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HOMOTOPY TYPES OF TRUNCATED PROJECTIVE RESOLUTIONS

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Abstract

We work over an arbitrary ring R. Given two truncated projective resolutions of equal length for the same module, we consider their underlying chain complexes. We show they may be stabilized by projective modules to obtain a pair of complexes of the same homotopy type.

1. Introduction

Truncated projective resolutions are of interest in both algebraic geometry and algebraic topology. If the modules in a resolution of length n are assumed to be free, then the n^{th} homology group is the n^{th} syzygy of the module being resolved. The minimal possible dimensions of the modules in such resolutions were of interest to mathematicians such as Hilbert and Milnor (see [**2**]).

In algebraic topology, truncated projective resolutions arise as the algebraic complexes associated to $(n-1)$-connected universal covers of CW-complexes of dimension n. Of particular interest is the case $n = 2$, as classification of the homotopy types of these truncated resolutions is closely related to Wall's D2 problem (see the introduction to [**1**]).

Given two truncated projective resolutions of the same module (of equal length), their final modules may be stabilized to produce homotopy equivalent algebraic complexes. This is a generalization of Schanuel's lemma which merely equates the final homology groups. The work of mathematicians such as Milnor, Whitehead and Wall suggests they were familiar with this basic homological result. Indeed, Wall's obstruction is suggestive of the modules required to stabilize the complexes (see [**3**], §3).

Given one truncated projective resolution of a module, this result provides a handle on all other possible truncated projective resolutions of the same length. Our purpose in this paper is to provide a simple proof of the result by explicitly constructing the desired homotopy equivalence between the two stabilized algebraic complexes.

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Formally, let R be a ring with identity and let M be a module over R. We assume a right action on all modules. Suppose we have exact sequences:

$$P_n \xrightarrow{\partial_n} P_{n-1} \xrightarrow{\partial_{n-1}} \cdots \xrightarrow{\partial_2} P_1 \xrightarrow{\partial_1} P_0 \overset{\epsilon}{\dashrightarrow} M \dashrightarrow 0$$

and

$$Q_n \xrightarrow{\partial'_n} Q_{n-1} \xrightarrow{\partial'_{n-1}} \cdots \xrightarrow{\partial'_2} Q_1 \xrightarrow{\partial'_1} Q_0 \overset{\epsilon'}{\dashrightarrow} M \dashrightarrow 0$$

with the P_i and Q_i all projective modules. Our main result is:

Theorem 1.1. *The complexes*

$$P_n \oplus S_n \xrightarrow{\partial_n \oplus 0} P_{n-1} \xrightarrow{\partial_{n-1}} \cdots \xrightarrow{\partial_2} P_1 \xrightarrow{\partial_1} P_0 \tag{1}$$

and

$$Q_n \oplus T_n \xrightarrow{\partial'_n \oplus 0} Q_{n-1} \xrightarrow{\partial'_{n-1}} \cdots \xrightarrow{\partial'_2} Q_1 \xrightarrow{\partial'_1} Q_0 \tag{2}$$

are chain homotopy equivalent, where the projective modules T_i, S_i are defined inductively by $T_0 \cong P_0$, $S_0 \cong Q_0$ and for $i = 1, \ldots, n$, $T_i \cong S_{i-1} \oplus P_i$, $S_i \cong T_{i-1} \oplus Q_i$.

Given maps $f\colon A \to C$, $g\colon B \to C$, the notation $f \oplus g$ will always be used to denote the map $f \oplus g\colon A \oplus B \to C$ given by $f \oplus g\colon (a, b) \mapsto f(a) + g(b)$.

2. Construction of chain homotopy equivalence

For each $i \in 1, \ldots, n$ we have natural inclusions of summands:

$$\iota_i\colon P_i \to T_i (\cong S_{i-1} \oplus P_i) \qquad \iota'_i\colon Q_i \to S_i (\cong T_{i-1} \oplus Q_i).$$

Let $\iota_0\colon P_0 \to T_0$ and $\iota'_0\colon Q_0 \to S_0$ both be the identity map.

For $i = 1, \ldots, n$, we define

$$\begin{gathered} \delta_i\colon T_i (\cong P_i \oplus S_{i-1}) \to T_{i-1} \oplus S_{i-1} \\ \text{and } \delta'_i\colon S_i (\cong Q_i \oplus T_{i-1}) \to S_{i-1} \oplus T_{i-1} \end{gathered}$$

by

$$\delta_i = \begin{pmatrix} \iota_{i-1}\partial_i & 0 \\ 0 & 1 \end{pmatrix} \qquad \delta'_i = \begin{pmatrix} \iota'_{i-1}\partial'_i & 0 \\ 0 & 1 \end{pmatrix}.$$

For $r = 0, \ldots, n-1$, let $\mathcal{C}_r$ denote the chain complex

$$P_n \oplus S_n \xrightarrow{\partial_n \oplus 0} \cdots \xrightarrow{\partial_{r+2}} P_{r+1} \xrightarrow{\iota_r \partial_{r+1}} T_r \xrightarrow{\delta_r} T_{r-1} \oplus S_{r-1} \xrightarrow{\delta_{r-1} \oplus 0} \cdots \xrightarrow{\delta_1 \oplus 0} T_0 \oplus S_0.$$

Also let $\mathcal{C}_n$ denote the chain complex

$$T_n \oplus S_n \xrightarrow{\delta_n \oplus 0} T_{n-1} \oplus S_{n-1} \xrightarrow{\delta_{n-1} \oplus 0} \cdots \xrightarrow{\delta_2 \oplus 0} T_1 \oplus S_1 \xrightarrow{\delta_1 \oplus 0} T_0 \oplus S_0.$$

Clearly $\mathcal{C}_0$ is the chain complex (1). For $r = 0, \ldots n-1$, the chain complex $\mathcal{C}_{r+1}$

is obtained from $\mathcal{C}_r$ by replacing

$$\xrightarrow{\partial_{r+2}} P_{r+1} \xrightarrow{\iota_r \partial_{r+1}} T_r \xrightarrow{\delta_r}$$

with

$$\xrightarrow{\iota_{r+1}\partial_{r+2}} P_{r+1} \oplus S_r \xrightarrow{\delta_{r+1}} T_r \oplus S_r \xrightarrow{\delta_r \oplus 0} .$$

This is a simple homotopy equivalence, so $\mathcal{C}_{r+1}$ is chain homotopy equivalent to $\mathcal{C}_r$.

Similarly, for $r = 0, \ldots, n-1$, let $\mathcal{D}_r$ denote the chain complex

$$Q_n \oplus T_n \xrightarrow{\partial'_n \oplus 0} \cdots \xrightarrow{\partial'_{r+2}} Q_{r+1} \xrightarrow{\iota'_r \partial'_{r+1}} S_r \xrightarrow{\delta'_r} S_{r-1} \oplus T_{r-1} \xrightarrow{\delta'_{r-1} \oplus 0} \cdots \xrightarrow{\delta'_1 \oplus 0} S_0 \oplus T_0.$$

Again let $\mathcal{D}_n$ denote the chain complex

$$S_n \oplus T_n \xrightarrow{\delta'_n \oplus 0} S_{n-1} \oplus T_{n-1} \xrightarrow{\delta'_{n-1} \oplus 0} \cdots \xrightarrow{\delta'_2 \oplus 0} S_1 \oplus T_1 \xrightarrow{\delta'_1 \oplus 0} S_0 \oplus T_0.$$

Clearly $\mathcal{D}_0$ is the chain complex (2). As before, for $r = 0, \ldots n-1$, the chain complex $\mathcal{D}_{r+1}$ is chain homotopy equivalent to $\mathcal{D}_r$.

We have (1) chain homotopy equivalent to $\mathcal{C}_n$ and (2) chain homotopy equivalent to $\mathcal{D}_n$. We complete the proof of the theorem by showing that $\mathcal{C}_n$ is chain isomorphic to $\mathcal{D}_n$.

Lemma 2.1. *There exist inverse pairs of maps h_i, k_i making the following diagram commute:*

$$\begin{array}{ccccccccccccc}
T_n \oplus S_n & \xrightarrow{\delta_n \oplus 0} & T_{n-1} \oplus S_{n-1} & \xrightarrow{\delta_{n-1} \oplus 0} & \cdots & \xrightarrow{\delta_2 \oplus 0} & T_1 \oplus S_1 & \xrightarrow{\delta_1 \oplus 0} & T_0 \oplus S_0 & \dashrightarrow^{\epsilon \oplus 0} & M & \dashrightarrow & 0 \\
\downarrow h_n & & \downarrow h_{n-1} & & & & \downarrow h_1 & & \downarrow h_0 & & \downarrow 1 & & \\
S_n \oplus T_n & \xrightarrow{\delta'_n \oplus 0} & S_{n-1} \oplus T_{n-1} & \xrightarrow{\delta'_{n-1} \oplus 0} & \cdots & \xrightarrow{\delta'_2 \oplus 0} & S_1 \oplus T_1 & \xrightarrow{\delta'_1 \oplus 0} & S_0 \oplus T_0 & \dashrightarrow^{\epsilon' \oplus 0} & M & \dashrightarrow & 0 \\
\downarrow k_n & & \downarrow k_{n-1} & & & & \downarrow k_1 & & \downarrow k_0 & & \downarrow 1 & & \\
T_n \oplus S_n & \xrightarrow{\delta_n \oplus 0} & T_{n-1} \oplus S_{n-1} & \xrightarrow{\delta_{n-1} \oplus 0} & \cdots & \xrightarrow{\delta_2 \oplus 0} & T_1 \oplus S_1 & \xrightarrow{\delta_1 \oplus 0} & T_0 \oplus S_0 & \dashrightarrow^{\epsilon \oplus 0} & M & \dashrightarrow & 0.
\end{array}$$

Proof. As T_0, S_0 are projective, we may pick f_0, g_0 so that the following diagrams commute:

$$\begin{array}{ccc}
T_0 & \xrightarrow{\epsilon} & M \\
\downarrow f_0 & & \downarrow 1 \\
S_0 & \xrightarrow{\epsilon'} & M
\end{array}
\qquad\qquad
\begin{array}{ccc}
T_0 & \xrightarrow{\epsilon} & M \\
\uparrow g_0 & & \uparrow 1 \\
S_0 & \xrightarrow{\epsilon'} & M.
\end{array}
\tag{3}$$

Define $h_0 \colon T_0 \oplus S_0 \to S_0 \oplus T_0$ and $k_0 \colon S_0 \oplus T_0 \to T_0 \oplus S_0$ by

$$h_0 = \begin{pmatrix} f_0 & 1 - f_0 g_0 \\ 1 & -g_0 \end{pmatrix} \qquad\qquad k_0 = \begin{pmatrix} g_0 & 1 - g_0 f_0 \\ 1 & -f_0 \end{pmatrix}.$$

Direct calculation shows that $h_0 k_0 = 1$ and $k_0 h_0 = 1$.

Also from commutativity of (3), we deduce:

$$(\epsilon' \quad 0)\begin{pmatrix} f_0 & 1-f_0g_0 \\ 1 & -g_0 \end{pmatrix} = (\epsilon' f_0 \quad \epsilon'(1-f_0g_0)) = (\epsilon \quad 0)$$

and

$$(\epsilon \quad 0)\begin{pmatrix} g_0 & 1-g_0f_0 \\ 1 & -f_0 \end{pmatrix} = (\epsilon g_0 \quad \epsilon(1-g_0f_0)) = (\epsilon' \quad 0).$$

Hence the following diagrams commute:

$$\begin{array}{ccc} T_0 \oplus S_0 & \xrightarrow{\epsilon\oplus 0} & M \\ \downarrow h_0 & & \downarrow 1 \\ S_0 \oplus T_0 & \xrightarrow{\epsilon'\oplus 0} & M \end{array} \qquad \begin{array}{ccc} T_0 \oplus S_0 & \xrightarrow{\epsilon\oplus 0} & M \\ \uparrow k_0 & & \uparrow 1 \\ S_0 \oplus T_0 & \xrightarrow{\epsilon'\oplus 0} & M. \end{array}$$

Now suppose that for some $0 < i \leqslant n$, we have defined $h_j\colon T_j \oplus S_j \to S_j \oplus T_j$ and $k_j\colon S_j \oplus T_j \to T_j \oplus S_j$ for $j = 0, \dots, i-1$, so that for each j, we have $h_jk_j = 1$ and $k_jh_j = 1$. We proceed by induction.

As before, pick f_i, g_i so that the following diagrams commute:

$$\begin{array}{ccc} T_i & \xrightarrow{\delta_i} & T_{i-1} \oplus S_{i-1} \\ \downarrow f_i & & \downarrow h_{i-1} \\ S_i & \xrightarrow{\delta'_i} & S_{i-1} \oplus T_{i-1} \end{array} \qquad \begin{array}{ccc} T_i & \xrightarrow{\delta_i} & T_{i-1} \oplus S_{i-1} \\ \uparrow g_i & & \uparrow k_{i-1} \\ S_i & \xrightarrow{\delta'_i} & S_{i-1} \oplus T_{i-1}. \end{array} \tag{4}$$

Define $h_i\colon T_i \oplus S_i \to S_i \oplus T_i$ and $k_i\colon S_i \oplus T_i \to T_i \oplus S_i$ by

$$h_i = \begin{pmatrix} f_i & 1-f_ig_i \\ 1 & -g_i \end{pmatrix} \qquad k_i = \begin{pmatrix} g_i & 1-g_if_i \\ 1 & -f_i \end{pmatrix}.$$

Direct calculation shows that $h_ik_i = 1$ and $k_ih_i = 1$.

Recall $h_{i-1}k_{i-1} = 1$ and $k_{i-1}h_{i-1} = 1$. From commutativity of (4) we deduce:

$$(\delta'_i \quad 0)\begin{pmatrix} f_i & 1-f_ig_i \\ 1 & -g_i \end{pmatrix} = (\delta'_i f_i \quad \delta'_i(1-f_ig_i)) = h_{i-1}(\delta_i \quad 0)$$

and

$$(\delta_i \quad 0)\begin{pmatrix} g_i & 1-g_if_i \\ 1 & -f_i \end{pmatrix} = (\delta_i g_i \quad \delta_i(1-g_if_i)) = k_{i-1}(\delta'_i \quad 0).$$

Hence the following diagrams commute:

$$\begin{array}{ccc} T_i \oplus S_i & \xrightarrow{\delta_i\oplus 0} & T_{i-1} \oplus S_{i-1} \\ \downarrow h_i & & \downarrow h_{i-1} \\ S_i \oplus T_i & \xrightarrow{\delta'_i\oplus 0} & S_{i-1} \oplus T_{i-1} \end{array} \qquad \begin{array}{ccc} T_i \oplus S_i & \xrightarrow{\delta_i\oplus 0} & T_{i-1} \oplus S_{i-1} \\ \uparrow k_i & & \uparrow k_{i-1} \\ S_i \oplus T_i & \xrightarrow{\delta'_i\oplus 0} & S_{i-1} \oplus T_{i-1}. \end{array}$$

So we may construct the h_i, k_i as required. □

We know the h_i, $i = 0, \dots, n$ constitute a chain map $h\colon \mathcal{C}_n \to \mathcal{D}_n$. Also the k_i constitute a chain map $k\colon \mathcal{D}_n \to \mathcal{C}_n$. As h and k are mutually inverse we have that $\mathcal{C}_n$ and $\mathcal{D}_n$ are chain isomorphic. Hence (1) and (2) are chain homotopy equivalent as required.

3. Injective Resolutions

Finally we note that dual arguments may be used in the same way to prove the dual result:

Theorem 3.1. *Let* (I_r, ∂_r) *and* (J_r, ∂'_r) *be injective resolutions for some module* M*, truncated after the* n^{th} *terms (so* $M \cong \mathrm{Ker}(\partial_0 \colon I_0 \to I_1) \cong \mathrm{Ker}(\partial'_0 \colon J_0 \to J_1)$*). Then stabilizing the final modules,* I_n *and* J_n*, with the appropriate injective modules results in chain homotopy equivalent complexes.*

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