

## SIMPLICIAL POLYTOPE COMPLEXES AND DELOOPINGS OF $K$ -THEORY

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### *Abstract*

This paper is a continuation of the author's previous paper on scissors congruence, in which we defined the notion of a polytope complex and its  $K$ -theory. In this paper we produce formulas for the delooping of a simplicial polytope complex and the cofiber of a morphism of simplicial polytope complexes. Along the way we also prove that the (classical and higher) scissors congruence groups of polytopes in a homogeneous  $n$ -manifold (with sufficient geometric data) are determined by its local properties.

### 1. Introduction

This paper is a continuation of the work started in [8], which introduced the concept of a polytope complex and its  $K$ -theory. The goal of that paper was to define  $K$ -theory in such a way that on  $K_0$  we had the scissors congruence group of the polytope complex. More concretely, a polytope complex  $\mathcal{C}$  is a small double category, which vertically has a Grothendieck topology, and horizontally is a groupoid. Given a polytope complex  $\mathcal{C}$ , we can produce a Waldhausen category  $\mathrm{SC}(\mathcal{C})$  such that  $K_0(\mathrm{SC}(\mathcal{C}))$  is the free abelian group generated by objects of  $\mathcal{C}$  under the two relations  $[A] = [B]$  if  $A$  is horizontally isomorphic to  $B$ , and  $[A] = \sum_{i=1}^n [A_i]$  if the  $A_i$ 's are disjoint (have the vertically initial object as their product) and  $\{A_i \twoheadrightarrow A\}_{i=1}^n$  is a covering family of  $A$  in the vertical topology in  $\mathcal{C}$ .

One fundamental question we can ask about this construction is which spectra appear as the  $K$ -theory of a polytope complex. We know, for example, that Segal's  $\Gamma$ -spaces model all connective spectra, and we may hope that polytope complexes do, as well. In this paper we start answering this question; along the way we produce several computational results for the  $K$ -theory of polytope complexes. The majority of this paper is an analysis of Waldhausen's  $S$ -construction in the particular case of the  $K$ -theory of a polytope complex. In this case it turns out that for a polytope complex  $\mathcal{C}$  we can find a polytope complex  $s_n\mathcal{C}$  such that  $|wS_n\mathrm{SC}(\mathcal{C})| \simeq |w\mathrm{SC}(s_n\mathcal{C})|$ . We can make this construction compatible with the simplicial structure maps from Waldhausen's  $S$ -construction, and therefore construct an  $S$ -construction directly on the polytope level.

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However, as the  $S$ -construction adds an extra simplicial dimension, it becomes necessary to be able to define the  $K$ -theory of a simplicial polytope complex  $\mathcal{C}$ . (Here, as well as in the rest of this paper, we consider a simplicial polytope complex to be a simplicial object in the category of polytope complexes; see section 3 for more details.) As the definition of  $K$ -theory relies on geometric realizations, we can define  $K(\mathcal{C})$  to be the spectrum defined by

$$K(\mathcal{C})_n = |w \underbrace{S \cdots S}_n \text{SC}(\mathcal{C})|.$$

From this definition, we can immediately conclude the following:

**Theorem 1.1.** *All suspension spectra arise as the  $K$ -theory of a simplicial polytope complex.*

In addition, by analyzing the  $S$ -construction on  $\text{SC}(\mathcal{C})$  we obtain the following computation of the delooping of the  $K$ -theory of  $\mathcal{C}$ :

**Theorem 1.2.** *Let  $\mathcal{C}$  be a simplicial polytope complex. Let  $\sigma\mathcal{C}$  be the simplicial polytope complex given by the bar construction. More concretely, we define*

$$(\sigma\mathcal{C})_n = \underbrace{\mathcal{C}_n \vee \mathcal{C}_n \vee \cdots \vee \mathcal{C}_n}_n,$$

*with the simplicial structure maps defined analogously to the usual bar construction. Then  $\Omega K(\sigma\mathcal{C}) \simeq K(\mathcal{C})$ .*

See section 6 and corollary 8.8 for more details.

Unfortunately for our computational aspirations, the definition of the  $K$ -theory of a polytope complex is problematic from a computational standpoint, as it relies on a Waldhausen category which does not come from an exact category (in the sense of Quillen, [3]), does not have a cylinder functor (as in [7], section 1.6) and which is not good (in the sense of Toën, [5]). This means that very few computational techniques are directly available for analyzing this problem, as most approaches covered in the literature depend on one of these properties. The one tool for Waldhausen categories which does not depend in any way on extra assumptions is Waldhausen's cofiber theorem, which, given a functor  $G: \mathcal{E} \rightarrow \mathcal{E}'$  between Waldhausen categories constructs a simplicial Waldhausen category  $S.G$  whose  $K$ -theory is the cofiber of the map  $K(G): K(\mathcal{E}) \rightarrow K(\mathcal{E}')$ . By passing this computation down through the polytope complex construction of  $S$  we find the following formula for the cofiber of a morphism of simplicial polytope complexes.

**Theorem 1.3.** *Let  $g: \mathcal{C} \rightarrow \mathcal{D}$  be a morphism of simplicial polytope complexes. We define a simplicial polytope complex  $(\mathcal{D}/g)$  by setting*

$$(\mathcal{D}/g)_n = \mathcal{D}_n \vee \underbrace{\mathcal{C}_n \vee \mathcal{C}_n \vee \cdots \vee \mathcal{C}_n}_n.$$

*The simplicial structure maps are defined as for  $\mathcal{D} \vee \sigma\mathcal{C}$ , except that  $\partial_0$  is induced by*

the three morphisms

$$\partial_0: \mathcal{D}_n \longrightarrow \mathcal{D}_{n-1} \quad \partial_0 g_n: \mathcal{C}_n \longrightarrow \mathcal{D}_{n-1} \quad 1: \mathcal{C}^{\vee n-1} \longrightarrow \mathcal{C}^{\vee n-1}.$$

Then

$$K(\mathcal{C}.) \xrightarrow{K(g)} K(\mathcal{D}.) \longrightarrow K((\mathcal{D}/g).)$$

is a cofiber sequence of spectra.

See section 7 and corollary 8.8 for more details.

As a consequence of the techniques in section 8 we also get the following proposition:

**Proposition 1.4.** *Let  $X$  and  $Y$  be homogeneous geodesic  $n$ -manifolds with a preferred open cover in which the geodesic connecting any two points in a single set is unique. If there exist preferred open subsets  $U \subseteq X$  and  $V \subseteq Y$  and an isometry  $\varphi: U \rightarrow V$  then the scissors congruence spectra of  $X$  and  $Y$  are equivalent.*

The organization of this paper is as follows. Section 2 covers the notation we use, as well as a basic summary of Waldhausen’s  $S$ -construction and the results we use about it. Section 3 defines the category of polytope complexes. Sections 4 and 5 concern the construction of  $s\mathcal{C}$ . Section 6 describes the fundamental computation necessary for the first theorem, and section 7 the second. Section 8 wraps up the paper with a basic approximation result about simplicial polytope complexes, which allows us to simplify the formulas computed in sections 6 and 7 to the ones used here.

## 2. Preliminaries

### 2.1. Notation

In this paper,  $\mathcal{C}$  and  $\mathcal{D}$  will always denote polytope complexes, while  $\mathcal{E}$  will be a general Waldhausen category. We denote the category of polytope complexes and polytope functors by **PolyCpx**. In a double category (or a polytope complex) we will denote vertical morphism by dotted arrows  $A \dashrightarrow B$  and horizontal morphisms by solid arrows  $A \rightarrow B$ ; functors and morphisms not in a double category or  $\text{SC}(\mathcal{C})$  will be denoted  $X \rightarrow Y$ . If a vertical morphism in  $\text{Tw}(\mathcal{C}_p)$  is a covering sub-map we will denote it by  $A \dashrightarrow B$ . (For more details on polytope complexes and the functors  $\text{Tw}$  and  $\text{SC}$  see [8]; for convenience, we also recall the definition in the next section.)

For any two polytope complexes  $\mathcal{C}$  and  $\mathcal{D}$ ,  $\mathcal{C} \vee \mathcal{D}$  denotes the polytope complex obtained by identifying the two initial objects. For a nonnegative integer  $n$ , we write  $\mathcal{C}^{\vee n}$  for  $\bigvee_{j=1}^n \mathcal{C}$ ;  $\mathcal{C}^{\vee 0}$  will be the trivial polytope complex with no noninitial objects.

We will often be discussing commutative squares. Sometimes, in order to save space, we will write a commutative square

$$\begin{array}{ccc} A & \longrightarrow & B \\ f \downarrow & & \downarrow g \\ C & \longrightarrow & D \end{array}$$

as

$$f, g: (A \longrightarrow B) \longrightarrow (C \longrightarrow D).$$

In this paper, whenever we refer to an  $n$ -simplicial category we will always be referring to a functor  $(\Delta^{\text{op}})^n \longrightarrow \mathbf{Cat}$ , rather than an enriched category. In order to distinguish simplicial objects from non-simplicial objects, we will add a dot as a subscript to a simplicial object; thus  $\mathcal{C}$  is a polytope complex, but  $\dot{\mathcal{C}}$  is a simplicial polytope complex. For any functor  $F$  we will write  $F^{(n)}$  for the  $n$ -fold application of  $F$ .

## 2.2. A quick review of polytope complexes

In this section we will give a very brief summary of the definition of a polytope complex and the associated constructions. These definitions are discussed in much more detail in [8].

Given any double category  $\mathcal{C}$  we define  $F_{\mathcal{C}}: \mathbf{FinSet}^{\text{op}} \longrightarrow \mathbf{DbCat}$  to be the functor  $I \longmapsto \mathcal{C}^I$ . We then define  $\text{Tw}(\mathcal{C}) = \left( \int_{\mathbf{FinSet}^{\text{op}}} F_{\mathcal{C}}^{\text{op}} \right)^{\text{op}}$ .

**Definition 2.1.** A *polytope complex* is a double category  $\mathcal{C}$  satisfying the following properties:

- (V) Vertically,  $\mathcal{C}$  is a preorder which has a unique initial object and is closed under pullbacks. In addition,  $\mathcal{C}$  has a Grothendieck topology.
- (H) Horizontally,  $\mathcal{C}$  is a groupoid.
- (P) For any pair of morphisms  $P: B' \dashrightarrow B$  and  $\Sigma: A \longrightarrow B$ , where  $P$  is vertical and  $\Sigma$  horizontal, there exists a unique commutative square

$$\begin{array}{ccc} \Sigma^* B' & \longrightarrow & B' \\ \downarrow \Sigma^* P & & \downarrow P \\ A & \xrightarrow{\Sigma} & B \end{array}$$

The functor  $\Sigma^*: (\mathcal{C}_v \downarrow B) \longrightarrow (\mathcal{C}_v \downarrow A)$  is an equivalence of categories.

- (C) If  $\{X_{\alpha} \dashrightarrow X\}_{\alpha \in A}$  is a set of vertical morphisms which is a covering family of  $X$ , and  $\Sigma: Y \longrightarrow X$  is any horizontal morphism, then  $\{\Sigma^* X_{\alpha} \dashrightarrow Y\}_{\alpha \in A}$  is a covering family of  $Y$ .
- (E) If  $\{X_{\alpha} \dashrightarrow X\}_{\alpha \in A}$  is a covering family such that for some  $\alpha_0 \in A$  we have  $X_{\alpha_0} = \emptyset$ , then the family  $\{X_{\alpha} \dashrightarrow X\}_{\alpha \neq \alpha_0}$  is also a covering family.

A *polytope* is a non-initial object of  $\mathcal{C}$ . The full double subcategory of polytopes of  $\mathcal{C}$  will be denoted  $\mathcal{C}_p$ . We will say that two polytopes  $a, b \in \mathcal{C}$  are *disjoint* if there exists an object  $c \in \mathcal{C}$  with vertical morphisms  $a \dashrightarrow c$  and  $b \dashrightarrow c$  such that the pullback  $a \times_c b$  is the vertically initial object.

A *polytope functor* is a double functor between polytope complexes that preserves vertical and mixed pullbacks.

We denote the category of polytope complexes and polytope functors by **PolyCpx**. For convenience, we call horizontal morphisms in  $\text{Tw}(\mathcal{C})$  *shuffles*. A vertical morphism  $f: \{a_i\}_{i \in I} \dashrightarrow \{b_j\}_{j \in J} \in \text{Tw}(\mathcal{C})$  satisfying the condition that for any  $i, i' \in I$  such that  $f(i) = f(i')$ ,  $a_i \times b_{f(i)} a_{i'} = \emptyset$  is called a *sub-map*; a sub-map satisfying the extra

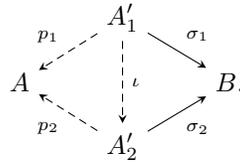
condition that for all  $j \in J$  the family  $\{a_i \dashrightarrow b_j\}_{i \in f^{-1}(j)}$  is a covering family is called a *covering sub-map*.

The functor  $\text{SC}: \mathbf{PolyCpx} \rightarrow \mathbf{WaldCat}$  (discussed in more detail in [8] section 4) is given by the following definition:

**Definition 2.2.** The category  $\text{SC}(\mathcal{C})$  is defined to have  $\text{ob SC}(\mathcal{C}) = \text{ob}(\text{Tw}(\mathcal{C}_p))$ . The morphisms of  $\text{SC}(\mathcal{C})$  are equivalence classes of diagrams in  $\text{Tw}(\mathcal{C}_p)$

$$A \xleftarrow{\quad p \quad} A' \longrightarrow B$$

where two diagrams are considered equivalent if there is a vertical isomorphism  $\iota: A'_1 \dashrightarrow A'_2 \in \text{Tw}(\mathcal{C}_p)_v$  which makes the following diagram commute:



We will generally refer to a morphism as a *pure sub-map* (resp. *pure shuffle*) if in some representing diagram the shuffle (resp. sub-map) component is the identity.

We say that a morphism  $A \xleftarrow{\quad p \quad} A' \xrightarrow{\quad \sigma \quad} B$  is a

**cofibration** if  $p$  is a covering sub-map and  $\sigma$  has an injective set map, and a

**weak equivalence** if  $p$  is a covering sub-map and  $\sigma$  has a bijective set map.

The composition of two morphisms  $f: A \rightarrow B$  and  $g: B \rightarrow C$  represented by

$$A \xleftarrow{\quad p \quad} A' \xrightarrow{\quad \sigma \quad} B \quad \text{and} \quad B \xleftarrow{\quad q \quad} B' \xrightarrow{\quad \tau \quad} C$$

is defined to be the morphism represented by the sub-map  $A' \times_B B' \dashrightarrow A' \dashrightarrow A$  and the shuffle  $A' \times_B B' \rightarrow B' \xrightarrow{\quad \tau \quad} C$ .

**2.3. The  $K$ -theory of a Waldhausen category**

This section contains a brief review of Waldhausen’s  $S$ . construction for  $K$ -theory, originally introduced in [7], as well as some results which are surely well-known to experts, but for which we could not find a reference. We omit all proofs in this section, as the mentioned results are either well-known in the literature (see, for example, [7]) or simple to prove from the definitions.

Given a Waldhausen category  $\mathcal{E}$ , we define  $S_n \mathcal{E}$  to be the category of commutative triangles defined as follows. An object  $A$  is a triangle of objects  $A_{ij}$  for pairs  $1 \leq i \leq j \leq n$ . The diagram consists of cofibrations  $A_{ij} \hookrightarrow A_{(i+1)j}$  and morphisms  $A_{ij} \rightarrow A_{i(j+1)}$  such that for every pair  $i < j$  and any  $1 \leq k \leq n - i$  the induced diagram

$$A_{ij} \hookrightarrow A_{(i+k)j} \longrightarrow A_{(i+k)(j+k)}$$

is a cofiber sequence. A morphism  $\varphi: A \rightarrow B$  consists of morphisms  $\varphi_{ij}: A_{ij} \rightarrow B_{ij}$  making the induced diagram commute. Note that  $S_0 \mathcal{E}$  is the trivial category with one object and one morphism, and  $S_1 \mathcal{E} = \mathcal{E}$ .

The  $S_n\mathcal{E}$ 's assemble into a simplicial category by letting the  $k$ -th face map remove all objects  $A_{ij}$  with  $i = k$  or  $j = k + 1$ , and the  $k$ -th degeneracy repeat a row and column appropriately. We can assemble the  $S_n\mathcal{E}$ 's into a simplicial Waldhausen category in the following manner. A morphism  $\varphi: A \rightarrow B \in S_n\mathcal{E}$  is a weak equivalence if  $\varphi_{ij}$  is a weak equivalence for all  $i < j$ .  $\varphi$  is a cofibration if for all  $i < j$  the induced morphism

$$B_{ij} \cup_{A_{ij}} A_{(i+1)j} \longrightarrow B_{(i+1)j}$$

is a cofibration in  $\mathcal{E}$ . Note that this means that in particular for all  $i < j$  the morphism  $\varphi_{ij}$  is a cofibration in  $\mathcal{E}$ .

We obtain the  $K$ -theory spectrum of a Waldhausen category  $\mathcal{E}$  by defining

$$K(E)_n = \Omega |wS^{(n)}\mathcal{E}|.$$

From proposition 1.5.3 in [7] we know that above level 0 this will be an  $\Omega$ -spectrum.

We now turn our attention to some tools for computing with Waldhausen categories. An exact functor of Waldhausen categories  $F: \mathcal{E} \rightarrow \mathcal{E}'$ , naturally yields a functor between  $S$  constructions, and therefore between the  $K$ -theory spectra. We are interested in several cases of such functors which produce equivalences on the  $K$ -theory level.

The first two examples we consider will be simply inclusions of subcategories. While a Waldhausen category can contain a lot of morphisms which are neither cofibrations nor weak equivalences, most of these are not important. We will say that a Waldhausen subcategory  $\tilde{\mathcal{E}}$  of a Waldhausen category  $\mathcal{E}$  is a *simplification of  $\mathcal{E}$*  if it contains all objects, weak equivalences, and cofibrations of  $\mathcal{E}$ . As the  $S$  construction only really looks at these morphisms, it is clear that the inclusion  $\tilde{\mathcal{E}} \rightarrow \mathcal{E}$  induces the identity map  $K(\tilde{\mathcal{E}}) \rightarrow K(\mathcal{E})$ .

Now suppose that  $\hat{\mathcal{E}}$  is a subcategory of  $\mathcal{E}$  with the property that any morphism  $f \in \mathcal{E}$  can be factored as  $hg$ , with  $h$  an isomorphism and  $g \in \hat{\mathcal{E}}$ , and such that  $\hat{\mathcal{E}}$  contains the zero object of  $\mathcal{E}$ . Then  $\hat{\mathcal{E}}$  is a Waldhausen category. Let  $\hat{S}_n\mathcal{E}$  be the full subcategory of  $S_n\mathcal{E}$  containing all objects in  $S_n\hat{\mathcal{E}}$ . Then  $\hat{S}_n\mathcal{E}$  is an equivalent Waldhausen subcategory of  $S_n\mathcal{E}$ , and thus that for all  $n \geq 1$ ,

$$|wS^{(n-1)}\hat{S}_n\mathcal{E}| \simeq |wS^{(n)}\mathcal{E}|.$$

Thus we can compute the  $K$ -theory of  $\mathcal{E}$  using only morphisms from  $\hat{\mathcal{E}}$  in the first level of the  $S$  construction.

Now we consider pairs of adjoint functors between Waldhausen categories. Suppose that we have an adjoint pair of exact functors  $F: \mathcal{E} \rightleftarrows \mathcal{E}': G$ ; these produce a pair of maps  $K(F): K(\mathcal{E}) \rightleftarrows K(\mathcal{E}') : K(G)$ . Generally an adjoint pair of functors produces a homotopy equivalence on the classifying space level, so naively we might expect these to be inverse homotopy equivalences. Unfortunately, in the  $S$  construction we always restrict our attention to weak equivalences in the category, so we need more information than just an adjoint pair of exact functors. If both the unit and counit of our adjunction is a weak equivalence then we are fine, however, as the adjunction must also restrict to an adjunction on the subcategories of weak equivalences. We call an adjoint pair of exact functors satisfying this extra condition an *exact adjoint pair*, and we say that  $F$  is *exactly left adjoint* to  $G$ . Given any exact adjoint pair we get a pair of inverse equivalences on the  $K$ -theory level.

We finish up this section with a short discussion of a simplification of the  $S$ -construction.  $S_n$  can be defined more informally as the category whose objects are all choices of  $n - 1$  composable cofibrations, together with the choices of all cofibers. As the cofiber of a cofibration  $A \hookrightarrow B \in \mathcal{E}$  is a pushout, any object  $A \in S_n\mathcal{E}$  is defined, up to isomorphism, by the diagram

$$A_{11} \hookrightarrow A_{21} \hookrightarrow \cdots \hookrightarrow A_{n1},$$

and any morphism  $\varphi$  by its restriction to this row. We will denote the category of such objects  $F_n\mathcal{E}$ . We can clearly make  $F_n\mathcal{E}$  into a Waldhausen category in a way analogous to the way we made  $S_n\mathcal{E}$  into a Waldhausen category. However, these do not assemble easily into a simplicial Waldhausen category, as  $\partial_0$ , the 0-th face map, must take cofibers, and this is only defined up to isomorphism. Thus while  $F_n\mathcal{E}$  is easier to work with on each level,  $S_n\mathcal{E}$  is often easier to work with when working with the simplicial structure. (Note that if in  $\mathcal{E}$  all cofibrations come with a canonical choice of cofiber that satisfies the condition that for a cofibration sequence  $A \hookrightarrow B \hookrightarrow C$  we have  $C/B = (C/A)/(B/A)$  then the  $F_n$ 's automatically assemble into a simplicial Waldhausen category. This will be exactly the case that we will be considering later in the paper.)

### 3. Thickenings

**Definition 3.1.** Let  $\mathcal{C}$  be a polytope complex. We define the polytope complex  $\mathcal{C}^{\boxtimes}$  to be the full subcategory of  $\text{Tw}(\mathcal{C}_p)$  containing all objects  $\{a_i\}_{i \in I} \in \text{Tw}(\mathcal{C}_p)$  such that for all distinct  $i, j \in I$  there exists an  $a \in \mathcal{C}$  such that  $a_i \times_a a_j = \emptyset$ . The topology on  $\mathcal{C}_v^{\boxtimes}$  is defined pointwise. More precisely, let  $X = \{x_i\}_{i \in I}$ , and  $X_\alpha = \{x_j^{(\alpha)}\}_{j \in J_\alpha}$ . We say that  $\{p^\alpha: X_\alpha \dashrightarrow X\}_{\alpha \in A}$  is a covering family if for each  $i \in I$  the family  $\{P_j^\alpha: x_j^{(\alpha)} \dashrightarrow x_i\}_{j \in (p^\alpha)^{-1}(i), \alpha \in A}$  is a covering family in  $\mathcal{C}$ .

It is easy to check that  $-^{\boxtimes}$  is in fact a functor  $\mathbf{PolyCpx} \rightarrow \mathbf{PolyCpx}$ . It will turn out that  $-^{\boxtimes}$  is a monad on  $\mathbf{PolyCpx}$ , and that  $\text{SC}: \mathbf{PolyCpx} \rightarrow \mathbf{WaldCat}$  factors through the inclusion  $\mathbf{PolyCpx} \rightarrow \text{Kl}(-^{\boxtimes})$  (the Kleisli category of this monad). This factorization provides us with extra morphisms between polytopes, which will be exactly the morphisms we need later when we start doing calculations with face maps in the  $S$ -construction.

We start by considering the monad structure of  $-^{\boxtimes}$ . There exists a natural inclusion  $\eta_{\mathcal{C}}: \mathcal{C} \rightarrow \mathcal{C}^{\boxtimes}$  which includes  $\mathcal{C}$  into  $\mathcal{C}^{\boxtimes}$  as the singleton sets; these assemble into a natural transformation  $\eta: 1 \Rightarrow -^{\boxtimes}$ . This transformation is not a natural isomorphism, even though, morally speaking,  $\mathcal{C}^{\boxtimes}$  ought to have the same  $K$ -theory as  $\mathcal{C}$  (as it contains objects which are formal sums of objects of  $\mathcal{C}$ ). It turns out that once we pass to  $\mathbf{WaldCat}$  by  $\text{SC}$  we can find a natural ‘‘almost inverse’’: an exact left adjoint.

**Lemma 3.2.** *The functor  $-^{\boxtimes}$  is a monad on  $\mathbf{PolyCpx}$ .*

*Proof.* In order to make  $-^{\boxtimes}$  into a monad, we need to define a unit and a multiplication. The unit  $\eta: 1_{\mathbf{PolyCpx}} \rightarrow (-^{\boxtimes})$  will be the natural transformation defined on each polytope complex  $\mathcal{C}$  by the natural inclusion  $\mathcal{C} \rightarrow \mathcal{C}^{\boxtimes}$  given by including  $\mathcal{C}$  as

the singleton sets indexed by the set  $\{*\}$ . The multiplication  $\mu: (-^\boxtimes)^\boxtimes \rightarrow (-^\boxtimes)$  is given by the functor  $\mathcal{C}^{\boxtimes\boxtimes} \rightarrow \mathcal{C}^\boxtimes$  given on objects by

$$\{\{a_j^{(i)}\}_{j \in J_i}\}_{i \in I} \longmapsto \{a_j^{(i)}\}_{(i,j) \in \coprod_{i \in I} J_i}.$$

In order for these definitions to make  $-^\boxtimes$  into a monad, we need to make sure that the way we choose the coproduct of indexing sets satisfies the following conditions:

$$\coprod_{k \in K} J_k = \coprod_{k \in K} \coprod_{j \in J_k} I_{j,k} \quad \text{and} \quad \coprod_{i \in I} \{*\} = \coprod_{\{*\}} I = I.$$

We do this in the following manner. For any elements  $a, b$  in a finite set  $I$ , we define  $a \circ b$  to be the tuple  $(a, b)$  if neither  $a$  nor  $b$  is itself a tuple. If  $a = (x_1, \dots, x_n)$  and  $b$  is not a tuple, then  $a \circ b = (x_1, \dots, x_n, b)$ ; if  $a$  is not a tuple and  $b$  is a tuple  $(y_1, \dots, y_m)$  then  $a \circ b = (a, y_1, \dots, y_m)$ . If both  $a$  and  $b$  are tuples, as above, then  $a \circ b = (x_1, \dots, x_n, y_1, \dots, y_m)$ . We then define

$$\coprod_{i \in I} J_i = \begin{cases} I & \text{if } J_i = \{*\} \text{ for all } i \in I, \\ J_i & \text{if } I = \{*\}, \\ \{i \circ j \mid i \in I, j \in J_i\} & \text{otherwise.} \end{cases}$$

It is easy to check that this satisfies the conditions we need.  $\square$

**Lemma 3.3.** *There exists a natural transformation  $\nu: \text{SC}(-^\boxtimes) \Rightarrow \text{SC}(-)$  which for every polytope complex  $\mathcal{C}$  is exactly left adjoint to  $\text{SC}(\eta_{\mathcal{C}}): \text{SC}(\mathcal{C}) \rightarrow \text{SC}(\mathcal{C}^\boxtimes)$ . The counit of this adjunction will be the identity transformation.*

*Proof.* Fix a polytope complex  $\mathcal{C}$ , and let  $G = \text{SC}(\eta_{\mathcal{C}})$ . To show that  $G$  has a left adjoint  $F$  it suffices to show that for any  $B \in \text{SC}(\mathcal{C}^\boxtimes)$ ,  $(B \downarrow G)$  has an initial object. If we write  $B = \{B_j\}_{j \in J}$ , where  $B_j = \{b_k^j\}_{k \in K_j}$  then the pure covering sub-map

$$\{B_j\}_{j \in J} \longleftarrow \{\{b_k^j\}\}_{(j,k) \in \coprod_{j \in J} K_j}$$

is the desired object; we define  $\nu_{\mathcal{C}}$  to be the adjoint where  $\nu_{\mathcal{C}}(B) = \{b_k^j\}_{(j,k) \in \coprod_{j \in J} K_j}$ . Then the unit is objectwise a pure covering sub-map — thus a weak equivalence — and the counit is the identity, as desired. To see that these assemble into a natural transformation, note that  $\nu_{\mathcal{C}}$  “flattens” each set of sets by covering it with a set of singletons. By purely set-theoretic observations it is clear that this commutes with applying a functor pointwise to each set element, so  $\nu$  does, indeed, assemble into a natural transformation.

It remains to show that  $\nu_{\mathcal{C}}$  is exact. As left adjoints commute with colimits and  $\text{SC}(\mathcal{C})$  has all pushouts,  $\nu_{\mathcal{C}}$  preserves all pushouts. The fact that  $F$  preserves cofibrations and weak equivalences follows from the definition of  $F$  and the fact that covering sub-maps in  $\mathcal{C}^\boxtimes$  are defined pointwise.  $\square$

Now consider the Kleisli category of this monad,  $\text{Kl}(-^\boxtimes)$ . Informally speaking,  $\text{Kl}(-^\boxtimes)$  is the category of sets of polytopes that can be “added”, in the sense that we can think of a covering sub-map  $\{a_i\}_{i \in I} \dashrightarrow \{b_j\}_{j \in J}$  as expressing the relation  $\sum_{i \in I} a_i = \sum_{j \in J} b_j$ . There exists a natural inclusion  $\iota: \mathbf{PolyCpx} \rightarrow \text{Kl}(-^\boxtimes)$  which is the identity on objects, and takes a polytope functor  $F: \mathcal{C} \rightarrow \mathcal{D}$  to the functor

$\eta_{\mathcal{D}}F$ . Using the functor given by lemma 3.3 we can extend SC to a functor on  $\text{Kl}(-^{\boxtimes})$  rather than just on **PolyCpx**.

**Lemma 3.4.** *The functor  $\text{SC}: \text{PolyCpx} \rightarrow \text{WaldCat}$  factors through  $\iota$ .*

*Proof.* We define a functor  $\widetilde{\text{SC}}: \text{Kl}(-^{\boxtimes}) \rightarrow \text{WaldCat}$  by setting  $\widetilde{\text{SC}}(\mathcal{C}) = \text{SC}(\mathcal{C})$  on polytope complexes  $\mathcal{C} \in \text{Kl}(-^{\boxtimes})$ , and by

$$\widetilde{\text{SC}}(F: \mathcal{C} \longrightarrow \mathcal{D}) = \nu_{\mathcal{D}}\text{SC}(F): \text{SC}(\mathcal{C}) \longrightarrow \text{SC}(\mathcal{D}^{\boxtimes}) \longrightarrow \text{SC}(\mathcal{D}).$$

Note that given any polytope functor  $F: \mathcal{C} \rightarrow \mathcal{D}$ ,

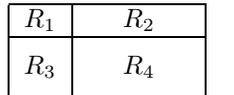
$$\widetilde{\text{SC}}(\iota(F)) = \nu_{\mathcal{D}}\text{SC}(\eta_{\mathcal{D}})SC(F) = \text{SC}(F),$$

as  $\nu_{\mathcal{D}}$  is left adjoint to  $\text{SC}(\eta_{\mathcal{D}})$  and the counit of the adjunction is the identity. Thus  $\widetilde{\text{SC}}\iota = \text{SC}$ , as desired.  $\square$

By abusing notation we will write SC for the extension  $\text{Kl}(-^{\boxtimes}) \rightarrow \text{WaldCat}$ .

We finish up with an example of a polytope complex which is an algebra over  $-^{\boxtimes}$ , and a polytope complex which is not an algebra over  $-^{\boxtimes}$ . Consider  $\mathcal{G}_{E^n}$ , the polytope complex of  $n$ -dimensional Euclidan polytopes. We can define a functor  $\mathcal{G}_{E^n}^{\boxtimes} \rightarrow \mathcal{G}_{E^n}$  by mapping any set of pairwise disjoint polytopes to the union of that set (which is well-defined if we define a polytope to be a nonempty union of simplices). It is easy to check that this does, in fact, make  $\mathcal{G}_{E^n}$  into an algebra over  $-^{\boxtimes}$ .

Now let  $\mathcal{C}$  be the polytope complex of rectangles in  $\mathbb{R}^2$  whose sides are parallel to the coordinate axes, with the group of translations acting on it. We claim that this is *not* an algebra over  $-^{\boxtimes}$ . Indeed, suppose that it were, so we have a functor  $F: \mathcal{C}^{\boxtimes} \rightarrow \mathcal{C}$ . Consider a rectangle  $R$  split into four sub-rectangles:



We know that  $F(\{R\}) = R$  and  $F(\{R_i\}) = R_i$ . Now consider  $F(\{R_1, R_4\})$ . This must sit inside  $R$ , and also contain both  $R_1$  and  $R_4$ , so it must be  $R$ . Similarly,  $F(\{R_2, R_3\}) = R$ . But then

$$R = F(\{R_1, R_4\}) \times_{F(\{R\})} F(\{R_2, R_3\}) = F(\{R_1, R_4\} \times_{\{R\}} \{R_2, R_3\}) = F(\emptyset) = \emptyset.$$

Contradiction. So  $\mathcal{C}$  is not an algebra over  $-^{\boxtimes}$ .

### 4. Filtered polytopes

The  $S$ . construction considers sequences of objects included into one another. In this section we will look at filtered objects where all of the cofibrations are actually acyclic cofibrations.

Let  $W_n\text{SC}(\mathcal{C})$  be the full subcategory of  $F_n\text{SC}(\mathcal{C})$  which contains all objects

$$A_1 \xrightarrow{\sim} A_2 \xrightarrow{\sim} \dots \xrightarrow{\sim} A_n.$$

We can make  $W_n\text{SC}(\mathcal{C})$  into a Waldhausen category by taking the structure induced from  $F_n\text{SC}(\mathcal{C})$ . Then  $W_n\text{SC}(\mathcal{C})$  contains  $\widetilde{W}_n\text{SC}(\mathcal{C})$  — the full subcategory of  $W_n\text{SC}(\mathcal{C})$  of all such objects which can be represented by only pure sub-maps — as an equivalent subcategory.

Our goal for this section is to define a polytope complex  $f_n\mathcal{C}$  such that  $\text{SC}(f_n\mathcal{C})$  is equivalent (as a Waldhausen category) to  $W_n\text{SC}(\mathcal{C})$ .

**Definition 4.1.** Let  $f_n\mathcal{C}$  be the following polytope complex. An object  $A \in f_n\mathcal{C}$  is a diagram

$$A_1 \leftarrow\!\!\!\! \leftarrow A_2 \leftarrow\!\!\!\! \leftarrow \cdots \leftarrow\!\!\!\! \leftarrow A_n$$

in  $\text{Tw}(\mathcal{C}_p)$  such that each  $A_i \in \mathcal{C}^{\boxtimes}$  and  $A_1$  is indexed by a singleton set. The vertical morphisms  $p: A \twoheadrightarrow B$  are diagrams

$$\begin{array}{ccccccc} A_1 & \leftarrow\!\!\!\! \leftarrow & A_2 & \leftarrow\!\!\!\! \leftarrow & \cdots & \leftarrow\!\!\!\! \leftarrow & A_n \\ \downarrow p_1 & & \downarrow p_2 & & & & \downarrow p_n \\ B_1 & \leftarrow\!\!\!\! \leftarrow & B_2 & \leftarrow\!\!\!\! \leftarrow & \cdots & \leftarrow\!\!\!\! \leftarrow & B_n \end{array}$$

in  $\mathcal{C}^{\boxtimes}$ , and the horizontal morphisms are defined analogously. We put a topology on  $f_n\mathcal{C}$  by defining a family  $\{X_\alpha \twoheadrightarrow X\}_{\alpha \in A}$  to be a covering family if for each  $i = 1, \dots, n$  the family  $\{X_{\alpha i} \twoheadrightarrow X_i\}_{\alpha \in A}$  is a covering family in  $\mathcal{C}^{\boxtimes}$ .

Now we construct the functors which give an equivalence of categories between  $\widetilde{W}_n\text{SC}(\mathcal{C})$  and  $\text{SC}(f_n\mathcal{C})$ . The functor  $H: \text{SC}(f_n\mathcal{C}) \rightarrow \widetilde{W}_n\text{SC}(\mathcal{C})$  simply takes an object of  $\text{SC}(f_n\mathcal{C})$  to the sequence of its levelwise unions. More formally, given an object  $\{a_i\}_{i \in I}$  in  $\text{SC}(f_n\mathcal{C})$ , where for each  $i \in I$  we have

$$a_i = a_i^1 \leftarrow\!\!\!\! \leftarrow a_i^2 \leftarrow\!\!\!\! \leftarrow \cdots \leftarrow\!\!\!\! \leftarrow a_i^n,$$

with  $a_i^j \in \mathcal{C}^{\boxtimes}$ , we define an object  $H(\{a_i\}_{i \in I}) \in \widetilde{W}_n\text{SC}(\mathcal{C})$  by

$$A_1 \hookrightarrow A_2 \hookrightarrow \cdots \hookrightarrow A_n$$

where  $A_j = \coprod_{i \in I} a_i^j \in \text{Tw}(\mathcal{C}_p)$ . In other words, we consider each object  $a_i$  to be a diagram in  $\text{Tw}(\mathcal{C}_p)$  and we take the coproduct of all of these diagrams.

To construct an inverse  $G: \widetilde{W}_n\text{SC}(\mathcal{C}) \rightarrow \text{SC}(f_n\mathcal{C})$  to this functor we take a diagram in  $\widetilde{W}_n\text{SC}(\mathcal{C})$  and turn it into a coproduct of pure covering sub-maps in  $\text{Tw}(\mathcal{C}_p)$ . It will turn out that each of these diagrams represents an object of  $\text{SC}(f_n\mathcal{C})$ , which will give us the desired functor. Given an object  $A \in \widetilde{W}_n\text{SC}(\mathcal{C})$  represented by

$$A_1 \hookrightarrow A_2 \hookrightarrow \cdots \hookrightarrow A_n$$

we know that we can write every acyclic cofibration in this diagram as a pure covering sub-map. When a morphism can be represented in this way the representation is

unique, so we can in fact consider this object to be a diagram

$$A_1 \leftarrow\!\!\!\leftarrow A_2 \leftarrow\!\!\!\leftarrow \cdots \leftarrow\!\!\!\leftarrow A_n$$

in  $\text{Tw}(\mathcal{C}_p)$ . This sits above an analogous diagram in **FinSet**. Given any such diagram in **FinSet** we can write it as a coproduct of fibers over the indexing set  $I$  of  $A_1$ . Consequently we can write  $A$  as

$$\coprod_{i \in I} (A_1^i \leftarrow\!\!\!\leftarrow A_2^i \leftarrow\!\!\!\leftarrow \cdots \leftarrow\!\!\!\leftarrow A_n^i).$$

We will show that each of these component diagrams actually represents an object of  $f_n\mathcal{C}$ . Indeed, we know by definition that  $A_1^i$  is a singleton set  $\{a_i\}$ . Thus if we write  $A_j^i$  as  $\{b_k\}_{k \in K}$ , from the fact that each of the morphisms in the diagram is a sub-map we know that for  $K, k' \in K$ ,  $b_k \times_{a_i} b_{k'} = \emptyset$ , so each  $A_j^i$  is an object of  $\mathcal{C}^{\boxtimes}$ . Thus this diagram is an object of  $f_n\mathcal{C}$  as desired. This definition extends directly to the morphisms as well.

We need to prove that these functors are exact and inverses up to choice of indexing set. It is easy to see that they are inverses on objects, so we focus our attention on the morphisms in the categories. To this end we define two projection functors  $\pi_1: W_n\text{SC}(\mathcal{C}) \rightarrow \text{SC}(\mathcal{C})$  and  $P_1: f_n\mathcal{C} \rightarrow \mathcal{C}$  which will help us analyze the situation.

**Lemma 4.2.** *Let  $P_1: f_n\mathcal{C} \rightarrow \mathcal{C}$  take a diagram  $A_1 \leftarrow\!\!\!\leftarrow \cdots \leftarrow\!\!\!\leftarrow A_n$  to the unique element of  $A_1$ . Then the functor  $\text{SC}(P_1)$  is faithful.*

*Proof.* It suffices to show that given any diagram

$$\begin{array}{ccc} A_1 & \xrightarrow{\sim} & A_2 \\ f \downarrow & & \\ B_1 & \xrightarrow{\sim} & B_2 \end{array}$$

there exists at most one morphism  $g: A_2 \rightarrow B_2$  that makes the diagram commute. In particular, if we consider the diagram in  $\text{Tw}((f_n\mathcal{C})_p)$  representing such a commutative square, we have

$$\begin{array}{ccccc} A_2 & \leftarrow\!\!\!\leftarrow & A'_2 & \longrightarrow & B_2 \\ \vdots & & \vdots & & \vdots \\ \downarrow & & \downarrow & & \downarrow \\ A_1 & \leftarrow\!\!\!\leftarrow & A'_1 & \xrightarrow{\sigma} & B_1 \end{array}$$

where the morphism  $A'_2 \dashrightarrow A'_1$  is a covering sub-map because the square commutes. In particular, this means that  $A'_2 = \sigma^*B_2$ . Thus we can complete the square exactly when we have a sub-map  $\sigma^*B_2 \dashrightarrow A_2$  which makes the left-hand square commute, of which there is at most one.  $\square$

And, completely analogously, we can prove a symmetric statement about  $\pi_1$ .

**Lemma 4.3.** *Let  $\pi_1: \widetilde{W}_n\text{SC}(\mathcal{C}) \rightarrow \text{SC}(\mathcal{C})$  be the exact functor which takes an object  $A_1 \xrightarrow{\sim} \dots \xrightarrow{\sim} A_n$  to  $A_1$ . Then  $\pi_1$  is faithful.*

We can now prove the main result of this section.

**Proposition 4.4.**  *$W_n\text{SC}(\mathcal{C})$  is exactly equivalent to  $\text{SC}(f_n\mathcal{C})$ .*

*Proof.* We will show that  $G$  and  $H$  induce inverse equivalences between  $\widetilde{W}_n\text{SC}(\mathcal{C})$  and  $\text{SC}(f_n\mathcal{C})$ , which will show the result as  $\widetilde{W}_n\text{SC}(\mathcal{C})$  is exactly equivalent to  $W_n\text{SC}(\mathcal{C})$ .

It is clear that  $GH$  and  $HG$  are the identity up to choice of indexing set on objects, so it remains to show that they are identities on morphisms. From the definitions it is easy to see that  $\text{SC}(P_1)G = \pi_1$  and that  $\pi_1H = \text{SC}(P_1)$ , so that

$$\text{SC}(P_1)GH = \pi_1H = \text{SC}(P_1) \quad \text{and} \quad \pi_1HG = \text{SC}(P_1)G = \pi_1.$$

As  $\text{SC}(P_1)$  and  $\pi_1$  are both faithful, if we consider these on hom-sets we see that  $G$  and  $H$  are mutual inverses on any hom-set. Thus  $\widetilde{W}_n\text{SC}(\mathcal{C})$  is isomorphic to  $\text{SC}(f_n\mathcal{C})$ .

It remains to show that  $G$  and  $H$  are exact functors. We already know that they preserve pushouts, so all it remains to show is that they preserve cofibrations and weak equivalences. Note that we know by definition that  $\pi_1$  and  $\text{SC}(P_1)$  are exact functors; thus in order to show that  $G$  and  $H$  are exact it suffices to show that  $\pi_1$  and  $\text{SC}(P_1)$  reflect cofibrations and weak equivalences.

For both of these cases it suffices to show that in  $\text{Tw}(\mathcal{C}_p)$  if

$$\begin{array}{ccccc} A_1 & \xleftarrow{p} & A'_1 & \xrightarrow{\sigma} & B_1 \\ \uparrow \hat{i} & & & & \uparrow \hat{j} \\ A_2 & \xleftarrow{q} & \sigma^*B_2 & \xrightarrow{\tilde{\sigma}} & B_2 \end{array}$$

commutes and  $\sigma$  has an injective set-map, then  $q$  is a covering sub-map and  $\tilde{\sigma}$  has an injective set-map. The first of these is true because  $q$  is the pullback along  $i$  of  $jp$ , which is a covering sub-map; the second of these is true because pullbacks preserve injectivity of set-maps. So we are done.  $\square$

*Remark 4.5.* If we define  $P_n$  and  $\pi_n$  analogously to  $P_1$  and  $\pi_1$  we see that  $\text{SC}(P_n)$  and  $\pi_n$  are exact equivalences of categories. Thus  $\text{SC}(f_n\mathcal{C})$  and  $W_n\text{SC}(\mathcal{C})$  are exactly equivalent, as they are both equivalent to  $\text{SC}(\mathcal{C})$ . We do not use these functors because they are not compatible with the simplicial maps of  $S_n\text{SC}(\mathcal{C})$ , and thus will not give inverse equivalences on the  $K$ -theory.

## 5. Combing

Let  $f: A \hookrightarrow B \in \text{SC}(\mathcal{C})$  be a cofibration. We define the *image* of  $f$  to be the cofiber of the canonical cofibration  $B/A \hookrightarrow B$  (for a discussion of this cofibration, see [8], corollary 6.7). We will write the image of  $f$  as  $\text{im}(f)$ ; when the cofibration

is clear from context we will often write it as  $\text{im}_B(A)$ . Note that we have an acyclic cofibration

$$A \hookrightarrow \text{im}_B(A).$$

More concretely, if we write  $A = \{a_i\}_{i \in I}$  and  $B = \{b_j\}_{j \in J}$ , and if  $f$  can be represented by the covering sub-map  $p$  and the shuffle  $\sigma$ ,  $\text{im}_B(A) = \{b_j\}_{j \in \text{im } \sigma}$ .

Now suppose that we are given an object  $A = (A_1 \hookrightarrow \cdots \hookrightarrow A_n) \in F_n\text{SC}(\mathcal{C})$ . Then we define the  $i$ -th *strand* of  $A$ ,  $\text{St}_i(A)$  to be the diagram

$$A_i/A_{i-1} \hookrightarrow \text{im}_{A_{i+1}}(A_i/A_{i-1}) \hookrightarrow \cdots \hookrightarrow \text{im}_{A_n}(A_i/A_{i-1}).$$

We can consider  $\text{St}_i(A)$  to be an object of  $F_n\text{SC}(\mathcal{C})$  by padding the front with sufficiently many copies of the zero object; then we can canonically write  $A = \coprod_{i=1}^n \text{St}_i(A)$ .

**Definition 5.1.** We will say that a morphism  $f: A \rightarrow B \in F_n\text{SC}(\mathcal{C})$  is *layered* if for all  $1 \leq i < k \leq n$  the diagram

$$\begin{array}{ccc} A_k/A_i & \hookrightarrow & A_k \\ f_k/f_i \downarrow & & \downarrow f_k \\ B_k/B_i & \hookrightarrow & B_k \end{array}$$

commutes. We define  $L_n\text{SC}(\mathcal{C})$  to be the subcategory of  $F_n\text{SC}(\mathcal{C})$  containing all layered morphisms.

Not all morphisms are layered. For example, let  $X$  be a nonzero object, and let  $g: X \hookrightarrow Y$  be any cofibration in  $\text{SC}(\mathcal{C})$ . Then  $\emptyset \hookrightarrow Y$  and  $X \hookrightarrow Y$  are both objects of  $F_2\text{SC}(\mathcal{C})$  and we have a non-layered morphism

$$\begin{array}{ccc} \emptyset & \hookrightarrow & Y \\ \downarrow & & \parallel \\ X & \hookrightarrow & Y \end{array}$$

between them. As all cofibers of acyclic cofibrations are trivial, all morphisms of  $W_n\text{SC}(\mathcal{C})$  are layered. In fact, if we let  $I_{ni}: F_{n-i+1}\text{SC}(\mathcal{C}) \rightarrow F_n\text{SC}(\mathcal{C})$  be the functor which pads a diagram with  $i$  copies of  $\emptyset$  at the beginning, then the restriction of  $I_{ni}$  to  $L_{n-i+1}\text{SC}(\mathcal{C})$  has its image in  $L_n\text{SC}(\mathcal{C})$ .

**Lemma 5.2.**

1.  $f$  is layered if and only if for all  $1 \leq i < n$ , the morphism

$$f_{i,i+1}: (A_i \hookrightarrow A_{i+1}) \longrightarrow (B_i \hookrightarrow B_{i+1}) \in F_2\text{SC}(\mathcal{C})$$

is layered.

2. For any commutative square  $(A_1 \hookrightarrow A_2) \longrightarrow (B_1 \hookrightarrow B_2)$  there is an induced commutative square  $(\text{im}_{A_2}(A_1) \hookrightarrow A_2) \longrightarrow (\text{im}_{B_2}(B_1) \hookrightarrow B_2)$ . If the commutative square  $(A_1 \hookrightarrow A_2) \longrightarrow (B_1 \hookrightarrow B_2)$  is layered then so is the commutative square

$$(A_2/A_1 \hookrightarrow A_2) \longrightarrow (B_2/B_1 \hookrightarrow B_2).$$

*Proof.*

1. The forwards direction is trivial, so it suffices to prove the backwards direction. We will prove it by induction on  $k$ . For  $k = i + 1$  this is given. Now suppose that it is true up to  $k$ . Then we have the following diagram

$$\begin{array}{ccccc}
 & & A_k & \longleftarrow & A_k/A_i \\
 & & \downarrow & \searrow & \downarrow \\
 A_{k+1}/A_k & \longleftarrow & A_{k+1} & \longleftarrow & A_{k+1}/A_i \\
 \downarrow & & \downarrow & & \downarrow \\
 & & B_k & \longleftarrow & B_k/B_i \\
 & & \downarrow & \searrow & \downarrow \\
 B_{k+1}/B_k & \longleftarrow & B_{k+1} & \longleftarrow & B_{k+1}/B_i
 \end{array}$$

in which we know that every face other than the front one commutes; we want to show that the front face also commutes. Let

$$\alpha: A_{k+1}/A_i \hookrightarrow A_{k+1} \longrightarrow B_{k+1}$$

and

$$\beta: A_{k+1}/A_i \longrightarrow B_{k+1}/B_i \hookrightarrow B_{k+1};$$

we want to show that  $\alpha = \beta$ . As  $A_{k+1}/A_i$  is the pushout of the diagram

$$A_{k+1} \longleftarrow A_k \longrightarrow A_k/A_i,$$

it suffices to show that  $f\alpha = f\beta$  and  $g\alpha = g\beta$  for  $f: A_k/A_i \hookrightarrow A_{k+1}/A_i$  and  $g: A_{k+1} \longrightarrow A_{k+1}/A_i$ . The first of these follows directly from the fact that all faces of the cube but the front one commute. For the second of these, note that we have a weak equivalence  $A_k \amalg A_{k+1}/A_k \xrightarrow{\sim} A_{k+1}$  and weak equivalences are epimorphisms, so in fact it suffices to show that  $g_1\alpha = g_1\beta$  and  $g_2\alpha = g_2\beta$  for

$$g_1 = A_k \hookrightarrow A_{k+1} \longrightarrow A_{k+1}/A_i$$

and

$$g_2 = A_{k+1}/A_k \hookrightarrow A_{k+1} \longrightarrow A_{k+1}/A_i.$$

The first of these follows from a simple diagram chase, keeping in mind that all horizontal cofibrations in this cube are actually sections of cofiber maps. The second of these also turns into a simple diagram chase after noting that for any sequence of cofibrations  $X \hookrightarrow Y \hookrightarrow Z$  in  $\text{SC}(\mathcal{C})$  we have

$$Z/Y \hookrightarrow Z \longrightarrow Z/X \hookrightarrow Z = Z/Y \hookrightarrow Z.$$

2. Note that if we have a commutative square  $(A_1 \hookrightarrow A_2) \longrightarrow (B_1 \hookrightarrow B_2)$  it can be represented by the following commutative diagram in  $\text{Tw}(\mathcal{C}_p)$ :

$$\begin{array}{ccccc}
 A_1 & \leftarrow \cdots & A'_1 & \longrightarrow & A_2 \\
 \uparrow & & \uparrow & & \uparrow \\
 A''_1 & \leftarrow \cdots & X & \longrightarrow & A'_2 \\
 \downarrow & & \downarrow & & \downarrow \\
 B_1 & \leftarrow \cdots & B'_1 & \longrightarrow & B_2
 \end{array}$$

where the starred squares are pullbacks. We know  $A'_1 \cong \text{im}_{A_2}(A_1)$  and  $B'_1 \cong \text{im}_{B_2}(B_1)$  and the middle column in the diagram represents a morphism between them. In fact, the right-hand half of this diagram is — up to isomorphism — exactly the square that the lemma states exists. The second part of the statement follows because the cofiber of  $A_2/A_1 \hookrightarrow A_2$  is exactly  $\text{im}_{A_2}(A_1)$ . □

**Lemma 5.3.**  *$L_n\text{SC}(\mathcal{C})$  is a Waldhausen category. The cofibrations (resp. weak equivalences) in  $L_n\text{SC}(\mathcal{C})$  are exactly the morphisms which are levelwise cofibrations (resp. weak equivalences).  $L_n\text{SC}(\mathcal{C})$  is a simplification of  $F_n\text{SC}(\mathcal{C})$ .*

We postpone the proof of this lemma until section 9 as it is technical and not particularly illuminating.

**Lemma 5.4.**  *$\text{St}_i$  is an exact functor  $L_n\text{SC}(\mathcal{C}) \rightarrow W_{n-i+1}\text{SC}(\mathcal{C})$ . We have a natural transformation  $\eta_i: I_{ni}\text{St}_i \rightarrow \text{id}$  induced by the inclusions  $\text{im}_{A_k}(A_i/A_{i-1}) \hookrightarrow A_k$ . On  $W_{n-i+1}\text{SC}(\mathcal{C})$  we have for  $i \neq j$*

$$\text{St}_i I_{ni} = \text{id} \quad \text{and} \quad \text{St}_j I_{ni} \text{St}_i = 0.$$

*Proof.* Let  $f: A \rightarrow B \in L_n\text{SC}(\mathcal{C})$ . We claim that the morphism

$$\begin{array}{ccccccc}
 A_i/A_{i-1} & \hookrightarrow & A_{i+1} & \hookrightarrow & \cdots & \hookrightarrow & A_n \\
 f_i/f_{i-1} \downarrow & & \downarrow f_{i+1} & & & & \downarrow f_n \\
 B_i/B_{i-1} & \hookrightarrow & B_{i+1} & \hookrightarrow & \cdots & \hookrightarrow & B_n
 \end{array}$$

in  $F_{n-i+1}\text{SC}(\mathcal{C})$  is also layered. By lemma 5.2(1) we know that it suffices to check that each square in this diagram satisfies the layering condition. All squares but the first one satisfy it because  $f$  is layered. The first square can be factored as

$$\begin{array}{ccccc}
 A_i/A_{i-1} & \hookrightarrow & A_i & \hookrightarrow & A_{i+1} \\
 f_i/f_{i-1} \downarrow & & \downarrow f_i & & \downarrow f_{i+1} \\
 B_i/B_{i-1} & \hookrightarrow & B_i & \hookrightarrow & B_{i+1}
 \end{array}$$

The right-hand square satisfies the layering condition because  $f$  is layered; the left-hand square satisfies it by lemma 5.2(2). If we let  $T_i: L_n\text{SC}(\mathcal{C}) \rightarrow L_{n-i+1}\text{SC}(\mathcal{C})$  be the functor taking an object to this truncation then  $T_i$  is exact, as by lemma 5.3 layered cofibrations are exactly levelwise. Note that  $T_i I_{ni} = \text{id}$  and we have a natural transformation  $\eta': I_{ni} T_i \rightarrow \text{id}$ .

We can write  $\text{St}_i = \text{St}_1 T_i$ ; thus if we can prove the lemma for  $i = 1$  we will be done. The fact that  $f$  is layered implies that  $\text{St}_1$  is a functor  $L_n\text{SC}(\mathcal{C}) \rightarrow W_n\text{SC}(\mathcal{C})$  (as it is obtained by taking levelwise cofibers in a commutative diagram). As colimits commute past one another, we see that this preserves pushouts along cofibrations. Thus to see that  $\text{St}_1$  is exact it remains to show that it preserves cofibrations and weak equivalences, which is true because both weak equivalences and cofibrations are preserved by taking cofibers, and  $\text{St}_1$  simply takes two successive cofibers.

The natural transformation  $\eta_1$  is obtained by factoring each cofibration  $A_1 \hookrightarrow A_k$  through the weak equivalence  $A_1 \xrightarrow{\simeq} \text{im}_{A_k}(A_1)$ . By the discussion in the proof of lemma 5.2(2) this will in fact be a natural transformation.

Now we show the last part of the lemma. It is a simple computation to see that  $\text{St}_i|_{W_m\text{SC}(\mathcal{C})}$  is the identity if  $i = 1$ , and 0 otherwise. Thus  $\text{St}_i I_{ni} = \text{St}_1 T_i I_{ni} = \text{St}_1$  is the identity. If  $j < i$  then the  $j$ -th component of  $I_{ni} \text{St}_i$  is  $\emptyset$ , so  $\text{St}_j I_{ni} \text{St}_i = 0$  trivially. If  $j > i$  then  $\text{St}_j I_{ni} \text{St}_i = \text{St}_{j-i+1} T_i I_{ni} \text{St}_i = \text{St}_{j-i+1} \text{St}_1 = 0$  because  $j - i + 1 > 1$ . Thus we are done.  $\square$

**Proposition 5.5.** *We define  $CP: \prod_{m=1}^n W_m\text{SC}(\mathcal{C}) \rightarrow L_n\text{SC}(\mathcal{C})$  to be the functor which takes an  $n$ -tuple  $(X_1, \dots, X_n)$  to  $\prod_{i=1}^n I_{n(n-i+1)}(X_i)$ . Then there is an exact equivalence of categories*

$$\text{St}: L_n\text{SC}(\mathcal{C}) \xleftrightarrow{\quad} \prod_{m=1}^n W_m\text{SC}(\mathcal{C}) CP,$$

where  $\text{St}$  is induced by the functors  $\text{St}_m$  for  $m = 1, \dots, n$ .

*Proof.* We first show that these form an equivalence of categories. From lemma 5.4 above, we know that the composition  $\text{St} \circ CP$  is the identity on each component (as  $\text{St}_i \text{St}_j A$  is the zero object for  $i \neq j$ ), and thus the identity functor. On the other hand, the composition  $CP \circ \text{St}$  comes with a natural transformation

$$\eta = \prod_{m=1}^n \eta_{n-m+1}: CP \circ \text{St} \longrightarrow \text{id};$$

it remains to show that  $\eta$  is in fact a natural isomorphism. However, for every object  $A$ ,  $\eta_A$  is simply the natural morphism  $\prod_{i=1}^n \text{St}_i(A) \rightarrow A$ , which is clearly an isomorphism. So these are in fact inverse equivalences.

As each component of  $CP$  is exact (as cofibrations and weak equivalences in  $L_n\text{SC}(\mathcal{C})$  are levelwise) we know that  $CP$  is exact. On the other hand,  $\text{St}_i$  is exact for all  $i$ , so  $\text{St}$  is exact. So we are done.  $\square$

The functor  $\text{St}$  ‘‘combs’’ an object of  $L_n\text{SC}(\mathcal{C})$  by separating all of the strands of different lengths.

## 6. Simplicial polytope complexes

Our goal for this section is to assemble the  $f_i\mathcal{C}$  into a simplicial polytope complex which will mimic Waldhausen's  $S$ -construction.

Given  $1 \leq i \leq n$  we define a morphism  $\partial_i^{(n)}: f_n\mathcal{C} \rightarrow f_{n-1}\mathcal{C}$  in  $\text{Kl}(-^\boxtimes)$  induced by skipping the  $i$ -th term. If  $i > 1$  this functor comes from **PolyCpx**; if  $i = 1$  then we cut off the singleton element from the front, and therefore have to split the rest of the object into fibers over the different polytopes in the (newly) first set. (This is why  $\partial_i^{(n)}$  is a morphism in  $\text{Kl}(-^\boxtimes)$  rather than in **PolyCpx**.) We define the morphism  $\sigma_i^{(n)}: f_n\mathcal{C} \rightarrow f_{n+1}\mathcal{C}$  to be the morphism of  $\text{Kl}(-^\boxtimes)$  given by the polytope functor which repeats the  $i$ -th stage. For  $i \leq 0$  we define the morphisms  $\sigma_i^{(n)}: f_n\mathcal{C} \rightarrow f_n\mathcal{C}$  and  $\partial_i^{(n)}: f_n\mathcal{C} \rightarrow f_n\mathcal{C}$  to be the identity on  $f_n\mathcal{C}$ . Note that the only one of these morphisms that does not come from **PolyCpx** is  $\partial_1^{(n)}$ .

**Definition 6.1.** Let  $s_n\mathcal{C} = \bigvee_{i=1}^n f_i\mathcal{C}$ . We define simplicial structure maps between these by

$$\partial_0 = (0: f_n\mathcal{C} \longrightarrow s_{n-1}\mathcal{C}) \vee \bigvee_{i=1}^{n-1} (1: f_i\mathcal{C} \longrightarrow f_i\mathcal{C}),$$

where  $0$  is the polytope functor sending everything to the initial object  $\emptyset$ ,

$$\partial_i = \bigvee_{j=1}^n \partial_{n-j+i}^{(j)} \quad \text{for } i \geq 1,$$

and

$$\sigma_i: s_n\mathcal{C} \longrightarrow s_{n+1}\mathcal{C} = \bigvee_{j=1}^n \sigma_{n-j+i}^{(j)} \quad \text{for } i \geq 0.$$

It is easy to see that with the  $\partial_i$ 's as the face maps and the  $\sigma_i$ 's as the degeneracy maps,  $s\mathcal{C}$  becomes a simplicial polytope complex.

Proposition 5.5 and proposition 4.4 give us an exact equivalence

$$L_n\text{SC}(\mathcal{C}) \longrightarrow \prod_{m=1}^n \text{SC}(f_m\mathcal{C}).$$

However,  $\prod_{m=1}^n \text{SC}(f_m\mathcal{C})$  is exactly equivalent to  $\text{SC}(\bigvee_{m=1}^n f_m\mathcal{C}) = \text{SC}(s_n\mathcal{C})$ . Thus we have proved the following:

**Corollary 6.2.** *Let  $H_m: \text{SC}(f_m\mathcal{C}) \rightarrow \widetilde{W}_m\text{SC}(\mathcal{C})$  be the functor in proposition 4.4, let  $\iota_m$  be the natural inclusion  $\widetilde{W}_m\text{SC}(\mathcal{C}) \rightarrow W_m\text{SC}(\mathcal{C})$ , and let  $CP_n$  be the functor from proposition 5.5. Then  $F_n = CP_n \circ (\prod_{m=1}^n \iota_m \circ H_m)$  is an exact equivalence of categories.*

We know that  $L\text{SC}(\mathcal{C})$  is a simplicial Waldhausen category, and  $\text{SC}(s\mathcal{C})$  is a simplicial Waldhausen category.  $F$  is a levelwise exact equivalence; we would like to show that it commutes with the simplicial maps, and therefore assembles to a functor of simplicial Waldhausen categories. This will allow us to conclude that the two constructions give equivalent  $K$ -theory spectra, and thus that we can work directly with the  $\text{SC}(s\mathcal{C})$  definition.

**Proposition 6.3.** *The functor  $F: \text{SC}(s\mathcal{C}) \rightarrow L.\text{SC}(\mathcal{C})$  is an exact equivalence of simplicial Waldhausen categories.*

*Proof.* First we will show that  $F$  is, in fact, a functor of simplicial Waldhausen categories. In particular, it suffices to show that the following two diagrams commute for each  $i$ :

$$\begin{array}{ccc} \text{SC}(s_n\mathcal{C}) & \xrightarrow{\sigma_i} & \text{SC}(s_{n+1}\mathcal{C}) \\ F_n \downarrow & & \downarrow F_{n+1} \\ L_n\text{SC}(\mathcal{C}) & \xrightarrow{\sigma_i} & L_{n+1}\text{SC}(\mathcal{C}) \end{array}$$

and

$$\begin{array}{ccccc} \text{SC}(s_n\mathcal{C}) & \xrightarrow{\partial_i} & \text{SC}((s_{n-1}\mathcal{C})^{\boxtimes}) & \xrightarrow{\nu_{s_{n-1}\mathcal{C}}} & \text{SC}(s_{n-1}\mathcal{C}) \\ F_n \downarrow & & & & \downarrow F_{n-1} \\ L_n\text{SC}(\mathcal{C}) & \xrightarrow{\partial_i} & & & L_{n-1}\text{SC}(\mathcal{C}) \end{array}$$

where the first diagram is a square because all  $\sigma_i$ 's come from morphisms in **PolyCpx**. Both of these diagrams commute by simple computations, since  $F_n$  takes “levelwise unions”.

Now by corollary 6.2 we know that levelwise  $F_n$  is an exact equivalence of Waldhausen categories. In addition, propositions 5.5 and 4.4 give us formulas for the levelwise inverse equivalences; an analogous proof shows that these also assemble into a functor of simplicial Waldhausen categories. Thus  $F$  is an equivalence of simplicial Waldhausen categories, as desired.  $\square$

Suppose that  $\mathcal{C}$  is a simplicial polytope complex. We define the  $K$ -theory spectrum of  $\mathcal{C}$  by

$$K(\mathcal{C})_n = |NwS^{(n)}\text{SC}(\mathcal{C}): (\Delta^{\text{op}})^{n+2} \longrightarrow \mathbf{Sets}|.$$

(Note that this definition is compatible with the  $K$ -theory of a polytope complex, if we consider a polytope complex as a constant simplicial complex.)

**Lemma 6.4.**  *$K(\mathcal{C})$  is a spectrum, which is an  $\Omega$ -spectrum above level 0.*

In the proof of this lemma we use the following obvious generalization of lemma 5.2 in [6]. A *fiber sequence* of multisimplicial categories is a sequence which is a fibration sequence up to homotopy after geometric realization of the nerves.

**Lemma 6.5.** ([6], 5.2) *Let*

$$X\cdots \longrightarrow Y\cdots \longrightarrow Z\cdots$$

*be a diagram of  $n$ -simplicial categories. Suppose that the following three conditions hold:*

- *the composite morphism is constant,*

- $Z_{\dots m}$  is connected for all  $m \geq 0$ , and
- $X_{\dots m} \rightarrow Y_{\dots m} \rightarrow Z_{\dots m}$  is a fiber sequence for all  $m \geq 0$ .

Then  $X_{\dots} \rightarrow Y_{\dots} \rightarrow Z_{\dots}$  is a fiber sequence.

We now prove lemma 6.4.

*Proof of lemma 6.4.* Suppose that  $X_{\dots}$  is an  $n$ -simplicial object; we will write  $PX_{\dots}$  for the  $n$ -simplicial object in which  $PX_{m_1 \dots m_n} = X_{(m_1+1)m_2 \dots m_n}$ .

Consider the following sequence of functors.

$$wS_1 S^{(n-1)}SC(\mathcal{C}) \longrightarrow PwS^{(n)}SC(\mathcal{C}) \longrightarrow wS^{(n)}SC(\mathcal{C}),$$

where the first functor is the constant inclusion as the 0-space, and the second is the contraction induced by  $\partial_0$  on the outermost simplicial level. As  $S_0\mathcal{E}$  is constant for any Waldhausen category  $\mathcal{E}$ , the composite of the diagram is constant. Similarly, for any  $m \geq 0$   $wS^{(n)}SC(\mathcal{C}_m)$  is connected, as if we plug in 0 to any of the  $S$ -directions we get a constant category. In addition, by proposition 1.5.3 of [7], this is a fiber sequence if  $n \geq 2$ . Thus by lemma 6.5 for  $n + 1$ -simplicial categories, the original diagram was a fiber sequence.

As  $S_1\mathcal{E} = \mathcal{E}$  for all Waldhausen categories  $\mathcal{E}$ , this fiber sequence gives us, for every  $n \geq 2$ , an induced map  $K(\mathcal{C})_{n-1} \rightarrow \Omega K(\mathcal{C})_n$  which is a weak equivalence. It remains to show that we have a morphism  $K(\mathcal{C})_0 \rightarrow \Omega K(\mathcal{C})_1$ . Considering the above sequence for  $n = 1$  we have

$$wS_1 SC(\mathcal{C}) \longrightarrow PwS SC(\mathcal{C}) \longrightarrow wS SC(\mathcal{C}).$$

While the third criterion from lemma 6.5 no longer applies, the composition is still constant and  $PwS SC(\mathcal{C})$  is still contractible, so we have a well-defined (up to homotopy) morphism  $K(\mathcal{C})_0 \rightarrow \Omega K(\mathcal{C})_1$ , as desired.  $\square$

For all  $n$   $L SC(\mathcal{C})$  is a simplification of  $S SC(\mathcal{C})$ , so  $K(\mathcal{C})_n = |NwL^{(n)}SC(\mathcal{C})|$ . Now let

$$\tilde{K}(\mathcal{C})_n = |NwSC(s^{(n)}\mathcal{C}) : (\Delta^{\text{op}})^{n+2} \longrightarrow \mathbf{Sets}|.$$

(This is clearly a spectrum, as the proof of 6.4 translates directly to this case.) By proposition 6.3 we have a morphism  $\tilde{K}(\mathcal{C}) \rightarrow K(\mathcal{C})$  induced by  $F$ , which is levelwise an equivalence (and thus an equivalence of spectra). In particular we can take  $\tilde{K}(\mathcal{C})$  to be the definition of the  $K$ -theory of a simplicial polytope complex.

The main advantage of passing to simplicial polytope complexes is that it allows us to start the  $S$ -construction at any level, and thus compute deloopings of our  $K$ -theory spectra on the polytope complex level.

**Corollary 6.6.** *Let  $\mathcal{C}$  be a simplicial polytope complex, and let  $\sigma\mathcal{C}$  be the simplicial polytope complex with*

$$(\sigma\mathcal{C})_k = s_k\mathcal{C}_k.$$

*Then  $\Omega K(\sigma\mathcal{C}) \simeq K(\mathcal{C})$ .*

*Proof.* Geometric realizations on multisimplicial sets simply look at the diagonal, so

$$\tilde{K}(\mathcal{C})_n = |[k] \mapsto NwSC(s_k^{(n)}\mathcal{C}_k)|.$$

Thus

$$\begin{aligned} \tilde{K}(\sigma\mathcal{C})_n &= |[k] \mapsto NwSC(s_k^{(n)}(s_k\mathcal{C}_k))| \\ &= |[k] \mapsto NwSC(s_k^{(n+1)}\mathcal{C}_k)| = \tilde{K}(\mathcal{C})_{n+1}. \end{aligned}$$

Thus  $\tilde{K}(\sigma\mathcal{C})$  is a spectrum which is a shift of  $\tilde{K}(\mathcal{C})$ , so  $\tilde{K}(\mathcal{C}) \simeq \Omega\tilde{K}(\sigma\mathcal{C})$ . As  $\tilde{K}(\mathcal{C}) \simeq K(\mathcal{C})$ , the desired result follows.  $\square$

Using this corollary we can compute a polytope complex model of every sphere. The polytope complex  $\mathbb{S} = \emptyset \dashrightarrow *$  has  $K(\mathbb{S})$  equal to the sphere spectrum (up to stable equivalence). (For more on this, see [8], section 5.) In order to get  $S^1$  we need to deloop  $\mathbb{S}$ . Note that  $f_n\mathbb{S} = \mathbb{S}$  for all  $n$ , so  $s_n\mathbb{S} = \mathbb{S}^{\vee n}$ . So the simplicial polytope complex which gives  $S^1$  on  $K$ -theory is

$$(\mathbb{S}^{\vee 0}, \mathbb{S}^{\vee 1}, \mathbb{S}^{\vee 2}, \dots, \mathbb{S}^{\vee n}, \dots).$$

Since  $f_n(\mathcal{C} \vee \mathcal{D}) = f_n\mathcal{C} \vee f_n\mathcal{D}$  we know that  $f_n\mathbb{S}^{\vee m} = \mathbb{S}^{\vee m}$ , so we compute that the simplicial polytope complex which gives  $S^2$  on  $K$ -theory is

$$(\mathbb{S}^{\vee 0}, \mathbb{S}^{\vee 1}, \mathbb{S}^{\vee 2^2}, \mathbb{S}^{\vee 3^2}, \dots, \mathbb{S}^{\vee n^2}, \dots).$$

In general we obtain  $S^k$  as the  $K$ -theory of

$$(\mathbb{S}^{\vee 0^k}, \mathbb{S}^{\vee 1^k}, \mathbb{S}^{\vee 2^k}, \mathbb{S}^{\vee 3^k}, \dots, \mathbb{S}^{\vee n^k}, \dots).$$

Note that in fact this works for  $k = 0$  as well, as long as we interpret  $0^0$  to be 1.

We also get the following model for the suspension spectrum of any simplicial set:

**Corollary 6.7.** *For any set  $A$ , let  $\hat{A} = \bigvee_A \mathbb{S}$ . Given a simplicial set  $X$ , write  $\hat{X}$  for the simplicial polytope complex with  $(\hat{X})_n = \hat{X}_n$ . Then*

$$\Sigma_+^\infty X \simeq K(\hat{X}).$$

*Proof.* We have

$$\begin{aligned} \Sigma_+^\infty X &\simeq \operatorname{hocolim}_{\Delta^{\text{op}}} \Sigma_+^\infty X_n \simeq \operatorname{hocolim}_{\Delta^{\text{op}}} \bigvee_{X_n} QS^0 \\ &\simeq \operatorname{hocolim}_{\Delta^{\text{op}}} \bigvee_{X_n} K(\mathbb{S}) \simeq \operatorname{hocolim}_{\Delta^{\text{op}}} K(\hat{X}_n) \simeq K(\hat{X}), \end{aligned}$$

as desired.  $\square$

## 7. Cofibers

Waldhausen's cofiber lemma (see [7], corollary 1.5.7) gives the following formula for the cofiber of a functor  $G: \mathcal{E} \rightarrow \mathcal{E}'$ . We define  $S_n G$  to be the pullback of the

diagram

$$S_n \mathcal{E} \xrightarrow{S_n G} S_n \mathcal{E}' \xleftarrow{\partial_0} S_{n+1} \mathcal{E}'.$$

Define  $K(S.G)$  by

$$K(S.G)_n = |wS^{(n)} S.G|.$$

Then the sequence  $K(\mathcal{E}) \rightarrow K(\mathcal{E}') \rightarrow K(S.G)$  is a homotopy cofiber sequence.

Our goal for this section is to compute a version of this for polytope complexes.

**Definition 7.1.** Let  $g: \mathcal{C} \rightarrow \mathcal{D}$  be a morphism in  $\text{Kl}(-^\infty)$ . We define  $\mathcal{D}/g$  to be the simplicial polytope complex with  $(\mathcal{D}/g)_n = f_{n+1} \mathcal{D} \vee s_n \mathcal{C}$  and the following structure maps. For all  $i > 0$ ,  $\partial_i: (\mathcal{D}/g)_n \rightarrow (\mathcal{D}/g)_{n-1}$  is induced by the two morphisms

$$\partial_{i+1}^{(n+1)}: f_{n+1} \mathcal{D} \longrightarrow f_n \mathcal{D} \quad \text{and} \quad \partial_i: s_n \mathcal{C} \longrightarrow s_{n-1} \mathcal{C}.$$

Similarly, for all  $i \geq 0$ ,  $\sigma_i: (\mathcal{D}/g)_n \rightarrow (\mathcal{D}/g)_{n+1}$  is induced by the morphisms

$$\sigma_{i+1}^{(n+1)}: f_{n+1} \mathcal{D} \longrightarrow f_{n+2} \mathcal{D} \quad \text{and} \quad \sigma_i: s_n \mathcal{C} \longrightarrow s_{n+1} \mathcal{C}.$$

$\partial_0$ , on the other hand, is induced by the three morphisms

$$\partial_1^{(n+1)}: f_{n+1} \mathcal{D} \longrightarrow f_n \mathcal{D} \quad f_n g: f_n \mathcal{C} \longrightarrow f_n \mathcal{D} \quad 1: s_{n-1} \mathcal{C} \longrightarrow s_{n-1} \mathcal{C}.$$

When  $g$  is clear from context we will often write  $\mathcal{D}/\mathcal{C}$  instead of  $\mathcal{D}/g$ . For every  $n \geq 0$  we have a diagram of polytope complexes

$$\mathcal{D} \longrightarrow (\mathcal{D}/g)_n \longrightarrow s_n \mathcal{C}$$

given by the inclusion  $\mathcal{D} \rightarrow f_{n+1} \mathcal{D}$  (as the constant objects) and the projection down to  $s_n \mathcal{C}$ . Then  $\text{SC}(\mathcal{D}/g)$  is the pullback of

$$\text{SC}(s.\mathcal{C}) \xrightarrow{\text{SC}(s.g)} \text{SC}(s.\mathcal{D}) \xleftarrow{\partial_0} \text{PSC}(s.\mathcal{D}),$$

which exactly mirrors the construction of  $S.G$ . (This is clear from an analysis of  $S.\text{SC}(g)$  analogous to that of section 5.) In particular we have from [7] proposition 1.5.5 and corollary 1.5.7 that

$$wS^{(n)} \text{SC}(\mathcal{C}) \longrightarrow wS^{(n)} \text{SC}(\mathcal{D}) \longrightarrow wS^{(n)} \text{SC}(\mathcal{D}/g) \longrightarrow wS^{(n)} \text{SC}(s.\mathcal{C})$$

is a fiber sequence of  $n+1$ -simplicial categories.

Generalizing this to simplicial polytope complexes, we have the following proposition.

**Proposition 7.2.** *Let  $g: \mathcal{C} \rightarrow \mathcal{D}$ . be a morphism of simplicial polytope complexes, and write  $(\mathcal{D}/g)$ . for the simplicial polytope complex where  $(\mathcal{D}/g)_n = (\mathcal{D}_n/g_n)_n$ . Then we have a cofiber sequence of spectra*

$$K(\mathcal{C}.) \longrightarrow K(\mathcal{D}.) \longrightarrow K((\mathcal{D}/g).),$$

where the first map is induced by  $g$ ., and the second is induced for each  $n$  by the inclusion  $\mathcal{D}_n \rightarrow (\mathcal{D}_n/g_n)_n$  as the constant objects of  $f_{n+1} \mathcal{D}_n$ .

*Proof.* As all cofiber sequences in spectra are also fiber sequences, it suffices to show that this is a fiber sequence. As homotopy pullbacks in spectra are level-wise (see, for example, [2], section 18.3), it suffices to show that for all  $n \geq 0$ ,  $K(\mathcal{C})_n \rightarrow K(\mathcal{D})_n \rightarrow K((\mathcal{D}/g)_n)$  is a homotopy fiber sequence. However, as we know that above level 0 all of these are  $\Omega$ -spectra it in fact suffices to show this for  $n > 0$ .

Thus in particular we want to show that for all  $n > 0$  the sequence

$$wS^{(n)}\text{SC}(\mathcal{C}) \longrightarrow wS^{(n)}\text{SC}(\mathcal{D}) \longrightarrow wS^{(n)}\text{SC}((\mathcal{D}/g))$$

is a homotopy fiber sequence of  $n + 1$ -simplicial categories. Let  $\mathcal{D}/g$  be the bisimplicial polytope complex where the  $(k, \ell)$ -th polytope complex is  $(\mathcal{D}_k/g_k)_\ell$ . It will suffice to show that

$$wS^{(n)}\text{SC}(\mathcal{D}) \longrightarrow wS^{(n)}\text{SC}(\mathcal{D}/g) \longrightarrow wS^{(n)}\text{SC}(s\mathcal{C})$$

is a fiber sequence of  $n + 2$ -simplicial categories (where  $\mathcal{D}$  is now considered a bisimplicial polytope complex); in this diagram the second morphism is induced by the projection  $(\mathcal{D}_k/g_k)_\ell \rightarrow s_\ell \mathcal{C}_k$  for all pairs  $(k, \ell)$ . Then by comparing this sequence for the functor  $1: \mathcal{C} \rightarrow \mathcal{C}$  to the functor  $g$  we will be able to conclude the desired result. (In this approach we follow Waldhausen in [7], 1.5.6.)

We show this by applying lemma 6.5, where we fix the index of the simplicial direction of  $\mathcal{C}$  and  $\mathcal{D}$ . The composition of the two functors is constant, as we first include  $\mathcal{D}$  and then project away from it, and as we do not fix any of the  $S$  indices the last space will be connected. Thus we want

$$wS^{(n)}\text{SC}(\mathcal{D}_k) \longrightarrow wS^{(n)}\text{SC}(\mathcal{D}_k/g_k) \longrightarrow wS^{(n)}\text{SC}(s\mathcal{C}_k)$$

to be a fiber sequence, which holds by our discussion above. So we are done. □

### 8. Wide and tall subcategories

We now take a slight detour into a more computational direction. Consider the case of a polytope complex  $\mathcal{D}$ , together with a subcomplex  $\mathcal{C}$ . We know that the inclusion  $\mathcal{C} \rightarrow \mathcal{D}$  induces a map  $K(\mathcal{C}) \rightarrow K(\mathcal{D})$ . The goal of this section is to give sufficient conditions on  $\mathcal{C}$  which will ensure that this map is an equivalence.

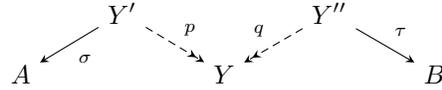
We start off the section with an easy computational result which will make later proofs much simpler.

**Lemma 8.1.** *For any object  $Y \in w\text{SC}(\mathcal{C})$ ,  $(Y \downarrow w\text{SC}(\mathcal{C}))$  is a cofiltered preorder.*

*Proof.* In order to see that  $(Y \downarrow w\text{SC}(\mathcal{C}))$  is a preorder it suffices to show that given any diagram

$$A \xleftarrow{\sim} Y \xrightarrow{\sim} B$$

in  $w\text{SC}(\mathcal{C})$  there exists at most one morphism  $A \xrightarrow{\sim} B$  that makes the diagram commute. This diagram is represented by a diagram in  $\text{Tw}(\mathcal{C}_p)$



where  $\sigma$  and  $\tau$  are isomorphisms. Then morphisms  $h: A \rightarrow B$  such that  $g = hf$  correspond exactly to factorizations of  $q$  through  $p$ ; as  $(\text{Tw}(\mathcal{C}_p)^{\text{Sub}} \downarrow T)$  is a preorder, there is at most one of these and we are done.

Thus it remains to show that this preorder is cofiltered; in particular, we want to find an object below  $A$  and  $B$  under  $Y$ . Given a shuffle  $\sigma'$ , let  $f_{\sigma'} \in \text{SC}(\mathcal{C})$  be the pure shuffle defined by  $\sigma'$ ; similarly, for a sub-map  $p'$  let  $f^{p'} \in \text{SC}(\mathcal{C})$  be the pure sub-map defined by  $p'$ . Let  $Z = Y' \times_Y Y''$  be the vertical pullback of  $p$  and  $q$ . Then, the pullback of

$$A \xrightarrow{\sigma^{-1}} Y' \leftarrow \text{----} Z$$

gives a weak equivalence  $A \xrightarrow{\sim} Z$ , and analogously we have a weak equivalence  $B \xrightarrow{\sim} Z$ . As these commute under  $Y$  we see that  $(Y \downarrow w\text{SC}(\mathcal{C}))$  is cofiltered, as desired. □

The first condition that we need in order to have an equivalence on  $K$ -theory is that we must have the same  $K_0$ ; more specifically, we need every object of  $\text{SC}(\mathcal{D})$  to be weakly equivalent to something in  $\text{SC}(\mathcal{C})$ . As a condition on polytope complexes, this turns into the following definition.

**Definition 8.2.** Suppose that  $\mathcal{D}$  is a polytope complex and  $\mathcal{C} \rightarrow \mathcal{D}$  is an inclusion of polytope complexes. We say that  $\mathcal{C}$  has *sufficiently many covers* if for every object  $B \in \mathcal{D}$  there exists a finite covering family  $\{B_\alpha \dashrightarrow B\}_{\alpha \in A}$  such that the  $B_\alpha$  are pairwise disjoint, and such that every  $B_\alpha$  is horizontally isomorphic to an object of  $\mathcal{C}$ .

Our first approximation result is almost obvious: if we cover all weak equivalence classes of objects, and all morphisms between these objects, then we must have an equivalence on  $K$ -theory. More formally, we have the following:

**Lemma 8.3.** *Suppose that  $\mathcal{C}$  has sufficiently many covers, and that  $\text{SC}(\mathcal{C})$  sits inside  $\text{SC}(\mathcal{D})$  as a full subcategory. Then the induced map  $|\text{wSC}(\mathcal{C})| \rightarrow |\text{wSC}(\mathcal{D})|$  is a homotopy equivalence.*

*Proof.* Using Quillen’s Theorem A (from [3]) it suffices to show that for all  $Y \in \text{wSC}(\mathcal{D})$  the category  $(Y \downarrow \text{wSC}(\mathcal{C}))$  is contractible. Now as  $(Y \downarrow \text{wSC}(\mathcal{D}))$  is a preorder (by lemma 8.1) and  $\text{SC}(\mathcal{C})$  is a subcategory of  $\text{SC}(\mathcal{D})$ , we know that  $(Y \downarrow \text{wSC}(\mathcal{C}))$  is also a preorder; thus to show that it is contractible we only need to know that it is cofiltered. In addition, as  $\text{SC}(\mathcal{C})$  is a full subcategory of  $\text{SC}(\mathcal{D})$ , it in fact suffices to show that we have enough objects for it to be cofiltered, so it suffices to show that this category is nonempty for all  $Y$ .

So let us show that for all  $Y \in \text{wSC}(\mathcal{D})$  the category  $(Y \downarrow \text{wSC}(\mathcal{C}))$  is nonempty. We need to show that for any  $Y \in \text{wSC}(\mathcal{D})$  there exists a  $Z \in \text{wSC}(\mathcal{C})$  an a weak equivalence  $Y \xrightarrow{\sim} Z$ . Write  $Y = \{y_i\}_{i \in I}$ . For each  $i \in I$ , let  $\{y_\alpha^{(i)} \dashrightarrow y_i\}_{\alpha \in A_i}$  be the cover

guaranteed by the sufficient covers condition, and let  $\beta_\alpha^{(i)}: y_\alpha^{(i)} \rightarrow z_\alpha^{(i)}$  be the horizontal isomorphisms guaranteed by the sufficient covers condition. Then the induced vertical morphism  $\{y_\alpha^{(i)}\}_{i \in I, \alpha \in A_i} \dashrightarrow \{y_i\}_{i \in I}$  is a covering sub-map and the horizontal morphism  $\beta: \{y_\alpha^{(i)}\}_{i \in I, \alpha \in A_i} \rightarrow \{z_\alpha^{(i)}\}_{i \in I, \alpha \in A_i}$  is a horizontal isomorphism, so the morphism in  $\text{SC}(\mathcal{D})$  represented by this is a weak equivalence. But by definition  $\{z_\alpha^{(i)}\}_{i \in I, \alpha \in A_i}$  is in  $\text{SC}(\mathcal{C})$ , so we are done.  $\square$

In the statement of the previous lemma we had two conditions. One was a condition on  $\mathcal{C}$ , and one was a condition on  $\text{SC}(\mathcal{C})$ . We would like to get those conditions down to conditions just about  $\mathcal{C}$ , as that will make using this kind of result easier. In order for a morphism of  $\text{SC}(\mathcal{D})$  to be in  $\text{SC}(\mathcal{C})$  we need some representative of the morphism to come from a diagram in  $\text{Tw}(\mathcal{C})$ ; in particular, this means that both the representing object, and the morphisms which are the components of the vertical and horizontal components, must be in  $\mathcal{C}$ .

If  $\mathcal{C}$  is not a full subcomplex of  $\mathcal{D}$  then much of this analysis becomes much more difficult, so for the rest of this section we will assume that  $\mathcal{C}$  is a full subcomplex of  $\mathcal{D}$ . This means that as long as we know that a representing object of the morphism is in  $\text{Tw}(\mathcal{C})$ , it is sufficient to conclude that the morphism will be in  $\text{SC}(\mathcal{C})$ . In particular, we want to be able to conclude that just because the source and target of a morphism are in  $\text{SC}(\mathcal{C})$  then the morphism must be, as well. We can translate this into the following condition.

**Definition 8.4.** Suppose that  $\mathcal{C}$  is a full subcomplex of  $\mathcal{D}$ . We say that  $\mathcal{C}$  is *wide* (respectively, *tall*) if for any horizontal (resp. vertical) morphism  $A \rightarrow B \in \mathcal{D}$ , if  $B$  is in  $\mathcal{C}$  then so is  $A$ .

If  $\mathcal{C}$  is a full subcomplex of  $\mathcal{D}$  then we know that  $\text{Tw}(\mathcal{C})$  is a full subcategory of  $\text{Tw}(\mathcal{D})$ . If  $\mathcal{C}$  happens to also be wide, we know something even stronger: given any horizontally connected component of  $\text{Tw}(\mathcal{D})$ , either that entire component is in  $\text{Tw}(\mathcal{C})$ , or nothing in the component is in  $\text{Tw}(\mathcal{C})$ . Analogously, if  $\mathcal{C}$  is tall we can say the same thing for vertically connected components. This lets us conclude that  $\text{SC}(\mathcal{C})$  is a full subcategory of  $\text{SC}(\mathcal{D})$ .

**Lemma 8.5.** *Let  $\mathcal{C}$  be a full subcomplex of  $\mathcal{D}$ . If  $\mathcal{C}$  is wide or tall then  $\text{SC}(\mathcal{C})$  is a full subcategory of  $\text{SC}(\mathcal{D})$ .*

*Proof.* Let  $\{a_i\}_{i \in I}, \{b_j\}_{j \in J} \in \text{SC}(\mathcal{C})$ , and let  $f: \{a_i\}_{i \in I} \rightarrow \{b_j\}_{j \in J}$  be a morphism in  $\text{SC}(\mathcal{D})$ . This morphism is represented by a diagram

$$\{a_i\}_{i \in I} \xleftarrow{p} \{a'_k\}_{k \in K} \xrightarrow{\sigma} \{b_j\}_{j \in J}.$$

In order for  $f$  to be in  $\text{SC}(\mathcal{C})$  it suffices to show that each  $a'_k$  is in  $\mathcal{C}$ , as  $\mathcal{C}$  is a full subcategory of  $\mathcal{D}$ . Now if  $\mathcal{C}$  is wide then for all  $k \in K$  we have a horizontal morphism  $\Sigma_k: a_k \rightarrow b_{\sigma(k)}$ . As  $\mathcal{C}$  is wide and each  $b_j \in \mathcal{C}$  we must have  $a'_k \in \mathcal{C}$  for all  $k$ . So  $\text{SC}(\mathcal{C})$  is a full subcategory of  $\text{SC}(\mathcal{D})$ . If, on the other hand,  $\mathcal{C}$  is tall then for each  $k \in K$  we consider the vertical morphism  $P_k: a'_k \dashrightarrow a_{p(k)}$ . As  $a_i \in \mathcal{C}$  for all  $i \in I$  we must also have  $a'_k \in \mathcal{C}$  for all  $k' \in K$ . So  $\text{SC}(\mathcal{C})$  is a full subcategory of  $\text{SC}(\mathcal{D})$ , and we are done.  $\square$

Which leads us to the following approximation result.

**Proposition 8.6.** *Suppose that  $\mathcal{C}$  is a subcomplex of  $\mathcal{D}$  with sufficiently many covers. If  $\mathcal{C}$  is wide or tall, the inclusion  $\mathcal{C} \rightarrow \mathcal{D}$  induces an equivalence  $K(\mathcal{C}) \rightarrow K(\mathcal{D})$ .*

*Proof.* Lemma 8.3 shows that  $K(\mathcal{C})_i \rightarrow K(\mathcal{D})_i$  is an equivalence for  $i = 0$ . If we can show that for all  $n$ ,  $s_n\mathcal{C}$  is a wide or tall subcomplex of  $s_n\mathcal{D}$  with sufficiently many covers we will be done, as we will be able to induct on  $i$  to see that the induced morphism is an equivalence on all levels. In fact, note that it suffices to show that  $f_n\mathcal{C}$  is a wide (resp. tall) subcomplex of  $f_n\mathcal{D}$  with sufficiently many covers.

First we show that  $f_n\mathcal{C}$  has sufficiently many covers in  $f_n\mathcal{D}$ . Consider an object  $D$

$$D_1 \leftarrow \dots \leftarrow D_n$$

of  $f_n\mathcal{D}$ . As  $\mathcal{C}$  has sufficiently many covers in  $\mathcal{D}$  there exists a covering family of  $D_n$ ,  $\{B_\alpha \rightarrow D_n\}_{\alpha \in A}$ , in which every object is horizontally isomorphic to an object of  $\mathcal{C}$ . Given an object  $X \in \mathcal{D}$ , let  $\bar{X} \in f_n\mathcal{D}$  be the constant object where  $\bar{X}_k = \{X\}$ . Then the family  $\{\bar{B}_\alpha \rightarrow \bar{D}\}_{\alpha \in A}$  is a covering family of  $D$ . As each  $B_\alpha$  was horizontally isomorphic to an object in  $\mathcal{C}$ , each  $\bar{B}_\alpha$  is horizontally isomorphic to something in  $f_n\mathcal{C}$ , and we are done.

The fact that if  $\mathcal{C}$  was a tall (resp. wide) subcomplex of  $\mathcal{D}$  then  $f_n\mathcal{C}$  is a tall (resp. wide) subcomplex of  $\mathcal{D}$  follows directly from the definition of  $f_n\mathcal{C}$  and  $f_n\mathcal{D}$ .  $\square$

Finally, we can generalize this result to simplicial polytope complexes.

**Corollary 8.7.** *Suppose that  $\mathcal{C} \rightarrow \mathcal{D}$  is a morphism of simplicial polytope complexes. If for each  $n$ , the morphism  $\mathcal{C}_n \rightarrow \mathcal{D}_n$  is an inclusion of  $\mathcal{C}_n$  as a subcomplex into  $\mathcal{D}_n$  and satisfies the conditions of lemma 8.6, then the map  $K(\mathcal{C}) \rightarrow K(\mathcal{D})$  is an equivalence.*

We finish up this section with a couple of applications of this result.

**More explicit formula for suspensions and cofibers.**

For any polytope complex  $\mathcal{C}$  and any positive integer  $n$  we have a polytope functor  $\mathcal{C} \rightarrow f_n\mathcal{C}$  given by including and object  $a$  as the constant object

$$\bar{a} = \{a\} \leftarrow \dots \leftarrow \{a\}$$

This includes  $\mathcal{C}$  as a wide subcomplex of  $f_n\mathcal{C}$ . In fact,  $\mathcal{C}$  also has sufficiently many covers. Given any object

$$A = A_1 \leftarrow \dots \leftarrow A_n,$$

write  $A_n = \{a_i\}_{i \in I}$ . Then the family  $\{\bar{a}_i \rightarrow A\}_{i \in I}$  is a covering family, and each  $\bar{a}_i \in \mathcal{C}$ . Thus we have an inclusion  $\mathcal{C}^{\vee n} \rightarrow f_n\mathcal{C}$  which induces an equivalence on  $K$ -theory.

In fact, this is an equivalence on the  $K$ -theory of simplicial polytope complexes, as this inclusion commutes with the simplicial structure maps. Thus  $s\mathcal{C}$  can be considered to be a bar construction on  $\mathcal{C}$ , as the structure maps of  $s\mathcal{C}$ , when restricted to the constant objects, exactly mirror the morphisms of the bar construction. (The 0-th face map forgets the first one, the next  $n - 1$  glue successive copies of  $\mathcal{C}$  together, and the  $n$ -th one forgets the last one, exactly as the bar construction does. The degeneracies each skip one of the  $\mathcal{C}$ 's in  $s_{n+1}\mathcal{C}$ .)

Generalizing to simplicial polytope complexes, this gives the following simplifications of the formulas for  $\sigma\mathcal{C}$ . and  $(\mathcal{D}/g)$ . from corollary 6.6 and proposition 7.2:

**Corollary 8.8.** *Let  $g: \mathcal{C} \rightarrow \mathcal{D}$ . be a morphism of simplicial polytope complexes. Let  $\sigma\mathcal{C}$ . and  $(\mathcal{D}/g)$ . be the simplicial polytope complexes defined by*

$$(\sigma\mathcal{C})_n = \mathcal{C}_n^{\vee n} \quad \text{and} \quad (\mathcal{D}/g)_n = \mathcal{D}_n \vee \mathcal{C}_n^{\vee n}.$$

Then  $\Omega K(\sigma\mathcal{C}) \simeq K(\mathcal{C})$  and

$$K(\mathcal{C}) \longrightarrow K(\mathcal{D}) \longrightarrow K((\mathcal{D}/g).)$$

is a cofiber sequence of spectra.

It is necessary to check that these inclusions commute with the simplicial maps, but it is easy to see that they do. Note that on  $(\mathcal{D}/g)_n$ ,  $\partial_0$  is induced by the three morphisms

$$\partial_0: \mathcal{D}_n \longrightarrow \mathcal{D}_{n-1} \quad g\partial_0: \mathcal{C}_n \longrightarrow \mathcal{D}_{n-1} \quad \partial_0^{\vee n-1}: \mathcal{C}_n^{\vee n-1} \longrightarrow \mathcal{C}_{n-1}^{\vee n-1}.$$

#### Local data on homogeneous manifolds.

Let  $X$  be a geodesic  $n$ -manifold with a preferred open cover  $\{U_\alpha\}_{\alpha \in A}$  such that for any  $\alpha \in A$  and any two points  $x, y \in U_\alpha$  there exists a unique geodesic connecting  $x$  and  $y$ . (For example,  $X = E^n$ ,  $S^n$ , or  $\mathbb{H}^n$  are examples of such  $X$ . In the first and third case we take our open cover to be the whole space; in the second case we take it to be the set of open hemispheres.) We then define a polytope complex  $\mathcal{C}_X$  in the following manner. Define a *simplex* of  $X$  to be a convex hull of  $n + 1$  points all sitting inside some  $U_\alpha$  with nonempty interior, and a *polytope* of  $X$  to be a finite union of simplices. We then define  $\mathcal{C}_{X_v}$  to be the poset of polytopes of  $X$  under inclusion with the obvious topology. Given two polytopes  $P$  and  $Q$ , we define a *local isometry of  $P$  onto  $Q$*  to be a triple  $(U, V, \varphi)$  such that  $U$  and  $V$  are open subsets of  $X$  with  $P \subseteq U$  and  $Q \subseteq V$ ,  $\varphi: U \rightarrow V$  is an isometry of  $U$  into  $V$ , and  $\varphi(P) = Q$ . Then we define a horizontal morphism  $P \rightarrow Q$  to be an equivalence class of local isometries of  $P$  onto  $Q$ , with  $(U, V, \varphi) \sim (U', V', \varphi')$  if  $\varphi|_{U \cap U'} = \varphi'|_{U \cap U'}$ . Under these definitions it is clear that  $\mathcal{C}_X$  is a polytope complex.

Now let  $U \subseteq X$  be any preferred open subset of  $X$  with the preferred cover  $\{U\}$ . Then  $\mathcal{C}_U$  is also a polytope complex and we have an obvious inclusion map  $\mathcal{C}_U \rightarrow \mathcal{C}_X$ .

**Lemma 8.9.** *If  $X$  is homogeneous then the inclusion  $\mathcal{C}_U \hookrightarrow \mathcal{C}_X$  induces an equivalence  $K(\mathcal{C}_U) \rightarrow K(\mathcal{C}_X)$ .*

*Proof.* Clearly  $\mathcal{C}_U$  is a tall subcomplex of  $\mathcal{C}_X$ . Given any polytope  $P \subseteq \mathcal{C}_X$  we can triangulate it by triangles small enough to be in a single chart. Once we are in a single chart we can subdivide each triangle by barycentric subdivision until the diameter of every triangle in the triangulation is small enough that the triangle can fit inside  $U$ . As  $X$  is homogeneous there is a local isometry of any such triangle into  $U$ , and thus  $\mathcal{C}_U$  has sufficiently many covers. Thus by proposition 8.6 the induced map  $K(\mathcal{C}_U) \rightarrow K(\mathcal{C}_X)$  is an equivalence.  $\square$

Any isometry  $X \rightarrow Y$  which takes preferred open sets into preferred open sets induces a polytope functor  $\mathcal{C}_X \rightarrow \mathcal{C}_Y$  (which is clearly an isomorphism). Thus the

statement of proposition 1.4 is exactly that all morphisms in the diagram

$$\mathcal{C}_X \longleftarrow \mathcal{C}_U \xrightarrow{c_\varphi} \mathcal{C}_V \longrightarrow \mathcal{C}_Y$$

are equivalences, which follows easily from the above lemma.

**Non-examples**

We conclude this section with a couple of non-examples. First, take any polytope complex  $\mathcal{C}$  and consider the polytope complex  $\mathcal{C} \vee \mathcal{C}$ .  $\mathcal{C}$  sits inside this (as the left copy, for example) and is tall by definition, but the K-theories of these are not equivalent as the left copy of  $\mathcal{C}$  does not contain sufficiently many covers. (In particular, it can't cover anything in the right copy of  $\mathcal{C}$ .) However, if we added “twist” isomorphisms — horizontal isomorphisms between corresponding objects in the left and right copies of  $\mathcal{C}$  — then the left  $\mathcal{C}$  would contain sufficiently many covers, and the K-theories of these would be equal.

As our second non-example we will look at ideals of a number field. Let  $K$  be a number field with Galois group  $G$ . Let the objects of  $\mathcal{C}$  be the ideals of  $K$ . We will have a vertical morphism  $I \dashrightarrow J$  whenever  $I|J$ , and we will have our horizontal morphisms induced by the action of  $G$ . The  $K$ -theory of this will be countably many spheres wedged together, one for each prime power ideal of  $K$ . The prime ideals sit inside  $\mathcal{C}$  as a wide subcomplex, but they do not give an equivalence because if  $\mathfrak{p}^k$  is a prime power ideal for  $k > 1$  then it can't be covered by prime ideals. The K-theories of these two will in fact be equivalent, since they are both countably many spheres wedged together, but the inclusion does not induce an equivalence.

**9. Proof of lemma 5.3**

This section concerns the proof of lemma 5.3:

*Lemma 5.3.*  $L_n\text{SC}(\mathcal{C})$  is a Waldhausen category. The cofibrations (resp. weak equivalences) in  $L_n\text{SC}(\mathcal{C})$  are exactly the morphisms which are levelwise cofibrations (resp. weak equivalences).  $L_n\text{SC}(\mathcal{C})$  is a simplification of  $F_n\text{SC}(\mathcal{C})$ .

A morphism  $A \rightarrow B \in F_n\text{SC}(\mathcal{C})$  is represented by a diagram

$$\begin{array}{ccccccc} A_1 & \hookrightarrow & A_2 & \hookrightarrow & \dots & \hookrightarrow & A_n \\ \downarrow & & \downarrow & & & & \downarrow \\ B_1 & \hookrightarrow & B_2 & \hookrightarrow & \dots & \hookrightarrow & B_n \end{array}$$

and by lemma 5.2(1) will be layered exactly when for each  $i = 1, \dots, n - 1$  the diagram

$$\begin{array}{ccc} A_i/A_{i-1} & \hookrightarrow & A_i \\ \downarrow & & \downarrow \\ B_i/B_{i-1} & \hookrightarrow & B_i \end{array}$$

commutes. As each square is considered separately, for all of the proofs in this section we will assume that  $n = 2$ , as for all other values of  $n$  the proofs will be equivalent, and it saves on having an extra variable floating around.

**Lemma 9.1.** *Any layered morphism which is levelwise a cofibration is a cofibration.*

*Proof.* We want to show that if

$$f_1, f_2: (A_1 \xrightarrow{i} A_2) \hookrightarrow (B_1 \xrightarrow{j} B_2)$$

is layered, then the induced morphism  $\varphi: A_2 \cup_{A_1} B_1 \rightarrow B_2$  is a cofibration. The layering condition ensures that the square  $(\text{im}_{A_2} A_1 \hookrightarrow A_2) \rightarrow (\text{im}_{B_2} B_1 \hookrightarrow B_2)$  is layered. Thus, it suffices to consider the case when  $i$  and  $j$  are weak equivalences. But then  $B_1 \xrightarrow{\sim} A_2 \cup_{A_1} B_1$  is a weak equivalence, and by 2-of-3 (lemma 6.9 in [8]) so is  $\varphi$ . As all weak equivalences are cofibrations, we are done.  $\square$

Now we turn our attention to showing that  $L_n \text{SC}(\mathcal{C})$  is a simplification of  $F_n \text{SC}(\mathcal{C})$ . We first develop a little bit of computational machinery for layering, which will allow us to work with cofibrations more easily.

Given any object  $A = \{a_i\}_{i \in I} \in \text{SC}(\mathcal{C})$ , we say that  $A'$  is a *subobject* of  $A$  if  $A' = \{a_i\}_{i \in I'}$  for some subset  $I' \subseteq I$ . If  $A', A''$  are two subobjects of  $A$ , we will write  $A' \cap A''$  for  $\{a_i\}_{i \in I' \cap I''}$ , and we will write  $A' \subseteq A''$  if  $I' \subseteq I''$ . Suppose that  $f: A \rightarrow B$  is a morphism in  $\text{SC}(\mathcal{C})$ . Pick a representation of this by a sub-map  $p$  and a shuffle  $\sigma$ , and write  $B = \{b_j\}_{j \in J}$ . Then  $\text{im}_B A = \{b_j\}_{j \in \text{im}\sigma}$ . Note that this agrees with the previous definition of image when  $f$  is a cofibration, and  $\text{im}_B A$  is a subobject of  $B$ . If we write  $A = A_1 \amalg A_2$  then  $A_1$  and  $A_2$  are subobjects of  $A$ , and  $\text{im}_B A = \text{im}_B A_1 \cup \text{im}_B A_2$ . If  $f$  were a cofibration, we also have  $\text{im}_B A_1 \cap \text{im}_B A_2 = \emptyset$ ; if  $f$  were a weak equivalence then  $\text{im}_B A = B$ . (For example,  $\text{im}_B A \cap (B/A) = \emptyset$ .) Given a second morphism  $g: B \rightarrow C$ ,  $\text{im}_C A \subseteq \text{im}_C B$ .

Now consider a commutative square

$$f_1, f_2: (A_1 \hookrightarrow A_2) \longrightarrow (B_1 \hookrightarrow B_2).$$

This square satisfies the layering condition exactly when

$$\text{im}_{B_2}(A_2/A_1) \subseteq \text{im}_{B_2}(B_2/B_1) = B_2/B_1,$$

or equivalently when  $\text{im}_{B_2}(A_2/A_1) \cap \text{im}_{B_2} B_1 = \emptyset$ . We will use this restatement in our computations.

**Lemma 9.2.** *Cofibrations are layered.*

*Proof.* If  $A \hookrightarrow B$  is a cofibration, then by definition  $A_2 \cup_{A_1} B_1 \hookrightarrow B_2$  is a cofibration. But we have an acyclic cofibration  $(A_2/A_1) \amalg B_1 \xrightarrow{\sim} A_2 \cup_{A_1} B_1$ , so we see that  $\text{im}_{B_2}(A_2/A_1) \cap \text{im}_{B_2} B_1 = \emptyset$ , as desired.  $\square$

**Lemma 9.3.** *Layered morphisms are closed under pushouts. More precisely, given any commutative square*

$$\begin{array}{ccc} A & \hookrightarrow & B \\ \downarrow & & \downarrow \\ C & \longrightarrow & D \end{array}$$

in which all morphisms are layered, the induced morphisms

$$C \hookrightarrow B \cup_A C \quad B \longrightarrow B \cup_A C \quad B \cup_A C \longrightarrow D$$

are all layered.

*Proof.* The first of these is clearly layered as it is a cofibration.

Write  $X_i = B_i \cup_{A_i} C_i$ . Keep in mind that for all  $i$ , we have an acyclic cofibration  $(B_i/A_i) \amalg C_i \xrightarrow{\sim} X_i$ .

For the second, we need to show that  $\text{im}_{X_2}(B_2/B_1) \cap \text{im}_{X_2}(X_1) = \emptyset$ . We have

$$\begin{aligned} \text{im}_{X_2}(B_2/B_1) \cap \text{im}_{X_2} X_1 &= \text{im}_{X_2}(B_2/B_1) \cap (\text{im}_{X_2} C_1 \cup \text{im}_{X_2}(B_1/A_1)) \\ &= (\text{im}_{X_2}(B_2/B_1) \cap \text{im}_{X_2} C_1) \cup \\ &\quad \cup (\text{im}_{X_2}(B_2/B_1) \cap \text{im}_{X_2}(B_1/A_1)). \end{aligned}$$

Consider the first of the two sets we are unioning. By the definition of  $X_2$ ,  $\text{im}_{X_2} C_2 \cap \text{im}_{X_2} B_2 = \text{im}_{X_2} A_2$ . Thus,

$$\text{im}_{X_2}(B_2/B_1) \cap \text{im}_{X_2} C_1 \subseteq \text{im}_{X_2}(B_2/B_1) \cap \text{im}_{X_2} C_2 \subseteq \text{im}_{X_2} A_2.$$

On the other hand,

$$\text{im}_{X_2}(C_1) \cap \text{im}_{X_2}(A_2) = \text{im}_{X_2}(\text{im}_{C_2} C_1 \cap \text{im}_{C_2} A_2) = \text{im}_{X_2}(\text{im}_{C_2} A_1) = \text{im}_{X_2} A_1$$

as  $A \rightarrow C$  is layered. Thus we want to show that  $\text{im}_{X_2}(B_2/B_1) \cap \text{im}_{X_2} A_1 = \emptyset$ . It suffices to show this inside  $B_2$ , where it is obvious. Now consider the second part. As  $B_1/A_1 \rightarrow X_2$  and  $B_2/B_1 \rightarrow X_2$  both factor through  $B_2$ , it suffices to show that  $(B_2/B_1) \cap \text{im}_{B_2}(B_1/A_1) = \emptyset$ , which is clear by definition.

It remains to show that the last of these morphisms is layered. In particular, we need to show that

$$\text{im}_{D_2}(X_2/X_1) \cap \text{im}_{D_2}(D_1) = \emptyset.$$

But it is easy to see that

$$\begin{aligned} \text{im}_{D_2}(X_2/X_1) \cap \text{im}_{D_2}(D_1) &= \text{im}_{D_2}((C_2 \amalg B_2/A_2)/(C_1 \amalg B_1/A_1)) \cap \text{im}_{D_2}(D_1) \\ &= \text{im}_{D_2}(C_2/C_1 \amalg (B_2/A_2)/(B_1/A_1)) \cap \text{im}_{D_2}(D_1) \\ &= (\text{im}_{D_2}(C_2/C_1) \cap \text{im}_{D_2}(D_1)) \cup \\ &\quad \cup (\text{im}_{D_2}((B_2/A_2)/(B_1/A_1)) \cap \text{im}_{D_2}(D_1)). \end{aligned}$$

The first term on the right-hand side is empty because  $C \rightarrow D$  is layered. The second term is empty because  $\text{im}_{D_2}((B_2/A_2)/(B_1/A_1)) \subseteq \text{im}_{D_2}(B_2/B_1)$ , and the intersection of this with  $\text{im}_{D_2}(D_1)$  is empty because  $B \rightarrow D$  is layered. So we are done.  $\square$

Now we are ready to prove lemma 5.3.

*Proof of lemma 5.3.* Firstly we will show that all weak equivalences of  $F_n\text{SC}(\mathcal{C})$  are layered. In particular, it suffices to show that any weak equivalences of  $F_n\text{SC}(\mathcal{C})$  is also a cofibration, since we already know by lemma 9.2 that all cofibrations are layered. In particular, if we have a commutative square

$$(A_1 \hookrightarrow A_2) \xrightarrow{\sim} (B_1 \hookrightarrow B_2)$$

we want to show that the induced morphism  $A_2 \cup_{A_1} B_1 \rightarrow B_2$  is a cofibration. As weak equivalences are preserved under pushouts we know that  $A_2 \xrightarrow{\sim} A_2 \cup_{A_1} B_1$  is a weak equivalence, as is  $A_2 \xrightarrow{\sim} B_2$ . As weak equivalences in  $\text{SC}(\mathcal{C})$  satisfy 2-of-3 in this direction we are done (see lemma 6.9 in [8]).

All morphisms  $A \rightarrow *$  are in  $L_n\text{SC}(\mathcal{C})$ , as these are trivially layered. As lemma 9.3 showed that  $L_n\text{SC}(\mathcal{C})$  is closed under pushouts, we see that  $L_n\text{SC}(\mathcal{C})$  is, in fact, a simplification of  $F_n\text{SC}(\mathcal{C})$ . Weak equivalences of  $L_n\text{SC}(\mathcal{C})$  are levelwise because weak equivalences in  $F_n\text{SC}(\mathcal{C})$  are levelwise, and cofibrations are levelwise by lemma 9.1.  $\square$

## References

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