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DIFFERENTIAL GRADED ALGEBRAS OVER SOME REDUCTIVE GROUPS

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Abstract

In this paper, we study the general properties of commutative differential graded algebras in the category of representations over a reductive algebraic group with an injective central cocharacter. Besides describing the derived category of differential graded modules over such an algebra, we also provide a criterion for the existence of a t-structure on the derived category together with a characterization of the coordinate ring of the Tannakian fundamental group of its heart.

1. Introduction

In [**11**], Kriz and May developed a general theory about Adams graded commutative differential graded algebras (cdga) over $\mathbb{Q}$. Given an Adams graded cdga A, they considered the bounded derived category $\mathcal{D}^f_A$ of A-modules and proved a number of formal properties of it. Under the assumption that A is cohomologically connected, that is, its negative cohomological degree part vanishes and its zero-th degree part is isomorphic to $\mathbb{Q}$, they showed that $\mathcal{D}^f_A$ has a t-structure with heart $\mathcal{H}^f_A$. Moreover, they used the reduced bar construction (see Section 9) to give a description of this heart. Since the reduced bar construction $\bar{B}(A)$ of A is a differential graded Hopf algebra, taking the zero-th cohomology gives us a Hopf algebra $H^0(\bar{B}(A))$, which corresponds to a pro-affine group scheme over $\mathbb{Q}$ with $\mathbb{G}_m$-action, denoted by G_A. Then $\mathcal{H}^f_A$ is equivalent to the category of graded representations of G_A.

Given any field k, let us take A to be Bloch's cycle complex $\mathcal{N}_k$. The theory of Kriz and May implies that if $\mathcal{N}_k$ is cohomologically connected (e.g. k a number field), then the heart $\mathcal{H}^f_{\mathcal{N}_k}$ exists. It turns out that this heart coincides with an earlier construction of Bloch and Kriz of mixed Tate motives [**3**]. For the definition of Bloch's cycle complex $\mathcal{N}_k$, we refer to [**12**]. Later on Spitzweck relates $\mathcal{D}^f_{\mathcal{N}_k}$ to Voevodsky's motives by constructing a functor

$$\theta_k \colon \mathcal{D}^f_{\mathcal{N}_k} \to DMT(k, \mathbb{Q})$$

over any field k [**14**], where $DMT(k, \mathbb{Q})$ is the full rigid tensor subcategory of Voevod-

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sky's triangulated category of geometric motives generated by Tate objects. The precise definition of $DMT(k, \mathbb{Q})$ can be found in [**12**].

If we identify Adams cdgas with cdgas in the category of representations over $\mathbb{G}_m$, the above example serves as a motivation to study more general motives by replacing $\mathbb{G}_m$ by a reasonably general reductive group. Using the theory of cdgas over GL_2, we generalize Spitzweck's equivalence to the case of motives for an elliptic curve without complex multiplication in [**5**].

We now state our main results and describe the outline of this paper. Assume G is a reductive group with an injective central cocharacter and A a cdga over G (see Definition 2.4). As in the case of Adams graded cdgas, we defined the derived category of dg A-modules over G, denoted by $\mathcal{D}_A^G$, and studied their properties, which are the contents of Section 2 to Section 7. If we further assume that A is cohomologically connected and let us denote by $\mathcal{D}_A^{G,f}$ the full subcategory of $\mathcal{D}_A^G$ consisting of compact objects, then we proved:

Theorem 1.1 (Theorem 8.3 and Theorem 11.2)**.** *Suppose A is cohomologically connected. There exists a non-degenerate t-structure on $\mathcal{D}_A^{G,f}$ with heart $\mathcal{H}_A^{G,f}$. Furthermore,*

- *There is a functor $\rho\colon D^b(\mathcal{H}_A^{G,f}) \longrightarrow \mathcal{D}_A^{G,f}$.*
- *The functor ρ constructed above is an equivalence of triangulated categories if and only if A is* 1*-minimal.*

The proof of the first part of the theorem is contained in Section 8. The next two sections are denoted to the proof of the second part. In it, we used the reduced bar construction to give several equivalent descriptions of $\mathcal{H}_A^f$. The proof is completed in Section 11. For the case $G = \mathbb{G}_m$, the above theorem reduces to that of Kriz and May. One observation is that $\mathcal{H}_A^f$ is a special kind of Tannakian category (see Definition 12.3), whose Tannakian fundamental group is a semidirect product of a prounipotent algebraic group and a reductive group. In the final section, we gave a description of the corresponding coordinate rings of the Tannakian fundamental groups of these special Tannakian categories by framed objects [**1**].

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2. Basic definitions

Convention 2.1. Let G be a reductive algebraic group over $\mathbb{Q}$ and $w\colon \mathbb{G}_m \to G$ be a central cocharacter – that is, the image of w is contained in the center of G. We assume that w is injective. Using the map w, we can define the weight of representations of G.

Definition 2.2. Let V be a rational G-representation. For any $r \in \mathbb{Z}$, define the weight r part of V to be a sub representation of V:

$$V\{r\} = \{x \in V \mid w(\lambda) \cdot x = \lambda^r x \quad \text{for any} \quad \lambda \in \mathbb{G}_m(k)\}.$$

A rational G-representation V is called pure of weight r if $V\{r\} = V$.

Convention 2.3. In order to be compatible with the Adams grading in May and Kriz's theory, we define the Adams degree for a pure weight r representation W of G over $\mathbb{Q}$ is defined to be $-r$. Caution: To distinguish the symbol of the weight, given a complex of linear G-representations A^*, we denote the Adams degree r part of A^* by $A^*(r)$. We call the category of linear G-representations over $\mathbb{Q}$ simply as the category of G-representations, denote it by $Rep(G)$. $\vee$ is used to denote the dual object according to the context.

Definition 2.4. A cdga $(A^*, d, \cdot)$ over G consists of a complex (A^*, d) in the category of G-representations, where $d = \oplus_n d^n \colon A^n \to A^{n+1}$ is a homomorphism between G-representations, satisfying:

- there exists a homomorphism of complexes of G-representations: $\cdot \colon A^* \otimes A^* \to A^*$, which is unital, graded commutative and associative.
- $d^{n+m}(a \cdot b) = d^n a \cdot b + (-1)^n a \cdot d^m b$, where $a \in A^n, b \in A^m$.
- the Adams grading gives a decomposition of A^* into $A^* = \oplus_{r \in \mathbb{Z}} A^*(r)$ (as subcomplexes) and $\mathbb{Q}$ (the trivial G-representation) is a direct summand of $A^*(0)$.

A^* is called Adams connected if the Adams decomposition satisfies $A^* = \oplus_{r \geqslant 0} A^*(r)$ and $A^*(0) = \mathbb{Q}$. Furthermore, A^* is called connected (resp. cohomologically connected) if $A^n = 0$ for $n < 0$ and $A^0 = \mathbb{Q}$ (resp. $H^n(A^*) = 0$ for $n < 0$ and $H^0(A^*) = \mathbb{Q}$).

For $x \in A^n(r)$, we call n the cohomological degree of x, denoted by $n = deg(x)$, and r the Adams degree of x, denoted by $r = |x|$.

Definition 2.5. Let A be a cdga over G. A dg A-module (M^*, d) over G consists of a complex M^* of G-representations with the differential d, together with a map $A^* \otimes M^* \to M^*, a \otimes m \to a \cdot m$, which makes M^* into an A^*-module, and satisfies the Leibniz rule

$$d(a \cdot m) = da \cdot m + (-1)^{deg a} a \cdot dm; a \in A^*, m \in M^*.$$

Remark 2.6. We fix a finite dimensional faithful representation $\mathbf{F}$ of G with positive weights. For the existence of such $\mathbf{F}$, we refer to Corollary 2.5 in [**7**]. By definition, there exists a decomposition of M^* into subcomplexes $M^* = \oplus_s M^*(s)$ satisfying $A^*(r) \cdot M^*(s) \subset M^*(r+s)$, which is called the Adams decomposition of M^*.

Definition 2.7. Let M and N be two dg A-modules. A morphism f between M and N is a morphism between the underlying complexes of G-representations of M and N such that $a \cdot f(m) = f(a \cdot m)$ for any $a \in A$ and $m \in M$.

Example 2.8. Let $A[n]$ denote the A^*-module which is A^{m+n} in degree m, with a natural action of A^* by multiplication. Given A^* a cdga over G, we let $A\langle r\rangle[n]$ be A^*-module which is $\bigoplus_{t \in \mathbb{Z}} A^{m+n}(t) \otimes \mathbf{F}^{\otimes r}(s-t)$ in bi-degree (m, s), with the action given by multiplication. More generally, given any G-representation W, $A[n] \otimes W$, with $\bigoplus_{t \in \mathbb{Z}} A^{m+n}(t) \otimes W(s-t)$ in degree (m, s), is also a dg A-module over G. When W

is a rational representation of G, $A[n]\otimes W$ is called the generalized sphere A-modules for any $n\in\mathbb{Z}$.

Definition 2.9. A dg A-module M is a cell module if

1. There is an isomorphism of A-modules in the category of G-representations:
$$\oplus_{j\in J}A[-n_j]\otimes V_j\to M,$$
where all the V_j are rational representations of G and all n_j are integers. That is, there is a set J and elements $b_j\in M^{n_j}\otimes(V_j)^\vee, j\in J$ such that the maps
$$A[-n_j]\otimes V_j\xrightarrow{\cdot b_j}M\otimes(V_j)^\vee\otimes V_j\xrightarrow{id_M\times ev}M$$
induce the above isomorphism, where ev is the evaluation map $(V_j)^\vee\otimes V_j\to\mathbb{Q}$.
2. There is a filtration on the index set J: $J_{-1}=\emptyset\subset J_0\subset J_1\cdots\subset J$ such that $J=\bigcup_{n=0}^\infty J_n$ and for $j\in J_n$, $db_j=\sum_{i\in J_{n-1}}a_{ij}b_i$, where $b_i\in M^{n_j+1}\otimes(V_i)^\vee$ for some V_i and $a_{ij}\in A\otimes(V_k)^\vee$ for those k such that $V_j\subset V_i\otimes V_k$ as G-modules. Here d as a differential map on $M\otimes(V_j)^\vee$ is the tensor product of the differential map on M and the identity map on $(V_j)^\vee$.

A finite cell module is a cell module with finite index set J.

Remark 2.10. Given M a cell module, using the condition 1 and 2 in Definition 2.9, we can construct a filtration of sub cell modules M_n, where M_n is isomorphic to $\oplus_{j\in J_n}A[-n_j]\otimes V_j$ as complexes of G-representations$\{M_n\}_{n\in\mathbb{Z}_{\geqslant 0}}$ is called the sequential filtration of M.

We denote the category of dg A-modules over G by $\mathcal{M}_A^G$, the category of cell A-modules by $\mathcal{CM}_A^G$ and the category of finite cell modules by $\mathcal{CM}_A^{G,f}$.

3. The derived category of dg modules

Let A be a cdga over G and let M and N be dg A-modules. Let $\mathcal{H}om_A(M,N)$ be the dg A-module over G with $\mathcal{H}om_A(M,N)^n$ consisting of linear maps $f\colon M\to N$ with $f(M^a)\subset N^{a+n}$, $f(am)=(-1)^{np}af(m)$ for $a\in A^p$ and $m\in M^a$, and with the differential d defined by $df(m)=d(f(m))-(-1)^nf(dm)$ for $f\in\mathcal{H}om_A(M,N)^n$.

Definition 3.1. For $f\colon M\to N$ a morphism of dg A-modules, we let Cone(f) be the dg A-module with:
$$\mathrm{Cone}(f)^n(r)=N^n(r)\oplus M^{n+1}(r)$$
and the differential is given by $d(n,m)=(dn+f(m),-dm)$.

Given M a dg A-module, we let $M[1]$ denote a dg A-module such that $M[1]^n=M^{n+1}$ with the differential $-d$, where d is the differential of M. Then we have the following sequence:
$$M\xrightarrow{f}N\xrightarrow{i}\mathrm{Cone}(f)\to M[1],$$
which is called a cone sequence.

Definition 3.2. We let $\mathcal{K}_A^G$ denote the homotopy category of the category of dg A-modules over G. The objects are the same as $\mathcal{M}_A^G$ and

$$\mathrm{Hom}_{\mathcal{K}_A^G}(M, N) = \mathrm{Hom}_G(\mathbb{Q}, H^0(\mathcal{H}om_A(M, N))).$$

The derived category $\mathcal{D}_A^G$ of dg A-modules over G is the localization of $\mathcal{K}_A^G$ with respect to quasi-isomorphisms between dg A-modules, which are defined as morphisms $M \to N$ being quasi-isomorphic on the underlying complexes of $\mathbb{Q}$-vector spaces.

Given $M, N \in \mathcal{M}_A^G$ (resp. $\mathcal{CM}_A^G$), we can define their direct sum to be the direct sum $M \oplus N$ of the chain complexes of G-representations which is equipped with a natural A-module structure (resp. cell A-module structure). Furthermore, the infinite direct sum exists in both $\mathcal{M}_A^G$ and $\mathcal{CM}_A^G$.

Lemma 3.3. *The infinite direct sums defined above is the categorical sum in $\mathcal{K}_A^G$.*

Convention 3.4. Let I be the complex $\mathbb{Q} \xrightarrow{\delta} \mathbb{Q} \oplus \mathbb{Q}$ with a free $\mathbb{Q}$-generator $[I]$ in degree -1, two free $\mathbb{Q}$-generators $[0], [1]$ in degree 0 and $\delta[I] = [0] - [1]$. We have two inclusions $i_0, i_1 \colon \mathbb{Q} \to I$ sending 1 to $[0], [1]$, respectively.

For M a dg A-module, we let $CM = \mathrm{Cone}(id_M)$. Notice that the cone CM is the quotient module $M \otimes (I/\mathbb{Q}[1])$.

Using the same idea of the proof in [**11**], we can show the following theorems.

Theorem 3.5. *Let L be a cell A-submodule of a cell A-module M. Let $e \colon N \to P$ be a quasi-isomorphism of dg A-modules. Then given maps $f \colon M \to P, g \colon L \to N$, and $h \colon L \otimes I \to P$ such that $f|_L = h \circ i_0$ and $e \circ g = h \circ i_1$, there are maps $\widehat{g}, \widehat{h}$ that make the following diagram commute.*

$$\begin{tikzcd}
L \arrow[rr, "i_0"] \arrow[dd] & & L \otimes I \arrow[dl, "h"'] \arrow[dd] & & L \arrow[ll, "i_1"'] \arrow[dl, "g"] \arrow[dd] \\
& P & & N \arrow[ll, "e"'] & \\
M \arrow[ur, "f"] \arrow[rr, "i_0"] & & M \otimes I \arrow[ul, "\widehat{h}"'] & & M \arrow[ll, "i_1"'] \arrow[ul, "\widehat{g}"]
\end{tikzcd}$$

Proof. Using Remark 2.10, we do the induction on the length of the sequential filtration $\{M_n\}$ on M. Because L is a sub cell A-module, we may get a compatible sequential filtration $\{L_n\}$ on L. The way of constructing $L_{n+1} \to M_{n+1}$ from $L_n \to M_n$ is to attach the cells not in L to M_{n+1}. So we may assume that $M \cong C(A[n] \otimes W)$ and $L \cong A[n] \otimes W$. Then using the semi-simplicity of the category of G-representations, we can further assume that W is an irreducible G-representation. We denote the generator of W by w^n.

Let $u = w^n \otimes [0]$ and $v = w^n \otimes [I]$ be the generators of $C(A[n] \otimes W)$. By definition, we have $d(v) = (-1)^n u$. We also have: $e \circ g(w^n) = h(w^n \otimes [1])$ and $f(u) = h(u)$. Therefore

$$\begin{aligned} d(h(w^n\otimes[I])-f(v)) &= hd(w^n\otimes[I])-f(dv)\\ &= h(d(w^n)\otimes[I]+(-1)^n h(w^n\otimes([0]-[1])))-(-1)^n f(u)\\ &= (-1)^n h(w^n\otimes[0])+(-1)^{n+1}h(w^n\otimes[1])-(-1)^n f(u)\\ &= (-1)^{n+1}h(w^n\otimes[1])=(-1)^{n+1}e\circ g(w^n). \end{aligned}$$

Because $e\circ g(w^n)$ is a coboundary and e induces a quasi-isomorphism, we know that $g(w^n)$ is also a coboundary, i.e., there exist $\tilde{n}\in N^{n-1}$ such that $d(\tilde{n})=g(w^n)$. Then $p=e(\tilde{n})+h(w^n\otimes[I])-f(v)$ is a cocycle. Then using the quasi-isomorphism at $n-1$, there exist a cocycle $n\in N$ and a chain $q\in P$ such that $d(q)=p-e(n)$. We define $\widehat{g}(j)=(-1)^n(\tilde{n}-n)$ and $\widehat{h}(j\otimes[I])=q$. □

Theorem 3.6 (Whitehead)**.** *If M is a cell A-module and $e\colon N\to P$ is a quasi-isomorphism of A-modules, then*

$$e_*\colon \mathrm{Hom}_{\mathcal{K}_A^G}(M,N)\to \mathrm{Hom}_{\mathcal{K}_A^G}(M,P)$$

is an isomorphism. So a quasi-isomorphism between cell A-modules is a homotopy equivalence.

Proof. The surjectivity is coming from Theorem 3.5, when we take $L=0$. The injectivity can be checked when we replace M and L by $M\otimes_{\mathbb{Q}} I$ and $M\otimes_{\mathbb{Q}}(\partial I)$ respectively. When N,P are both cell A-modules, taking $M=P$, we get a map $f\colon P\to N$ which corresponds to id_P. From the functoriality, f is the homotopy inverse of e. □

Corollary 3.7. *Let M,N be two dg A-modules, and $f\colon M\to N$ be a morphism between dg A-modules. Let $\widehat{M}$ and $\widehat{N}$ be two cell A-modules such that $\widehat{M}\xrightarrow{r_M}M$ and $\widehat{N}\xrightarrow{r_N}N$ are quasi-isomorphisms. Then there exists a morphism between cell A-modules, up to homotopy $\widehat{f}\in\mathcal{K}_A^G\colon\widehat{M}\to\widehat{N}$ lifting f.*

Proof. From Theorem 3.6, we know that $r_{N*}\colon \mathrm{Hom}_{\mathcal{K}_A^G}(\widehat{M},\widehat{N})\to\mathrm{Hom}_{\mathcal{K}_A^G}(\widehat{M},N)$ is an isomorphism. Therefore $f\circ r_M\in\mathcal{K}_A^G(\widehat{M},N)$ have a preimage, which is just $\widehat{f}$. □

Remark 3.8. From the above proof, we also know that: Given a cell module M and an arbitrary dg A-module N, we have:

$$\mathrm{Hom}_{\mathcal{D}_A^G}(M,N)\cong\mathrm{Hom}_{\mathcal{K}_A^G}(M,\widehat{N})\cong\mathrm{Hom}_{\mathcal{K}_A^G}(M,N),$$

where $\widehat{N}$ is a cell module and $\widehat{N}\to N$ is a quasi-isomorphism between dg A-modules.

Theorem 3.9 (Approximation by cell modules)**.** *For any dg A-module M, there is a cell A-module N and a quasi-isomorphism $e\colon N\to M$.*

Proof. We will construct a sequential filtration N_n and compatible maps $e_n\colon N_n\to M$ inductively. More precisely, we need to construct cell modules N_n, whose index set is denoted by J_n, satisfy the condition 2 in the definition of cell modules. For every pair (q,r), we decompose $H^q(M)(r)\cong\oplus_i V_i$ as the direct sum of irreducible G-representations V_i with the Adams degree r. Choosing a splitting of $Ker(M^q(r)\xrightarrow{d}M^{q+1}(r))\twoheadrightarrow H^q(M)(r)$, we think V_i as sub G-representations in $M^q(r)$, because of the semi-simplicity of the category of G-representations. Then we take $N_1=\oplus_{(q,r)}\oplus_i A[-q]\otimes V_i$ with trivial differential. There is a morphism between dg A-modules: $N_1\to M$, which is epimorphic on the cohomologies. Inductively, assume that $e_n\colon N_n\to M$ has been constructed. Consider the set of the pair of cocycles consisting the pairs of unequal cohomology classes on N_n and mapping under $(e_n)^*$ to the same element of $H^*(M)$. Choose a pair $W_1^q(r)$ and $W_2^q(r)$ that live in the bidegree (q,r) satisfying the above condition, i.e., we can view $W_1^q(r) \oplus W_2^q(r)$ as the kernel of the morphism $(e_n)^*$ on the cohomology of bidegree (q,r). (Here one needs to take a sign for the second component.) Simply denote $W_1^q(r) \oplus W_2^q(r)$ by W_1. There is a morphism between dg A-modules $A[-q] \otimes W_1$ to N_n extending the map between G-representations $W_1 \to H^q(N)(r)$. Take N_{n+1} to be the pushout of N_n and $A[-q] \otimes W_1 \oplus A[-q] \otimes W_1[1]$ over $A[-q] \otimes W_1$. Then we have $0 \to W_1 \to H^q(N_n)(r) \to H^q(N_{n+1})(r) \to 0$. We get N_{n+1} by attaching N_n with a generalized sphere dg A-module $A[-q] \otimes W_1[1]$, which implies N_{n+1} is a cell A-module. It is easy to see the differentials on N_{n+1} satisfy the condition 2 in the definition of cell modules. Now we have a distinguished triangle of dg A-modules: $A[-q] \otimes W_1 \xrightarrow{i} N_n \to N_{n+1} \to (A[-q] \otimes W_1)[1]$.

Note that $\mathrm{Hom}_A(A[-q] \otimes W_1, M) \cong \mathrm{Hom}_{D(G)}(W_1, M[q]) \cong \mathrm{Hom}_G(W_1, H^q(M))$. Therefore we have:

$$\mathrm{Hom}_A(N_{n+1}, M) \to \mathrm{Hom}_A(N_n, M) \xrightarrow{i} \mathrm{Hom}_A(A[-q] \otimes W_1, M) \cong \mathrm{Hom}_G(W_1, H^q(M)).$$

Because W_1 as a G-representation maps to zero in the cohomology group $H^q(M)(r)$, i.e., $i(e_n) = 0$ in $\mathrm{Hom}_G(W_1, H^q(M))$, one may find $e_{n+1} \in \mathrm{Hom}_A(N_{n+1}, M)$, which extends e_n. Let N be the direct limit of the N_n. Then N is a cell module and the morphism $N \to M$ is a quasi-isomorphism by the construction. □

Putting together with all previous results, we get:

Theorem 3.10. *Let A be a cdga over G. Then the functor $\mathcal{KCM}_A^G \to \mathcal{D}_A^G$ is an equivalence of triangulated categories.*

Definition 3.11. We define $\mathcal{D}_A^{G,f}$ to be the full subcategory of $\mathcal{D}_A^G$ whose objects are quasi-isomorphic to some finite cell A-module in $\mathcal{D}_A^G$.

Remark 3.12. From the proof of Theorem 3.10, we can know that $\mathcal{KCM}_A^{G,f} \to \mathcal{D}_A^{G,f}$ is an equivalence of triangulated categories.

Example 3.13. Let $A = \mathbb{Q}$, then $\mathcal{KCM}_A^{G,f}$ is just the bounded derived category of the category of rational representations of G, denoted by $D^b(G)$.

4. The weight filtration for dg modules

In this section, we assume that A is an Adams connected cdga over G.

Definition 4.1. A dg A-module M is called almost free, if there exists a family of irreducible G-representations $\{V_j\}_{j\in J}$ and morphisms of graded A-modules $\phi_j \colon A \otimes V_j \to M$, such that the induced morphism: $\oplus_{j\in J} A \otimes V_j \xrightarrow{\oplus\phi_j} M$ is an isomorphism of graded A-modules, which means that, forgetting the differentials, this is an isomorphism between G-representations. We call such $\{V_j, \phi_j\}_{j\in J}$ the generating data for M.

Example 4.2. All cell A-modules are almost free. Conversely, any cell A-module is obtained from the generating data together with suitable differentials.

We let M be an almost free dg A-module with a fixed generating data $\{V_j, \phi_j\}_{j\in J}$. Given a (V_j, ϕ_j), we assume that I is the smallest index subset of J such that

$d(\phi_j(V_j)) \subset \oplus_{i\in I}\phi_i(A\otimes V_i)$. Here we restrict ϕ_j to $A^*(0)\otimes V_j \cong V_j$. Because the differential map has Adams degree zero, the Adams degree $|V_j|$ of the left hand side is larger than or equal to the Adams degree of the right hand side $|V_i|$. Hence we have the subcomplex $W_n^J M = \oplus_{\{j,|V_j|\leqslant n\}}\phi_j(A\otimes V_j)$ of M.

Remark 4.3. The subcomplex of $W_n^J M$ is independent of the choice of the family $\{V_j, \phi_j\}_{j\in J}$. This is because if we choose another generating data $\{V_{j'}, \phi_{j'}\}_{j'\in J'}$ and if $|V_{j'}| = n$, then there exists $I \subset J$ such that $\phi_{j'}(V_{j'}) \subset \phi_i(A\otimes V_i)$ with $|V_i| \leqslant |V_{j'}|$. It follows that $\phi_{j'}(V_{j'}) \in W_n^J M$ and hence $W_n^{J'} M \subset W_n^J M$. By symmetry, we get the result. So we delete the J in the definition.

This gives us the increasing filtration as a dg A-module $W_*M\colon \cdots \subset W_nM \subset W_{n+1}M \subset \cdots \subset M$ with $M = \cup_n W_nM$. In the same way, we can define $W_{n/n'}M$ as the cokernel of the inclusion $W_{n'}M \to W_nM$ for $n \geqslant n'$. Write gr_n^W for $W_{n/n-1}$ and $W^{>n}$ for $W_{\infty/n}$. W_n defines an endofunctor in $\mathcal{CM}_A^G$. Furthermore, $\{W_n\}_{n\in\mathbb{Z}}$ form a functorial tower of endofunctors on $\mathcal{KCM}_A^G$: $\cdots \to W_n \to W_{n+1} \to \cdots \to id$.

Remark 4.4.

- The endofunctor W_n is exact for all n.
- For $m \leqslant n \leqslant \infty$, the sequence of endofunctors $W_m \to W_n \to W_{n/m}$ can extend to a distinguished triangle of endofunctors, i.e., for any $M \in \mathcal{KCM}_A^G$, we have a distinguished triangle $W_mM \to W_nM \to W_{n/m}M \to$ in $\mathcal{KCM}_A^G$.
- Using the isomorphism of categories between $\mathcal{KCM}_A^G$ and $\mathcal{D}_A^G$, we could define the tower of exact endofunctors on $\mathcal{D}_A^G$: $\cdots \to W_n \to W_{n+1} \to \cdots \to id$. Similarly we define $W_{n/n'}$, gr_n^W and $W^{>n}$ on $\mathcal{D}_A^G$.

5. Derived tensor product

Recall that the Hom functor $\mathcal{H}om_A(M,N)$ defines a bi-exact bi-functor:

$$\mathcal{H}om_A\colon (\mathcal{KCM}_A^G)^{op} \otimes \mathcal{KCM}_A^G \to \mathcal{D}_A^G,$$

which gives a well-defined derived functor of $\mathcal{H}om_A$ between derived categories of dg A-modules (also the derived categories of finite cell modules) by Proposition 3.10:

$$R\mathcal{H}om_A\colon (\mathcal{D}_A^G)^{op} \otimes \mathcal{D}_A^G \to \mathcal{D}_A^G.$$

Given two dg A-modules M and N, we let $M\otimes_A N$ be the dg A-module with underlying module $M\otimes_A N$ and with differential $d(m\otimes n) = dm\otimes n + (-1)^{degm} m\otimes dn$. The tensor product functor defines a bi-exact bi-functor:

$$\otimes_A\colon \mathcal{KCM}_A^G \otimes \mathcal{KCM}_A^G \to \mathcal{KCM}_A^G.$$

Via Theorem 3.9, we get a well-defined derived functor of $\otimes_A$:

$$\otimes_A^{\mathbb{L}}\colon \mathcal{D}_A^G \otimes \mathcal{D}_A^G \to \mathcal{D}_A^G.$$

Remark 5.1. We collect some facts without the proof.

- These bi-functors are adjoint, i.e.,

$$R\mathcal{H}om_A(M\otimes_A^{\mathbb{L}} N, K) \cong R\mathcal{H}om_A(M, R\mathcal{H}om_A(N,K)).$$

- The derived tensor product makes $\mathcal{D}_A^G$ into a triangulated tensor category with unit A and $\mathcal{D}_A^{G,f}$ as a triangulated tensor subcategory of $\mathcal{D}_A^G$.

These properties allow us to apply the category duality theory developed in [**13**].

Convention 5.2. Denote $M^\vee = R\mathcal{H}om_A(M, A)$.

Definition 5.3. An object $M \in \mathcal{D}_A^G$ is called rigid, if there exists an $N \in \mathcal{D}_A^G$ and morphisms $\delta\colon A \to M \otimes_A^{\mathbb{L}} N$ and $\epsilon\colon N \otimes_A^{\mathbb{L}} M \to A$ such that:

$$(id_M \otimes \epsilon) \circ (\delta \otimes id_M) = id_M, \qquad (id_N \otimes \delta) \circ (\epsilon \otimes id_N) = id_N.$$

Definition 5.4. An object $M \in \mathcal{D}_A^G$ is finite if there is a coevaluation map $\tilde{\eta}\colon A \to M \otimes^{\mathbb{L}} M^\vee$ such that the diagram

$$\begin{tikzcd} A \arrow[r, "\tilde{\eta}"] \arrow[d, "\eta"'] & M \otimes^{\mathbb{L}} M^\vee \arrow[d, "\gamma"] \\ R\mathcal{H}om_A(M, M) & M^\vee \otimes^{\mathbb{L}} M \arrow[l, "\mu"'] \end{tikzcd}$$

commutes. Here η and μ are given by the adjunction. γ changes the places of these two modules.

Remark 5.5. By Theorem 1.6 of [**13**], M is rigid if and only if the function

$$\epsilon_*\colon \mathrm{Hom}_{\mathcal{D}_A^G}(W, Z \otimes_A^{\mathbb{L}} N) \to \mathrm{Hom}_{\mathcal{D}_A^G}(W \otimes_A^{\mathbb{L}} M, Z)$$

is a bijection for all W and Z, where $\epsilon_*(f)$ is the composite

$$W \otimes_A^{\mathbb{L}} M \xrightarrow{f \otimes 1} Z \otimes_A^{\mathbb{L}} N \otimes_A^{\mathbb{L}} M \xrightarrow{1 \otimes \epsilon} Z \otimes_A^{\mathbb{L}} A \cong Z.$$

These conditions are also equivalent to saying that M is finite.

In the following, we will discuss the relations between finite objects in $\mathcal{D}_A^G$ and finite cell modules.

Definition 5.6. We say that a cell module N is a summand of a cell module M in $\mathcal{D}_A^G$ if there is a homotopy equivalence of A-modules between M and $N \oplus N'$ for some cell A-module N'.

Following the same proof as Theorem 5.7 in Part III of [**11**], we can get:

Lemma 5.7. *A cell module M is rigid if and only if it is a summand of a finite cell module in $\mathcal{D}_A^G$.*

Remark 5.8. Let $\mathcal{FCM}_A^G$ be the full subcategory of $\mathcal{CM}_A^G$ whose objects are the direct summands up to homotopy of finite cell A-modules. Then the homotopy category $\mathcal{KFCM}_A^G$ is the idempotent completion of $\mathcal{D}_A^{G,f}$. The above lemma implies that $\mathcal{KFCM}_A^G$ is the largest rigid tensor subcategory of the derived category $\mathcal{D}_A^G$. See section 5 in Part III of [**11**]. In particular, $\mathcal{D}_A^{G,f}$ is a rigid tensor subcategory of $\mathcal{KFCM}_A^G$.

Theorem 5.9. *Let A be an Adams connected cdga over G. Then $M \in \mathcal{D}_A^G$ is rigid if and only if $M \in \mathcal{D}_A^{G,f}$, which implies that there is an equivalence between $\mathcal{D}_A^{G,f}$ and $\mathcal{KFCM}_A^G$.*

Proof. It depends on the following lemma. $\square$

Lemma 5.10. *Assume that A is an Adams connected cdga over G. Let M be a finite cell A-module. Suppose N is a summand of M in $\mathcal{D}_A^G$. Then there is a finite A-cell module M' with $N \cong M'$ in $\mathcal{D}_A^G$.*

Proof. Via Theorem 3.9, we assume further that N is a cell module. By our assumption, we have $M = N \oplus N'$ in $\mathcal{KCM}_A^G$. Since M is finite, there is a minimal n such that $W_n M \neq 0$. Thus $W_{n-1}N$ is homotopy equivalent to zero. We may assume that $W_{n-1}N = 0$ in $\mathcal{CM}_A^G$. Similarly we assume that $M = W_{n+r}M, N = W_{n+r}N$ in $\mathcal{CM}_A^G$ for some $r \geqslant 0$. Now we proceed by induction on r.

Choose generating data $\{V_j, \phi_j\}_{j\in J}$ for $W_n M$. Let us prove that $W_n M = A \otimes V$ for a finite complex of G-representations V. In fact, by the definition of the weight functor and $W_{n-1}M = 0$, we can get an isomorphism: $W_n M = \oplus_{|V_j|=n}\phi_j(A \otimes V_j)$. Notice that $d(\phi_j(V_j)) \subset \oplus_i \phi_i(A \otimes V_i)$ and all these $|V_i|$'s have the same value. Using $A^*(0) = \mathbb{Q}$, we get $d(\phi_j(V_j)) \subset \oplus \phi_i(V_i)$. Set $V = \oplus_{j\in J}\phi_j(V_j)$, which is a complex of G-representations. So we have $W_n M = A \otimes V$ as dg A-modules. Because the category of G-representations is semisimple, we can assume that all differentials of V are zero. Let $p\colon M \to M$ be the composition of the projection $M \to N$ and the inclusion $N \to M$. Then we can see $W_n p = id \otimes q$, where $q\colon V \to V$ is an idempotent. V is a direct sum of G-representations with some shifts. Thus $W_n N \cong A \otimes im(q)$. We finish the case of $r = 0$. Using the distinguished triangle $W_n N \to N \to W_{n+r/n}N \to W_n N[1]$, we can replace N with the shifted cone of the map $W_{n+r/n}N \to A \otimes im(q)[1]$. Since $W_{n+r/n}N$ is a summand of $W_{n+r/n}M$, by induction, we get that $W_{n+r/n}$ is homotopy equivalent to a finite cell module. So the cone of $W_{n+r/n}N \to A \otimes im(q)$ is also homotopy equivalent to a finite cell module. □

Corollary 5.11. *Assume A is an Adams connected cdga over G. Then $\mathcal{D}_A^{G,f}$ is idempotent complete.*

6. Base change

Lemma 6.1. *Let N be a cell module. Then the functor $M \otimes_A N$ preserves exact sequences and quasi-isomorphisms in the variable M.*

Proof. By the induction of the sequential filtration of the cell module N, we reduce to the case that N is a generalized sphere module. Lemma 6.1 is true for generalized sphere modules trivially. □

We let $\phi\colon A \to B$ be a homomorphism of cdgas over G. Then there is a functor $\otimes_A B\colon \mathcal{M}_A^G \to \mathcal{M}_B^G$, which induces a functor on cell modules and the homotopy category $\phi_*\colon \mathcal{KCM}_A^G \to \mathcal{KCM}_B^G$. So we have a base change functor on the derived categories level: $\phi_*\colon \mathcal{D}_A^G \to \mathcal{D}_B^G$.

Remark 6.2. The restriction of ϕ_* on finite objects gives the functor on the bounded case.

Proposition 6.3. *If ϕ is a quasi-isomorphism, then ϕ_* is an equivalence of tensor triangulated categories.*

Proof. There is an isomorphism: $\mathrm{Hom}_{\mathcal{M}_B^G}(B \otimes_A M, N) \cong \mathrm{Hom}_{\mathcal{M}_A^G}(M, \phi^* N)$, for $M \in \mathcal{M}_A^G$ and $N \in \mathcal{M}_B^G$. Here ϕ^* is the pullback functor, which means that, for a given dg B-module, there is a natural dg A-module structure. Then we have: $\mathrm{Hom}_{\mathcal{K}_B^G}(B \otimes_A M, N) \cong \mathrm{Hom}_{\mathcal{K}_A^G}(M, \phi^* N)$. Using Remark 3.8, we get:

$$\begin{aligned}\mathrm{Hom}_{\mathcal{D}_B^G}(B \otimes_A M, N) \cong \mathrm{Hom}_{\mathcal{K}_B^G}(B \otimes_A \widehat{M}, N) \cong \mathrm{Hom}_{\mathcal{K}_A^G}(B \otimes_A \widehat{M}, \phi^* N)\\ \cong \mathrm{Hom}_{\mathcal{D}_A^G}(B \otimes_A M, \phi^* N),\end{aligned}$$

where $\widehat{M}$ is a cell A-module quasi-isomorphic to M.

Next we check that the unit of the adjunction and the counit are both quasi-isomorphisms. For the unit of the adjunction, given M a cell dg A-module, we need to show that $\phi \otimes Id \colon M \cong A \otimes_A M \to \phi^*(B \otimes_A M)$ is a quasi-isomorphism of A-modules. Assume that $M = A$ firstly. By assumption, we know that $\phi^* B$ is quasi-isomorphic to A as a dg A-module. Then assume $M = A[n] \otimes W$ for W a G-representation. $\phi^*(B \otimes_A A[n] \otimes W)$ is the same as $\phi^*(B[n] \otimes W)$. The latter is naturally quasi-isomorphic to $A[n] \otimes W$. If M is a cell module, using the induction on the length of its sequential filtration, we can get the desired quasi-isomorphism. For the counit part, given N a dg B-module, and choosing a quasi-isomorphism of dg B-module $\widehat{N} \to N$, where $\widehat{N}$ is a cell B-module, then we have $B \otimes_A \widehat{N} \to B \otimes_B N \cong N$, which is also a quasi-isomorphism. □

Corollary 6.4. *Assume that A and B are Adams connected cdgas over G. If ϕ is a quasi-isomorphism, then $\phi_* \colon \mathcal{D}_A^{G,f} \to \mathcal{D}_B^{G,f}$ is an equivalence of triangulated tensor categories.*

Proof. Notice that an equivalence between tensor triangulated categories induces an equivalence on the subcategories of rigid objects. By Proposition 6.3, we know that $\mathcal{D}_A^G$ and $\mathcal{D}_B^G$ are equivalent. Then by Theorem 5.9, we know that ϕ induces an equivalence between $\mathcal{D}_A^{G,f}$ and $\mathcal{D}_B^{G,f}$. □

Remark 6.5. For any cdga A over G, we have a morphism $\delta \colon \mathbb{Q} \to A$, which sends $A^*(0)$ to A. Then, for any $M \in \mathcal{M}_{\mathbb{Q}}^G$ and $N \in \mathcal{M}_A^G$, we have: $\mathrm{Hom}_{\mathcal{D}_A^G}(A \otimes M, N) \cong \mathrm{Hom}_{\mathcal{D}_{\mathbb{Q}}^G}(M, \delta^* N)$. Here δ^* is the forgetful functor, which forgets the A-module structure.

7. Minimal models

In the rest of this paper, we always assume that the cdgas are Adams connected, which implies that these cdgas take $\mathbb{Q}$ in Adams degree zero as a direct summand.

Definition 7.1. A cdga A over G is said to be generalized nilpotent if:

- A is a free commutative graded algebra over G, i.e., $A = Sym^* E$ for some $\mathbb{Z}_{>0}$-graded G-representations E. (Or a complex of G-representations concentrated in positive degrees and with zero differentials).
- For $n \geqslant 0$, let $A\langle n\rangle \subset A$ be the subalgebra generated by the elements of degree $\leqslant n$. Set $A\langle n+1, 0\rangle = A\langle n\rangle$ and for $q \geqslant 0$ define $A\langle n+1, q+1\rangle$ inductively as the subalgebra generated by $A\langle n\rangle$ and $A\langle n+1, q+1\rangle^{n+1} = \{x \in A\langle n+1\rangle | dx \in A\langle n+1, q\rangle\}$. Then for all $n \geqslant 0$, $A\langle n+1\rangle = \cup_{q \geqslant 0} A\langle n+1, q\rangle$.

A cdga A over G is called nilpotent, if for each $n \geqslant 1$, there is a $q_n \in \mathbb{Z}_{\geqslant 0}$ such that $A\langle n\rangle = A\langle n, q_n\rangle$ in the second condition above.

Definition 7.2. A connected cdga A over G is minimal if it is a free commutative graded algebra over G with decomposable differential: $d(A) \subset (IA)^2$. IA is the fundamental ideal, i.e., $IA = Ker(A \to \mathbb{Q} \cong A^0(0))$.

Convention 7.3. For a cdga A over G, we let QA be $IA/(IA \cdot IA)$.

Proposition 7.4. *If a connected cdga A over G is generalized nilpotent, then it is minimal. Conversely, if A is a minimal connected cdga over G and $A^q(r) = 0$ unless $2r \geqslant q$, then A is generalized nilpotent.*

Proof. The proof is the same as Proposition 2.3 in Part of IV of [**11**]. □

Definition 7.5. Let A be a cdga over G. Given a positive integer n, an n-minimal model of A over G is a map of cdgas over G:

$$s\colon A\{n\} \longrightarrow A,$$

with $A\{n\}$ generalized nilpotent and generated as an algebra in degrees $\leqslant n$, such that s induces an isomorphism on H^m for $1 \leqslant m \leqslant n$ and an injection on H^{n+1}. Here n is also allowed to be ∞. In this case, we mean that there is a map of cdgas over G: $A\{\infty\} \xrightarrow{s} A$ with $A\{\infty\}$ generalized nilpotent and s a quasi-isomorphism.

Proposition 7.6. *Let A be a cohomologically connected cdga over G. Then for each $n = 1, 2, \ldots, \infty$, there is an n-minimal model $A\{n\}$ over G: $A\{n\} \to A$.*

Proof. Following the idea of Proposition 2.4.9 in [**12**], we proceed by a double induction, first with respect to the Adams degree and then with respect to the cohomological degree. Because A is Adams connected, we have a canonical decomposition $A = \mathbb{Q} \oplus IA$, where IA is the kernel of the canonical map between cdgas over G: $A \to \mathbb{Q}$. Let $E_{10}(1) \subset I^1(1)$ be the G-representation $H^1(I)(1)$, thought of as a sub-module of $I^1(1)$. We give it cohomological degree 1 and Adams degree 1. Then we have a natural inclusion $E_{10}(1) \to A$, which extends to $Sym^* E_{10}(1) \to A$ using the algebra structure of A. In fact, this is a map between cdgas over G and induces an isomorphism on $H^1(-)(1)$. Then one can adjoin elements in cohomological degree 1 and Adams degree 1 to kill elements in the kernel of the map on $H^2(-)(1)$. So we have a $\mathbb{Z}$-graded G-representation $E_1(1)$ of Adams degree 1 and cohomological degree 1, a generalized nilpotent cdga $A_{1,1} = Sym^* E_1(1)$ over G and a map of cdgas over G: $A_{1,1} \to A$, which induces an isomorphism on $H^1(-)(1)$ and an injection on $H^2(-)(1)$.

We have a canonical decomposition of $A_{1,1} = \mathbb{Q} \oplus I_{1,1}$. Note that $H^p(I_{1,1}(r)) = 0$ for $r > 1, p \leqslant 1$. This is because that the lowest degree of cohomology of $I_{1,1}(r)$ is coming from $Sym^r E_1(1)$ and all the elements of $E_1(1)$ have cohomological degree 1. Iterating this process, one can construct the Adams degree $\leqslant 1$ part of the n-minimal model in case $n > 1$. This gives us a generalized nilpotent cdga over G: $A_{1,n} = Sym^* E_n(1)$, with $E_n(1)$ in Adams degree 1 and cohomological degrees $1, 2, \ldots, n$ together with a map over G: $A_{1,n} \to A$, which induces an isomorphism on $H^i(-)(1)$ for $1 \leqslant i \leqslant n$ and an injection for $i = n + 1$. In addition, letting $A_{1,n} = \mathbb{Q} \oplus I_{1,n}$, we have $H^p(I_{1,n}(r)) = 0$ for $r > 1, p \leqslant 1$.

Suppose by induction we have constructed $\mathbb{Z}$-graded G-representations:

$$E_n(1) \subset E_n(2) \subset \cdots \subset E_n(m),$$

where $E_n(j)$ have Adams degrees $1, \ldots, j$ and cohomological degrees $1, \ldots, n$, a differential on $A_{n,m} = Sym^* E_n(m)$ making $A_{m,n}$ a generalized nilpotent cdga over G, and a map $A_{m,n} \to A$ of cdgas over G that is an isomorphism on $H^i(-)(j)$ for $1 \leqslant i \leqslant n$, $j \leqslant m$, and an injection for $i = n+1, j \leqslant m$. We decompose $A_{m,n} = \mathbb{Q} \oplus I_{m,n}$ and then $H^p(I_{m,n}(r)) = 0$ for $r > m, p \leqslant 1$. Extending $E_n(m)$ to $E_n(m+1)$ by repeating the construction for $E_n(1)$ above. Then one can check the above conditions still hold. The induction goes through. Taking $E_n = \cup_m E_n(m)$, we have a differential on $A\{n\} = Sym^* E_n$ making $A\{n\}$ a generalized nilpotent cdga over G, and a map $A\{n\} \to A$ of cdgas over G that is an isomorphism on $H^i(-)$ for $1 \leqslant i \leqslant n$ and an injection for $i = n+1$. □

Remark 7.7. If $f\colon A \to B$ is a quasi-isomorphism of cdgas over G, and $s\colon A\{n\} \to A$, $t\colon B\{n\} \to B$ are n-minimal models, then there is an isomorphism of cdgas over G: $g\colon A\{n\} \to B\{n\}$ such that $g \circ s$ is homotopic to $t \circ f$. The proof is the same as the case without G-action in Chapter 4 of [**4**].

8. The t-structure of $\mathcal{D}_A^{G,f}$

The aim of this section is to define a t-structure on $\mathcal{D}_A^{G,f}$ if A is a cohomologically connected cdga over G. There is a canonical augmentation $\epsilon\colon A \to \mathbb{Q}$ by the projection onto $A^0(0) = \mathbb{Q}$. So we have a functor $q = \epsilon_*\colon \mathcal{CM}_A^G \to \mathcal{M}_{\mathbb{Q}}^G$ defined by $q(M) = M \otimes_A \mathbb{Q}$ and an induced exact tensor functor $q\colon \mathcal{D}_A^G \to \mathcal{D}_{\mathbb{Q}}^G$.

Remark 8.1. We recall that $\mathcal{D}_{\mathbb{Q}}^{G,f}$ is the derived category of finite dimensional G-representations. There is a canonical t-structure for $\mathcal{D}_{\mathbb{Q}}^{G,f}$. The idea is to use q to get the induced t-structure for $\mathcal{D}_A^{G,f}$ when A is a cohomologically connected cdga over G, which comes from the following general fact: Let $\phi\colon A \to B$ be a map of cohomologically connected cdgas over G. Then $\phi_*\colon \mathcal{D}_A^{G,f} \to \mathcal{D}_B^{G,f}$ is conservative, i.e., $\phi_*(M) \cong 0$ implies $M \cong 0$.

Proof. Take a non-zero object $M \in \mathcal{D}_A^{G,f}$. Then we can find a cell module P and a quasi-isomorphism $P \to M$ such that $W_{n-1}P = 0$, but W_nP is not acyclic. We choose generating data $\{V_j, \phi_j\}_{j \in J}$ for P, such that $|V_j| \geqslant n$ for $j \in J$. Because n is the minimal integer of the possible Adams degree, the same proof of Lemma 5.10, implies that $W_nP \otimes_A \mathbb{Q}$ is not acyclic. Notice that $W_n(P \otimes_A B) = W_nP \otimes_A B$ and $W_nP \otimes_A \mathbb{Q} = (W_nP \otimes_A B) \otimes_B \mathbb{Q}$. Therefore $P \otimes_A B$ is not isomorphic to zero in $\mathcal{KCM}_B^G$ and ϕ_*M is non-zero in $\mathcal{D}_B^{G,f}$. □

Define full subcategories $\mathcal{D}_A^{G,f,\leqslant 0}, \mathcal{D}_A^{G,f,\geqslant 0}$ and $\mathcal{H}_A^{G,f}$ of $\mathcal{D}_A^{G,f}$:

$$\begin{aligned}
\mathcal{D}_A^{G,f,\leqslant 0} &= \{M \in \mathcal{D}_A^{G,f} | H^n(qM) = 0 \ for \ n > 0\},\\
\mathcal{D}_A^{G,f,\geqslant 0} &= \{M \in \mathcal{D}_A^{G,f} | H^n(qM) = 0 \ for \ n < 0\},\\
\mathcal{H}_A^{G,f} \quad &= \{M \in \mathcal{D}_A^{G,f} | H^n(qM) = 0 \ for \ n \neq 0\}.
\end{aligned}$$

Remark 8.2. The functor q above coincides with the functor $gr_*^W = \prod_{n \in \mathbb{Z}} gr_n^W$. See Remark 1.9.1 in [**12**]. Then $M \in \mathcal{D}_A^{G,f,\leqslant 0}$ if and only if $H^m(gr_n^W M) = 0$ for all $m > 0$ and n.

Theorem 8.3. *Suppose A is cohomologically connected. Then $(\mathcal{D}_A^{G,f,\leqslant 0}, \mathcal{D}_A^{G,f,\geqslant 0})$ is a non-degenerate t-structure on $\mathcal{D}_A^{G,f}$ with heart $\mathcal{H}_A^{G,f}$.*

Proof. The proof is close to the corresponding proof of Theorem 1.1. 26 in [**12**]. Using Proposition 7.6 and Corollary 6.4, we can assume that A is connected after replacing by its minimal model. The proof will divide into the following lemmas.

Lemma 8.4. *Suppose that A is connected. Let $M \in \mathcal{D}_A^{G,f,\leqslant 0}$ (resp. $M \in \mathcal{D}_A^{G,f,\geqslant 0}$). Then there is a cell A-module $P \in \mathcal{CM}_A^{G,f}$ with generating data $\{V_j, \phi_j\}_{j\in J}$ such that $deg(\phi_j) \leqslant 0$ for all $j \in J$(resp. $deg(\phi_j) \geqslant 0$ for all $j \in J$) and a quasi-isomorphism $P \to M$.*

Proof. We prove the case $M \in \mathcal{D}_A^{G,f,\leqslant 0}$ firstly. Choose a quasi-isomorphism $Q \to M$ with $Q \in \mathcal{CM}_A^{G,f}$. Let $\{V_j, \phi_j\}_{j\in J}$ be generating data for Q. We can decompose d_Q with two parts d_Q^0 and d_Q^+, where d_Q^0 maps $\phi_j(V_j)$ to the submodule whose generating data (ϕ_i, V_i) have the Adams degree $|V_j|$ and d_Q^+ map to the complement part. After choosing suitable generating data, we may assume the collection S_0 of (V_j, ϕ_j) with $deg(\phi_j) = 0$ and $d_Q^0(\phi_j(V_j)) = 0$ forms a basis of $ker(d^0\colon \oplus_{deg(\phi_j)=0}\, \phi_j(V_j) \to \oplus_{deg(\phi_i)=1}\phi_i(V_i))$. Let $\tau^{\leqslant 0}Q$ be the sub A-module of Q with the generating data of $S = \{(V_j, \phi_j) | deg(\phi_j) < 0\} \bigcup S_0$.

Claim: $\tau^{\leqslant 0}Q$ is a subcomplex of Q. Given $\{V_\alpha, \phi_\alpha\}$, then

$$d_Q(\phi_\alpha(V_\alpha)) = d_Q^0(\phi_\alpha(V_\alpha)) \oplus d_Q^+(\phi_\alpha(V_\alpha)).$$

Using the connected condition of A, we know that:

1. If there exists $\phi_\beta(V_\beta) \subset d_Q^+(\phi_\alpha(V_\alpha))$, then $deg(\phi_\beta) \leqslant deg(\phi_\alpha)$; or
2. If there exists $\phi_\beta(V_\beta) \subset d_Q^0(\phi_\alpha(V_\alpha))$, then $deg(\phi_\beta) = deg(\phi_\alpha) + 1$.

If $(V_\alpha, \phi_\alpha) \in S$ with $deg(\phi_\alpha) \leqslant -1$, via $(d_Q^0)^2 = 0$ every summand of $d_Q^0(\phi_\alpha(V_\alpha))$ lies in S_0. So we only need to consider elements in S_0. Suppose that $(V_\alpha, \phi_\alpha) \in S_0$. Then we have:

$$d_Q(\phi_\alpha(V_\alpha)) \subset \bigoplus_{deg(\phi_\beta)=0} \phi_\beta(A \otimes V_\beta) \oplus \bigoplus_{deg(\phi_\gamma)\leqslant -1} \phi_\gamma(A \otimes V_\gamma).$$

Using $d_Q^2(\phi_\alpha(V_\alpha)) = 0$, we know that $d^0(\phi_\beta(V_\beta)) = 0$ for $deg(\phi_\beta) = 0$. This implies that $d_Q(\phi_\alpha(V_\alpha)) \subset \tau^{\leqslant 0}Q$. Next we show that $\tau^{\leqslant 0}Q \to Q$ is a quasi-isomorphism. Via Remark 8.1, we need only to check that $q\tau^{\leqslant 0}Q \to qQ$ is a quasi-isomorphism. This is clear because of $qQ \cong qM$ and $M \in \mathcal{D}_A^{f,\leqslant 0}$. For the case $M \in \mathcal{D}_A^{G,f,\geqslant 0}$, we need to check the proof of Theorem 3.9 carefully, where one may add the extra conditions on the degrees of the generating data. See Lemma 1.6.2 in [**12**]. □

Lemma 8.5. *Suppose that A is connected. Then* $\mathrm{Hom}_{\mathcal{D}_A^{G,f}}(M, N[-1]) = 0$ *for $M \in \mathcal{D}_A^{G,f,\leqslant 0}$ and $N \in \mathcal{D}_A^{G,f,\geqslant 0}$.*

Proof. By Lemma 8.4, we may assume that M and $N[-1]$ are cell A-modules with the generating data $\{(V_\alpha, \phi_\alpha)\}_{deg(\phi_\alpha)\leqslant 0}$ and $\{(V_\beta, \phi_\beta)\}_{deg(\phi_\beta)\geqslant 1}$. Recall that we have:

$$\mathrm{Hom}_{\mathcal{D}_A^{G,f}}(M, N[-1]) = \mathrm{Hom}_{\mathcal{KCM}_A^{G,f}}(M, N[-1]).$$

If $\phi\colon M \to N[-1]$, then $deg(\phi) = 0$. Therefore $f(\phi_\alpha(V_\alpha)) \subset \oplus_\beta \phi_\beta(A \otimes V_\beta)$. If we compute the cohomological degrees of both sides, since $A^i = 0$ when $i < 0$, this is impossible. □

Lemma 8.6. *Suppose that A is connected. For $M \in \mathcal{D}_A^{G,f}$, there is a distinguished triangle $M^{\leqslant 0} \to M \to M^{>0} \to M^{\leqslant 0}[1]$ with $M^{\leqslant 0} \in \mathcal{D}^{G,f,\leqslant 0}$ and $M^{>0} \in \mathcal{D}^{G,f,\geqslant 0}[-1]$.*

Proof. Via the same proof of Lemma 8.4, we can get a cell sub A-module $\tau^{\leqslant 0}M$ of M such that:

- $\tau^{\leqslant 0}M$ have generating data $\{(V_\alpha, \phi_\alpha)\}_{deg(\phi_\alpha)\leqslant 0}$.
- The map $q\tau^{\leqslant 0}M \to qM$ gives an isomorphism on H^n for $n \leqslant 0$.

Let $M^{\leqslant 0} = \tau^{\leqslant 0}M$ and let $M^{>0}$ be the cone of $\tau^{\leqslant 0}M \to M$. This gives us the distinguished triangle in $\mathcal{D}_A^G$: $M^{\leqslant 0} \to M \to M^{>0} \to M^{\leqslant 0}[1]$. Because $M \in \mathcal{D}_A^{G,f}$, then $gr_n^W M \in D_{\mathbb{Q}}^{G,f}$ for all n and is isomorphic to zero for all but finitely many n. The distinguished triangle and our assumption implies that $gr_n^W M^{\leqslant 0}, gr_n^W M^{>0} \in D_{\mathbb{Q}}^{G,f}$ and these graded quotients are isomorphic to zero for all but finitely many n. Via the weight filtration and by induction, we know that $M^{\leqslant 0}$ and $M^{>0}$ are all in $\mathcal{D}_A^{G,f}$. (Lemma 1.9.2 in [**12**].) By construction, $M^{\leqslant 0} \in \mathcal{D}^{G,f,\leqslant 0}$. Then applying the functor q to the distinguished triangle, we have $qM^{\leqslant 0} \to qM \to qM^{>0} \to qM^{\leqslant 0}[1]$. The second condition of $\tau^{\leqslant 0}M$ and Remark 8.2 implies that $M^{>0} \in \mathcal{D}^{G,f,\geqslant 0}[-1]$.

The only thing left to check is non-degeneracy for the t-structure. If we take $M \in \bigcap_{n\leqslant 0}\mathcal{D}^{\leqslant n}$, then $H^n(qM) = 0$ for all n, i.e., $qM \cong 0$ in $\mathcal{D}_{\mathbb{Q}}^{G,f}$. By the conservative property of the functor q, we know that $A \cong 0$ in $\mathcal{D}_A^{G,f}$. Another case is similar. □

Proposition 8.7. *$\mathcal{H}_A^{G,f}$ is a neutral Tannakian category over $\mathbb{Q}$.*

Proof. The derived tensor product makes $\mathcal{H}_A^{G,f}$ into an abelian tensor category. First we give a description about $\mathcal{H}_A^{G,f}$.

Lemma 8.8. *$\mathcal{H}_A^{G,f}$ is the smallest abelian subcategory of $\mathcal{H}_A^{G,f}$ containing the objects $A \otimes V$, where V is any rational G-representation, and closed under extensions in $\mathcal{H}_A^{G,f}$.*

Proof. (Induction on the weight filtration.) Let $\mathcal{H}_A^{G,T}$ be the full abelian subcategory containing all the objects $A \otimes V$, where V is any rational G-representation, and closed under extensions in $\mathcal{H}_A^{G,f}$. Let $M \in \mathcal{H}_A^{G,f}$ and $N = min\{n | W_n M \neq 0\}$. Then we have an exact sequence:

$$0 \to gr_N^W M \to M \to W^{>N}M \to 0.$$

By Lemma 5.10, we have $gr_N^W M \cong A \otimes C$, where C is in $D^b(G)$. Because the category of representations of G is semisimple, we view C as a direct sum of rational G-representations with some shifts. Assume there exists a summand $W[i]$ of C with shift $i \neq 0$. Then applying q, we get that $0 \neq H^i(q(gr_N^W M)) \subset H^i(qM)$, which is a contradiction of our choice of $M \in \mathcal{H}_A^{G,f}$. This implies that $gr_N^W M \in \mathcal{H}_A^{G,T}$. By induction on the length of the weight filtration, $W^{>N}M$ is in $\mathcal{H}_A^{G,T}$. So $M \in \mathcal{H}_A^{G,T}$ and $\mathcal{H}_A^{G,T} = \mathcal{H}_A^{G,f}$. □

Since $(A \otimes V)^\vee = A \otimes V^\vee$, where $V^\vee$ is the dual representation of V, it follows from the above description that $M \to M^\vee$ restricts from $\mathcal{D}_A^{G,f}$ to an exact involution on $\mathcal{H}_A^{G,f}$. $\mathcal{H}_A^{G,f}$ is rigid because $\mathcal{D}_A^{G,f}$ is rigid. The identity for the tensor product is A and $\mathcal{H}_A^{G,f}$ is $\mathbb{Q}$-linear. Furthermore we have a rigid tensor functor $q\colon \mathcal{H}_A^{G,f} \to \mathcal{H}_{\mathbb{Q}}^{G,f}$. Notice that $\mathcal{H}_{\mathbb{Q}}^{G,f}$ is equivalent to the category of rational representations of G. Because there is a faithful forgetful functor $w\colon \mathcal{H}_{\mathbb{Q}}^{G,f} \to Vec_{\mathbb{Q}}$, to show the existence of a fiber functor on $\mathcal{H}_A^{G,f}$ is sufficient to prove that q is faithful.

Recall that we can identify q with $gr_*^W = \prod gr_n^W$. Let $f\colon M \to N$ be a map in $\mathcal{H}_A^f$ such that $gr_n^W(f) = 0$ for all n. We need to show that $f = 0$. Again do the induction on the length of the weight filtration. We may assume that $W^n f = 0$, where n is the minimal integer such that $W_n M \oplus W_n N \neq 0$. Thus f is given by a map $\tilde{f}\colon W^{>n} M \to gr_n^W N$. We claim that $\tilde{f} = 0$. Using the induction on the weight filtration, it is enough to show the following statement: Given V and W pure weight rational G-representations such that $|V| > |W|$, then we have:

$$\mathrm{Hom}_{\mathcal{H}_A^{G,f}}(A \otimes V, A \otimes W) \cong 0.$$

In fact, since A is connected, then $H^0(A \otimes W \otimes V^\vee) \cong W \otimes V^\vee$. The latter is a rational representation of G with Adams degree strictly smaller than zero, which implies that $\mathrm{Hom}_G(\mathbb{Q}, H^0(A \otimes W \otimes V^\vee)) \cong 0$. Using

$$\begin{aligned} \mathrm{Hom}_{\mathcal{H}_A^{G,f}}(A \otimes V, A \otimes W) &\cong \mathrm{Hom}_{\mathcal{D}_A^{G,f}}(A \otimes V, A \otimes W) \\ &\cong \mathrm{Hom}_{D(G)}(V, A \otimes W) \cong \mathrm{Hom}_G(\mathbb{Q}, H^0(A \otimes W \otimes V^\vee)) \cong 0, \end{aligned}$$

we know that q is faithful. □

9. The bar construction

Let A be a cdga over G and let M, N be two dg A-modules. Then we define:

$$T^G(N, A, M) = N \otimes T(A) \otimes M,$$

where $T(A) = \mathbb{Q} \oplus A \oplus (A \otimes A) \oplus \cdots = \oplus_{r \geqslant 0} T^r(A)$ is the tensor algebra generated by A. It is spanned by the elements of the form $n[a_1|\cdots|a_r]m$. Note that $T^G(N, A, M)$ is a simplicial graded abelian group with $N \otimes T^r(A) \otimes M$ in degree r, whose face maps are:

$$\begin{aligned} \delta_0(n[a_1|\cdots|a_r]m) &= na_1[a_2|\cdots|a_r]m, \\ \delta_i(n[a_1|\cdots|a_r]m) &= na_1[a_2|\cdots|a_i a_{i+1}|\cdots|a_r]m,\ 1 \leqslant i \leqslant r-1, \\ \delta_r(n[a_1|\cdots|a_{r-1}|a_r]m) &= n[a_1|\cdots|a_{r-1}]a_r m, \end{aligned}$$

and degeneracies are: $s_i(n[a_1|\cdots|a_r]m) = n[a_1|\cdots|a_{i-1}|1|a_i|\cdots|a_r]m$. We define:

$$\delta = \sum_{0 \leqslant i \leqslant r} (-1)^i \delta_i\colon N \otimes T^r(A) \otimes M \to N \otimes T^{r-1}(A) \otimes M.$$

Let $D^G(N, A, M)$ be the degenerate elements, those elements are spanned by the images of the s_i for every i.

Definition 9.1. Define the bar complex of M and N to be:

$$B^G(N, A, M) = T^G(N, A, M)/D^G(N, A, M).$$

Note that $B^G(N, A, M)$ is a bicomplex. The total differential is defined by

$$d(n[a_1|\cdots|a_r]m) = \partial(n[a_1|\cdots|a_r]m)) + (-1)^{deg(n)+deg(m)+\sum deg(a_i)}\delta(n[a_1|\cdots|a_r]m),$$

where ∂ denotes the usual differential of A. We will consider the following special case that $M = N = \mathbb{Q}$, which is denoted by $\bar{B}^G(A)$, called the reduced bar construction. The properties of reduced bar construction such as the shuffle product, coproduct and involution ([**12**] Section 1.2) make $\bar{B}^G(A)$ a graded-commutative differential graded Hopf algebra in the category of G-representations. Similarly $\chi_A = H^0(\bar{B}^G(A))$ is an Adams graded Hopf algebra over G i.e., a graded Hopf algebra object in Rep_G.

Definition 9.2. Define $\gamma_A = I_{\chi_A}/(I_{\chi_A})^2$, where I_{χ_A} is the augmentation ideal of χ_A.

Lemma 9.3. *γ_A determines a structure of a cdga over G.*

Remark 9.4. Recall the definition of co-Lie algebras firstly. A co-Lie algebra is a k-module γ with a cobracket map $\gamma \to \gamma \otimes \gamma$ such that the dual $\gamma^\vee$ is a Lie algebra via the dual homomorphism. Sullivan showed that (Lemma 2.7 in [**11**] or p.279 in [**15**]): A co-Lie algebra γ determines and is determined by a structure of DGA on $\wedge(\gamma[-1])$. The proof of the above lemma comes from this fact directly.

Lemma 9.5. *Let A be a cdga over G. Then $H^*(\bar{B}^G(A))$ and χ_A is functorial in A and is a quasi-isomorphism invariant in A.*

Proof. Use the Eilenberg-Moore spectral sequence. See Lemma 2.21 in [**3**]. ☐

Theorem 9.6. *If A is a cohomologically connected cdga over G, then the 1-minimal model $A\{1\}$ of A is isomorphic to $\wedge(\gamma_A[-1])$.*

Proof. Follow the proof of [**3**]. From Lemma 9.5, we can assume that A is a generalized nilpotent cdga over G. A generalize nilpotent cdga A over G is a direct limit (A_α) of nilpotent cdga's. So we can assume that A is a nilpotent cdga over G with a free generator E which is a complex of G-representations. We need to use the following lemma, whose proof is totally the same as Lemma 2.32 in [**3**].

Lemma 9.7. *Assume that A is as above with free generator G-representation V. Fix an integer $s > 0$. We let the decreasing filtration $F^k\bar{B}^G(A)$ on $\bar{B}^G(A)$ be*

$$\langle x_{11}\cdots x_{1n_1} \otimes \cdots \otimes x_{m1}\cdots x_{mn_m} | s\sum \deg x_{ij} + s\sum n_j - (2s-1)m \geqslant k\rangle.$$

Then, for a sufficiently large s depending on A, the resulting spectral sequence satisfies

$$\wedge(V[1]) \cong E_{2s} \cong E_\infty \cong Gr_F H^*(\bar{B}^G(A)). \tag{1}$$

Let us prove the theorem via the lemma. Consider the projection $\pi_n\colon \bar{B}^G(A) \to A^{\otimes n}$, which maps $A^{\otimes m}$ to 0 if $m \neq n$. By definition of the differential map of the reduces bar construction, the boundaries of the reduced bar construction are the direct sums of the boundaries of the complex $A^{\otimes n}$ and decomposable elements. Via Proposition 7.4 and our assumption, A is minimal, i.e., the boundaries of A are decomposable. Then π_1 induces a map $\phi\colon H^*(\bar{B}^G(A)) \to (QA)[1]$. Hence we get a map between (graded) vector spaces:

$$\phi\colon QH^*(\bar{B}^G(A)) \to (QA)[1].$$

Using the isomorphism (1) in the above lemma and taking the indecomposable parts of both hands sides of it, we know that ϕ induces an isomorphism between vector spaces:

$$\tilde{\phi}\colon QH^0(\bar{B}^G(A)) \to (QA)[1].$$

Moreover, this is an isomorphism between co-Lie algebras. Note that the co-Lie algebra structure of LHS is obtained from the coproduct of the Hopf algebra structure of $H^0(\bar{B}^G(A))$ and the co-Lie algebra structure of RHS is obtained from the quadratic part of the differential map on A with respect to the generator space QA. We can use the commutative diagram in Theorem 6.3 (iii) in [**11**] to check that $\tilde{\phi}$ is an isomorphism between the co-Lie algebras. Then we have a map:

$$\gamma_A[-1] = QH^0(\bar{B}^G(A))[-1] \to QA,$$

hence a map $\wedge(\gamma_A[-1]) \to A$, which is just the 1-minimal model of A. $\square$

10. Alternative identifications of the heart

In this section, we collect the different identifications of the category $\mathcal{H}_A^{G,f}$ based on the construction in the usual Adams graded case.

Definition 10.1. An Adams degree bounded below cell A-module is minimal if it is almost free and $d(M) \subset (IA)M$.

Definition 10.2. Let M be an Adams degree bounded below A-module. We define the nilpotent filtration $\{F_tM\}$ by letting $F_0M = 0$ and inductively letting F_tM be the sub A-module generated by $F_{t-1}M \cup \{m|dm \in F_{t-1}M\}$.

Remark 10.3. The minimal cell modules have the similar properties as the connected minimal cdgas. We can also define the generalized nilpotent A-modules. Since the proof of the following properties is the same as Part IV, section 3 in [**11**], we only list the main properties:

- A bounded below A-module M is generalized nilpotent if and only if it is a minimal cell A module.
- Let N be a dg A-module. Then there is a quasi-isomorphism $e\colon M \to N$, where M is a minimal A-module. This is unique up to the homotopy.

Next we want to use another way to describe cell A-modules, which is called the connection matrix. See [**11**] (called the twisting matrix also) or [**12**].

Definition 10.4. Let (M, d_M) be a complex of G-representations. An A-connection for M is a map $\Gamma\colon M \to IA \otimes M$ of G representations and cohomological degree 1. We say Γ is flat if $d\Gamma + \Gamma^2 = 0$. Here $d\Gamma = d_{IA\otimes M} \circ \Gamma + \Gamma \circ d_M$ and we extend Γ to $\Gamma\colon IA \otimes M \to IA \otimes M$ by the Leibniz rule.

Remark 10.5. Given a connection $\Gamma\colon M \to IA \otimes M$, we define

$$d_0\colon M \to A \otimes M = M \oplus IA \otimes M, m \to d_M m \oplus \Gamma m$$

and extend d_0 to $d_\Gamma\colon A \otimes M \to A \otimes M$ by the Leibniz rule. The above equation is equivalent to saying that $d_\Gamma^2 = 0$.

Definition 10.6. We call an A-connection Γ for M nilpotent if M admits a filtration by complexes of G-representations: $0 = M_{-1} \subset M_0 \subset \cdots \subset M_n \subset \cdots \subset M$ such that $M = \cup_n M_n$ and such that $d_M(M_n) \subset M_{n-1}$ and $\Gamma(M_n) \subset IA \otimes M_{n-1}$ for every $n \geqslant 0$.

Remark 10.7. Let $\Gamma\colon M \to IA \otimes M$ be a flat nilpotent connection. Then the dg A-module $(A \otimes M, d_\Gamma)$ is a cell module.

Lemma 10.8. *Let $\Gamma\colon M \to IA \otimes M$ be a flat connection. Suppose there is an integer r_0 such that $|m| \geqslant r_0$ for all $m \in M$. Then Γ is nilpotent.*

Proof. The proof is the same as Lemma 1.13.3 in [**12**]. □

Definition 10.9. A morphism $f\colon (M, d_M, \Gamma_M) \to (N, d_N, \Gamma_N)$ is a map of complexes of G-representations: $f = f_0 + f^+\colon M \to A \otimes N = N \oplus IA \otimes N$ such that $d_{\Gamma_N} f = f d_{\Gamma_M}$.

Definition 10.10. We denote the category of flat nilpotent connections over A by $Conn_A^G$ and denote the full subcategory of flat nilpotent connections on M with M a bounded complex of rational G-representations by $Conn_A^{G,f}$.

We can define a tensor operation on $Conn_A$ by

$$(M, \Gamma) \otimes (M', \Gamma') = (M \otimes M', \Gamma \otimes id + id \otimes \Gamma').$$

Complexes of $\mathbb{Q}$-vector spaces act on $Conn_A$ by: $(M, \Gamma) \otimes K = (M, \Gamma) \otimes (K, 0)$. We recall that I is the complex $\mathbb{Q} \xrightarrow{\delta} \mathbb{Q} \oplus \mathbb{Q}$ with $\mathbb{Q}$ in degree -1 and with connection 0. We have the two inclusions $i_0, i_1\colon \mathbb{Q} \to I$.

Definition 10.11. Two maps $f, g\colon (M, \Gamma) \to (M', \Gamma')$ are homotopic if there is a map $h\colon (M, \Gamma) \otimes I \to (M', \Gamma')$ satisfying $f = h \circ (id \otimes i_0), g = h \circ (id \otimes i_1)$.

Definition 10.12. Denote the homotopy category of $Conn_A^G$ by $\mathcal{H}Conn_A^G$, which has the same objects as $Conn_A^G$ and morphisms are homotopy classes of maps in $Conn_A^G$.

Remark 10.13. When we pass to homotopy classes and given a cell A-module M, it is totally determined by the underlying G-representation M_0, i.e., $M_0 = M \otimes_A \mathbb{Q}$.

We list the main properties of flat connections. The proof is the same as in Section 1.14 in [**12**].

(**A**) The category of A-cell modules is equivalent to the category of flat nilpotent A-connections.

(**B**) The equivalence in (**A**) passes to an equivalence of $\mathcal{H}Conn_A^G$ with the homotopy category $\mathcal{KCM}_A^G$ as triangulated tensor categories.

(**C**) If A is connected, the equivalence in (**B**) defines an equivalence of Tannakian categories $\mathcal{H}_A^{G,f}$ and the category of flat connections on G-representations $Conn_A^{G,f}$.

Assume that A is a generalized nilpotent connected cdga over G. Let M be a complex of rational G-representations and $\Gamma\colon M \to IA \otimes M$ a flat connection. The argument after Remark 1.14.6 in [**12**] implies that Γ is a map $M \to A^1 \otimes M$. The flatness of Γ makes M into an Adams graded co-module for the co-Lie algebra γ_A over G. In this case, one can show that there is an equivalence $Conn_A^{G,f} \cong co-rep^{G,f}(\gamma_A)$ (See Remark 1.14.8 in [**12**]), hence using the equivalence in (**C**), we have $\mathcal{H}_A^{G,f} \cong Conn_A^{G,f} \cong co-rep^{G,f}(\gamma_A)$. Since $\mathcal{H}_A^{G,f}$ and $co-rep^{G,f}(\gamma_A)$ are invariant under quasi-isomorphisms of cohomologically connected cdga$'$s, if we assume that A is cohomologically connected, then we get an equivalence $\mathcal{H}_A^{G,f} \cong co-rep^{G,f}(\gamma_A)$.

11. The main theorem

Lemma 11.1. *Let $\mathcal{D}$ be a triangulated category with t-structure $(\mathcal{D}^{\leqslant 0}, \mathcal{D}^{\geqslant 0})$. We denote its heart by $\mathcal{H}$. Assume that there is a triangulated functor $\rho\colon D^b(\mathcal{H}) \to \mathcal{D}$ such that:*

- *$\rho|_{\mathcal{H}[i]}$ is an inclusion for any $i \in \mathbb{Z}$;*
- *$\mathcal{D}$ is bounded, i.e., for any $M \in \mathcal{D}$, there exist $a \leqslant b \in \mathbb{Z}$ satisfying $M \in \mathcal{D}^{[a,b]} = \mathcal{D}_{\geqslant a} \cap \mathcal{D}_{\leqslant b}$;*
- *For any $M, N \in \mathcal{H}$ and $n \in \mathbb{Z}$, ρ induces an isomorphism*

$$\mathrm{Hom}_{D^b(\mathcal{H})}(M, N[n]) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{D}}(\rho(M), \rho(N)[n]).$$

Then ρ is an equivalence between triangulated categories.

Proof. We do the induction on the length of the object. Given an object A in $\mathcal{D}$, there exist the minimal a and maximal b such that $A \in \mathcal{D}_{\geqslant a} \cap \mathcal{D}_{\leqslant b}$. Then we define the length of A to be $b-a$. Firstly, we prove the following: For any $A, B \in D^b(\mathcal{H})$ and $n \in \mathbb{Z}$, we have:

$$\mathrm{Hom}_{D^b(\mathcal{H})}(M, N[n]) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{D}}(\rho(M), \rho(N)[n]). \tag{2}$$

By induction we assume that, for any $A \in D^b(\mathcal{H})^{a,b}$, $B \in D^b(\mathcal{H})^{c,d}$ and $max\{b-a, d-c\} \leqslant m-1$, the above is true. Take any A with the length smaller than m, and B with the length $m = b-a$. There is a distinguished triangle: $\tau_{\geqslant a}\tau_{\leqslant b-1}B \to B \to \tau_{\geqslant b}\tau_{\leqslant b}B \to \tau_{\geqslant a}\tau_{\leqslant b-1}B[1] \to$. Then we have a long exact sequence:

$$\begin{aligned}\mathrm{Hom}(A, \tau_{\geqslant a}\tau_{\leqslant b-1}B) \to \mathrm{Hom}(A, B) \to \mathrm{Hom}(A, \tau_{\geqslant b}\tau_{\leqslant b}B)& \\ \to \mathrm{Hom}(A, \tau_{\geqslant a}\tau_{\leqslant b-1}B[1]) \to \cdots .&\end{aligned}$$

Compare the above sequence with:

$$\begin{aligned}\mathrm{Hom}(\rho(A), \rho(\tau_{\geqslant a}\tau_{\leqslant b-1}B)) \to \mathrm{Hom}(\rho(A), \rho(B)) \to \mathrm{Hom}(\rho(A), \rho(\tau_{\geqslant b}\tau_{\leqslant b}B))& \\ \to \mathrm{Hom}(\rho(A), \rho(\tau_{\geqslant a}\tau_{\leqslant b-1}B[1])) \to \cdots .&\end{aligned}$$

We know (2) holds for A, B by the five lemma and induction. Then we assume both A and B have length m. Using the similar method and induction again, we can know that (2) is true, i.e., ρ is fully faithful.

Next we want to use induction to show that ρ is essentially surjective. It is enough to show that, for any object $B \in \mathcal{D}$, there exists $A \in D^b(\mathcal{H})$ such that $\rho(A) \cong B$. Take any element $B \in \mathcal{D}$ with length m. Then we have: $\tau_{\geqslant a}\tau_{\leqslant b-1}B \to B \to \tau_{\geqslant b}\tau_{\leqslant b}B \to \tau_{\geqslant a}\tau_{\leqslant b-1}B[1] \to$, i.e., $\tau_{\geqslant b}\tau_{\leqslant b}B[-1] \xrightarrow{f} \tau_{\geqslant a}\tau_{\leqslant b-1}B \to B \to \tau_{\geqslant b}\tau_{\leqslant b}B \to$. By assumption, we have A_1 and $A_2 \in D^b(\mathcal{H})$ map to $\tau_{\geqslant b}\tau_{\leqslant b}B[-1]$ and $\tau_{\geqslant a}\tau_{\leqslant b-1}B$ respectively.

By (2), we know that there exists a map g from A_1 to A_2, whose image under ρ is just f. We take $A = cone(g)$. Then by the axiom of triangulated categories, there exists a map $\rho(A) \to B$. By the five lemma and Yoneda lemma, applying the functor of type $\mathrm{Hom}(\tilde{B},)$, where $\tilde{B} \in \mathcal{D}$, we know that $\rho(A) \cong B$. □

Theorem 11.2. *Let A be a cohomologically connected cdga over G. Then*

- *There is a functor: $\rho\colon D^b(\mathcal{H}_A^{G,f}) \longrightarrow \mathcal{D}_A^{G,f}$.*
- *The functor ρ constructed above is an equivalence of triangulated categories if and only if A is 1-minimal.*

Proof. Construct the functor ρ first. Let $M^* = \{M^n, \delta^n\colon M^n \to M^{n+1}\}$ be in $\mathcal{H}_A^{G,f}$. Assume that each M^n is minimal. Furthermore, we assume that M^n is given by generating data $\{V_{j_m^n}, \phi_{j_m^n}\}_{j_m^n \in J^n}$ and the connection matrix is denoted by Γ^n. Then we define ρM^* with generating data $\{V_{j_m^n}, \phi_{j_m^n}[n]\}_{j_m^n \in J^n}$ and its differential given by: $d|_{\phi_{j_m^n}[n](V_{j_m^n})} = \Gamma^n[n] + \delta^n[n]$. If $f^*\colon M^* \to N^*$ is a quasi-isomorphism of chain complexes, then $\rho(f^*)$ is a quasi-isomorphism of A-modules.

For the second statement, we assume that A is 1-minimal, i.e., $A \cong \wedge^*(\gamma[-1])$, where γ is the co-Lie algebra consisted by indecomposable elements of $H^0(\bar{B}^G(A))$. In order to apply the above result to our case ($\mathcal{D} = \mathcal{D}_A^{G,f}$ and $\mathcal{H} = \mathcal{H}_A^{G,f}$), we need to check the conditions in Lemma 11.1. The first and second condition are automatic. We check the third condition. Notice that $\mathcal{H}_A^{G,f}$ can be identified with the category of co-representations of γ in the category of G-representations. In fact, given a finite dimensional co-representation V, we can associate it with a cell module $A \otimes V$.

We recall the following basic facts (Lemma 23.1, Example 1 (p. 315) and pp. 319, 320 in [**8**]). Given a differential graded Lie algebra L, we have:

$$Ext_L^n(\mathbb{Q},\mathbb{Q}) \cong Ext_{UL}^n(\mathbb{Q},\mathbb{Q}) \cong H^n((\wedge^*(L[-1]))^\vee),$$
$$Ext_L^n(\mathbb{Q},V) \cong Ext_{UL}^n(\mathbb{Q},V) \cong H^n((\wedge^*(L[-1]))^\vee \otimes V).$$

UL is the universal enveloping Lie algebra of L and V is any L-module. $L[-1]_k = L_{k-1}$. Applying to the co-Lie algebra γ, we get: $Ext_\gamma^n(\mathbb{Q},\mathbb{Q}) \cong H^n(\wedge^*(\gamma[-1]))$. In fact, the proof of this isomorphism can be extended to the following case. Given a co-Lie algebra γ over G and a γ co-representation V, we have: $Ext_\gamma^n(\mathbb{Q}, V) \cong \mathrm{Hom}_G(\mathbb{Q}, H^n(\wedge^*(\gamma[-1]) \otimes V))$. Notice that the left-hand side computes the extension groups in the category of γ-representations. Therefore we have:

$$\begin{aligned}
&\mathrm{Hom}_{D^b(\mathcal{H}_A^{G,f})}(\wedge^*(\gamma[-1]) \otimes V, \wedge^*\gamma[-1] \otimes W[n]) \\
&\quad\cong Ext^n_{\mathcal{H}_A^{G,f}}(\wedge^*(\gamma[-1]) \otimes V, \wedge^*(\gamma[-1]) \otimes W) \cong Ext_\gamma^n(\mathbb{Q}, V^\vee \otimes W) \\
&\quad\cong \mathrm{Hom}_G(\mathbb{Q}, H^n(\wedge^*(\gamma[-1]) \otimes V^\vee \otimes W)) \cong \mathrm{Hom}_G(\mathbb{Q}, H^0(\wedge^*(\gamma[-1]) \otimes V^\vee \otimes W[n])) \\
&\quad\cong \mathrm{Hom}_{\mathcal{D}_A^{G,f}}(\mathbb{Q}, \wedge^*(\gamma[-1]) \otimes V^\vee \otimes W[n]) \\
&\quad\cong \mathrm{Hom}_{\mathcal{D}_A^{G,f}}(\wedge^*(\gamma[-1]) \otimes V, \wedge^*(\gamma[-1]) \otimes W[n]).
\end{aligned}$$

One can check that the composition of these isomorphisms is $\rho\colon D^b(\mathcal{H}_A^{G,f}) \longrightarrow \mathcal{D}_A^{G,f}$.

Conversely, we assume that ρ is an equivalence. Without loss of generality, we assume that A is generalized nilpotent. The above computation tells us that $H^n(A \otimes V)(0) \cong Ext_\gamma^n(\mathbb{Q}, V)$, where γ is the co-Lie algebra consisting of the indecomposable elements in $H^0(\bar{B}(A))$. Let us consider the map $A \to \wedge^*(\gamma[-1])$, Applying the functor

$\mathrm{Hom}_G(V[n], \cdot)$ for any $n \in \mathbb{Z}$ and any G-representation V, we get: $\mathrm{Hom}_G(V[n], A) \cong H^n(A \otimes V)(0) \cong Ext^n_\gamma(\mathbb{Q}, V) \cong \mathrm{Hom}_G(V[n], \wedge^*(\gamma[-1]))$. This implies that, viewed as G-representations, the map is a quasi-isomorphism. Therefore A is 1-minimal. □

Corollary 11.3. *Let A be a cohomologically connected cdga over G. Then*

- *There is a functor: $\rho\colon D^b(co - rep^{G,f}_{\mathbb{Q}}(\chi_A)) \longrightarrow \mathcal{D}^{G,f}_A$. Furthermore, ρ induces a functor on the hearts $\mathcal{H}(\rho)\colon co - rep^{G,f}(\chi_A) \to \mathcal{H}^{G,f}_A$, which is an equivalence of Tannakian categories.*
- *The functor ρ is an equivalence of triangulated categories if and only if A is 1-minimal.*

12. Split Tannakian categories over a reductive group

In this section, we describe a special kind of Tannakian categories. Fix a reductive group R.

Definition 12.1. We say that C is a neutral Tannakian category over R if C is a neutral Tannakian category over $\mathbb{Q}$ and there exists an exact faithful $\mathbb{Q}$-linear tensor functor $\widetilde{\omega}\colon C \to Rep(R)$, whose composition with the forgetful functor $F\colon Rep(R) \to Vec_{\mathbb{Q}}$ is the fiber functor $\omega\colon C \to Vec_{\mathbb{Q}}$.

Example 12.2. Assume that R satisfies Convection 2.1 and that A is a cohomologically connected cdga over R. From Theorem 8.3, we know the existence of the heart $\mathcal{H}^{R,f}_A$ of $\mathcal{D}^{R,f}_A$, which is a neutral Tannakian category over R. Another example is the relative completion $\mathcal{G}$ with respect a given map from an abstract group $\Gamma \xrightarrow{\rho} R(\mathbb{Q})$. For the definition of the relative completion, we refer to [**9**].

Motivated by the above examples, we define:

Definition 12.3. A neutral Tannakian category C over R is split if the full subcategory of C consisting of semi-simple objects is isomorphic to $Rep(R)$.

Remark 12.4. The Tannakian fundamental group of a neutral Tannakian category C with a tensor generator is isomorphic to a linear proalgebraic group. See Proposition 2.20 in [**7**]. If we assume further that this Tannakian category is split over G and its Tannakian fundamental group is connected, then it will be the form $U \rtimes R$, where U is a prounipotent algebraic group. Example 12.2 satisfy these conditions.

In the end, we want to use the method of framed objects (Section 6 of [**3**] for example) to give a description of the coordinate ring of the Tannakian fundamental group of any split neutral Tannakian category C with a tensor generator. We assume that the Tannakian fundamental group is connected. As explained in Remark 12.4, it is enough to determine the coordinate ring of the prounipotent radical as a Hopf algebra object in $Rep(R)$.

Definition 12.5. A framed object in C is an object X in C together with an element $u \in \mathrm{Hom}_R(V, \widetilde{\omega}(X))$ (called the frame vector) and an element of $v \in \mathrm{Hom}_R(\widetilde{\omega}(X), \mathbb{Q})$ (called the frame covector), where V is an irreducible R-representation. We denote this object by (X, u, v).

Definition 12.6. Two framed objects $X, Y \in C$ are identified if there is a mapping $X \to Y$ compatible with the framing.

Notice that such pairs X, Y define a relation $\mathcal{R}$ on the set of all framed objects. Then χ_C is defined as the set of equivalence classes of the smallest equivalence relation containing $\mathcal{R}$. By our definition χ_C is graded over the irreducible rational representations of G. For a given rational R- representation V, we denote the V-graded piece of χ_C by $\chi_C(V)$.

Claim 12.7. *χ_C is a Hopf algebra in Rep(R).*

In order to explain our claim, first we recall a fundamental result – Proposition 3.1 in [**9**]. If $(V_\alpha)_\alpha$ is a set of irreducible rational R-representations, viewed right R-modules, then, as an (R, R) bimodule, $\mathcal{O}(R)$ is canonically isomorphic to $\bigoplus_\alpha (V_\alpha)^\vee \boxtimes V_\alpha$. In other words, there exists a Hopf algebraic structure on $\bigoplus_\alpha (V_\alpha)^\vee \boxtimes V_\alpha$. If we denote the generator of $(V_\alpha)^\vee \boxtimes V_\alpha$ by $v_\alpha^\vee \boxtimes v_\alpha$, then:

(1) Let m be the product map of the Hopf algebra. Then we have

$$m((v_\alpha^\vee \boxtimes v_\alpha) \otimes (v_\beta^\vee \boxtimes v_\beta)) = \oplus_\gamma m_{\alpha,\beta}^\gamma (v_\gamma^\vee \boxtimes v_\gamma),$$

where $m_{\alpha,\beta}^\gamma$ is the corresponding multiplicity. In fact, the index set runs through the irreducible representations V_γ appearing in the tensor product of $V_\alpha \otimes V_\beta$.

(2) Let Δ be the coproduct map of the Hopf algebra. Then we have

$$\Delta(v_\alpha^\vee \boxtimes v_\alpha) = \sum_{\beta,\gamma} n_\alpha^{\beta,\gamma}((v_\beta)^\vee \boxtimes v_\beta) \otimes ((v_\gamma)^\vee \boxtimes v_\gamma), \tag{3}$$

where $n_\alpha^{\beta,\gamma}$ is the corresponding multiplicity.

Now we move to our claim.

- The sum on χ_C is the Baer sum.
- The product is defined by the tensor product of underlying objects together with the tensor product of the framings. Let $(X_1, u_1, v_1), (X_2, u_2, v_2)$ be two framed objects, which represent two classes $[(X_1, u_1, v_1)]$ and $[(X_2, u_2, v_2)]$ in $\chi_C(V_\alpha)$ and $\chi_C(V_\beta)$ respectively. Then we define: $[(X_1, u_1, v_1)] \cdot [(X_2, u_2, v_2)] = [(X_1 \otimes X_2, u_1 \otimes u_2, v_1 \otimes v_2)] \in \chi_C(V_\alpha \otimes V_\beta)$.
- The coproduct $\psi = \oplus_\alpha \psi_\alpha = \oplus_\alpha \oplus_{\beta,\gamma} \psi_\alpha^{\beta,\gamma}$, where $\psi_\alpha^{\beta,\gamma} \colon \chi_C(V_\alpha) \to \chi_C(V_\beta) \otimes \chi_C(V_\gamma)$, the first index set α runs over all irreducible representations V_α and the second index set β, γ is the same as the index set appearing in the coproduct (3), is defined as follows. We let $[(M, u, v)] \in \chi_C(V_\alpha)$ and let $\sum x_i \otimes y_i = 1 \in \mathrm{Hom}_R(\widetilde{\omega}(M), V_\gamma) \otimes \mathrm{Hom}_R(V_\gamma, \widetilde{\omega}(M))$. Here 1 denotes the identity map under the isomorphism:
$$\mathrm{Hom}_R(\widetilde{\omega}(M), V_\gamma) \otimes \mathrm{Hom}_R(V_\gamma, \widetilde{\omega}(M)) \cong End(\mathrm{Hom}_R(V_\gamma, \widetilde{\omega}(M))).$$
Then we put $\psi_\alpha^{\beta,\gamma}([(M, u, v)]) = \sum_{\beta,\gamma} n_\alpha^{\beta,\gamma}[(M \otimes (V_\gamma)^\vee, u, x_i)] \otimes [(M, y_i, v)]$. We view u on the right-hand side as the adjunction of $V_\beta \otimes V_\gamma \to V_\alpha \xrightarrow{u} M$.
- The involution $S \colon \chi_C(V_\alpha) \to \chi_C(V_\alpha)$ is defined by sending $[(M, u, v)]$ to $[(M^\vee \otimes V_\alpha, v, u)]$.

One may check these operations satisfy the axioms of a Hopf algebra. Motivated by the result in Section 1.6 of [**1**], we have:

Proposition 12.8. *Let C be a split neutral Tannakian category over R with a tensor generator and the Tannakian fundamental group of C is connected. Then C is* *equivalent to the category of finite dimensional χ_C-comodules in Rep(R).*

Proof. Because $Rep(R)$ is a full subcategory of C, we consider $V \in Rep(R)$ as an element in C. Given any object $M \in C$, we assign it to a χ_C-comodule $\Phi(M)$ such that:

- the underlying R-mod structure is
$$\Phi(M) = \bigoplus_\alpha \Phi(M)(V_\alpha) = \bigoplus_\alpha \mathrm{Hom}_R(V_\alpha, \widetilde{\omega}(M)) \otimes V_\alpha,$$
where the index set α runs through all the isomorphic classes of irreducible R-representations.
- the χ_C-comodule structure is defined by the map $\Phi(M) \otimes \Phi(M)^\vee \to \chi_C$, which sends $u \otimes v$ to $[(M, u, v)]$. If v doesn't lie in $\mathrm{Hom}_R(\mathbb{Q}, \widetilde{\omega}(M))$, then consider it to be zero.

This defines a functor Φ from C to the category of finite dimensional χ_C-comodules in $Rep(R)$. Notice that the coproduct on χ_C is conilpotent. For the definition of a conilpotent coproduct, we refer to Section 3.8 in [**6**]. Using Theorem 3.9.1 in [**6**], we know that $\mathcal{U}_C$, which is defined to be $Spec(\chi_C)$, is a prounipotent algebraic group over $\mathbb{Q}$. On the other hand, C is isomorphic to the category of representations over a linear proalgebraic group $\mathcal{G}$, which is a semi-product of a prounipotent algebraic group $\mathcal{U}$ with R. Hence Φ induces a map from $H^i(\mathcal{U}) \cong H^i(\mathfrak{u}) \to H^i(\mathcal{U}_C) \cong H^i(\mathfrak{u}_C)$, where $\mathfrak{u}$ (resp. $\mathfrak{u}_C$) is the Lie algebra of $\mathcal{U}$ (resp. $\mathcal{U}_C$).

We recall the following basic properties about extension groups in C or the category of finite dimensional $\mathcal{G}$-representations (Section 5.1 in [**10**]). We have an isomorphism in $Rep(R)$:
$$H^i(\mathfrak{u}) \cong H^i(\mathcal{U}) \cong \bigoplus_\alpha H^i(\mathcal{G}, (V_\alpha)^\vee) \otimes V_\alpha \cong \bigoplus_\alpha Ext^i_C(1, (V_\alpha)^\vee) \otimes V_\alpha,$$
where the index set runs over all irreducible R-representations.

Note that $\mathfrak{u}_C \cong \mathrm{Hom}(I_C/(I_C)^2, \mathbb{Q})$ where I_C is the fundamental ideal of χ_C. Using Chevalley-Eilenberg cochains, one may show that Φ induces an isomorphism $H^1(\mathfrak{u}_C) \cong \bigoplus_\alpha Ext^1_C(1, (V_\alpha)^\vee) \otimes V_\alpha$. Then following the same idea in the proof of Theorem A.6 in appendix of [**10**], Φ induces an embedding from $\bigoplus_\alpha Ext^2_C(1, (V_\alpha)^\vee) \otimes V_\alpha$ to $H^2(\mathfrak{u}_C)$. Via Proposition 2.1 in [**10**], we know that: $\Phi\colon \mathfrak{u}_C \to \mathfrak{u}$ is a R-equivariant isomorphism between pronilpotent Lie algebras. Therefore Φ induces an equivalence of Tannakian categories between C and the category of finite dimensional χ_C-comodules in $Rep(R)$. □

Remark 12.9. If we assume that R satisfies Convention 2.1, then one can prove Proposition 12.8 via the properties of mixed categories developed in Section 2 of [**2**] without difficulty.

Example 12.10. Let F be a field finitely generated over a prime field and let l be a prime number. The category $MTM_{F,l}$ of mixed Tate representations of $Gal(\bar{F}/F)$ is a split neutral Tannakian category over $\mathbb{G}_m$. Proposition 12.8 implies that $MTM_{l,F}$ is isomorphic to the category of finite dimensional $\chi_{MTM_{l,F}}$ comodules in $Rep(\mathbb{G}_m)$. This has been shown in [**1**]. If F is a number field, the abelian category $MTM(F, \mathbb{Q})$ of mixed Tate motives with rational coefficients over F exists which is a split neutral

Tannakian category over $\mathbb{G}_m$. One can use Proposition 12.8 to describe such category. Moreover, the full rigid tensor subcategory of the abelian category of mixed motives generated by the motive of a fixed smooth projective variety is conjecturally a split neutral Tannakian categories over some reductive group.

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