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COFIBRANT OBJECTS AMONG HIGHER-DIMENSIONAL CATEGORIES

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Abstract

We define a notion of cofibration among ∞-categories and show that the cofibrant objects are exactly the free ones, that is, those generated by polygraphs.

1. Introduction

Polygraphs [**3, 4**], or computads [**11, 12**] are structured systems of generators for ∞-categories, extending the familiar notion of presentation by generators and relations beyond monoids or groups, and have recently proved extremely well-adapted to higher-dimensional rewriting [**6, 7**].

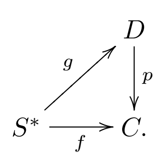
They also lead to a simple definition of a homology for ∞-categories [**8, 10**], based on the following construction: a *polygraphic resolution* of an ∞-category C is a pair (S, p) where

- S is a polygraph, generating a free ∞-category S^*;
- the morphism $p\colon S^* \to C$ is a trivial fibration (see 6.1).

S gives rise to an chain complex $\mathbb{Z}S$, whose homology only depends on C, so that we may define a polygraphic homology by

$$\mathrm{H}_*^{\mathrm{pol}}(C) =_{\mathrm{def}} \mathrm{H}_*(\mathbb{Z}S).$$

Here the main property of free ∞-categories is that they are *cofibrant.* In other words, given a polygraph S and a trivial fibration $p\colon D \to C$, any morphism $f\colon S^* \to C$ lifts to a morphism $g\colon S^* \to C$:



The main purpose of the present work is to prove the converse, namely that all cofibrant ∞-categories are freely generated by polygraphs, thus establishing a simple, abstract characterization of the free objects, otherwise defined by a rather complex inductive construction.

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We first review the basic categories in play (Sections 2 to 4): **Glob**, $\mathbf{Cat}_\infty$ and **Pol** stand respectively for the category of globular sets, ∞-categories and polygraphs. Section 5 investigates the technical notion of context, which we need later on. Section 6 defines trivial fibrations, cofibrations, and shows that the free ∞-categories are cofibrant. We then turn to the main result, proving that cofibrant ∞-categories are free (Section 7). Here the keypoint is that the full subcategory of $\mathbf{Cat}_\infty$ whose objects are freely generated by polygraphs is Cauchy-complete, which means that its idempotent endomorphisms split. The Cauchy-completeness argument is the essential part of this work and will be easier to follow if we keep in mind the simpler case of monoids: thus, let **Mon** denote the category of monoids, and **Fmon** the full subcategory of **Mon** whose objects are the free monoids. A submonoid of a free monoid is not necessarily free itself: consider for example the submonoid of $(\mathbb{N}, +)$ generated by $\{2, 3\}$. However, if $M = S^*$ is the free monoid on the alphabet S and $h\colon M \to M$ is an *idempotent* endomorphism of M, then the submonoid $\mathrm{Fix}(h) = \{m \in M \mid h(m) = m\}$ of fixpoints of h *is* free, which easily leads to a splitting of h in **Fmon**, hence to the fact that **Fmon** is Cauchy-complete. The idea is to find a set of generators of $\mathrm{Fix}(h)$ without non-trivial relations in M. A simple way to build such a set is by considering the subset $S_1 \subset S$ of those $s \in S$ such that $h(s) = usv$ where $h(u) = h(v) = 1$. Then we define $T = \{h(s) \mid s \in S_1\}$. It turns out that the obvious inclusion $T^* \to M$ sends T^* isomorphically to $\mathrm{Fix}(h)$, as shown by the existence of a retraction $M \to T^*$. Now the same ideas carry into higher dimensions, with ∞-categories instead of monoids and polygraphs instead of generating sets, but the general case involves additional technicalities, due to the presence of higher-dimensional compositions.

Let us finally point out that our cofibrant ∞-categories are actually the cofibrant objects in a Quillen model structure on $\mathbf{Cat}_\infty$ recently discovered by Yves Lafont, Krzysztof Worytkiewicz and the author [**9**].

2. Globular sets

Let **O** be the small category defined as follows:

- the objects of **O** are integers $0, 1, \ldots$;
- the arrows are generated by composition of $\mathrm{s}_n, \mathrm{t}_n\colon n \to n{+}1$, $n \in \mathbb{N}$ subject to the following equations

$$\mathrm{s}_{n+1} \circ \mathrm{s}_n = \mathrm{t}_{n+1} \circ \mathrm{s}_n,$$
$$\mathrm{s}_{n+1} \circ \mathrm{t}_n = \mathrm{t}_{n+1} \circ \mathrm{t}_n.$$

As a consequence, $\mathbf{O}(m, n)$ has exactly two elements if $m < n$, namely $\mathrm{s}_{m,n} = \mathrm{s}_{n-1} \circ \cdots \circ \mathrm{s}_m$ and $\mathrm{t}_{m,n} = \mathrm{t}_{n-1} \circ \cdots \circ \mathrm{t}_m$. $\mathbf{O}(m, n) = \emptyset$ if $m > n$, and contains the unique element id_m if $m = n$.

Definition 2.1. A *globular set* is a presheaf on **O**.

In other words, a globular set is a functor from $\mathbf{O}^{\mathrm{op}}$ to **Sets**. Globular sets and natural transformations form a category **Glob**. The Yoneda embedding

$$\mathbf{O} \to \mathbf{Glob}$$

takes each integer n to the *standard globe* $O[n]$. We still denote by $\mathrm{s}_n, \mathrm{t}_n \colon O[n] \to O[n{+}1]$ the morphisms of globular sets representing the corresponding arrows from n to $n{+}1$.

Let X be a globular set and p an integer, the set $X(p)$ will be denoted by X_p, and its elements called *cells of dimension* p or p*-cells*. Hence $O[n]$ has exactly two p-cells for $p < n$, exactly one n-cell, and no p-cells for $p > n$. Let $\partial O[n]$ be the globular set with the same cells as $O[n]$ except for $(\partial O[n])_n = \emptyset$, and

$$\mathrm{i}_n \colon \partial O[n] \to O[n]$$

the canonical injection: $\partial O[n]$ has two p-cells for $p < n$ and no other cells. We denote by σ_n and τ_n the maps $X(\mathrm{s}_n)$ and $X(\mathrm{t}_n)$ respectively. Hence a double sequence of maps

$$\sigma_n, \tau_n \colon X_n \leftleftarrows X_{n+1}$$

satisfying the *boundary conditions*:

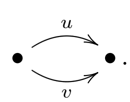
$$\sigma_n \circ \sigma_{n+1} = \sigma_n \circ \tau_{n+1},$$
$$\tau_n \circ \sigma_{n+1} = \tau_n \circ \tau_{n+1}.$$

If $m < n$, we set $\sigma_{m,n} = \sigma_m \circ \cdots \circ \sigma_{n-1}$ and $\tau_{m,n} = \tau_m \circ \cdots \circ \tau_{n-1}$. Let $0 \leqslant i < n$, we say that the n-cells $x, y \in X_n$ are i*-composable* if $\tau_{i,n} x = \sigma_{i,n} y$, a relation we denote by $x \triangleright_i y$.

Now let X be a globular set, Yoneda's lemma yields a natural equivalence

$$X_n \cong \mathbf{Glob}(O[n], X). \tag{1}$$

If $u \in X_n$ and $\sigma_{n-1}(u) = x$, $\tau_{n-1}(u) = y$, x and y are respectively the *source* and the *target* of u, which we simply denote by $u \colon x \to y$. Likewise, if $\sigma_{i,n} u = x$ and $\tau_{i,n} u = y$, we shall write $u \colon x \to_i y$. In case $u \colon x \to y$ and $v \colon x \to y$, we say that u, v are *parallel*, which we denote by $u \parallel v$:



Any two 0-cells are also considered to be parallel. Let $X_n^{\shortparallel}$ denote the set of ordered pairs of parallel n-cells in X. We get a natural equivalence

$$X_n^{\shortparallel} \cong \mathbf{Glob}(\partial O[n{+}1], X) \tag{2}$$

similar to (1). The equivalences (1) and (2) assert that, for each n, the functors $X \mapsto X_n$ and $X \mapsto X_n^{\shortparallel}$ from **Glob** to **Sets** are representable, the representing objects being respectively $O[n]$ and $\partial O[n{+}1]$.

For each integer n, let $\mathbf{O}_n$ denote the full subcategory of $\mathbf{O}$ whose objects are $0, \ldots, n$. The presheaves on $\mathbf{O}_n$ are the n*-globular sets*, and form a category we denote by $\mathbf{Glob}_n$. For each $n < m$, the inclusion $\mathbf{O}_n \to \mathbf{O}_m$ induces a truncation functor from $\mathbf{Glob}_m$ to $\mathbf{Glob}_n$. Likewise, we get a truncation functor from **Glob** to $\mathbf{Glob}_n$.

3. ∞-categories

Recall that an ∞-category is a globular set C endowed with

- a *product* $u *_{n-1} v \colon x \to z$ defined for all $u \colon x \to y$ and $v \colon y \to z$ in C_n;
- a *product* $u *_i v \colon x *_i y \to z *_i t$ defined for all $u \colon x \to z$ and $v \colon y \to t$ in C_n with $i < n{-}1$ and $u \triangleright_i v$;
- a *unit* $1_{n+1}(x) \colon x \to x$ defined for all $x \in C_n$.

These operations satisfy the conditions of *associativity*, *left and right unit*, *composition of units* and *exchange*:

- $(x *_i y) *_i z = x *_i (y *_i z)$ for all $x \triangleright_i y \triangleright_i z$ in C_n with $i < n$;
- $1_{n,i}(x) *_i u = u = u *_i 1_{n,i}(y)$ for all $u \colon x \to_i y$ in C_n with $i < n$, where $1_{n,i} = 1_n \circ 1_{n-2} \circ \cdots \circ 1_{i+1}$;
- $1_{n+1}(x *_i y) = 1_{n+1}(x) *_i 1_{n+1}(y)$ for all $x, y \in C_n$ with $i < n$ and $x \triangleright_i y$;
- $(x *_i y) *_j (z *_i t) = (x *_j z) *_i (y *_j t)$ for all $x, y, z, t \in C_n$ with $i < j < n$ and $x \triangleright_i y$, $x \triangleright_j z$, $y \triangleright_j t$.

Let C, D be ∞-categories. A *morphism* $f \colon C \to D$ is a morphism of the underlying globular sets preserving units and products. ∞-categories and morphisms build a category $\mathbf{Cat}_\infty$, and there is a forgetful functor

$$\mathcal{U} \colon \mathbf{Cat}_\infty \to \mathbf{Glob}.$$

Its left adjoint $\mathbf{Glob} \to \mathbf{Cat}_\infty$ associates to each globular set X the *free ∞-category* X^* generated by it. From this adjunction and the natural equivalences (1) and (2) we get

$$C_n \cong \mathbf{Cat}_\infty(O[n]^*, C), \tag{3}$$

$$C_n^{\shortparallel} \cong \mathbf{Cat}_\infty(\partial O[n{+}1]^*, C). \tag{4}$$

Note that $\mathbf{Glob}$ is a topos of presheaves and that the functor $\mathcal{U}$ is finitary monadic over $\mathbf{Glob}$. Hence $\mathbf{Cat}_\infty$ is complete and cocomplete, and we shall take limits and colimits in $\mathbf{Cat}_\infty$ without further explanations (see also [**1, 2, 13**]).

Likewise, an n-globular set endowed with products and units as above, up to dimension n, determines an *n-category*; n-categories and morphisms build a category $\mathbf{Cat}_n$. As in the case of globular sets, we get a truncation functor

$$\mathcal{T}_n^m \colon \mathbf{Cat}_m \to \mathbf{Cat}_n$$

whenever $n < m$, and likewise

$$\mathcal{T}_n^\infty \colon \mathbf{Cat}_\infty \to \mathbf{Cat}_n.$$

Remark that $\mathbf{Cat}_0 = \mathbf{Sets}$ whereas $\mathbf{Cat}_1$ amounts to the category of small categories. Now $\mathcal{T}_n^m$ admits a left adjoint $\mathcal{F}_n^m \dashv \mathcal{T}_n^m$, for $0 \leqslant n < m \leqslant \infty$, which simply extends the n-category C by adding units in all dimensions k for $n < k \leqslant m$:

$$\mathcal{F}_n^m C \colon C_0 \leftleftarrows \cdots \leftleftarrows C_n \leftleftarrows C_n \leftleftarrows \cdots .$$

In particular, if C is an ∞-category and n an integer, we may define the *n-skeleton of C* by

$$C^{(n)} = \mathcal{F}_n^\infty \mathcal{T}_n^\infty C.$$

It will be convenient to extend this notation by setting $C^{(-1)} = 0$, the initial ∞-category with no cells. There is a canonical inclusion

$$j^{(n)} \colon C^{(n)} \to C^{(n+1)}.$$

Here again $j^{(-1)}$ denotes the unique morphism $0 \to C^{(0)}$. The following result is then an easy consequence of the definitions:

Lemma 3.1. *Any ∞-category C is the colimit of its n-skeleta:*

$$C^{(-1)} \xrightarrow{j^{(-1)}} C^{(0)} \xrightarrow{j^{(0)}} \cdots \xrightarrow{j^{(n-1)}} C^{(n)} \xrightarrow{j^{(n)}} \cdots .$$

4. Polygraphs

We recall the construction of polygraphs, following the presentation of [**4**].

4.1. Attaching cells

Let us first define a category $\mathbf{Cat}_n^+$ of n-categories with attached additional $n{+}1$-cells:

- objects of $\mathbf{Cat}_n^+$ are pairs (C, G) where C is an n-category and G is a graph $\sigma_n, \tau_n \colon C_n \leftleftarrows S_{n+1}$ such that σ_n, τ_n satisfy the *boundary conditions*
$$\sigma_{n-1} \circ \sigma_n = \sigma_{n-1} \circ \tau_n,$$
$$\tau_{n-1} \circ \sigma_n = \tau_{n-1} \circ \tau_n;$$
- if $C^+ = (C, C_n \leftleftarrows S_{n+1})$ and $D^+ = (D, D_n \leftleftarrows T_{n+1})$ are objects of $\mathbf{Cat}_n^+$, then a morphism $f \in \mathbf{Cat}_n^+(C^+, D^+)$ is a pair (g, u) where $g \in \mathbf{Cat}_n(C, D)$ and u is a map $S_{n+1} \to T_{n+1}$ such that (g_n, u) is a morphism of graphs; that is
$$g_n \circ \sigma_n = \sigma_n \circ u,$$
$$g_n \circ \tau_n = \tau_n \circ u.$$

Let $C^+ = (C, G)$ be an object of $\mathbf{Cat}_n^+$; the first projection $(C, G) \mapsto C$ determines a functor

$$\mathcal{A}_n \colon \mathbf{Cat}_n^+ \to \mathbf{Cat}_n.$$

On the other hand there is a functor

$$\mathcal{R}_n \colon \mathbf{Cat}_{n+1} \to \mathbf{Cat}_n^+$$

taking the $n{+}1$-category C to the pair $\left(\mathcal{T}_n^{n+1} C, C_n \leftleftarrows C_{n+1}\right)$: $\mathcal{R}_n$ forgets all information about compositions and identities in dimension $n{+}1$, keeping only the set C_{n+1} of $n{+}1$-cells with their respective sources and targets in C_n. Clearly

$$\mathcal{A}_n \mathcal{R}_n = \mathcal{T}_n^{n+1}.$$

Now the key fact is that $\mathcal{R}_n$ admits a left-adjoint

$$\mathcal{L}_n \colon \mathbf{Cat}_n^+ \to \mathbf{Cat}_{n+1}.$$

For example, $\mathbf{Cat}_0^+$ is the category of graphs and $\mathcal{L}_0$ associates to each graph the free category it generates. It is convenient to extend our notation by defining $\mathbf{Cat}_{-1}^+$

as $\mathbf{Cat}_0(=\mathbf{Sets})$ and $\mathcal{L}_{-1}$ as the identity functor. Let us describe $\mathcal{L}_n$ in some detail. Given $C^+ = (C, C_n \leftleftarrows S_{n+1})$ in $\mathbf{Cat}_n^+$, we first define a formal language $\mathbf{E}$ consisting of:

- a constant $\mathbf{c}_\alpha$ for each $\alpha \in S_{n+1}$, and a constant $\mathbf{i}_c$ for each $c \in C_n$;
- a binary function symbol $\star_i$ for each $i \in \{1, \dots, n\}$.

Thus $\mathbf{E}$ is the smallest set of expressions containing all constants and having the property that $(e \star_i f) \in \mathbf{E}$ whenever $e \in \mathbf{E}$, $f \in \mathbf{E}$ and $0 \leqslant i \leqslant n$. A *type* is an ordered pair (x, y) of parallel cells in C_n, denoted in this context by $x \to y$. For any $e \in \mathbf{E}$, and type $x \to y$, the relation

$$e : x \to y,$$

which reads "e has type $x \to y$", is defined inductively by the following conditions:

- for each $\alpha \in S_{n+1}$, $\mathbf{c}_\alpha : \sigma_n\alpha \to \tau_n\alpha$;
- for each $c \in C_n$, $\mathbf{i}_c : c \to c$;
- if $e : x \to y$ and $f : y \to z$, then $(e \star_n f) : x \to z$;
- if $e : x \to y$, $f : z \to t$ and $x \rhd_i z$, then $(e \star_i f) : x *_i z \to y *_i t$, for $0 \leqslant i < n$.

An expression e is *typable* if there is at least one type $x \to y$ such that $e : x \to y$. Let $\mathbf{E}_T$ be the subset of $\mathbf{E}$ consisting of typable expressions. A key feature of this type system is that any typable expression has at most *one* type: in fact, structural induction shows that whenever $e : x \to y$ and $e : x' \to y'$ then $x' = x$ and $y' = y$. As a consequence, there are unique maps $\sigma, \tau \colon \mathbf{E}_T \to C_n$ such that $\sigma(\mathbf{c}_\alpha) = \sigma_n(\alpha)$ and $\tau(\mathbf{c}_\alpha) = \tau_n(\alpha)$ for each $\alpha \in S_{n+1}$, and $e : \sigma(e) \to \tau(e)$ for each $e \in \mathbf{E}_T$. By composition with the maps σ_i and τ_i for $i < n$, we get maps $\sigma_{i,n+1}, \tau_{i,n+1} \colon \mathbf{E}_T \to C_i$, so that we may still define a relation $\rhd_i$ on $\mathbf{E}_T$ by $e \rhd_i f$ if and only if $\tau_{i,n+1}(e) = \sigma_{i,n+1}(f)$. We define a relation $e \sim f$ on typable expressions by the following conditions:

- $(e \star_i (f \star_i g)) \sim ((e \star_i f) \star_i g)$ if $e \rhd_i f \rhd_i g$ in $\mathbf{E}_T$;
- $(\mathbf{i}_c \star_n e) \sim e$ if $e \in \mathbf{E}_T$, $c \in C_n$ and $\sigma(e) = c$. Likewise $(e \star_n \mathbf{i}_c) \sim e$ if $\tau(e) = c$;
- $\mathbf{i}_{c *_i d} \sim (\mathbf{i}_c \star_i \mathbf{i}_d)$ if $c, d \in C_n$, $0 \leqslant i < n$ and $c \rhd_i d$;
- $((e \star_j f) \star_i (g \star_j h)) \sim ((e \star_i g) \star_j (f \star_i g))$ if $e \rhd_j f$, $g \rhd_j h$, $e \rhd_i g$ and $0 \leqslant i < j \leqslant n$.

Let us denote by $\cong$ the congruence generated by $\sim$ on $\mathbf{E}_T$, and define

$$S^*_{n+1} = \mathbf{E}_T / \cong .$$

The canonical surjection $\mathbf{E}_T \to S^*_{n+1}$, $e \mapsto \langle e \rangle$ satisfies the expected compatibility conditions:

- $\sigma(e)$, $\tau(e)$ only depend on $\langle e \rangle$; whence the relation $e \rhd_i f$ only depends on $\langle e \rangle$ and $\langle f \rangle$;
- $\langle (e \star_i f) \rangle$ only depends on $\langle e \rangle$ and $\langle f \rangle$.

Therefore, we may define $\langle e\rangle *_i \langle f\rangle = \langle (e \star_i f)\rangle$ if $e \triangleright_i f$, $\sigma_n(\langle e\rangle) = \sigma(e)$, $\tau_n(\langle e\rangle) = \tau(e)$ and $1_{n+1}(c) = \langle \mathbf{i}_c\rangle$ for $e \in \mathbf{E}_T$ and $c \in C_n$. We finally set

$$\mathcal{L}_n C^+ =_{\mathrm{def}} C_0 \leftleftarrows C_1 \leftleftarrows \cdots \leftleftarrows C_n \leftleftarrows S^*_{n+1}.$$

We leave it as an exercise to check that all axioms of n+1-categories are satisfied and that the above construction acts on morphisms, making $\mathcal{L}_n$ a functor from $\mathbf{Cat}_n$ to $\mathbf{Cat}_{n+1}$. Clearly

$$\mathcal{T}_n^{n+1}\mathcal{L}_n = \mathcal{A}_n.$$

Moreover, there is a natural transformation

$$\eta_{C^+} \colon C^+ \to \mathcal{R}_n\mathcal{L}_n C^+$$

such that $\eta_{C^+} = \left(\eta^1_{C^+}, \eta^2_{C^+}\right)$ where $\eta^1_{C^+}$ is the identity on C and $\eta^2_{C^+} \colon S_{n+1} \to S^*_{n+1}$ is $\alpha \mapsto \langle \mathbf{c}_\alpha\rangle$. Note that $\eta^2_{C^+}$ is injective. By construction, $\mathcal{L}_n$ satisfies the universal property of Lemma 4.1 below; whence $\mathcal{L}_n \dashv \mathcal{R}_n$.

Lemma 4.1. *Let $C^+ = (C, C_n \leftleftarrows S_{n+1})$ in $\mathbf{Cat}^+_n$, D an n+1-category and*

$$f = (g, u) : C^+ \to \mathcal{R}_n D$$

a morphism in $\mathbf{Cat}^+_n$. There is a unique map $u^* \colon S^*_{n+1} \to D_{n+1}$ satisfying the following properties:

- *$u^* \circ \eta^2_{C^+} = u$;*
- *there is an $f^* \in \mathbf{Cat}_{n+1}(\mathcal{L}_n C^+, D)$ such that $\mathcal{T}_n^{n+1} f^* = g$ and $f^*_{n+1} = u^*$.*

4.2. The category of polygraphs

We now define the category $\mathbf{Pol}_n$ of *n-polygraphs* by induction on n. Precisely we define $\mathbf{Pol}_n$ together with a functor

$$\mathcal{J}_n \colon \mathbf{Pol}_n \to \mathbf{Cat}^+_{n-1}.$$

- $\mathbf{Pol}_0$ is just **Sets**, and $\mathcal{J}_0$ is the identity functor;
- Suppose $\mathcal{J}_n \colon \mathbf{Pol}_n \to \mathbf{Cat}^+_{n-1}$ has been defined. An n+1-polygraph is a pair $S = (S', C^+)$ where S' is an n-polygraph and C^+ an object of $\mathbf{Cat}^+_n$ such that $\mathcal{A}_n C^+ = \mathcal{L}_{n-1}\mathcal{J}_n S'$. We set $\mathcal{J}_{n+1} S = C^+$. If $S = (S', C^+)$ and $T = (T', D^+)$, a morphism $f \colon S \to T$ of n+1-polygraphs is a pair (f', u) where $f' \in \mathbf{Pol}_n(S', T')$, $u \in \mathbf{Cat}^+_n(C^+, D^+)$ and $\mathcal{A}_n u = \mathcal{L}_{n-1}\mathcal{J}_n f'$.

We denote by $\mathcal{I}_n^{n+1} \colon \mathbf{Pol}_{n+1} \to \mathbf{Pol}_n$ the first projection $(S', C^+) \mapsto S'$. The following commutative diagram summarizes the induction step:

$$\begin{array}{ccccc}
\mathbf{Pol}_{n+1} & \xrightarrow{\mathcal{J}_{n+1}} & \mathbf{Cat}^+_n & \xrightarrow{\mathcal{L}_n} & \mathbf{Cat}_{n+1} \\
\downarrow{\scriptstyle \mathcal{I}_n^{n+1}} & & & \searrow{\scriptstyle \mathcal{A}_n} & \downarrow{\scriptstyle \mathcal{T}_n^{n+1}} \\
\mathbf{Pol}_n & \xrightarrow[\mathcal{J}_n]{} & \mathbf{Cat}^+_{n-1} & \xrightarrow[\mathcal{L}_{n-1}]{} & \mathbf{Cat}_n.
\end{array}$$

Let $\mathcal{Q}_n = \mathcal{L}_{n-1}\mathcal{J}_n$; the above commutation yields

$$\mathcal{T}_n^{n+1}\mathcal{Q}_{n+1} = \mathcal{Q}_n\mathcal{I}_n^{n+1}. \tag{5}$$

We define, by induction on $n \geqslant 0$, a functor $\mathcal{P}_n \colon \mathbf{Cat}_n \to \mathbf{Pol}_n$, right-adjoint to $\mathcal{Q}_n$:

- for $n = 0$, $\mathcal{P}_0$ and $\mathcal{Q}_0$ are both the identity functor on $\mathbf{Pol}_0 = \mathbf{Cat}_0 = \mathbf{Sets}$;
- suppose $\mathcal{Q}_n \dashv \mathcal{P}_n$, and let D be an $n{+}1$-category. $D' = \mathcal{T}_n^{n+1} D$ is an n-category and by induction hypothesis, we get an n-polygraph $S' = \mathcal{Q}_n D'$. Moreover, the counit of the adjunction yields a morphism of n-categories
$$\epsilon\colon \mathcal{Q}_n \mathcal{P}_n D' \to D',$$
whose n-th component is a map
$$\epsilon_n\colon S_n'^* \to D_n'.$$
Now $\mathcal{P}_{n+1} D$ is by definition the polygraph $S = (S', C^+)$, where
$$C^+ = (\mathcal{Q}_n S', S_n'^* \leftleftarrows S_{n+1})$$
and S_{n+1} is the set of triples $(z, x, y) \in D_{n+1} \times S_n'^* \times S_n'^*$ such that $x \parallel y$ and $z\colon \epsilon_n(x) \to \epsilon_n(y)$. The source and target of (z, x, y) are x and y, respectively. Likewise, $\mathcal{P}_{n+1}$ acts on morphisms: we refer to [**10**] for details, and a complete proof that $\mathcal{Q}_{n+1} \dashv \mathcal{P}_{n+1}$.

Remark that, by construction,
$$\mathcal{I}_n^{n+1} \mathcal{P}_{n+1} = \mathcal{P}_n \mathcal{T}_n^{n+1}. \tag{6}$$

Definition 4.2. A *polygraph* S is a sequence $(S^n)_{n\in\mathbb{N}}$ such that, for each $n \geqslant 0$, S^n is an n-polygraph and $\mathcal{I}_n^{n+1} S^{n+1} = S^n$.

Likewise, if S and T are polygraphs, a *morphism* $f\colon S \to T$ amounts to a sequence $(f^n)_{n\in\mathbb{N}}$ such that $f^n\colon S^n \to T^n$ is a morphism of n-polygraphs and $\mathcal{I}_n^{n+1} f^{n+1} = f^n$. Polygraphs and morphisms build a category $\mathbf{Pol}$. For each polygraph S, let $\mathcal{I}_n^\infty S = S^n$, making $\mathcal{I}_n^\infty$ a functor from $\mathbf{Pol}$ to $\mathbf{Pol}_n$. From (5), (6) and $\mathcal{Q}_n \dashv \mathcal{P}_n$, we get a pair of adjoint functors
$$\mathcal{Q}\colon \mathbf{Pol} \to \mathbf{Cat}_\infty,$$
$$\mathcal{P}\colon \mathbf{Cat}_\infty \to \mathbf{Pol},$$
such that, for each $n \geqslant 0$,
$$\mathcal{T}_n^\infty \mathcal{Q} = \mathcal{Q}_n \mathcal{I}_n^\infty$$
and
$$\mathcal{I}_n^\infty \mathcal{P} = \mathcal{P}_n \mathcal{T}_n^\infty.$$
Thus, we may summarize the above construction by using the following less explicit, but simpler notation:

- a 0-polygraph is a set S_0, generating the 0-category (i.e. set) $S_0^* = S_0$;
- given an n-polygraph $S_0,\ S_0^* \leftleftarrows S_1, \ldots, S_{n-1}^* \leftleftarrows S_n$ with the free n-category $S_0^* \leftleftarrows \ldots \leftleftarrows S_n^*$ it generates, an $n{+}1$-polygraph is determined by a graph
$$\sigma_n, \tau_n\colon S_n^* \leftleftarrows S_{n+1}$$
satisfying the boundary conditions, and the free $n{+}1$-category generated by it is $S_0^* \leftleftarrows S_1^* \leftleftarrows \cdots S_n^* \leftleftarrows S_{n+1}^*$;
- a polygraph S is an infinite sequence $S_0, S_0^* \leftleftarrows S_1, \ldots, S_{n-1}^* \leftleftarrows S_n \ldots$ such that for each p, $S_0, \ldots, S_{p-1}^* \leftleftarrows S_p$ is a p-polygraph.

Likewise, a morphism $f\colon S \to T$ between polygraphs S, T amounts to a sequence of maps $f_n\colon S_n \to T_n$ such that for all $\xi\colon x \to y$ in S_n, $f_n(\xi)\colon f^*_{n-1}(x) \to f^*_{n-1}(y)$, where f^*_n is the unique extension of f_n which is compatible with products and units. From now on, for any polygraph S, we set $S^* = \mathcal{Q}S$. We call *generators of dimension* n, or n*-generators*, the elements of S_n. Each $\alpha \in S_n$ generates an *atomic* n-cell $\alpha^* \in S^*_n$ (see 4.1).

Remark that any globular set X can be viewed as a particular polygraph and that this identification makes **Glob** a full subcategory of **Pol**. Moreover the free ∞-category generated by a globular set is the same as the free ∞-category generated by the corresponding polygraph. However most free ∞-categories generated by polygraphs cannot be generated by globular sets alone.

For instance the globular sets $O[n]$ and $\partial O[n]$ can be viewed as polygraphs, and generate ∞-categories $O[n]^*$ and $\partial O[n]^*$. Remark that in this case, the free construction does not create new non-identity cells. Therefore, in the sequel, we drop the "$*$" in the notation of these ∞-categories. Likewise, i_n will denote a morphism of globular sets, polygraphs, or ∞-categories according to the context.

Let $C^+ = (C, C_n \leftleftarrows S_{n+1})$ in $\mathbf{Cat}^+_n$; the $n{+}1$-category $\mathcal{L}_n C^+$ has the same n-cells as C, hence an inclusion morphism $j\colon \mathcal{F}^\infty_n C \to \mathcal{F}^\infty_{n+1}\mathcal{L}_n C^+$. Each generator $\alpha \in S_{n+1}$ gives an $n{+}1$-cell in $\mathcal{L}_n C^+$, whose source and target give parallel n-cells in C. Hence by (3) and (4), we get two morphisms

$$\rho\colon \sum_{S_{n+1}} \partial O[n{+}1] \to \mathcal{F}^\infty_n C$$

and

$$\chi\colon \sum_{S_{n+1}} O[n{+}1] \to \mathcal{F}^\infty_{n+1}\mathcal{L}_n C^+,$$

making the following diagram commutative:

$$\begin{array}{ccc}
\sum_{S_{n+1}} \partial O[n{+}1] & \xrightarrow{\ \rho\ } & \mathcal{F}^\infty_n C \\
{\scriptstyle \sum_{S_{n+1}} \mathrm{i}_{n+1}}\downarrow & & \downarrow{\scriptstyle j} \\
\sum_{S_{n+1}} O[n{+}1] & \xrightarrow[\ \chi\ ]{} & \mathcal{F}^\infty_{n+1}\mathcal{L}_n C^+.
\end{array}$$

Now Lemma 4.1 implies that the above square is a pushout. In the particular case where S is a polygraph, $C = (S^*)^{(n)}$ and $C^+ = (C, S^*_n \leftleftarrows S_{n+1})$, we get the following result:

Lemma 4.3. *The diagram*

$$\begin{array}{ccc}
\sum_{S_{n+1}} \partial O[n{+}1] & \xrightarrow{\ \rho\ } & (S^*)^{(n)} \\
{\scriptstyle \sum_{S_n} \mathrm{i}_n}\downarrow & & \downarrow{\scriptstyle j^{(n)}} \\
\sum_{S_{n+1}} O[n{+}1] & \xrightarrow[\ \chi\ ]{} & (S^*)^{(n+1)}
\end{array}$$

is a pushout in $\mathbf{Cat}_\infty$.

4.3. Linearization

Let $n \geqslant 1$ and C an $n{-}1$-category. Given an abelian monoid $(A, +)$, we may extend C to an n-category $D = A \ltimes C$, as follows:

- $\mathcal{T}^n_{n-1} D = C$; that is, D coincides with C up to dimension $n{-}1$;
- $D_n = A \times C^{\shortparallel}_{n-1}$, with $(a, (x, y)) : x \to y$ for each $a \in A$ and each pair (x, y) of parallel cells in C_{n-1};
- let $x \parallel y \parallel z$ in C_{n-1}, and a, b in A, the composition $(a, (x, y)) *_{n-1} (b, (y, z))$ is by definition $(a + b, (x, z))$;
- let $u = (a, (x, y))$, $v = (b, (z, t))$ in D_n and $i \in \{0, \dots, n{-}2\}$ such that $u \triangleright_i v$. This implies $x \triangleright_i z$ and $y \triangleright_i t$ (in C), so that $x *_i z \parallel y *_i t$ and we may define $u *_i v = (a + b, (x *_i z, y *_i t))$;
- for each $x \in C_{n-1}$, $1_n(x) = (0, (x, x))$.

We leave it as an exercise to check the axioms of n-categories on $A \ltimes C$.

Let S be a polygraph; we apply the above construction to the particular case where $C = \mathcal{T}^{\infty}_{n-1} S^*$ and A is the free abelian group $\mathbb{Z}S_n$ on S_n. To each generator $\alpha \in S_n$ corresponds a generator $\tilde{\alpha}$ of $\mathbb{Z}S_n$. Elements of $\mathbb{Z}S_n$ are thus of the form

$$a = \sum_{\alpha \in S_n} n_\alpha \tilde{\alpha},$$

where $n_\alpha \in \mathbb{Z}$ and all but a finite number of coefficients are zero. Let $D = A \ltimes C$. There is a map $S_n \to D_n$, given by $\alpha \mapsto (\tilde{\alpha}, (x, y))$ for each n-generator $\alpha\colon x \to y$, which in turn determines a morphism $f\colon \big(C, S^*_{n-1} \leftleftarrows S_n\big) \to \mathcal{R}_{n-1} D$ in $\mathbf{Cat}^+_{n-1}$. Thus Lemma 4.1 applies, and we get a morphism

$$f^*\colon \mathcal{T}^{\infty}_n S^* \to D$$

in $\mathbf{Cat}_n$, whence a unique *linearization map*

$$\lambda\colon S^*_n \to \mathbb{Z}S_n$$

satisfying the following properties:

- for each $\alpha \in S_n$, $\lambda(\alpha^*) = \tilde{\alpha}$;
- if $0 \leqslant i \leqslant n{-}1$ and $x \triangleright_i y$ in S^*_n, then $\lambda(x *_i y) = \lambda(x) + \lambda(y)$;
- for each $x \in S^*_{n-1}$, $\lambda(1_n(x)) = 0$.

Now, for each $x \in S^*_n$, $\lambda(x)$ has a unique expression of the form

$$\lambda(x) = \sum_{\alpha \in S_n} \mathrm{w}_\alpha(x) \tilde{\alpha}, \tag{7}$$

where $\mathrm{w}_\alpha(x) \in \mathbb{Z}$ (in fact $\mathrm{w}_\alpha(x) \in \mathbb{N}$). Note that for each fixed n, the correspondence $S^* \mapsto \mathbb{Z}S_n$ is functorial. Precisely, let $\mathbf{Fcat}_\infty$ be the full subcategory of $\mathbf{Cat}_\infty$ whose objects are of the form S^*, where S is a polygraph. To each morphism $u\colon S^* \to T^*$ corresponds a linear map $\tilde{u}_n\colon \mathbb{Z}S_n \to \mathbb{Z}T_n$. As identities and compositions are preserved, we get a functor from $\mathbf{Fcat}_\infty$ to the category $\mathbf{Ab}$ of abelian groups, and by composing with the forgetful functor $\mathbf{Ab} \to \mathbf{Sets}$, also a functor $\mathcal{Z}\colon \mathbf{Fcat}_\infty \to \mathbf{Sets}$. Now there is a functor $\mathcal{Y}\colon \mathbf{Fcat}_\infty \to \mathbf{Sets}$ which associates to each S^* the set S^*_n of its n-cells. Here a useful observation is that linearization gives rise to a natural transformation from $\mathcal{Y}$ to $\mathcal{Z}$: let S, T be polygraphs, $u \in \mathbf{Fcat}_\infty(S^*, T^*)$, and λ_S, λ_T the respective linearization maps, the following diagram commutes:

$$\begin{array}{ccc} S_n^* & \xrightarrow{u_n} & T_n^* \\ {\scriptstyle\lambda_S}\downarrow & & \downarrow{\scriptstyle\lambda_T} \\ \mathbb{Z}S_n & \xrightarrow[\tilde{u}_n]{} & \mathbb{Z}T_n. \end{array}$$

In particular, for each n-cell x in S_n^*, we get

$$\lambda_T(u_n(x)) = \sum_{\alpha \in S_n} \mathrm{w}_\alpha(x) \lambda_T(u_n(\alpha^*)). \tag{8}$$

We call $\mathrm{w}_\alpha(x)$ the *weight of x at α*. As a consequence of (8), for each $x \in S_n^*$ and each generator $\beta \in T_n$,

$$\mathrm{w}_\beta(u_n(x)) = \sum_{\alpha \in S_n} \mathrm{w}_\alpha(x) \mathrm{w}_\beta(u_n(\alpha^*)). \tag{9}$$

As only finitely many of the coefficients $\mathrm{w}_\alpha(x)$ are non-zero, we may define the *total weight* of x as the non-negative integer

$$\mathrm{w}(x) = \sum_{\alpha \in S_n} \mathrm{w}_\alpha(x).$$

Looking back at the construction of $\mathcal{L}_n$ via formal expressions, we note that $\mathrm{w}_\alpha(x)$ is also the number of occurrences of the symbol $\mathbf{c}_\alpha$ in any expression representing x. Likewise, if $\mathrm{w}(x) = 0$, there is a unique $x' \in S_{n-1}^*$ such that $x = 1_n(x')$, and more generally, a unique choice of $k < n$ and $x'' \in S_k^*$ such that $x = 1_{n,k}(x'')$ and $\mathrm{w}(x'') > 0$.

5. Contexts

This purely technical section introduces contexts, a convenient way to formulate the two results we shall need later, namely equation (11) and Lemma 5.6.

5.1. Indeterminates

Let C be an ∞-category, and $n \geqslant 1$. Recall from Section 4.1 that an *n-type on C* is an ordered pair (x, y) of parallel cells in C_{n-1}, that is an element of $C_{n-1}^{|||}$. The *type* of an n-cell $x \in C_n$ is the pair $(\sigma_{n-1}x, \tau_{n-1}x)$. Hence the type of an n-cell is a particular n-type. Let S be a polygraph, $n \geqslant 1$, and $\xi = (x, y)$ an n-type on S^*. We build a new polygraph $T = S[\xi]$ by adjoining ξ as a new n-generator. Precisely, T coincides with S up to dimension $n-1$, $T_n = S_n + \{\xi\}$ and $T_{n-1}^* \leftleftarrows T_n$ extends $S_{n-1}^* \leftleftarrows S_n$ by

$$\sigma_{n-1}(\xi) = x,$$
$$\tau_{n-1}(\xi) = y.$$

Thus we get an inclusion map $S_n^* \to T_n^*$. Suppose $j \geqslant n$ and T has been defined up to dimension j together with an inclusion map $S_j^* \to T_j^*$. We set $T_{j+1} = S_{j+1}$. This yields $T_j^* \leftleftarrows T_{j+1}$ and by Lemma 4.1, a new inclusion $S_{j+1}^* \to T_{j+1}^*$. Now ξ generates an n-cell $\xi^* = \mathbf{x}$ of T^*, which we call an *n-indeterminate* of type ξ on S. We let boldface variables $\mathbf{x}, \mathbf{y}, \ldots$ range over indeterminates.

Definition 5.1. Let $\mathbf{x}$ be an n-indeterminate of type ξ on the polygraph S; an n-*context over* $\mathbf{x}$ is an n-cell u of $(S[\xi])^*$ such that $\mathrm{w}_\xi(u) = 1$.

We denote n-contexts over $\mathbf{x}$ by $c[\mathbf{x}]$, $d[\mathbf{x}]$, ... A context $c[\mathbf{x}]$ is *trivial* if $c[\mathbf{x}] = \mathbf{x}$. An n-cell z of S^* is *adapted* to the context $c[\mathbf{x}]$ if it has the same type as $\mathbf{x}$. Any adapted n-cell may be *substituted* to the indeterminate in a given context: let $\mathbf{x} = \xi^*$ be an n-indeterminate of type ξ and z an adapted n-cell. There is a map $u_z \colon S_n + \{\xi\} \to S_n^*$ defined by $u_z(\alpha) = \alpha^*$ if $\alpha \in S_n$ and $u_z(\xi) = z$. Lemma 4.1 applies and gives a morphism

$$\mathrm{sub}_z \colon (S[\xi])^* \to S^*$$

such that $\mathrm{sub}_z(\mathbf{x}) = z$. Likewise, for each context $c[\mathbf{x}]$ over $\mathbf{x}$, we define $c[z]$ as $\mathrm{sub}_z(c[\mathbf{x}])$. By applying (8) to sub_z, we get

$$\lambda_S(c[z]) = \lambda_S(z) + \sum_{\alpha \in S_n} \mathrm{w}_\alpha(c[\mathbf{x}])\tilde{\alpha}. \tag{10}$$

Let S, T be polygraphs, and $u \in \mathbf{Fcat}_\infty(S^*, T^*)$. To each n-type $\xi = (x, y)$ in S^* corresponds an n-type $\psi = (u(x), u(y))$. Let $\xi^* = \mathbf{x}$ and $\psi^* = \mathbf{y}$. Yet another application of Lemma 4.1 yields a unique morphism

$$\hat{u} \colon (S[\xi])^* \to (T[\psi])^*$$

such that $\hat{u}(\alpha^*) = u(\alpha^*)$ if $\alpha \in S_n$ and $\hat{u}(\mathbf{x}) = \mathbf{y}$. In this situation, we get the following result:

Lemma 5.2. *For each n-context $c[\mathbf{x}]$, $\hat{u}(c[\mathbf{x}])$ is an n-context over $\mathbf{y}$.*

Proof. We have to show that $\mathrm{w}_\psi(\hat{u}(c[\mathbf{x}])) = 1$. By (9),

$$\mathrm{w}_\psi(\hat{u}(c[\mathbf{x}])) = \sum_{\alpha \in S_n + \{\xi\}} \mathrm{w}_\alpha(c[\mathbf{x}])\mathrm{w}_\psi(\hat{u}_n(\alpha^*))$$

but, for each $\alpha \neq \xi$, $\hat{u}_n(\alpha^*) = u_n(\alpha^*)$ already belongs to T_n^* so that $\mathrm{w}_\psi(\hat{u}_n(\alpha^*)) = 0$; whence

$$\mathrm{w}_\psi(\hat{u}(c[\mathbf{x}])) = \mathrm{w}_\xi(c[\mathbf{x}])\mathrm{w}_\psi(\hat{u}_n(\xi^*)).$$

By definition $\mathrm{w}_\xi(c[\mathbf{x}])) = 1$, and $\hat{u}_n(\xi^*) = \psi^*$, so that $\mathrm{w}_\psi(\hat{u}_n(\xi^*)) = 1$ and we get the result. □

We denote by $c^u[\mathbf{y}]$ the context $\hat{u}(c[\mathbf{x}])$ just defined. Now for each adapted n-cell z in S^*,

$$u(c[z]) = c^u[u(z)]. \tag{11}$$

This amounts to the naturality of the substitution viewed in appropriate categories. In fact, consider the comma category $\mathbf{C} = O[n] \downarrow \mathbf{Fcat}_\infty$. Objects of $\mathbf{C}$ may be represented as pairs (S, z) where S is a polygraph and $z \in S_n^*$, whereas a morphism $u \colon (S, z) \to (T, z')$ is an $u \in \mathbf{Fcat}_\infty(S^*, T^*)$ such that $u(z) = z'$. Now there are two functors $\mathcal{B}, \mathcal{C} \colon \mathbf{C} \to \mathbf{Fcat}_\infty$ given by $\mathcal{B}(S, z) = S^*$ and $\mathcal{C}(S, z) = (S[\xi])^*$, where ξ is the type of z. For each $Z = (S, z)$ in $\mathbf{C}$, we get $\mathrm{sub}_z \colon \mathcal{C}Z \to \mathcal{B}Z$. This determines a natural transformation from $\mathcal{C}$ to $\mathcal{B}$. Thus for each $u\colon (S,z) \to (T,u(z))$, the following diagram commutes:

$$\begin{array}{ccc} (S[\xi])^* & \xrightarrow{\hat{u}} & (T[\psi])^* \\ \downarrow{\scriptstyle \mathrm{sub}_z} & & \downarrow{\scriptstyle \mathrm{sub}_{u(z)}} \\ S^* & \xrightarrow[u]{} & T^*, \end{array}$$

which implies (11).

5.2. Thin contexts

We pay special attention to contexts built on no other atomic n-cell but the indeterminate itself.

Definition 5.3. Let $\mathbf{x}$ be an indeterminate of type ξ on a polygraph S, and $c[\mathbf{x}]$ an n-context over $\mathbf{x}$. We call $c[\mathbf{x}]$ a *thin context* if $\mathrm{w}_\alpha(c[\mathbf{x}]) = 0$ for each $\alpha \in S_n$.

Given a polygraph S and $\mathbf{x}$ an n-indeterminate on S, we define a family $(\mathbf{C}_i^{\mathbf{x}})_{0 \leqslant i \leqslant n}$ of sets of n-contexts over $\mathbf{x}$ by induction on i:

- $\mathbf{C}_0^{\mathbf{x}} = \{\mathbf{x}\}$;
- $\mathbf{C}_i^{\mathbf{x}} = \{a *_{i-1} c[\mathbf{x}] *_{i-1} b \mid c[\mathbf{x}] \in \mathbf{C}_{i-1}^{\mathbf{x}}, a \in S_n^*, b \in S_n^*, a \triangleright_{i-1} c[\mathbf{x}] \triangleright_{i-1} b\}$ for each $i > 0$.

Observe that

- each n-context over $\mathbf{x}$ belongs to $\cup_{0 \leqslant i \leqslant n} \mathbf{C}_i^{\mathbf{x}}$;
- each *thin* n-context over $\mathbf{x}$ belongs to $\cup_{0 \leqslant i < n} \mathbf{C}_i^{\mathbf{x}}$.

In fact the exchange rule allows to perform higher-dimensional compositions outside lower-dimensional ones. Also remark that, if $c[\mathbf{x}] \in \mathbf{C}_i^{\mathbf{x}}$ and $j \geqslant i$, then, by induction on i,

$$\mathrm{w}(\sigma_{j,n}(c[\mathbf{x}])) \geqslant \mathrm{w}(\sigma_{j,n}(\mathbf{x})). \tag{12}$$

Lemma 5.4. *If $n > 1$ and $c[\mathbf{x}]$ is a thin n-context, then there is an $n{-}1$-context $\partial c[\mathbf{y}]$ over the indeterminate $\mathbf{y}$ of type $(\sigma_{n-2,n}(\mathbf{x}), \tau_{n-2,n}(\mathbf{x}))$, satisfying the following properties:*

- *for each adapted n-cell z, $\sigma_{n-1}(c[z]) = \partial c[\sigma_{n-1}(z)]$;*
- *if $\partial c[\mathbf{y}]$ is trivial, then so is $c[\mathbf{x}]$.*

Proof. Let $c[\mathbf{x}]$ be a thin n-context, with $n > 1$. The above remarks show that there is an $i < n$ such that $c[\mathbf{x}] \in \mathbf{C}_i^{\mathbf{x}}$. We show, by induction on the least such i, the existence of an $n{-}1$-context $\partial c[\mathbf{y}]$ over $\mathbf{y}$ of type $(\sigma_{n-2,n}(\mathbf{x}), \tau_{n-2,n}(\mathbf{x}))$ satisfying the following properties:

1. $\partial c[\mathbf{y}] \in \mathbf{C}_i^{\mathbf{y}}$;
2. for each adapted n-cell z in S^*, $\sigma_{n-1}(c[z]) = \partial c[\sigma_{n-1}(z)]$;
3. $\sigma_{i-1,n}(c[\mathbf{x}]) = \sigma_{i-1,n-1}(\partial c[\mathbf{y}])$ and $\tau_{i-1,n}(c[\mathbf{x}]) = \tau_{i-1,n-1}(\partial c[\mathbf{y}])$ if $i > 1$;
4. if $\partial c[\mathbf{y}]$ is trivial, so is $c[\mathbf{x}]$.

If $i = 0$, then $c[\mathbf{x}] = \mathbf{x}$ and we set $\partial c[\mathbf{y}] = \mathbf{y}$ of the appropriate type, so that conditions 1 to 4 hold. Suppose that $i > 0$ and the result holds up to $i{-}1$. Choose an n-context $d[\mathbf{x}] \in \mathbf{C}^{\mathbf{x}}_{i-1}$ and n-cells a, b in S^* such that $a \triangleright_{i-1} d[\mathbf{x}] \triangleright_{i-1} b$ and

$$c[\mathbf{x}] = a *_{i-1} d[\mathbf{x}] *_{i-1} b.$$

As $c[\mathbf{x}]$ is thin, $\mathrm{w}(a) = \mathrm{w}(b) = 0$ and there are $n{-}1$-cells a', b' such that $a = 1_n(a')$ and $b = 1_n(b')$. By the induction hypothesis we may choose an $n{-}1$-context $\partial d[\mathbf{y}] \in \mathbf{C}^{\mathbf{y}}_{i-1}$ satisfying the above conditions. In particular, condition 3 shows that

$$a' \triangleright_{i-1} \partial d[\mathbf{y}] \triangleright_{i-1} b',$$

so that we may define

$$\partial c[\mathbf{y}] = a' *_{i-1} \partial d[\mathbf{y}] *_{i-1} b'. \tag{13}$$

Conditions 1, 2 and 3 are straightforward. As for condition 4, suppose that $\partial c[\mathbf{y}]$ is trivial: this can only happen if $i = 0$. Otherwise, $\partial c[\mathbf{y}]$ is given by (13), so that

$$a' *_{i-1} \partial d[\mathbf{y}] *_{i-1} b' = \mathbf{y}. \tag{14}$$

There are unique integers j, k in $\{0, \ldots, n{-}1\}$, and non-identity cells $a'' \in S^*_j$, $b'' \in S^*_k$ such that $a' = 1_{n-1,j}(a'')$ and $b' = 1_{n-1,k}(b'')$. Two cases are possible:

- j and k are both $\leqslant i{-}1$, in which case a and b are respectively identities on the source and target of $d[\mathbf{x}]$, so that $c[\mathbf{x}] = d[\mathbf{x}]$ and $c[\mathbf{x}] \in \mathbf{C}^{\mathbf{x}}_{i-1}$, a contradiction, because of the minimality of i;
- at least one of j, k is $> i{-}1$, say $j > i{-}1$. By applying $\sigma_{j,n-1}$ to both members of (14), we get

  $$a'' *_{i-1} \sigma_{j,n-1}(\partial d[\mathbf{y}]) *_{i-1} \sigma_{j,n-1}b' = \sigma_{j,n-1}(\mathbf{y}),$$

  and by taking the weight (in S^*_j) on both sides,

  $$\mathrm{w}(a'') + \mathrm{w}(\sigma_{j,n-1}(\partial d[\mathbf{y}])) + \mathrm{w}(\sigma_{j,n-1}b') = \mathrm{w}(\sigma_{j,n-1}(\mathbf{y})),$$

  which, combined with (12), implies $\mathrm{w}(a'') = 0$. This contradicts the hypothesis that a'' is not an identity.

Hence i cannot be $\neq 0$, and $c[\mathbf{x}] = \mathbf{x}$. □

Lemma 5.5. *Let $c[\mathbf{x}]$ be an n-context and z an adapted n-cell. If $c[z] = z$, then $c[\mathbf{x}]$ is trivial.*

Proof. We proceed by induction on the dimension n. If $n = 1$, all contexts are trivial and we are done. Suppose now $n > 1$ and the result holds in dimension $n{-}1$. Let $c[\mathbf{x}]$ be an n-context and z an adapted n-cell such that

$$c[z] = z. \tag{15}$$

Thus $\lambda_S(c[z]) = \lambda_S(z)$ and because of (10),

$$\sum_{\alpha \in S_n} \mathrm{w}_\alpha(c[\mathbf{x}])\tilde{\alpha} = 0.$$

Therefore $c[\mathbf{x}]$ is thin, and by Lemma 5.4 we get an $n{-}1$-context $\partial c[\mathbf{y}]$ such that

$\sigma_{n-1}(c[z]) = \partial c[\sigma_{n-1}(z)]$. Hence, by taking the source on both sides of (15), we get

$$\partial c[\sigma_{n-1}(z)] = \sigma_{n-1}(z).$$

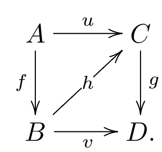
Thus, by the induction hypothesis, $\partial c[\mathbf{y}]$ is trivial and so is $c[\mathbf{x}]$ by Lemma 5.4. □

Lemma 5.6. *Let $c[\mathbf{x}]$ be a thin n-context, and z an adapted n-cell. If $c[z]$ is parallel to z, then $c[z] = z$.*

Proof. Suppose $c[\mathbf{x}]$ is a thin n-context, and z is an adapted n-cell such that $c[z] \parallel z$. If $n = 1$, then thin contexts are trivial and the result is immediate. Otherwise, $n > 1$ and by Lemma 5.4, there is an $n-1$-context $\partial c[\mathbf{y}]$ such that $\sigma_{n-1}(c[z]) = \partial c[\sigma_{n-1}(z)]$. As $c[z]$ is parallel to z, this implies $\partial c[\sigma_{n-1}(z)] = \sigma_{n-1}(z)$. By Lemma 5.5, $\partial c[\mathbf{y}]$ is trivial, and by Lemma 5.4 again, so is $c[\mathbf{x}]$. Hence $c[z] = z$. □

6. Two classes of morphisms

Let $\mathbf{C}$ be a category, and $f\colon A \to B$, $g\colon C \to D$ morphisms. f has the *left-lifting property* with respect to g (or, equivalently, g has the *right-lifting property* with respect to f) if, for each pair of morphisms $u\colon A \to C$, $v\colon B \to D$ such that $g \circ u = v \circ f$, there exists an $h\colon B \to C$ making the following diagram commutative:



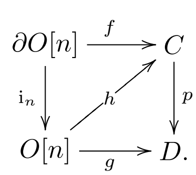
For any class $\mathbb{M}$ of morphisms in $\mathbf{C}$, ${}^{\pitchfork}\mathbb{M}$ (resp. $\mathbb{M}^{\pitchfork}$) denotes the class of morphisms in $\mathbf{C}$ which have the left- (resp. right-) lifting property with respect to all morphisms in $\mathbb{M}$.

6.1. Trivial fibrations

Let $\mathbb{I}$ be the set $\{\mathrm{i}_n | n \in \mathbb{N}\}$ as morphisms in $\mathbf{Cat}_\infty$.

Definition 6.1. A morphism of ∞-categories is a *trivial fibration* if and only it belongs to $\mathbb{I}^{\pitchfork}$.

In other words, $p\colon C \to D$ is a trivial fibration if for all n, $f\colon \partial O[n] \to C$, and $g\colon O[n] \to D$ such that $p \circ f = g \circ \mathrm{i}_n$, there is an $h\colon O[n] \to C$ making the following diagram commutative:



Definition 6.2. Let C be an ∞-category. A *polygraphic resolution* of C is a pair (S, p) where S is a polygraph and $p\colon S^* \to C$ is a trivial fibration.

It was shown in [**10**] that, for each ∞-category C, the counit of the adjunction $\mathcal{Q} \dashv \mathcal{P}$,

$$\epsilon_C : \mathcal{QP}C \to C,$$

is a trivial fibration. Hence $(\mathcal{P}C, \epsilon_C)$ is a polygraphic resolution of C, so that:

Proposition 6.3. *Each ∞-category C has a polygraphic resolution.*

6.2. Cofibrations

Definition 6.4. A morphism of ∞-categories is a *cofibration* if and only if it has the left-lifting property with respect to all trivial fibrations.

Hence the class of cofibrations is exactly ${}^{\pitchfork}(\mathbb{I}^{\pitchfork})$. Immediate examples of cofibrations are the maps i_n themselves. The following lemma summarizes standard properties of maps defined by left-lifting conditions (see [**5**]).

Lemma 6.5. *Let* $\mathbf{C}$ *be a category, and* $\mathbb{M}$ *an arbitrary class of morphisms of* $\mathbf{C}$*. Let* $\mathbb{L} = {}^{\pitchfork}\mathbb{M}$*. Then*

- $\mathbb{L}$ *is stable by direct sums: if* $f_i : X_i \to Y_i$, $i \in I$ *is a family of maps of* $\mathbb{L}$ *with direct sum* $f = \sum_{i\in I} f_i : \sum_{i\in I} X_i \to \sum_{i\in I} Y_i$*, then* $f \in \mathbb{L}$*;*
- $\mathbb{L}$ *is stable by pushout: whenever* $f \in \mathbb{L}$ *and*

$$\begin{array}{ccc} X & \longrightarrow & Z \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ Y & \longrightarrow & T \end{array}$$

  is a pushout square in $\mathbf{C}$*, then* $g \in \mathbb{L}$*;*
- *suppose*

$$X_0 \xrightarrow{l_0} \cdots \xrightarrow{l_{n-1}} X_n \xrightarrow{l_n} \cdots$$

  is a sequence of maps $l_n \in \mathbb{L}$*, with colimit* $(X, m_n : X_n \to X)$*. Then* $m_0 : X_0 \to X$ *belongs to* $\mathbb{L}$*.*

Definition 6.6. An ∞-category C is *cofibrant* if $0 \to C$ is a cofibration.

Proposition 6.7. *Free ∞-categories are cofibrant.*

Proof. Let S be a polygraph and $C = S^*$. By Lemma 4.3, for each $n \geqslant -1$, the canonical inclusion $j^{(n)} : C^{(n)} \to C^{(n+1)}$ is a pushout of $\sum_{S_n} \mathrm{i}_n$. Now Lemma 6.5 applies in the particular case where $\mathbb{L}$ is the class of cofibrations: by the first point, $\sum_{S_n} \mathrm{i}_n$ is a cofibration, and by the second point, so is $j^{(n)}$. By Lemma 3.1, C is a colimit of the sequence

$$C^{(-1)} \xrightarrow{j^{(-1)}} C^{(0)} \xrightarrow{j^{(0)}} \cdots \xrightarrow{j^{(n-1)}} C^{(n)} \xrightarrow{j^{(n)}} \cdots ;$$

hence the third point of Lemma 6.5 applies, with $X_n = C^{(n-1)}$ and $l_n = j^{(n-1)}$, so that $0 \to C$ is a cofibration. In other words, C is cofibrant. $\square$

7. Cauchy-completeness

We are now ready to establish the converse of Proposition 6.7. Recall from Section 4.3 that $\mathbf{Fcat}_\infty$ is the full subcategory of $\mathbf{Cat}_\infty$ whose objects are all ∞-categories freely generated by polygraphs. The core of our argument is the following theorem:

Theorem 7.1. $\mathbf{Fcat}_\infty$ *is Cauchy-complete.*

In other words, idempotent morphisms in $\mathbf{Fcat}_\infty$ split; that is, for each object C in $\mathbf{Fcat}_\infty$, and each endomorphism $h\colon C \to C$ such that $h \circ h = h$, there is an object D in $\mathbf{Fcat}_\infty$, together with morphisms $r\colon C \to D$, $u\colon D \to C$, satisfying $r \circ u = \mathrm{id}$ and $u \circ r = h$.

Proof. The proof will occupy most of this section. Let S be a polygraph, and let $h\colon S^* \to S^*$ be an idempotent morphism in $\mathbf{Cat}_\infty$. We need to build a polygraph T, together with morphisms $u\colon T^* \to S^*$ and $r\colon S^* \to T^*$, such that

$$r \circ u = \mathrm{id}, \tag{16}$$

$$u \circ r = h. \tag{17}$$

We shall define T, u and r inductively on the dimension. In dimension 0,

$$T_0 = \{h(x) \mid x \in S_0^* = S_0\},$$

u is the inclusion $T_0^* = T_0 \to S_0^* = S_0$, and for each $x \in S_0$, $r(x) = h(x)$. The equations (16) and (17) are clearly satisfied.

Suppose now that $n > 0$ and T, u, r have been defined up to dimension $n{-}1$, and satisfy the required conditions. We shall extend the $n{-}1$ polygraph T to an n-polygraph, and the morphisms u, r of $n{-}1$-categories to morphisms of n-categories still satisfying the above equations.

▷ *Step 1.* Let us split S_n in three subsets S_n^0, S_n^1 and S_n^2, according to the value of $h(\alpha^*)$, for $\alpha \in S_n$:

- $S_n^0 = \{\alpha \in S_n \mid \mathrm{w}(h(\alpha^*)) = 0\}$, hence S_n^0 is the set of generators whose image by h is an identity;
- S_n^1 is the set of generators $\alpha \in S_n$ such that $\mathrm{w}_\alpha(h(\alpha^*)) = 1$ and $\mathrm{w}_\beta(h(\alpha^*)) = 0$ if $\beta \notin S_n^0 \cup \{\alpha\}$;
- $S_n^2 = S_n \setminus S_n^0 \cup S_n^1$.

We may now define a set T_n by:

$$T_n = \{h(\alpha^*) \mid \alpha \in S_n^1\}.$$

By definition, there is an inclusion map

$$\upsilon\colon T_n \to S_n^*$$

such that

$$h \circ \upsilon = \upsilon. \tag{18}$$

Indeed, elements of T_n belong to the image of the idempotent h; hence they are fixed by h. We now define a graph $\sigma^T, \tau^T : T^*_{n-1} \leftleftarrows T_n$ by

$$\sigma^T = r \circ \sigma_{n-1} \circ \upsilon \tag{19}$$
$$\tau^T = r \circ \tau_{n-1} \circ \upsilon, \tag{20}$$

where σ_{n-1}, τ_{n-1} are the source and target maps in S^* and r is given by the induction hypothesis:

$$\begin{array}{ccc} T^*_{n-1} & \xleftarrow{\sigma^T,\tau^T} & T_n \\ {\scriptstyle r}\uparrow & & \downarrow{\scriptstyle \upsilon} \\ S^*_{n-1} & \xleftarrow[\sigma_{n-1},\tau_{n-1}]{} & S^*_n. \end{array}$$

By using the fact that r is a morphism up to dimension $n-1$, we see that for each $\theta \in T_n$, $\sigma^T(\theta) \parallel \tau^T(\theta)$ and the boundary conditions are satisfied. Thus, by Lemma 4.1, T extends to an n-polygraph and the free $n-1$-category T^* extends to a free n-category. We still denote these extensions by T, T^*, and the source and target maps $T^*_{n-1} \leftleftarrows T^*_n$ by σ^T, τ^T. On the other hand,

$$\begin{aligned} u \circ \sigma^T &= u \circ r \circ \sigma_{n-1} \circ \upsilon, \\ &= h \circ \sigma_{n-1} \circ \upsilon, \\ &= \sigma_{n-1} \circ h \circ \upsilon, \\ &= \sigma_{n-1} \circ \upsilon, \end{aligned}$$

and the following diagram commutes

$$\begin{array}{ccc} T^*_{n-1} & \xleftarrow{\sigma^T} & T_n \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle \upsilon} \\ S^*_{n-1} & \xleftarrow[\sigma_{n-1}]{} & S^*_n. \end{array}$$

Likewise

$$u \circ \tau^T = u \circ r \circ \tau_{n-1} \circ \upsilon.$$

Hence $\upsilon\colon T_n \to S^*_n$ gives rise to $u_n\colon T^*_n \to S^*_n$, extending u to a morphism of n-categories $T^* \to S^*$. Note that $h \circ u = u$. To sum up, we have extended T and u up to dimension n. Remark that the only property of T_n we needed so far is that its elements are fixed by h.

▷ *Step 2.* We introduce the auxiliary n-polygraph U such that

- U is identical to S up to dimension $n-1$;
- $U_n = S^0_n + S^1_n$ and the source and target maps $U^*_{n-1} \leftleftarrows U_n$ simply restrict those on S_n.

Thus we get an inclusion monomorphism of n-polygraphs $\iota\colon U \to S$, generating a monomorphism of n-categories $\iota^*\colon U^* \to S^*$. The restrictions of σ_{n-1} and τ_{n-1} to U^*_n will be denoted by σ^U and τ^U, as well as the corresponding maps on generators: $U^*_{n-1} \leftleftarrows U_n$.

Lemma 7.2. *There are morphisms of n-categories*

$$h'\colon U^* \to U^*, \quad k\colon S^* \to U^*,$$

such that the following diagram commutes:

$$\begin{array}{ccc} U^* & \xrightarrow{\iota^*} & S^* \\ {\scriptstyle h'}\downarrow & \swarrow{\scriptstyle k} & \downarrow{\scriptstyle h} \\ U^* & \xrightarrow[\iota^*]{} & S^*. \end{array}$$

Proof. The existence of h' making the outer square commutative follows from the remark that U^* is stable by h, so that h' is simply the restriction of h to U^*.

The existence of a factorization $h = \iota^* \circ k$ reduces to the fact that U_n contains all n-generators α such that $\mathrm{w}_\alpha(y) \neq 0$ for some n-cell y in the image of h. Thus, let $y = h(x)$ in S_n^*. Because h is idempotent, $h(y) = y$. Consider

$$Y = \{\alpha \in S_n \mid \alpha \notin S_n^0 \text{ and } \mathrm{w}_\alpha(y) > 0\}.$$

We just need to prove that $Y \subset S_n^1$. First note that, for each $\beta \in S_n$, $\mathrm{w}_\beta(y) = \mathrm{w}_\beta(h(y))$ so that, by using (9) from Section 4.3:

$$\mathrm{w}_\beta(y) = \sum_{\alpha \in S_n} \mathrm{w}_\alpha(y)\mathrm{w}_\beta(h(\alpha^*)). \tag{21}$$

If $\alpha \notin Y$, either $\mathrm{w}_\alpha(y) = 0$ or $\alpha \in S_n^0$, so that $\mathrm{w}(h(\alpha^*)) = 0$. In both cases, the product $\mathrm{w}_\alpha(y)\mathrm{w}_\beta(h(\alpha^*))$ vanishes. Hence (21) becomes

$$\mathrm{w}_\beta(y) = \sum_{\alpha \in Y} \mathrm{w}_\alpha(y)\mathrm{w}_\beta(h(\alpha^*)). \tag{22}$$

Now, if $\beta \in Y$, then $\mathrm{w}_\beta(y) > 0$ and the right member of (22) does not vanish either. Therefore, there is at least one $\alpha \in Y$ such that $\mathrm{w}_\beta(h(\alpha^*)) > 0$.

On the other hand, let us show that, for each $\alpha \in Y$, there is at least one $\gamma \in Y$ such that $\mathrm{w}_\gamma(h(\alpha^*)) > 0$. Suppose the contrary and let $\alpha \in Y$ such that for all $\gamma \in Y$, $\mathrm{w}_\gamma(h(\alpha^*)) = 0$. As by definition $\mathrm{w}(h(\alpha^*)) > 0$, there is at least one $\beta \in S_n \setminus Y$ such that $\mathrm{w}_\beta(h(\alpha^*)) > 0$. But $\mathrm{w}_\beta(h(\alpha^*)) = \mathrm{w}_\beta(h(h(\alpha^*)))$ and (9) gives

$$\mathrm{w}_\beta(h(\alpha^*)) = \sum_{\gamma \in S_n} \mathrm{w}_\gamma(h(\alpha^*))\mathrm{w}_\beta(h(\gamma^*)).$$

In the above sum, $\mathrm{w}_\gamma(h(\alpha^*)) = 0$ whenever $\gamma \in Y$ or $\mathrm{w}_\gamma(y) = 0$, whence

$$\mathrm{w}_\beta(h(\alpha^*)) = \sum_{\gamma \in S_n^0} \mathrm{w}_\gamma(h(\alpha^*))\mathrm{w}_\beta(h(\gamma^*));$$

but, $\gamma \in S_n^0$ implies $\mathrm{w}_\beta(h(\gamma^*)) = 0$. Hence $\mathrm{w}_\beta(h(\alpha^*)) = 0$, which is a contradiction. For each $\alpha \in y$, let

$$m_\alpha = \sum_{\beta \in Y} \mathrm{w}_\beta(h(\alpha^*)).$$

We have just shown that for each $\alpha \in Y$, $m_\alpha > 0$. By taking the sum over all generators β in Y in (22), we get

$$\sum_{\beta \in Y} \mathrm{w}_\beta(y) = \sum_{\alpha \in Y} \mathrm{w}_\alpha(y) m_\alpha,$$

which implies that $m_\alpha = 1$ for each $\alpha \in Y$. This determines a map $\omega\colon Y \to Y$ which to each $\alpha \in Y$ associates the unique $\beta = \omega(\alpha)$ in Y such that $\mathrm{w}_\beta(h(\alpha^*)) > 0$; in fact $\mathrm{w}_\beta(h(\alpha^*)) = 1$. We have shown earlier that ω is surjective. Finally, let $\alpha \in Y$ and $\beta = \omega(\alpha)$; we have as above

$$\mathrm{w}_\beta(h(\alpha^*)) = \sum_{\gamma \in S_n} \mathrm{w}_\gamma(h(\alpha^*)) \mathrm{w}_\beta(h(\gamma^*)),$$

where all terms in the sum vanish, but for $\gamma = \beta$; whence

$$\mathrm{w}_\beta(h(\alpha^*)) = \mathrm{w}_\beta(h(\alpha^*)) \mathrm{w}_\beta(h(\beta^*)).$$

This implies $\mathrm{w}_\beta(h(\beta^*)) = 1$. Therefore $\omega(\beta) = \beta$ and $\omega \circ \omega = \omega$. Being surjective, ω is necessarily the identity.

To sum up, for each $\alpha \in Y$, $\mathrm{w}_\alpha(h(\alpha^*)) = 1$, and $\mathrm{w}_\beta(h(\alpha^*)) = 0$ if $\beta \notin S_n^0 \cup \{\alpha\}$, that is $\alpha \in S_n^1$ and we are done. As for the upper-left triangle, $\iota^* \circ k \circ \iota^* = h \circ \iota^* = \iota^* \circ h'$, and because ι^* is a monomorphism, $k \circ \iota^* = h'$. □

Thus, let $u'\colon T^* \to U^*$ defined by $u' = k \circ u$, we get $\iota^* \circ u' = \iota^* \circ k \circ u = h \circ u = u$.

▷ *Step 3.* We now define a morphism $r'\colon U^* \to T^*$ which coincides with r in dimensions $i < n$. All we need is a map

$$\rho\colon U_n \to T_n^*$$

satisfying the boundary conditions. Thus, let $\alpha \in U_n$, we distinguish two cases, according as $\alpha \in S_n^0$ or $\alpha \in S_n^1$.

◇ *Case 1.* Let $\alpha \in S_n^0$. There is a unique $y \in S_{n-1}^*$ such that $h(\alpha^*) = 1_n(y)$. Now $r(y) \in T_{n-1}^*$, so that we may define $\rho(\alpha) = 1_n(r(y))$. The boundary conditions are straightforward in this case.

◇ *Case 2.* Let $\alpha \in S_n^1$. There is a unique generator $\theta \in T_n$ such that $h(\alpha^*) = \upsilon(\theta)$. We define $\rho(\alpha) = \theta^*$. By using the induction hypothesis on r and u, we get

$$\begin{aligned}
\sigma^T(\rho(\alpha)) &= \sigma^T(\theta^*) \\
&= r(\sigma_{n-1}(\upsilon(\theta))) \\
&= r(\sigma_{n-1}(h(\alpha^*))) \\
&= r(h(\sigma_{n-1}(\alpha^*))) \\
&= r(u(r(\sigma_{n-1}(\alpha^*)))) \\
&= r(\sigma_{n-1}(\alpha^*)) \\
&= r'(\sigma^U(\alpha)).
\end{aligned}$$

Hence $\sigma^T(\rho(\alpha)) = r'(\sigma^U(\alpha))$ and likewise $\tau^T(\rho(\alpha)) = r'(\tau^U(\alpha))$; the boundary conditions are satisfied.

Thus ρ gives rise to a morphism of ∞-categories $r' \colon U^* \to T^*$ extending r up to dimension n.

$\triangleright$ *Step 4.* Having defined $u' \colon T^* \to U^*$ and $r' \colon U^* \to T^*$, we first note that $u' \circ r' = h'$, which directly follows from our definition of r'. We now prove the following lemma:

Lemma 7.3. $r' \circ u' = \mathrm{id}$.

Proof. $r' \circ u'$ is an endomorphism of the ∞-category T^*. We know by the induction hypothesis that $r' \circ u' = r \circ u = \mathrm{id}$ in all dimensions $i < n$. Thus, it suffices to show that, for each generator $\theta \in T_n$,

$$r'(u'(\theta^*)) = \theta^*. \tag{23}$$

This follows from two facts:

- the two members of (23) are parallel cells,

  $$\sigma^T(r'(u'(\theta^*))) = r'(u'(\sigma^T(\theta^*))),$$

  because r', u' are morphisms. But $\sigma^T(\theta^*)$ has dimension $n-1$, where, by the induction hypothesis, $r' \circ u' = \mathrm{id}$, so that the above equation becomes

  $$\sigma^T(r'(u'(\theta^*))) = \sigma^T(\theta^*)$$

  and likewise

  $$\tau^T(r'(u'(\theta^*))) = \tau^T(\theta^*).$$

- there is a *thin* n-context $c[\mathbf{x}]$ in T^* such that

  $$r'(u'(\theta^*)) = c[\theta^*].$$

  In fact, by the definition of T_n, there is a generator $\alpha \in S_n^1$ such that $u'(\theta^*) = h(\alpha^*)$. Hence there is an n-context $d[\mathbf{y}]$ in U^* such that $u'(\theta^*) = d[\alpha^*]$ and $\mathrm{w}_\beta(d[\mathbf{y}]) = 0$ whenever $\beta \notin S_n^0$. Now by applying (11) of Section 5.1,

  $$\begin{aligned} r'(d[\alpha^*]) &= d^{r'}[r'(\alpha^*)] \\ &= d^{r'}[\rho(\alpha)] \\ &= d^{r'}[\theta^*]. \end{aligned}$$

  Define $c[\mathbf{x}] = d^{r'}[\mathbf{x}]$. For each generator $\psi \in T_n$, by (9),

  $$\mathrm{w}_\psi(c[\theta^*]) = \mathrm{w}_\psi(r'(d[\alpha^*])) = \sum_{\beta \in U_n} \mathrm{w}_\beta(d[\alpha^*]) \mathrm{w}_\psi(r'(\beta^*)).$$

  In the last sum, all terms vanish except for $\beta = \alpha$; hence

  $$\mathrm{w}_\psi(c[\theta^*]) = \mathrm{w}_\psi(\theta^*).$$

  By (10), this implies $\mathrm{w}_\psi(c[\mathbf{x}]) = 0$. Therefore $c[\mathbf{x}]$ is thin, and we are done.

$c[\mathbf{x}]$ is a thin context such that $c[\theta^*] \parallel \theta^*$. By Lemma 5.6, $c[\theta^*] = \theta^*$ and (23) is proved. $\square$

▷ *Step 5.* We complete the argument by defining $r = r' \circ k$. Hence r is a morphism $S^* \to T^*$ and

$$\begin{aligned} u \circ r &= \iota^* \circ u' \circ r' \circ k, \\ &= \iota^* \circ h' \circ k, \\ &= \iota^* \circ k \circ \iota^* \circ k, \\ &= h \circ h, \\ &= h. \end{aligned}$$

Also

$$\begin{aligned} r \circ u &= r' \circ k \circ \iota^* \circ u', \\ &= r' \circ h' \circ u', \\ &= r' \circ u' \circ r' \circ u', \\ &= \mathrm{id} \circ \mathrm{id}, \\ &= \mathrm{id}. \end{aligned}$$

Thus (16) and (17) hold in dimension n completing the proof of Theorem 7.1. □

This easily leads to our main result:

Theorem 7.4. *Any cofibrant ∞-category is isomorphic to a free one.*

Proof. Let C be a cofibrant ∞-category. By Proposition 6.3, C has a free resolution $p\colon S^* \to C$, with S^* an object of $\mathbf{Fcat}_\infty$. Because C is cofibrant, and p is a trivial fibration, the identity morphism $\mathrm{id}_C\colon C \to C$ lifts through p, whence a morphism $q\colon C \to S^*$ such that $p \circ q = \mathrm{id}_C$. Let $h = q \circ p$, $h \circ h = q \circ p \circ q \circ p = q \circ \mathrm{id}_C \circ p = q \circ p = h$; hence h is an idempotent endomorphism of S^*. By Theorem 7.1 on Cauchy-completeness, we get a polygraph T, and morphisms $r\colon S^* \to T^*$, $u\colon T^* \to S^*$ such that $r \circ u = \mathrm{id}_{T^*}$ and $u \circ r = h$. Now, let $f = p \circ u\colon T^* \to C$ and $g = r \circ q\colon C \to T^*$. We get

$$\begin{aligned} g \circ f &= r \circ q \circ p \circ u \\ &= r \circ h \circ u \\ &= r \circ u \circ r \circ u \\ &= \mathrm{id}_{T^*} \circ \mathrm{id}_{T^*} \\ &= \mathrm{id}_{T^*}. \end{aligned}$$

Likewise

$$\begin{aligned} f \circ g &= p \circ u \circ r \circ q \\ &= p \circ h \circ q \\ &= p \circ q \circ p \circ q \\ &= \mathrm{id}_C \circ \mathrm{id}_C \\ &= \mathrm{id}_C. \end{aligned}$$

Hence $f\colon T^* \to C$ is an isomorphism with inverse $g = f^{-1}$ so that C is isomorphic to a free object, as required. □

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