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OPERATIONS ON THE HOPF-HOCHSCHILD COMPLEX FOR MODULE-ALGEBRAS

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Abstract

It is proved that Kaygun's Hopf-Hochschild cochain complex for a module-algebra is a brace algebra with multiplication. As a result, an analogue of Deligne's Conjecture holds for module-algebras, and the Hopf-Hochschild cohomology of a module-algebra has a Gerstenhaber algebra structure.

1. Introduction

Let H be a bialgebra, and let A be an associative algebra. The algebra A is said to be an *H-module-algebra* if there is an H-module structure on A such that the multiplication on A becomes an H-module morphism. For example, if S denotes the Landweber-Novikov algebra [**15, 22**], then the complex cobordism $MU^*(X)$ of a topological space X is an S-module-algebra. Likewise, the singular mod p cohomology $H^*(X;\mathbf{F}_p)$ of a topological space X is an $\mathcal{A}_p$-module-algebra, where $\mathcal{A}_p$ denotes the Steenrod algebra associated to the prime p [**7, 20**]. Other similar examples from algebraic topology can be found in [**4**]. Important examples of module-algebras from Lie and Hopf algebras theory can be found in, e.g., [**12**, V.6].

In [**14**] Kaygun defined a Hochschild-like cochain complex $\mathrm{CH}^*_{\mathrm{Hopf}}(A,A)$ associated to an H-module-algebra A, called the *Hopf-Hochschild cochain complex*, that takes into account the H-linearity. In particular, if H is the ground field, then Kaygun's Hopf-Hochschild cochain complex reduces to the usual Hochschild cochain complex $C^*(A,A)$ of A [**11**]. Kaygun [**14**] showed that the Hopf-Hochschild cohomology of A shares many properties with the usual Hochschild cohomology. For example, it can be described in terms of derived functors, and it satisfies Morita invariance.

The usual Hochschild cochain complex $C^*(A,A)$ has a very rich structure. Namely, it is a brace algebra with multiplication [**10**]. Combined with a result of McClure and Smith [**19**] concerning the singular chain operad associated to the little squares operad $\mathcal{C}_2$, the brace algebra with multiplication structure on $C^*(A,A)$ leads to a positive solution of Deligne's Conjecture [**6**]. Also, passing to cohomology, the brace algebra with multiplication structure implies that the Hochschild cohomology modules $HH^*(A,A)$ form a Gerstenhaber algebra, which is a graded version of a Poisson algebra. This fact was first observed by Gerstenhaber [**8**].

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The purpose of this note is to observe that Kaygun's Hopf-Hochschild cochain complex $\mathrm{CH}^*_{\mathrm{Hopf}}(A, A)$ of a module-algebra A also admits the structure of a brace algebra with multiplication. In fact, the Hopf-Hochschild complex is a sub-complex of the usual Hochschild complex, and the former inherits the latter's structure. As in the classical case, this leads to a version of Deligne's Conjecture for module-algebras. Also, the Hopf-Hochschild cohomology modules $HH^*_{\mathrm{Hopf}}(A, A)$ form a Gerstenhaber algebra. When the bialgebra H is the ground field, these structures reduce to the ones in Hochschild cohomology.

A couple of remarks are in order. First, there is another cochain complex $C^*_{MA}(A)$ that can be associated to an H-module-algebra A [**24**]. The cochain complex $C^*_{MA}(A)$ is different from the Hopf-Hochschild complex $\mathrm{CH}^*_{\mathrm{Hopf}}(A, A)$. The former controls the deformations of A, in the sense of Gerstenhaber [**9**], with respect to both the H-module structure and the algebra structure on A. It is not known if there is any relationship between the two complexes.

Second, the results and arguments here can be adapted to module-coalgebras, comodule-algebras, and comodule-coalgebras. To do that, one replaces the crossed product algebra X (section 2.3) associated to an H-module-algebra A by a suitable crossed product (co)algebra [**1, 2, 3**] and replaces Kaygun's Hopf-Hochschild cochain complex by a suitable variant.

1.1. Organization

The rest of this paper is organized as follows.

In the following section, we recall the construction of the Hopf-Hochschild cochain complex $\mathrm{CH}^*_{\mathrm{Hopf}}(A, A)$ from Kaygun [**14**].

In Section 3, it is observed that $\mathrm{CH}^*_{\mathrm{Hopf}}(A, A)$ has the structure of an operad with multiplication (Theorem 3.1). This leads in Section 4 to the desired brace algebra with multiplication structure on $\mathrm{CH}^*_{\mathrm{Hopf}}(A, A)$ (Corollary 4.1). Explicit formulas of these brace operations are given (11).

In Section 5, it is observed that the brace algebra with multiplication structure on $\mathrm{CH}^*_{\mathrm{Hopf}}(A, A)$ leads to a homotopy G-algebra structure (Corollary 5.1). The differential from this homotopy G-algebra and the Hopf-Hochschild differential are then identified, up to a sign (Theorem 5.2).

Passing to cohomology, this leads in Section 6 to a Gerstenhaber algebra structure on the Hopf-Hochschild cohomology modules $HH^*_{\mathrm{Hopf}}(A, A)$ (Corollary 6.1). The graded associative product and the graded Lie bracket on $HH^*_{\mathrm{Hopf}}(A, A)$ are explicitly described (16).

In the final section, combining our results with a result of McClure and Smith [**19**], a version of Deligne's Conjecture for module-algebras is obtained (Corollary 7.1). This section can be read immediately after Section 4 and is independent of Sections 5 and 6.

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2. Hopf-Hochschild cohomology

In this section, we fix some notations and recall from [**14**, Section 3] the Hopf-Hochschild cochain complex associated to a module-algebra.

2.1. Notations

Fix a ground field K once and for all. Tensor product and vector space are all meant over K.

Let $H = (H, \mu_H, \Delta_H)$ denote a K-bialgebra with associative multiplication μ_H and coassociative comultiplication Δ_H. It is assumed to be unital and counital, with its unit and counit denoted by 1_H and $\varepsilon\colon H \to K$, respectively.

Let $A = (A, \mu_A)$ denote an associative, unital K-algebra with unit 1_A (or simply 1).

In a coalgebra (C, Δ), we use Sweedler's notation [**23**] for comultiplication:

$$\Delta(x) = \sum x_{(1)} \otimes x_{(2)}, \Delta^2(x) = \sum x_{(1)} \otimes x_{(2)} \otimes x_{(3)}, \text{etc.}$$

These notations will be used throughout the rest of this paper.

2.2. Module-algebra

Recall that the algebra A is said to be an *H-module-algebra* [**5, 12, 21, 23**] if and only if there exists a left H-module structure on A such that μ_A and the unit map $1_A\colon K \to A$ are both H-module morphisms, i.e.,

$$b(a_1a_2) = \sum (b_{(1)}a_1)(b_{(2)}a_2) \tag{1}$$

for $b \in H$ and $a_1, a_2 \in A$, and

$$b(1_A) = \varepsilon(b)1_A$$

for $b \in H$.

We will assume that A is an H-module-algebra for the rest of this paper.

2.3. Crossed product algebra

Define the vector space

$$X \overset{\text{def}}{=} A \otimes A \otimes H.$$

Define a multiplication on X [**14**, Definition 3.1] by setting

$$(a_1 \otimes a_1' \otimes b^1)(a_2 \otimes a_2' \otimes b^2) = \sum a_1 \left(b^1_{(1)}a_2\right) \otimes \left(b^1_{(3)}a_2'\right) a_1' \otimes b^1_{(2)}b^2$$

for $a_1 \otimes a_1' \otimes b^1$ and $a_2 \otimes a_2' \otimes b^2$ in X. It is shown in [**14**, Lemma 3.2] that X is an associative, unital K-algebra, called the *crossed product algebra*.

Note that if $H = K$ (= the trivial group bialgebra $K[\{e\}]$), then X is just the enveloping algebra $A \otimes A^{\text{op}}$, where A^{op} is the opposite algebra of A.

The algebra A is a left X-module via the action

$$(a \otimes a' \otimes b)a_0 = a(ba_0)a'$$

for $a \otimes a' \otimes b \in X$ and $a_0 \in A$. Likewise, the vector space $A^{\otimes(n+2)}$ is a left X-module via the action

$$(a \otimes a' \otimes b)(a_0 \otimes \cdots \otimes a_{n+1}) = \sum ab_{(1)}a_0 \otimes b_{(2)}a_1 \otimes \cdots \otimes b_{(n+1)}a_n \otimes b_{(n+2)}a_{n+1}a'$$

for $a_0 \otimes \cdots \otimes a_{n+1} \in A^{\otimes(n+2)}$.

2.4. Bar complex

Consider the chain complex $\mathrm{CB}_*(A)$ of vector spaces with

$$\mathrm{CB}_n(A) = A^{\otimes(n+2)},$$

and with differential

$$d_n^{\mathrm{CB}} = \sum_{j=0}^{n} (-1)^j \partial_j \colon \mathrm{CB}_n(A) \to \mathrm{CB}_{n-1}(A),$$

where

$$\partial_j(a_0 \otimes \cdots \otimes a_{n+1}) = a_0 \otimes \cdots \otimes (a_j a_{j+1}) \otimes \cdots \otimes a_{n+1}.$$

It is mentioned above that each vector space $\mathrm{CB}_n(A) = A^{\otimes(n+2)}$ is a left X-module. Using the module-algebra condition (1), it is straightforward to see that each ∂_j is X-linear. Therefore, $\mathrm{CB}_*(A)$ can be regarded as a chain complex of left X-modules.

Note that in the case $H = K$, the chain complex $\mathrm{CB}_*(A)$ of $A \otimes A^{\mathrm{op}}$-modules is the usual bar complex of A.

2.5. Hopf-Hochschild cochain complex

The *Hopf-Hochschild cochain complex of A with coefficients in A* is the cochain complex of vector spaces:

$$(\mathrm{CH}^*_{\mathrm{Hopf}}(A, A), d_{\mathrm{CH}}) \stackrel{\mathrm{def}}{=} \mathrm{Hom}_X((\mathrm{CB}_*(A), d^{\mathrm{CB}}), A). \tag{2}$$

Its nth cohomology module is denoted by $HH^n_{\mathrm{Hopf}}(A, A)$ and is called the *nth Hopf-Hochschild cohomology of A with coefficients in A*.

When $H = K$, the cochain complex $(\mathrm{CH}^*_{\mathrm{Hopf}}(A, A), d_{\mathrm{CH}})$ is the usual Hochschild cochain complex of A with coefficients in itself [**11**], and $HH^n_{\mathrm{Hopf}}(A, A)$ is the usual Hochschild cohomology module.

In what follows, we will use the notation $\mathrm{CH}^*_{\mathrm{Hopf}}(A, A)$ to denote (i) the Hopf-Hochschild cochain complex $(\mathrm{CH}^*_{\mathrm{Hopf}}(A, A), d_{\mathrm{CH}})$, (ii) the sequence $\{\mathrm{CH}^n_{\mathrm{Hopf}}(A, A)\}$ of vector spaces, or (iii) the graded vector space $\oplus_n \mathrm{CH}^n_{\mathrm{Hopf}}(A, A)$. It should be clear from the context what $\mathrm{CH}^*_{\mathrm{Hopf}}(A, A)$ means.

3. Algebraic operad

The purpose of this section is to show that the vector spaces $\mathrm{CH}^*_{\mathrm{Hopf}}(A, A)$ in the Hopf-Hochschild cochain complex of an H-module-algebra A with self coefficients form an operad with multiplication (Theorem 3.1).

3.1. Operads

Recall from [**16, 17, 18**] that a *non-*Σ *operad* $\mathcal{O} = \{\mathcal{O}(n), \gamma, \mathrm{Id}\}$ consists of a sequence of vector spaces $\mathcal{O}(n)$ $(n \geqslant 1)$ together with structure maps

$$\gamma\colon \mathcal{O}(k) \otimes \mathcal{O}(n_1) \otimes \cdots \otimes \mathcal{O}(n_k) \to \mathcal{O}(n_1 + \cdots + n_k),$$

for $k, n_1, \ldots, n_k \geqslant 1$ and an *identity element* $\mathrm{Id} \in \mathcal{O}(1)$, satisfying the following two axioms.

1. The structure maps γ are required to be *associative*, in the sense that
$$\gamma(\gamma(f; g_{1,k}); h_{1,N}) = \gamma(f; \gamma(g_1; h_{1,N_1}), \ldots, \gamma(g_i; h_{N_{i-1}+1,N_i}), \ldots, \gamma(g_k; h_{N_{k-1}+1,N_k})).$$
Here $f \in \mathcal{O}(k)$, $g_i \in \mathcal{O}(n_i)$, $N = n_1 + \cdots + n_k$, and $N_i = n_1 + \cdots + n_i$. Given elements $x_i, x_{i+1}, \ldots$, the symbol $x_{i,j}$ is the abbreviation for the sequence x_i, x_{i+1}, …, x_j or $x_i \otimes \cdots \otimes x_j$ whenever $i \leqslant j$.
2. The maps
$$\gamma(-; \mathrm{Id}, \ldots, \mathrm{Id})\colon \mathcal{O}(k) \to \mathcal{O}(k) \quad \text{and} \quad \gamma(\mathrm{Id}; -)\colon \mathcal{O}(k) \to \mathcal{O}(k)$$
are both equal to the identity map on $\mathcal{O}(k)$ for each $k \geqslant 1$.

For the rest of this paper, we will refer to non-Σ operads simply as operads.

3.2. Operad morphism

Let $\mathcal{O}$ and $\mathcal{P}$ be operads. An *operad morphism* $\phi\colon \mathcal{O} \to \mathcal{P}$ consists of a sequence of linear maps $\phi_n\colon \mathcal{O}(n) \to \mathcal{P}(n)$ such that

$$\phi_1(\mathrm{Id}_{\mathcal{O}}) = \mathrm{Id}_{\mathcal{P}}$$

and the diagram

$$\begin{array}{ccc} \mathcal{O}(k) \otimes \mathcal{O}(n_1) \otimes \cdots \otimes \mathcal{O}(n_k) & \xrightarrow{\gamma} & \mathcal{O}(N) \\ \Big\downarrow{\scriptstyle \phi_k \otimes \phi_{n_1} \otimes \cdots \otimes \phi_{n_k}} & & \Big\downarrow{\scriptstyle \phi_N} \\ \mathcal{P}(k) \otimes \mathcal{P}(n_1) \otimes \cdots \otimes \mathcal{P}(n_k) & \xrightarrow{\gamma} & \mathcal{P}(N) \end{array} \tag{3}$$

commutes for all $k, n_1, \ldots, n_k \geqslant 1$, where $N = n_1 + \cdots + n_k$.

3.3. Operad with multiplication

Let $\mathcal{O}$ be an operad. A *multiplication* on $\mathcal{O}$ [**10**, Section 1.2] is an element $m \in \mathcal{O}(2)$ that satisfies

$$\gamma(m; m, \mathrm{Id}) = \gamma(m; \mathrm{Id}, m). \tag{4}$$

In this case, $(\mathcal{O}, m)$ is called an *operad with multiplication*. A multiplication on $\mathcal{O}$ is equivalent to an operad morphism $\mathcal{A}s \to \mathcal{O}$, where $\mathcal{A}s$ is the operad for associative algebras. The associative operad $\mathcal{A}s$ is generated by an element $\mu \in \mathcal{A}s(2)$, whose image under an operad morphism $\mathcal{A}s \to \mathcal{O}$ is a multiplication on $\mathcal{O}$.

3.4. Operad with multiplication structure on $\mathrm{CH}^*_{\mathrm{Hopf}}(A, A)$

In what follows, in order to simplify the typography, we will sometimes write $\mathcal{C}(n)$ for the vector space $\mathrm{CH}^n_{\mathrm{Hopf}}(A, A)$. To show that the vector spaces $\mathrm{CH}^*_{\mathrm{Hopf}}(A, A)$

form an operad with multiplication, we first define the structure maps, the identity element, and the multiplication.

Structure maps For $k, n_1, \ldots, n_k \geqslant 1$, define a map

$$\gamma\colon \mathcal{C}(k) \otimes \mathcal{C}(n_1) \otimes \cdots \otimes \mathcal{C}(n_k) \to \mathcal{C}(N)$$

by setting

$$\begin{aligned} &\gamma(f; g_{1,k})(a_{0,N+1}) \\ &= f\left(a_0 \otimes g_1(1 \otimes a_{1,n_1} \otimes 1) \otimes \cdots \otimes g_i(1 \otimes a_{N_{i-1}+1,N_i} \otimes 1) \otimes \cdots \otimes a_{N+1}\right). \end{aligned} \tag{5}$$

Here the notation is as in the definition of an operad above, and each $a_i \in A$.

Identity element Let $\mathrm{Id} \in \mathcal{C}(1)$ be the element such that

$$\mathrm{Id}(a_0 \otimes a_1 \otimes a_2) = a_0 a_1 a_2. \tag{6}$$

This is indeed an element of $\mathcal{C}(1)$, since the identity map on A is H-linear.

Multiplication Let $\pi \in \mathcal{C}(2)$ be the element such that

$$\pi(a_0 \otimes a_1 \otimes a_2 \otimes a_3) = a_0 a_1 a_2 a_3. \tag{7}$$

This is indeed an element of $\mathcal{C}(2)$, since the multiplication map $A^{\otimes 2} \to A$ on A is H-linear.

Theorem 3.1. *The data*

$$\mathcal{C} = \{\mathcal{C}(n), \gamma, \mathrm{Id}\}$$

forms an operad. Moreover, $\pi \in \mathcal{C}(2)$ is a multiplication on the operad $\mathcal{C}$.

Proof. It is immediate from (5) and (6) that $\gamma(-; \mathrm{Id}^{\otimes k})$ and $\gamma(\mathrm{Id}; -)$ are the identity map on $\mathcal{C}(k)$ for each $k \geqslant 1$.

To prove associativity of γ, we use the notations in the definition of an operad and compute as follows:

$$\begin{aligned} &\gamma\left(\gamma(f; g_{1,k}); h_{1,N}\right)(a_0 \otimes \cdots \otimes a_{M+1}) \\ &\quad = \gamma(f; g_{1,k})\left(a_0 \otimes \cdots \otimes h_j(1 \otimes a_{M_{j-1}+1,M_j} \otimes 1) \otimes \cdots \otimes a_{M+1}\right) \\ &\quad = f\left(a_0 \otimes \cdots \otimes g_i(1 \otimes z_i \otimes 1) \otimes \cdots \otimes a_{M+1}\right) \\ &\quad = \gamma\left(f; \ldots, \gamma(g_i; h_{N_{i-1}+1,N_i}), \ldots\right)(a_0 \otimes \cdots \otimes a_{M+1}). \end{aligned}$$

Here the element z_i ($1 \leqslant i \leqslant k$) is given by

$$\begin{aligned} z_i &= \bigotimes_{l=N_{i-1}+1}^{N_i} h_l\left(1 \otimes a_{M_{l-1}+1,M_l} \otimes 1\right) \\ &= h_{N_{i-1}+1}\left(1 \otimes a_{M_{N_{i-1}}+1,M_{N_{i-1}+1}} \otimes 1\right) \otimes \cdots \otimes h_{N_i}\left(1 \otimes a_{M_{N_i-1}+1,M_{N_i}} \otimes 1\right). \end{aligned}$$

This shows that γ is associative and that $\mathcal{C} = \{\mathcal{C}(n), \gamma, \mathrm{Id}\}$ is an operad.

To see that $\pi \in \mathcal{C}(2)$ is a multiplication on $\mathcal{C}$, one observes that both

$$\gamma(\pi; \pi, \mathrm{Id})(a_0 \otimes \cdots \otimes a_4) \quad \text{and} \quad \gamma(\pi; \mathrm{Id}, \pi)(a_0 \otimes \cdots \otimes a_4)$$

are equal to the product $a_0 a_1 a_2 a_3 a_4$.

This finishes the proof of Theorem 3.1. ∎

3.5. Relationship with the Hochschild complex

Here we observe that the operad with multiplication $(\mathcal{C}, \pi)$ induces the usual operad with multiplication structure on the Hochschild complex.

Recall that the Hochschild complex

$$C^*(A, A) = \{C^n(A, A) = \mathrm{Hom}_K(A^{\otimes n}, A)\}$$

is an operad with multiplication μ_A [**10**]. Indeed, $C^*(A, A)$ is the endomorphism operad of A. Its structure maps γ are given by

$$\gamma(f; g_{1,k})(a_{1,N}) = f(\cdots \otimes g_i(a_{N_{i-1}+1,N_i}) \otimes \cdots)$$

for $f \in C^n(A, A)$, $g_i \in C^{n_i}(A, A)$, and $a_j \in A$. The notations are as in section 3.1 with $N_0 = 0$. The identity element in $C^1(A, A)$ is the identity map on A.

Denote by

$$\phi_{\mathrm{Hopf}} \colon \mathcal{A}s \to \mathrm{CH}^*_{\mathrm{Hopf}}(A, A) \quad \text{and} \quad \phi \colon \mathcal{A}s \to C^*(A, A)$$

the operad morphisms corresponding to the operads with multiplication $\mathrm{CH}^*_{\mathrm{Hopf}}(A, A)$ and $C^*(A, A)$, respectively. The following observation says that ϕ is induced by ϕ_{Hopf}.

Corollary 3.2. *The operad with multiplication $C^*(A, A)$ is induced by $\mathrm{CH}^*_{\mathrm{Hopf}}(A, A)$ in the sense that there is an operad morphism $\rho \colon \mathrm{CH}^*_{\mathrm{Hopf}}(A, A) \to C^*(A, A)$ such that the diagram*

$$\begin{array}{ccc} \mathcal{A}s & \xrightarrow{\phi_{\mathrm{Hopf}}} & \mathrm{CH}^*_{\mathrm{Hopf}}(A, A) \\ {\scriptstyle \phi}\downarrow & \swarrow {\scriptstyle \rho} & \\ C^*(A, A) & & \end{array} \tag{8}$$

commutes.

Proof. Let us abbreviate $C^n(A, A)$ to C^n and $\mathrm{CH}^n_{\mathrm{Hopf}}(A, A)$ to $\mathcal{C}(n)$. The component map $\rho \colon \mathcal{C}(n) \to C^n$ is defined as

$$\rho(g)(a_{1,n}) = g(1 \otimes a_{1,n} \otimes 1)$$

for $a_i \in A$. To see that the map ρ is an operad morphism, first observe that it preserves the identity elements because

$$\rho(\mathrm{Id})(a) = \mathrm{Id}(1 \otimes a \otimes 1) = a$$

by (6). Moreover, the commutativity of the diagram (3) for ρ follows from the fact that both $\rho\left(\gamma(f; g_{1,k})\right)(a_{1,N})$ and $\gamma\left(\rho(f); \rho(g_{1,k})\right)(a_{1,N})$ are equal to

$$f\left(1 \otimes \cdots \otimes g_i(1 \otimes a_{N_{i-1}+1,N_i} \otimes 1) \otimes \cdots \otimes 1\right).$$

Here $f \in \mathcal{C}(k)$, $g_i \in \mathcal{C}(n_i)$, $N = n_1 + \cdots + n_k$, $N_0 = 0$, and $a_j \in A$.

Now we show that the diagram (8) commutes. Note that the operad morphism ϕ is uniquely determined by the property $\phi(\mu) = \mu_A$, where $\mu \in \mathcal{A}s(2)$ is the generator for the associative operad. Likewise, we have $\phi_{\mathrm{Hopf}}(\mu) = \pi$. Since we have

$$\rho(\pi)(a_1 \otimes a_2) = \pi(1 \otimes a_1 \otimes a_2 \otimes 1) = a_1 a_2$$

by (7), we conclude that $\rho(\pi) = \mu_A$ and $\rho(\phi_{\mathrm{Hopf}}(\mu)) = \mu_A$. This implies by uniqueness that $\phi = \rho \circ \phi_{\mathrm{Hopf}}$. ∎

4. Brace algebra

The purpose of this section is to show that the graded vector space $\mathrm{CH}^*_{\mathrm{Hopf}}(A, A)$ admits the structure of a brace algebra with multiplication. Explicit formulas of the brace operations are given in (11).

4.1. Brace algebra

For a graded vector space $V = \oplus_{n=1}^{\infty} V^n$ and an element $x \in V^n$, set $\deg x = n$ and $|x| = n - 1$. Elements in V^n are said to have *degree n*.

Recall from [**10**, Definition 1] that a *brace algebra* is a graded vector space $V = \oplus V^n$ together with a collection of brace operations $x\{x_1, \ldots, x_n\}$ of degree $-n$, satisfying the associativity axiom:

$$x\{x_{1,m}\}\{y_{1,n}\} = \sum_{0 \leqslant i_1 \leqslant \cdots \leqslant i_m \leqslant n} (-1)^{\varepsilon} x\{y_{1,i_1}, x_1\{y_{i_1+1,j_1}\}, y_{j_1+1}, \ldots, y_{i_m}, \\ x_m\{y_{i_m+1,j_m}\}, y_{j_m+1,n}\}.$$

Here the sign is given by $\varepsilon = \sum_{p=1}^{m} \left(|x_p| \sum_{q=1}^{i_p} |y_q| \right)$.

4.2. Brace algebra with multiplication

Let $V = \oplus V^n$ be a brace algebra. A *multiplication* on V [**10**, Section 1.2] is an element $m \in V^2$ such that

$$m\{m\} = 0. \tag{9}$$

In this case, we call $V = (V, m)$ a *brace algebra with multiplication*.

4.3. Brace algebra from operad

Suppose that $\mathcal{O} = \{\mathcal{O}(n), \gamma, \mathrm{Id}\}$ is an operad. Define the following operations on the graded vector space $\mathcal{O} = \oplus \mathcal{O}(n)$:

$$x\{x_1, \ldots, x_n\} \stackrel{\mathrm{def}}{=} \sum (-1)^{\varepsilon} \gamma(x; \mathrm{Id}, \ldots, \mathrm{Id}, x_1, \mathrm{Id}, \ldots, \mathrm{Id}, x_n, \mathrm{Id}, \ldots, \mathrm{Id}). \tag{10}$$

Here the sum runs over all possible substitutions of $x_1, \ldots, x_n$ into $\gamma(x; \ldots)$ in the given order. The sign is determined by $\varepsilon = \sum_{p=1}^{n} |x_p| i_p$, where i_p is the total number of inputs in front of x_p. Note that

$$\deg x\{x_1, \ldots, x_n\} = \deg x - n + \sum_{p=1}^{n} \deg x_p,$$

so the operation (10) is of degree $-n$.

Proposition 1 in [**10**] establishes that the operations (10) make the graded vector space $\oplus \mathcal{O}(n)$ into a brace algebra. Moreover, a multiplication on the operad $\mathcal{O}$ in the sense of § 3.3 is equivalent to a multiplication on the brace algebra $\oplus \mathcal{O}(n)$. In fact, for an element $m \in \mathcal{O}(2)$, one has that

$$m\{m\} = \gamma(m; m, \mathrm{Id}) - \gamma(m; \mathrm{Id}, m).$$

It follows that the condition (4) is equivalent to (9). In other words, an operad with multiplication $(\mathcal{O}, m)$ gives rise to a brace algebra with multiplication $(\oplus \mathcal{O}(n), m)$. Combining this discussion with Theorem 3.1, we obtain the following result.

Corollary 4.1. *The graded vector space* $\mathrm{CH}^*_{\mathrm{Hopf}}(A, A)$ *is a brace algebra with brace operations as in* (10) *and multiplication* π (7).

The brace operations on $\mathrm{CH}^*_{\mathrm{Hopf}}(A, A)$ can be described more explicitly as follows. For $f \in \mathcal{C}(k)$ and $g_i \in \mathcal{C}(m_i)$ $(1 \leqslant i \leqslant n)$, we have

$$f\{g_1, \dots, g_n\} = \sum (-1)^{\varepsilon} \gamma(f; \mathrm{Id}^{r_1}, g_1, \mathrm{Id}^{r_2}, g_2, \dots, \mathrm{Id}^{r_n}, g_n, \mathrm{Id}^{r_{n+1}}),$$

where $\mathrm{Id}^r = \mathrm{Id}^{\otimes r}$. Here the r_j are given by

$$r_j = \begin{cases} i_1 & \text{if } j = 1, \\ i_j - i_{j-1} - 1 & \text{if } 2 \leqslant j \leqslant n, \\ k - i_n - 1 & \text{if } j = n+1, \end{cases}$$

and

$$\varepsilon = \sum_{p=1}^{n} (m_p - 1) i_p.$$

Write $M = \sum_{i=1}^{n} m_i$ and $M_j = \sum_{i=1}^{j} m_i$. For an element $a_{0,k+M-n+1}$ in $A^{\otimes(k+M-n)}$, we have

$$\begin{aligned} f\{g_{1,n}\}(a_{0,k+M-n+1}) = \sum (-1)^{\varepsilon} f(a_{0,i_1} &\otimes g_1(1 \otimes a_{i_1+1, i_1+m_1} \otimes 1) \otimes \cdots \\ &\otimes a_{i_{j-1}+M_{j-1}-(j-1)+2, i_j+M_{j-1}-j+1} \\ &\otimes g_j(1 \otimes a_{i_j+M_{j-1}-j+2, i_j+M_j-j+1} \otimes 1) \otimes \cdots \\ &\otimes a_{i_n+M-n+2, k+M-n+1}). \end{aligned} \tag{11}$$

5. Homotopy Gerstenhaber algebra

The purpose of this section is to observe that the brace algebra with multiplication structure on $\mathrm{CH}^*_{\mathrm{Hopf}}(A, A)$ induces a homotopy Gerstenhaber algebra structure (Corollary 5.1). The underlying cochain complex of this homotopy Gerstenhaber algebra is canonically isomorphic to the Hopf-Hochschild cochain complex of A (Corollary 5.3).

5.1. Homotopy G-algebra

Recall from [**10**, Definition 2] that a *homotopy G-algebra* $(V, d, \cup)$ consists of a brace algebra $V = \oplus V^n$, a degree $+1$ differential d, and a degree 0 associative $\cup$-product that makes V into a differential graded algebra, satisfying the following two conditions.

1. The $\cup$-product is required to satisfy the condition

$$(x_1 \cup x_2)\{y_{1,n}\} = \sum_{k=0}^{n} (-1)^{\varepsilon} x_1\{y_{1,k}\} \cup x_2\{y_{k+1,n}\},$$

where $\varepsilon = |x_2| \sum_{p=1}^{k} |y_p|$, for $x_i, y_j \in V$.

2. The differential is required to satisfy the condition

$$\begin{aligned} &d(x\{x_{1,n+1}\}) - (dx)\{x_{1,n+1}\} \\ &\quad - (-1)^{|x|}\sum_{i=1}^{n+1}(-1)^{|x_1|+\cdots+|x_{i-1}|}x\{x_1,\ldots,dx_i,\ldots,x_{n+1}\} \\ &= (-1)^{|x||x_1|+1}x_1 \cup x\{x_{2,n+1}\} - x\{x_{1,n}\} \cup x_{n+1} \\ &\quad + (-1)^{|x|}\sum_{i=1}^{n}(-1)^{|x_1|+\cdots+|x_{i-1}|}x\{x_1,\ldots,x_i \cup x_{i+1},\ldots,x_{n+1}\}. \end{aligned}$$

5.2. Homotopy G-algebra from brace algebra with multiplication

A brace algebra with multiplication $V = (V, m)$ gives rise to a homotopy G-algebra $(V, d, \cup)$ [**10**, Theorem 3], where the $\cup$-product and the differential are defined as:

$$\begin{aligned} x \cup y &= (-1)^{\deg x} m\{x, y\}, \\ d(x) &= m\{x\} - (-1)^{|x|}x\{m\}. \end{aligned} \tag{12}$$

In particular, this applies to the brace algebra $\mathrm{CH}^*_{\mathrm{Hopf}}(A, A)$ with multiplication π (Corollary 4.1).

Corollary 5.1. *For an H-module-algebra A, $\mathcal{C} = (\mathrm{CH}^*_{\mathrm{Hopf}}(A, A), d, \cup)$ is a homotopy G-algebra.*

5.3. Comparing differentials

There are two differentials on the graded vector space $\mathrm{CH}^*_{\mathrm{Hopf}}(A, A)$, namely, the differential d^n (12) induced by the multiplication π and the Hopf-Hochschild differential d^n_{CH} (2). The following result ensures that the cohomology modules defined by these two differentials are the same.

Theorem 5.2. *The equality*

$$d^n_{\mathrm{CH}} = (-1)^{n+1} d^n$$

holds for each n.

Proof. Pick $f \in \mathrm{CH}^n_{\mathrm{Hopf}}(A, A)$. Then we have

$$\begin{aligned} d^n f &= \pi\{f\} + (-1)^n f\{\pi\} \\ &= (-1)^{n-1}\gamma(\pi; \mathrm{Id}, f) + \gamma(\pi; f, \mathrm{Id}) + (-1)^n \sum_{i=1}^{n} (-1)^{i-1}\gamma(f; \mathrm{Id}^{i-1}, \pi, \mathrm{Id}^{n-i}). \end{aligned}$$

It follows that

$$(-1)^{n+1} d^n f = \gamma(\pi; \mathrm{Id}, f) + (-1)^{n+1}\gamma(\pi; f, \mathrm{Id}) + \sum_{i=1}^{n}(-1)^i \gamma(f; \mathrm{Id}^{i-1}, \pi, \mathrm{Id}^{n-i}). \tag{13}$$

Observe that [**14**]

$$g(a_{0,n+1}) = a_0 g(1 \otimes a_{1,n} \otimes 1) a_{n+1} \tag{14}$$

for $g \in \mathrm{CH}^n_{\mathrm{Hopf}}(A, A)$. Using (14) and applying the various terms in (13) to an element

$a_{0,n+2} \in \mathrm{CB}_{n+1}(A) = A^{\otimes(n+3)}$, we obtain

$$\begin{aligned} \gamma(\pi; \mathrm{Id}, f)(a_{0,n+2}) &= f(a_0 a_1 \otimes a_{2,n+2}), \\ \gamma(\pi; f, \mathrm{Id})(a_{0,n+2}) &= f(a_{0,n} \otimes a_{n+1} a_{n+2}), \\ \gamma(f; \mathrm{Id}^{i-1}, \pi, \mathrm{Id}^{n-i})(a_{0,n+2}) &= f(a_{0,i-1} \otimes a_i a_{i+1} \otimes a_{i+2,n+2}). \end{aligned} \tag{15}$$

The Theorem now follows immediately from (13) and (15). □

Corollary 5.3. *There is an isomorphism of cochain complexes,*

$$(\mathrm{CH}^*_{\mathrm{Hopf}}(A, A), d_{\mathrm{CH}}) \cong (\mathrm{CH}^*_{\mathrm{Hopf}}(A, A), d)$$
$$x \mapsto (-1)^{\frac{n(n+1)}{2}} x,$$

for $x \in \mathrm{CH}^n_{\mathrm{Hopf}}(A, A)$. *In particular, the cohomology modules on* $\mathrm{CH}^*_{\mathrm{Hopf}}(A, A)$ *defined by the differentials* d_{CH} *and* d *are equal.*

6. Gerstenhaber algebra

The purpose of this section is to observe that the homotopy G-algebra structure on $\mathrm{CH}^*_{\mathrm{Hopf}}(A, A)$ gives rise to a G-algebra structure on the Hopf-Hochschild cohomology modules $HH^*_{\mathrm{Hopf}}(A, A)$. Explicit formulas of these G-algebra operations are given in (16).

6.1. Gerstenhaber algebra

Recall from [**10**, Section 2.2] that a *G-algebra* $(V, \cup, [-,-])$ consists of a graded vector space $V = \oplus V^n$, a degree 0 associative $\cup$-product, and a degree -1 graded Lie bracket

$$[-,-] \colon V^m \otimes V^n \to V^{m+n-1},$$

satisfying the following two conditions:

$$x \cup y = (-1)^{\deg x \deg y} y \cup x,$$
$$[x, y \cup z] = [x, y] \cup z + (-1)^{|x| \deg y} y \cup [x, z].$$

In other words, the $\cup$-product is graded commutative, and the Lie bracket is a graded derivation for the $\cup$-product. In particular, a G-algebra is a graded version of a Poisson algebra. This algebraic structure was first studied by Gerstenhaber [**8**].

6.2. G-algebra from homotopy G-algebra

If $(V, d, \cup)$ is a homotopy G-algebra, one can define a degree -1 operation on V as

$$[x, y] \stackrel{\mathrm{def}}{=} x\{y\} - (-1)^{|x||y|} y\{x\}.$$

Passing to cohomology, $(H^*(V, d), \cup, [-,-])$ becomes a G-algebra ([**10**] Corollary 5 and its proof).

Combining the previous paragraph with Corollary 5.1 and Corollary 5.3, we obtain the following result.

Corollary 6.1. *The Hopf-Hochschild cohomology modules* $HH^*_{\mathrm{Hopf}}(A, A)$ *of an* H-*module-algebra* A *admits the structure of a* G-*algebra.*

This G-algebra can be described on the cochain level more explicitly as follows. Pick $\varphi \in \mathrm{CH}^n_{\mathrm{Hopf}}(A,A)$ and $\psi \in \mathrm{CH}^m_{\mathrm{Hopf}}(A,A)$. Then

$$\begin{aligned}(\psi \cup \varphi)(a_{0,m+n+1}) &= (-1)^{m+n-1}\psi(a_{0,m} \otimes 1)\varphi(1 \otimes a_{m+1,m+n+1}),\\ [\psi,\varphi] &= \psi\{\varphi\} - (-1)^{(m-1)(n-1)}\varphi\{\psi\}.\end{aligned} \tag{16}$$

Writing $a = a_{0,m+n}$, we have

$$\begin{aligned}\psi\{\varphi\}(a) &= \sum_{i=1}^{m} (-1)^{(i-1)(n-1)}\psi(a_{0,i-1} \otimes \varphi(1 \otimes a_{i,i+n-1} \otimes 1) \otimes a_{i+n,m+n}),\\ \varphi\{\psi\}(a) &= \sum_{j=1}^{n} (-1)^{(j-1)(m-1)}\varphi(a_{0,j-1} \otimes \psi(1 \otimes a_{j,j+m-1} \otimes 1) \otimes a_{j+m,m+n}).\end{aligned}$$

In particular, if $m = n = 1$, then the bracket operation

$$[\psi,\varphi](a_{0,2}) = \psi(a_0 \otimes \varphi(1 \otimes a_1 \otimes 1) \otimes a_2) - \varphi(a_0 \otimes \psi(1 \otimes a_1 \otimes 1) \otimes a_2)$$

gives $HH^1_{\mathrm{Hopf}}(A,A)$ a Lie algebra structure. There is another description of this Lie algebra in terms of (inner) derivations in [**14**, Proposition 3.9].

7. Deligne's Conjecture for module-algebras

The purpose of this section is to observe that a version of Deligne's Conjecture holds for the Hopf-Hochschild cochain complex of a module-algebra. The original Deligne's Conjecture for Hochschild cohomology is as follows.

Deligne's Conjecture ([**6**]). The Hochschild cochain complex $C^*(R,R)$ of an associative algebra R is an algebra over a suitable chain model of May's little squares operad $\mathcal{C}_2$ [**17**].

A positive answer to Deligne's conjecture was given by, among others, McClure and Smith [**19**, Theorem 1.1] and Kaufmann [**13**, Theorem 4.2.2]. There is an operad $\mathcal{H}$ whose algebras are the brace algebras with multiplication (section 4.2). For an associative algebra R, the Hochschild cochain complex $C^*(R,R)$ is a brace algebra with multiplication and hence an $\mathcal{H}$-algebra. McClure and Smith showed that $\mathcal{H}$ is quasi-isomorphic to the chain operad $\mathcal{S}$ obtained from the little squares operad $\mathcal{C}_2$ by applying the singular chain functor, thereby proving Deligne's Conjecture.

It has been observed that the Hopf-Hochschild cochain complex $\mathrm{CH}^*_{\mathrm{Hopf}}(A,A)$ is a brace algebra with multiplication (Corollary 4.1). Therefore, we can use the result of McClure and Smith [**19**, Theorem 1.1] to obtain the following version of Deligne's Conjecture for module-algebras.

Corollary 7.1 (Deligne's Conjecture for module-algebras)**.** *The Hopf-Hochschild complex* $\mathrm{CH}^*_{\mathrm{Hopf}}(A,A)$ *of an* H*-module-algebra* A *is an algebra over the McClure-Smith operad* $\mathcal{H}$ *that is a chain model for the little squares operad* $\mathcal{C}_2$.

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