing that G maps naturally to the group $\mathtt{DGAut}_\Gamma(\boldsymbol{B})$ of Γ-equivariant automorphisms of the DG-algebra $\boldsymbol{B}$: in fact, given $\sigma \in G$, we define $\tilde{\sigma} \in \mathtt{DGAut}_\Gamma(\boldsymbol{B})$ on generators by $\tilde{\sigma}(x) = \sigma(x), \tilde{\sigma}(y) = \sigma(y), \tilde{\sigma}(\nu) = \nu$. Each $\tilde{\sigma} \in \mathtt{DGAut}_\Gamma(\boldsymbol{B})$ yields an auto-equivalence $\tilde{\sigma_*}\colon \mathtt{DGMod}(\boldsymbol{B}) \to \mathtt{DGMod}(\boldsymbol{B})$ by twisting the action of $\boldsymbol{B}$ by $\tilde{\sigma}^{-1}$. It is clear that such auto-equivalences preserve the class of DG-models, since each axiom of Definition 2 is stable under twisting by $\tilde{\sigma} \in \mathtt{DGAut}_\Gamma(\boldsymbol{B})$. Moreover, if $\boldsymbol{L} \in \mathcal{M}(V,W)$, then $H^0(\boldsymbol{L}) \in \mathcal{R}(V,W)$ and hence $\sigma_*(H^0(\boldsymbol{L})) \in \mathcal{R}(V,W) \Longrightarrow \tilde{\sigma}_*(\boldsymbol{L}) \in \mathcal{M}(V,W)$ by Lemma 8.1. Thus, the above action of G on DG-models preserves each stratum $\mathcal{M}(V,W)$, and it is obvious that the bijections θ_1 and ω_1 are G-equivariant with respect to this action and the action of G on $\mathcal{R}(V,W)$ defined in Lemma 8.1.

Finally, it remains to define an action of G on the quiver varieties $\mathfrak{M}^\tau_\Gamma(V,W)$. To this end, as in Section 2, we represent the points of $\mathfrak{M}^\tau_\Gamma(V,W)$ by quadruples of matrices $(\bar{X}, \bar{Y}, \bar{i}, \bar{j})$ and let $\sigma.(\bar{X}, \bar{Y}, \bar{i}, \bar{j}) := (\sigma^{-1}(\bar{X}), \sigma^{-1}(\bar{Y}), \bar{i}, \bar{j})$. Since σ is Γ-equivariant and preserves the form $\omega = xy - yx$, this action is well-defined: the quadruple $\sigma.(\bar{X}, \bar{Y}, \bar{i}, \bar{j})$ satisfies the relations (2.7) and (2.8). Moreover, it is clear that $\sigma.(\bar{X}, \bar{Y}, \bar{i}, \bar{j})$ are precisely the Nakajima data corresponding to the "twisted" DG-model $\tilde{\sigma_*}(\boldsymbol{L})$ if $(\bar{X}, \bar{Y}, \bar{i}, \bar{j})$ corresponds to $\boldsymbol{L}$. Thus, we have an action of G on $\mathfrak{M}^\tau_\Gamma(V,W)$ such that the bijection θ_2, ω_2 are G-equivariant. Summing up, we have established the following

Theorem 8.2. *The maps (θ_1, ω_1) and (θ_2, ω_2) are G-equivariant bijective correspondences.*

9. Invariant subrings of the Weyl algebra

In this section we look at the simplest example of the algebra O^τ corresponding to $\tau = 1$. It is well-known that in this case the algebra B^τ is isomorphic to the crossed product $A_1(\mathbb{C}) * \Gamma$ and O^τ to the subring A_1^Γ of invariants of the first Weyl algebra $A_1(\mathbb{C}) = \mathbb{C}\langle x, y\rangle/(xy - yx - 1)$ under the action $x \mapsto \epsilon x$ and $y \mapsto \epsilon^{-1} y$. In fact,

$$B^\tau \cong \mathbb{C}\langle x, y\rangle * \Gamma/(xy - yx - 1) \cong (\mathbb{C}\langle x, y\rangle/(xy - yx - 1)) * \Gamma,$$
$$O^\tau \cong A_1^\Gamma(\mathbb{C}).$$

In the case of $\tau = 1$, from the relation (8), for any point $(\bar{X}, \bar{Y}, \bar{i}, \bar{j})$ of the Nakajima variety $\mathfrak{M}^\tau_\Gamma(V,W)$ we have

$$\bar{X}\bar{Y} - \bar{Y}\bar{X} + I = \bar{i}\bar{j}, \tag{68}$$

which is exactly the Calogero-Moser relation. Now, a pair $(\bar{X}, \bar{Y})$ satisfying this relation does not have common invariant subspace (see [**24**, Lemma 1.3]); hence condition (ii) of (6) in the definition of the Nakajima variety $\mathfrak{M}^{\tau}_{\Gamma}(V, W)$ is redundant. Thus, in the case of $\tau = 1$, the Nakajima variety is the space of equivalence classes of quadruples $(\bar{X}, \bar{Y}, \bar{i}, \bar{j})$ satisfying (68) and the following equations

$$\bar{X}\,\bar{G} = \epsilon(g)\,\bar{G}\,\bar{X}, \qquad \bar{Y}\,\bar{G} = \epsilon^{-1}(g)\,\bar{G}\,\bar{Y}. \tag{69}$$

Now we will give another description of the Nakajima variety. For this we remind the reader that $\{W_0, W_1, \dots, W_{m-1}\}$ is the complete set of irreducible Γ-modules such that the character of W_i is ϵ^i. Then, if $V \cong \bigoplus_{i=0}^{m-1} V_i \otimes W_i$ is the irreducible Γ-decomposition of V, we have

$$\mathtt{Hom}_{\Gamma}(V, V\otimes\epsilon) \cong \bigoplus_{i=0}^{m-1} \mathtt{Hom}(V_i, V_{i-1}), \qquad \mathtt{Hom}_{\Gamma}(V, V\otimes\epsilon^{-1}) \cong \bigoplus_{i=0}^{m-1} \mathtt{Hom}(V_i, V_{i+1}),$$

$$\mathtt{Hom}_{\Gamma}(W_n, V) \cong \mathtt{Hom}(\mathbb{C}, V_n), \qquad \mathtt{Hom}_{\Gamma}(V, W_n) \cong \mathtt{Hom}(V_n, \mathbb{C}).$$

We now introduce the following algebraic variety (see [**20**]):

$$\begin{aligned} D^n_{(k_0,\dots,k_{m-1})} := \Big\{ \Big(\bar{X}_0, \bar{X}_1, \dots, \bar{X}_{m-1}; \bar{Y}_0, \bar{Y}_1, \dots, \bar{Y}_{m-1}, \bar{i}_n, \bar{j}_n \Big) \Big| \qquad & \qquad (70)\\ \times \bar{X}_i \in \mathtt{Hom}(V_{i+1}, V_i),\ \bar{Y}_i \in \mathtt{Hom}(V_i, V_{i+1}), & \\ \times \bar{i}_n \in \mathtt{Hom}(\mathbb{C}, V_n)\,,\ \bar{j}_n \in \mathtt{Hom}(V_n, \mathbb{C}), & \\ \times \bar{X}_i \bar{Y}_i - \bar{Y}_{i-1}\bar{X}_{i-1} + Id_{k_i} = 0,\ i \neq n, & \\ \times \bar{X}_n \bar{Y}_n - \bar{Y}_{n-1}\bar{X}_{n-1} + Id_{k_n} = \bar{i}_n \bar{j}_n \Big\} \Big/\!\!\Big/ \prod_i GL(V_i), & \end{aligned}$$

where $k_i := \dim_{\mathbb{C}}(V_i)$. Then, due to equations (68) and (69), there is a well-defined map

$$\psi\colon \mathfrak{M}^{\tau}_{\Gamma}(V, W_n) \longrightarrow D^n_{(k_0,\dots,k_{m-1})},$$

$$\bar{X} \mapsto (\bar{X}_0, \bar{X}_1, \dots, \bar{X}_{m-1}),\ \ \bar{Y} \mapsto (\bar{Y}_0, \bar{Y}_1, \dots, \bar{Y}_{m-1}),\ \ \bar{i} \mapsto \bar{i}_n,\ \bar{j} \mapsto \bar{j}_n.$$

In fact one can easily prove the following result:

Theorem 9.1. *The map ψ is an isomorphism of algebraic varieties with the inverse map defined by*

$$(\bar{X}_0, \bar{X}_1, \dots, \bar{X}_{m-1}) \mapsto \bar{X},\ \ (\bar{Y}_0, \bar{Y}_1, \dots, \bar{Y}_{m-1}) \mapsto \bar{Y},\ \ \bar{i}_n \mapsto \bar{i},\ \bar{j}_n \mapsto \bar{j},$$

where $\bar{X}$ and $\bar{Y}$ are the following matrices:

$$\bar{X} := \begin{pmatrix} 0 & \bar{X}_0 & 0 & \dots & 0 \\ 0 & 0 & \bar{X}_1 & \dots & 0 \\ 0 & 0 & 0 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \bar{X}_{m-2} \\ \bar{X}_{m-1} & 0 & \dots & 0 & 0 \end{pmatrix}, \quad \bar{Y} := \begin{pmatrix} 0 & 0 & 0 & \dots & \bar{Y}_{m-1} \\ \bar{Y}_0 & 0 & 0 & \dots & 0 \\ 0 & \bar{Y}_1 & 0 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & \dots & \bar{Y}_{m-2} & 0 \end{pmatrix}.$$

Moreover,

$$\dim_{\mathbb{C}} D^n_{(k_0,k_1)} = 2(k_n - (k_0 - k_1)^2), \text{ for } m = 2, \tag{71}$$

and

$$\dim_{\mathbb{C}} D^n_{(k_0,\dots,k_{m-1})} = 2\bigg(k_n - \bigg(\sum_{i=0}^{m-1} k_i^2 - \sum_{i=0}^{m-1} k_i k_{i+1} - k_0 k_{m-1}\bigg)\bigg), \text{ for } m > 2. \tag{72}$$

Let us define the following set

$$N_n = \bigg\{(k_0, k_1, \dots, k_{m-1}) \in \mathbb{N}^m \,\bigg|\, k_n - \bigg(\sum_{i=0}^{m-1} k_i^2 - \sum_{i=0}^{m-1} k_i k_{i+1} - k_0 k_{m-1}\bigg) \geqslant 0\bigg\}.$$

From (72) we can see that this set consists exactly of those points of $\mathbb{N}^m$ for which the corresponding Nakajima variety $D^n_{(k_0,\dots,k_{m-1})}$ is nonempty. With this notation we can restate Theorem 2.5 as follows.

Corollary 9.2. *The set $\mathcal{R}(A_1^\Gamma)$ of isomorphism classes of ideals of A_1^Γ is in bijection with the union of algebraic varieties*

$$\bigsqcup_{n=0}^{m-1} \bigsqcup_{(k_0,\dots,k_{m-1})\in N_n} D^n_{(k_0,\dots,k_{m-1})}.$$

In the case $m = 2$, the varieties $D^n_{(k_0,\dots,k_{m-1})}$ have been introduced recently in [**18**] (see *loc. cit.*, Theorem 3) to classify the ideals of the $\mathbb{Z}_2$-invariant subring of $A_1(\mathbb{C})$. Thus our Corollary 9.2 may be viewed as a generalization of this description to the case of an arbitrary cyclic group $\mathbb{Z}_m$.

Appendix A. A_∞-morphisms of *DG*-modules

The *DG*-algebra $\boldsymbol{B}$, regarded as an A_∞-algebra, has only two nonzero structure maps $m_1^{\boldsymbol{B}} := d_{\boldsymbol{B}}$ and $m_2^{\boldsymbol{B}}$, the usual associative multiplication in $\boldsymbol{B}$. Any *DG*-module $\boldsymbol{L}$ over $\boldsymbol{B}$, viewed as an A_∞-module, also has only two nontrivial structure maps, which satisfy the Leibnitz rule:

$$m_1^{\boldsymbol{L}} m_2^{\boldsymbol{L}} = m_2^{\boldsymbol{L}}(m_1^{\boldsymbol{L}} \otimes 1) + m_2^{\boldsymbol{L}}(1 \otimes d_{\boldsymbol{B}}).$$

Now we recall the definition of morphisms of A_∞-modules (see [**14**]).

Definition A.1. A *morphism of A_∞-modules* $f\colon \boldsymbol{L} \to \boldsymbol{E}$ is a sequence of graded morphisms

$$f_n\colon \boldsymbol{L} \otimes \boldsymbol{B}^{\otimes n-1} \to \boldsymbol{E} \tag{73}$$

of degree $1 - n$ such that for each $n \geqslant 1$, we have

$$\sum (-1)^{r+st} f_u \circ (1^{\otimes r} \otimes m_s \otimes 1^{\otimes t}) = \sum (-1)^{(r+1)s} m_u \circ (f_r \otimes 1^{\otimes s}), \tag{74}$$

where the left-hand sum is taken over all decompositions $n = r + s + t, r, t \geqslant 0$, $s \geqslant 1$ and we put $u = r + 1 + t$. The right-hand sum is taken over all decompositions $n = r + s$, $r \geqslant 1$, $s \geqslant 0$ and we put $u = 1 + s$.

Lemma A.2. *Let $\boldsymbol{L}$ and $\boldsymbol{E}$ be DG-modules over $\boldsymbol{B}$, $\boldsymbol{L}$ having nonzero components only in degree* 0 *and* 1 *and $\boldsymbol{E}$ positively graded: $\boldsymbol{L} = L^0 \oplus L^1$ and $\boldsymbol{E} = E^0 \oplus E^1 \oplus E^2 \dots$.*

(*a*) *Any A_∞-morphism $f\colon \boldsymbol{L} \to \boldsymbol{E}$ is determined by two components (f_1, f_2) satisfying the relations:*

$$m_1^{\boldsymbol{E}} f_1^0 = f_1^1 m_1^{\boldsymbol{L}}, \tag{75}$$

$$f_1^0(m_2^{\boldsymbol{L}}(u,a)) - m_2^{\boldsymbol{E}}(f_1^0(u),a) = f_2(m_1^{\boldsymbol{L}}(u),a), \quad \forall\, u \in L^0, a \in R, \tag{76}$$

$$f_1^1(m_2^{\boldsymbol{L}}(v,a)) - m_2^{\boldsymbol{E}}(f_1^1(v),a) = m_1^{\boldsymbol{E}} f_2(v,a), \quad \forall\, v \in L^1, a \in R, \tag{77}$$

$$f_1^0(m_2^{\boldsymbol{L}}(v,c)) - m_2^{\boldsymbol{E}}(f_1^1(v),c) = -f_2(v,d_{\boldsymbol{B}}c), \quad \forall\, v \in L^1, c \in I, \tag{78}$$

$$f_2(v,ab) = m_2^{\boldsymbol{E}}(f_2(v,a),b) + f_2(m_2^{\boldsymbol{L}}(v,a),b), \quad \forall\, u \in L^0, a,b \in R. \tag{79}$$

(*b*) *If $m_1^{\boldsymbol{L}}$ is surjective then equations* (77)*–*(79) *are formal consequences of* (75) *and* (76)*.*

Proof. Relation (75) follows easily from (74) for $n = 1$. For $n = 2$ we get the equation

$$-f_2(1 \otimes d_{\boldsymbol{B}}) + f_1 \circ m_2^{\boldsymbol{L}} - f_2(m_1^{\boldsymbol{L}} \otimes 1) = m_2^{\boldsymbol{E}}(f_1 \otimes 1) + m_1^{\boldsymbol{E}} \circ f_2. \tag{80}$$

Since $\mathtt{deg}(f_2) = -1$ it has only one component $f_2\colon L^1 \otimes R \to E^0$ and therefore relations (76)–(78) are consequences of (80). For $n = 3$, equation (74) has the following form:

$$\begin{aligned} f_3(1 \otimes 1 \otimes d_{\boldsymbol{B}}) &+ f_3(1 \otimes d_{\boldsymbol{B}} \otimes 1) + f_3(m_1^{\boldsymbol{L}} \otimes 1 \otimes 1) \\ &- f_2(1 \otimes m_2^{\boldsymbol{B}}) + f_2(m_2^{\boldsymbol{L}} \otimes 1) + f_1 \circ m_3^{\boldsymbol{L}} \\ &= m_3^{\boldsymbol{E}}(f_1 \otimes 1 \otimes 1) - m_2^{\boldsymbol{E}}(f_2 \otimes 1) + m_1^{\boldsymbol{E}} \circ f_3. \end{aligned} \tag{81}$$

By the degree argument we can conclude that $f_n = 0$ for $n \geqslant 3$. Now since both $\boldsymbol{L}$ and $\boldsymbol{E}$ are *DG*-modules we have $m_3^{\boldsymbol{L}} = m_3^{\boldsymbol{E}} = 0$. Equation (81) can be simplified

$$-f_2(1 \otimes m_2^{\boldsymbol{B}}) + f_2(m_2^{\boldsymbol{L}} \otimes 1) = m_2^{\boldsymbol{E}}(f_2 \otimes 1) \tag{82}$$

which is equivalent to (78)–(79).

To prove part (*b*) we first apply m_1 to equation (76). Then using (75) and R-linearity of m_1 (i.e. $d_{\boldsymbol{B}}(a) = 0$ for any $a \in R$) we have

$$\begin{aligned} &m_1(f_1^0(m_2^{\boldsymbol{L}}(u,a))) - m_1(m_2^{\boldsymbol{E}}(f_1^0(u),a)) \\ &= f_1^1(m_2^{\boldsymbol{L}}(m_1(u),a)) - m_2^{\boldsymbol{E}}(f_1^1(m_1(u)),a) \\ &= m_1^{\boldsymbol{E}} f_2(m_1(u),a). \end{aligned}$$

Since m_1 is surjective this implies (77). Now let $a = d_{\boldsymbol{L}}c$ and $v = m_1^{\boldsymbol{L}}(u)$ in (76). Then

$$f_1^0(m_2^{\boldsymbol{L}}(u,d_{\boldsymbol{B}}c)) - m_2^{\boldsymbol{E}}(f_1^0(u),d_{\boldsymbol{B}}c) = f_2(m_1^{\boldsymbol{L}}(u),d_{\boldsymbol{B}}c). \tag{83}$$

Since $m_2^{\boldsymbol{L}}(u,c) = 0$ for all $c \in I$ and $u \in L^0$ by the Leibnitz rule we get

$$m_2^{\boldsymbol{L}}(m_1^{\boldsymbol{L}}(u),c) = -m_2^{\boldsymbol{L}}(u,d_{\boldsymbol{B}}c).$$

Similarly one can show that $m_2^{\boldsymbol{E}}(f_1^0(u),d_{\boldsymbol{B}}c) = -m_2^{\boldsymbol{E}}(f_1^1(v),c)$. By plugging the last two relations into (83) we obtain (78).

Now we will show that (76) implies (79). Let $v = m_1^{\boldsymbol{L}}(u)$; then from (76) we have

$$f_2(m_1^{\boldsymbol{L}}(u), ab) = f_1^0(m_2^{\boldsymbol{L}}(u, ab)) - m_2^{\boldsymbol{E}}(f_1^0(u), ab), \tag{84}$$

$$m_2^{\boldsymbol{E}}(f_2(m_1^{\boldsymbol{L}}(u), a), b) = m_2^{\boldsymbol{E}}(f_1^0(m_2^{\boldsymbol{L}}(u, a)), b) - m_2^{\boldsymbol{E}}(f_1^0(u), ab), \tag{85}$$

$$f_2(m_2^{\boldsymbol{L}}(m_1^{\boldsymbol{L}}(u), a), b) = f_1^0(m_2^{\boldsymbol{L}}(u, ab)) - m_2^{\boldsymbol{E}}(f_1^0(m_2^{\boldsymbol{L}}(u, a)), b). \tag{86}$$

Adding now (85) and (86) and using (84) we easily derive (79). □

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