PREPROJECTIVE ANALOGUE OF THE CONE CONSTRUCTION

BENJAMIN COOPER AND JOSHUA SUSSAN

(*communicated by Charles A. Weibel*)

Abstract

We formulate a relative, representation theoretic, notion of
the algebraic cone construction. This motivates a generalization
of the cone corresponding to a preprojective algebra.

1.   Introduction

One of the most important structures in homological algebra is the cone $C(f)$ of
a map f

$$f \rightsquigarrow C(f).$$

This paper begins with a generalization of this construction, to each finite dimensional
algebra B, there is an analogue of the association above. Specifically, if B is a finite
dimensional algebra and $\mathcal{C}$ is an A_∞-category, then there is a notion of a B-map
f_B in $\mathcal{C}$ and a B-cone construction $C_B(f_B)$ in $\mathcal{C}$. This definition is natural from the
perspective of the relative definition of the cone construction, which is reviewed in
Section 3.1. In particular, if the algebra B is the type A_2 Dynkin quiver path algebra,
then our construction specializes to the usual cone.

We now describe a particular finite dimensional algebra B which leads to an inter-
esting B-cone construction. To any quiver Q, one may form the path algebra kQ. For
each edge of the quiver Q, adding an edge oriented in the opposite direction produces
a new quiver $\overline{Q}$. The quotient of the path algebra $k\overline{Q}$ by a certain ideal (see Section 4)
is the preprojective algebra ΠQ associated to Q. The main task of our paper is to
study the new cone construction when the algebra B is the preprojective algebra ΠA_2
of the type A_2 Dynkin quiver

$$A_2 \quad \text{is to} \quad f \rightsquigarrow C(f) \quad \text{as} \quad \Pi A_2 \quad \text{is to} \quad \text{Section 4.}$$

This choice is inspired by the important relationship between A_n and ΠA_n in geomet-
ric representation theory. At the end of the paper, we relate these new preprojective
cones to the Fukaya categories of surfaces.

In the remainder of the introduction, we explain these ideas in more detail.
If $f\colon X \to Y$ is a chain map, then the algebraic cone of f is the chain complex

$C(f) = X[1] \oplus Y$ given by the direct sum of X with Y and differential

$$d_{C(f)} = \begin{pmatrix} -d_X & 0 \\ f & d_Y \end{pmatrix}.$$

This familiar gadget is central to many important concepts in mathematics. In their study of differential graded algebras, Bondal and Kapranov formulated the requirement that a map contain a cone in terms of the completeness of the Yoneda embedding [3]. In creating the foundations of A_∞-categories, Kontsevich and Seidel found a certain A_∞-category Δ which corepresents distinguished triangles, in particular, this category characterizes algebraic cones up to homotopy [7, 17]. In this paper, this relative characterization of cones is generalized by replacing the role of Δ by an A_∞-category associated to an arbitrary finite dimensional algebra B.

The rest of the paper investigates the implications of this definition when the algebra B is specialized to be the next most complicated choice after Δ. In more detail, since the category Δ is determined by the representation theory of the A_2-quiver, we study the analogue of the cone when the path algebra kA_2 is replaced by the preprojective algebra ΠA_2. The preprojective algebra ΠQ is the simplest non-trivial replacement for the path algebra kQ. In part, this is because kQ naturally embeds in ΠQ, and as kQ-module, ΠQ decomposes as a direct sum of preprojective kQ-modules. The moduli space of ΠQ finite dimensional representations naturally fibers over the corresponding moduli space of kQ representations and this relationship has important consequences in representation theory. For more details see [16].

In order to understand the preprojective cone, in Theorem 4.7 we use homotopy perturbation theory to compute the minimal model Π of the dg enhancement of the Ext-algebra of indecomposable objects in the category ΠA_2-mod, where

$$\Pi = Ext^*_{\Pi A_2}(I, I), \qquad I = \bigoplus M$$

and the direct sum is over all indecomposable ΠA_2-modules (up to isomorphism). This results in a complete description of the A_∞-structure on a tetrahedron with vertices labeled by indecomposable ΠA_2-modules. This A_∞-structure allows us to study the derived mapping spaces $\mathbb{R}Hom(\Pi, \mathcal{D})$ as $\mathcal{D}$ ranges over other A_∞-categories. Since preprojective distinguished triangles are homotopy classes of such functors $[\tilde{F}] \in Ob(H^0(\mathbb{R}Hom(\Pi, \mathcal{D})))$, we are able to formulate conclusions about them from the minimal model for Π.

In short, a preprojective map is a pair of cycles

$$f \colon A \leftrightarrows B \colon g$$

and the preprojective distinguished triangle is constructed from the algebraic cones $C(f)$ and $C(g)$ together with canonical maps between them

$$(12) \colon C(f) \leftrightarrows C(g) \colon (21).$$

In addition to the ordinary distinguished triangles, which are determined individually by f and g, the preprojective distinguished triangle determines families of Postnikov systems, or iterated cone constructions, of arbitrary length. The sense in which this data is precisely encoded by the A_∞-structure computed in Theorem 4.7 is discussed in Remark 4.14.

Finally, these observations are related to Fukaya categories of surfaces. In Proposition 4.19, we conclude by constructing a strict A_∞-functor from the wrapped Fukaya category of a pair of paints $\mathcal{W}(P)$ to a slight modification of the preprojective category Π.

This paper is part of an ongoing study of the relativization of structures in homological algebra and relationships to the categorification program in low dimensional topology. It has been made into a separate paper because it contains the lengthy computation of an A_∞-structure.

Acknowledgments

2. The language of A_∞-categories

An A_∞-category is a category in which the associativity of composition holds only up to coherent homotopy. The purpose of this section is to explain how the homotopy transfer theorem [**13**] implies that every differential graded category $\mathcal{C}$ uniquely determines an A_∞-category $H^0(\mathcal{C})$ and how homotopy classes of maps between dg categories can be computed from this A_∞-category [**6**].

2.1. A_∞-categories and A_∞-functors

Definition 2.1. An A_∞-*category* $\mathcal{C}$ consists of a collection of objects $Ob(\mathcal{C})$ and a $\mathbb{Z}$-graded k-module of morphisms $Hom(X,Y) = \bigoplus_{i\in\mathbb{Z}} Hom^i(X,Y)$ for each pair of objects $X, Y \in Ob(\mathcal{C})$ together with maps

$$m_d \colon Hom(X_{d-1}, X_d) \otimes \cdots \otimes Hom(X_0, X_1) \to Hom(X_0, X_d)[2-d], \quad d \geqslant 1,$$

which satisfy the relations

$$\sum_{l=0}^{d}\sum_{n=0}^{d-l}(-1)^{\ddagger_n} m_{d-l+1}(f_d, \ldots, f_{n+l+1}, m_l(f_{n+l}, \ldots, f_{n+1}), f_n, \ldots, f_1) = 0, \quad (1)$$

where $\ddagger_n = |f_n| + \cdots + |f_1| - n$ and $d \geqslant 1$.

An A_∞-category $\mathcal{C}$ is said to be *strictly unital* when there is a unique degree zero morphism $1_X \in Hom^0(X, X)$ for each $X \in Ob(\mathcal{C})$ which satisfies

$$\begin{aligned} m_2(f, 1_X) = f, \quad &(-1)^{|g|} m_2(1_X, g) = g, \\ \text{and} \quad m_d(\ldots, 1_X, \ldots) = 0, \quad &\text{when} \quad d \neq 2, \end{aligned} \quad (2)$$

for any maps $f\colon X \to A$ or $g\colon B \to X$ and any object $X \in Ob(\mathcal{C})$.

Example 2.2. Additive k-linear categories and differential graded categories are examples of A_∞-categories in which all of the higher multiplications m_d, for $d > 2$, vanish. Any A_∞-category $\mathcal{C}$ determines a dg category $\tau_{>2}\mathcal{C}$ which is obtained by forgetting the maps m_d for $d > 2$. Non-trivial examples of A_∞-categories appear in Section 4.

In an A_∞-category $\mathcal{C}$, there is a degree 1 map, $m_1\colon Hom(X_0, X_1) \to Hom(X_0, X_1)$, for each pair of objects $X_0, X_1 \in Ob(\mathcal{C})$, which satisfies $m_1 \circ m_1 = 0$; the simplest A_∞-relation above. Taking homology everywhere with respect to these maps produces the homotopy category defined below.

Definition 2.3. The *homotopy category* $H^0(\mathcal{C})$ of an A_∞-category $\mathcal{C}$ is the k-linear category with the same objects as $\mathcal{C}$ and morphisms given by homology classes of maps $[f] \in H^*(Hom(X, Y), m_1)$ for each $X, Y \in Ob(\mathcal{C})$. The composition is defined by

$$[f_2] \circ [f_1] = (-1)^{|f_1|}[m_2(f_2, f_1)].$$

Example 2.4. Suppose that R is ring and M is an R-module. If P is a projective resolution of M

$$P = [\cdots \to P_i \xrightarrow{d_i} P_{i+1} \to \cdots \to P_{-1} \to P_0] \to M \to 0$$

then the endomorphisms $End^*(P)$ of P form a differential graded category with one object.

In more detail, set $End^n(P) = \prod_{i\in\mathbb{Z}} Hom(P_i, P_{i+n})$. If $f = \{f_i\colon P_i \to P_{i+n}\}$ and $g = \{g_i\colon P_i \to P_{i+m}\}$ are maps of degree n and m, respectively, then the composite $g \circ f = \{g_{n+i} \circ f_i\colon P_i \to P_{i+n+m}\}$ is an endomorphism of degree $n + m$. This composition determines a multiplication μ. If $f = \{f_i\colon P_i \to P_{i+n}\}$ then $df = \{(df)_i\colon P_i \to P_{i+n+1}\}$ where

$$(df)_i = d_{i+n} \circ f_i - (-1)^n f_{i+1} d_i\colon P_i \to P_{i+n+1}.$$

The homotopy category $H^0(End^*(P))$ of $End^*(P)$ is the *Ext*-algebra of M

$$H^0(End^*(P)) \cong Ext_R^*(M).$$

Definition 2.5. An A_∞-*functor* $F\colon \mathcal{C} \to \mathcal{D}$ between A_∞-categories consists of a map $F\colon Ob(\mathcal{C}) \to Ob(\mathcal{D})$ and multilinear maps

$$F^d\colon Hom_\mathcal{C}(X_{d-1}, X_d) \otimes \cdots \otimes Hom_\mathcal{C}(X_0, X_1) \to Hom_\mathcal{D}(F(X_0), F(X_d))[1 - d]$$

for $d \geqslant 1$, which satisfy the equations

$$\sum_{r\geqslant 1} \sum_{s_1,\ldots,s_r} m_r^\mathcal{D}(F^{s_r}(f_d, \ldots, f_{d-s_r+1}), \ldots, F^{s_1}(f_{s_1}, \ldots, f_1)) =$$

$$\sum_{l,n}(-1)^{\ddagger_n} F^{d-l+1}(f_d, \ldots, f_{n+l+1}, m_l^\mathcal{C}(f_{n+l}, \ldots, f_{n+1}), f_n, \ldots, f_1).$$

The sign $\ddagger_n$ is as in Definition 2.1 and the first sum is over all partitions: $s_1 + \cdots + s_r = d$. The collection $\{F^d\}$ is also required to behave well with respect to units

$$F^1(1_X) = 1_{F(X)} \qquad \text{and} \qquad F^d(\ldots, 1_X, \ldots) = 0 \text{ for } d \geqslant 2.$$

Any such A_∞-functor $F\colon \mathcal{C} \to \mathcal{D}$ induces a map $H^0(F)\colon H^0(\mathcal{C}) \to H^0(\mathcal{D})$ between the associated homotopy categories. An A_∞-functor is a *quasi-isomorphism* or A_∞-*equivalence* when $H^0(F)$ is an equivalence of categories.

If $F\colon \mathcal{C} \to \mathcal{D}$ is an A_∞-functor then there is a dg functor $\tau_{>2}(F)\colon \tau_{>2}\mathcal{C} \to \tau_{>2}\mathcal{D}$ determined by the map F^1 between the truncations of $\mathcal{C}$ and $\mathcal{D}$, see Example 2.2.

Two A_∞-functors $F\colon \mathcal{C} \to \mathcal{D}$ and $G\colon \mathcal{D} \to \mathcal{E}$ can be composed to produce an A_∞-functor $G \circ F\colon \mathcal{C} \to \mathcal{E}$, the dth component of which is given by the equation

$$(G \circ F)^d(f_d, \ldots, f_1) = \sum_r \sum_{s_1, \ldots, s_r} G^r(F^{s_r}(f_d, \ldots, f_{d-s_r+1}), \ldots, F^{s_1}(f_{s_1}, \ldots, f_1)).$$

Definition 2.6. Let $\mathcal{A}_\infty(\mathcal{C}, \mathcal{D})$ denote the A_∞-category of A_∞-functors from $\mathcal{C}$ to $\mathcal{D}$. If $F, G\colon \mathcal{C} \to \mathcal{D}$ are two objects in this category, then a morphism (*pre-natural transformation*) $T \in Hom^g(F, G)$ of degree g from F to G is a sequence $T = (T^0, T^1, \ldots)$, where

$$T^d\colon Hom_\mathcal{C}(X_{d-1}, X_d) \otimes \cdots \otimes Hom_\mathcal{C}(X_0, X_1) \to Hom_\mathcal{D}(F(X_0), G(X_d))[g - d],$$

for all sequences of objects $(X_0, \ldots, X_d)$ in $\mathcal{C}$. There is an A_∞-structure on the collection of pre-natural transformations. For more details see [**17**, §(1d)].

2.2. Homotopy perturbation theory

Since an A_∞-category is a category in which the composition is associative only up to coherent homotopy, deforming an A_∞-category by a homotopy yields an A_∞-equivalent A_∞-category. The purpose of the homotopy transfer theorem is to make this precise, see [**17**, Prop. 1.12] or [**13**] for detailed arguments and signs.

If $\mathcal{C}$ and $\mathcal{D}$ are dg categories with the same set of objects, then we say that $\mathcal{D}$ is a *perturbation* of $\mathcal{C}$ when there are dg functors $f\colon \mathcal{C} \to \mathcal{D}$ and $g\colon \mathcal{D} \to \mathcal{C}$ so that f and g are identity maps on objects, $1_\mathcal{D} = fg$, and for each pair of objects $x, y \in \mathcal{C}$ there is a homotopy $h_{x,y}\colon Hom_\mathcal{C}(x, y) \to Hom_\mathcal{C}(x, y)$ of degree -1 which satisfies $dh_{x,y} - h_{x,y}d = 1 - gf$ where gf is the map induced by g and f between Hom-spaces.

Theorem 2.7. *Suppose that $\mathcal{C}$ is an A_∞-category, $\tau_{>2}\mathcal{C}$ is the dg category determined by forgetting the higher m_n-maps when $n \geqslant 3$ and $\mathcal{D}$ is a perturbation of $\tau_{>2}\mathcal{C}$ with dg functors*

$$f\colon \tau_{>2}\mathcal{C} \to \mathcal{D} \quad \text{and} \quad g\colon \mathcal{D} \to \tau_{>2}\mathcal{C}.$$

Then there is an A_∞-category $(\mathcal{D}', \{m_n^{\mathcal{D}'}\}_{n>3})$ and there are A_∞-functors

$$f'\colon \mathcal{C} \to \mathcal{D}' \quad \text{and} \quad g'\colon \mathcal{D}' \to \mathcal{C},$$

which determine an A_∞-equivalence $\mathcal{C} \simeq \mathcal{D}'$ and restrict to the initial data: $\tau_{>2}\mathcal{D}' = \mathcal{D}$, $\tau_{>2}(f') = f$ and $\tau_{>2}(g') = g$.

Since $m_1^\mathcal{D} = m_1^{\mathcal{D}'}$ and $m_2^\mathcal{D} = m_2^{\mathcal{D}'}$, one can view $\mathcal{D}'$ as an extension of the dg structure on $\mathcal{D}$.

An important special case of this theorem shows that every A_∞-category $\mathcal{C}$ is A_∞-equivalent to its own homotopy category $H^0(\mathcal{C})$, [**17**, Rmk. 1.13]. Following Example 2.4, when P is a projective resolution of an R-module M, there is an A_∞-structure on the *Ext*-algebra $Ext^*(M)$ making it A_∞-equivalent to the dg algebra $End^*(P)$ (with differential d and multiplication μ)

$$End^*(P) \cong Ext^*(M).$$

For this special case, following [**10**, Theorem 3.2], the equivalence of Theorem 2.7

may be expressed using maps

$$H \; \overset{\frown}{\underset{}{}} \; End^*(P) \underset{i}{\overset{p}{\rightleftarrows}} Ext^*(M), \tag{3}$$

where i and p are morphisms of degree zero and H is a homogeneous map of degree -1 such that
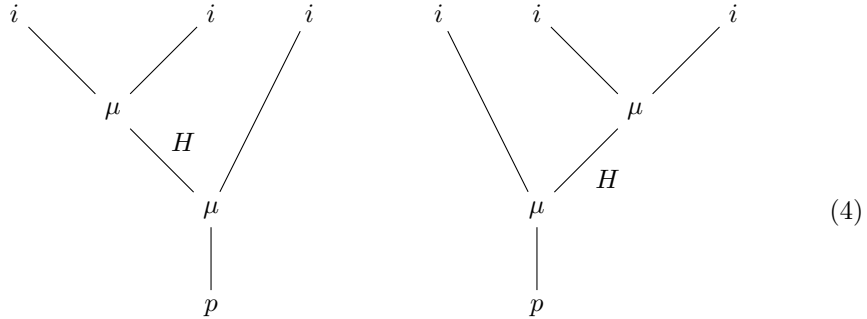
$$pi = 1, \quad 1 - ip = d(H), \quad H^2 = 0.$$

Then the A_∞-structure on $Ext^*(M)$ is given by

$$m_n = \sum_T m_n^T,$$

where T ranges over planar rooted binary trees with n leaves and m_n^T is given by composing the tree-shaped diagram obtained by labeling each leaf by i, each branch point by μ, each internal edge by H and the root by p.

For example, the two trees determining m_3 are pictured below.

$$\tag{4}$$

2.3. Homotopy classes of functors

The category of differential graded categories dgcat_k over k can be given the structure of a model category Hqe. A weak equivalence in this homotopy theory is a dg functor $f \colon \mathcal{C} \to \mathcal{D}$ which satisfies two properties:

1. $H^0(f) \colon H^0(\mathcal{C}) \to H^0(\mathcal{D})$ is essentially surjective.

2. $f_{x,y} \colon Hom_{\mathcal{C}}(x, y) \to Hom_{\mathcal{D}}(f(x), f(y))$ is a quasi-isomorphism for all $x, y \in Ob(\mathcal{C})$.

The homotopy category $Ho(\mathrm{dgcat}_k)$ of dg categories is the category obtained by formally inverting the weak equivalences above. Toën [19] proved that category $Ho(\mathrm{dgcat}_k)$ is closed and monoidal. In particular, there are dg categories $\mathcal{C} \otimes^{\mathbb{L}} \mathcal{D}$ and $\mathbb{R}Hom(\mathcal{D}, \mathcal{E})$ together with natural isomorphisms

$$Hom(\mathcal{C} \otimes^{\mathbb{L}} \mathcal{D}, \mathcal{E}) \xrightarrow{\sim} Hom(\mathcal{C}, \mathbb{R}Hom(\mathcal{D}, \mathcal{E})).$$

Unfortunately, Toën's description of the dg category $\mathbb{R}Hom(\mathcal{D}, \mathcal{E})$ is somewhat complicated. The purpose of this section is to discuss an alternative way to compute the derived space of mappings using A_∞-categories. G. Faonte proved the theorem below [6, Thm. 1.7]; the statement is sometimes attributed to Kontsevich.

Theorem 2.8. *If $\mathcal{D}$ and $\mathcal{E}$ are dg categories over a field of characteristic 0, then the derived mapping category $\mathbb{R}Hom(\mathcal{D}, \mathcal{E})$ is naturally isomorphic to the dg category of A_∞-functors from $\mathcal{D}$ to $\mathcal{E}$*

$$\mathbb{R}Hom(\mathcal{D}, \mathcal{E}) \xrightarrow{\sim} \mathcal{A}_\infty(\mathcal{D}, \mathcal{E}).$$

3.   Algebraic cones

If $f \colon X \to Y$ is a continuous map between topological spaces X and Y then the cone $C(f)$ of f is given by the pushout

$$C(f) = CX \sqcup_f Y, \qquad \text{where} \qquad CX = X \times [0,1]/X \times \{1\}.$$

This topological cone acts as a stand in for the quotient $Y/im(f)$ in the long exact sequence of homology groups associated to the quotient

$$\cdots \to H_n(X) \xrightarrow{f_*} H_n(Y) \to H_n(C(f)) \to H_{n-1}(X) \to \cdots$$

because $H_n(C(f)) \cong H_n(Y/im(f))$ for $n > 0$. After passing from topological spaces to cochain complexes, the cone $C(f)$ of a map $f \colon (X, d_X) \to (Y, d_Y)$ is the cochain complex formed by

$$C(f) = (X[1] \oplus Y, d_{C(f)}), \qquad \text{where} \qquad d_{C(f)} = \begin{pmatrix} -d_X & 0 \\ f & d_Y \end{pmatrix}. \tag{5}$$

In analogy with the relationship between the homology of quotient space and the topological cone, when $f \colon X \to Y$ is a map in an abelian category $\mathcal{A}$, the algebraic cone is isomorphic to the quotient in the derived category $D^b(\mathcal{A})$, $C(f) \cong Y/im(f)$. This is why the existence of objects equivalent to algebraic cones is a principal component of the definition of a triangulated category. For a similar discussion of cones see [**1**, §3.1]. In the remainder of this section we will reformulate the requirement that an algebraic cone exists in a manner which admits generalizations.

When an A_∞-category $\mathcal{C}$ is pretriangulated each cycle $f \in Hom_\mathcal{C}(X, Y)$, has a cone $C(f) \in Ob(\mathcal{C})$. Roughly speaking, a cone is said to exist in $\mathcal{C}$ when there is an object $C(f) \in Ob(\mathcal{C})$ representing the cone of f in the image of the Yoneda embedding. Let us explain precisely what we mean. For any A_∞-category $\mathcal{C}$, the category of modules $\mathcal{C}\text{-mod} = \mathcal{A}_\infty(\mathcal{C}, Ch_k)$ consists of A_∞-functors from $\mathcal{C}$ to the dg category of cochain complexes Ch_k. There is a Yoneda embedding of $\mathcal{C}$ into its associated category of modules $\mathcal{C}\text{-mod}$

$$\mathcal{Y} \colon \mathcal{C} \to \mathcal{C}\text{-mod}, \qquad X \mapsto \mathcal{Y}_X, \qquad \text{where} \qquad \mathcal{Y}_X(Y) = Hom_\mathcal{C}(X, Y).$$

This is an embedding in the sense that the associated functor $H^0(\mathcal{Y}) \colon H^0(\mathcal{C}) \to H^0(\mathcal{C}\text{-mod})$ between homotopy categories is full and faithful.

Definition 3.1. If $f \in Hom_\mathcal{C}(Y_0, Y_1)$ is a cycle, (so $m_1(f) = 0$), then the *cone* $\mathcal{C}(f) \in Ob(\mathcal{C}\text{-mod})$ *of* f is the $\mathcal{C}$-module determined by the assignment

$$\mathcal{C}(f)(X) = Hom(X, Y_0)[1] \oplus Hom(X, Y_1)$$

and structure maps

$$m_d^{\mathcal{C}(f)}((b_0, b_1), a_{d-1}, \ldots, a_1) = (m_d(b_0, a_{d-1}, \ldots, a_1), m_d(b_1, a_{d-1}, \ldots, a_1)$$
$$+ m_{d+1}(f, b_0, a_{d-1}, \ldots, a_1)).$$

Remark 3.2. The definition stems from the observation that for any A_∞-category $\mathcal{C}$, the homotopy category of modules $H^0(\mathcal{C}\text{-mod})$ is triangulated in the sense of Verdier and $\mathcal{C}(f) \cong C(\mathcal{Y}(f))$ where $\mathcal{Y}(f)\colon \mathcal{Y}_{Y_0} \to \mathcal{Y}_{Y_1}$.

Remark 3.3. When $\mathcal{C} = Ch_k$, so $m_d = 0$ for $d > 2$, the A_∞-cone in Definition 3.1 above is equivalent to the image of the algebraic cone in equation (5) under the Yoneda map.

$$\mathcal{C}(f) \cong \mathcal{Y}_{C(f)}.$$

This justifies the next definition.

Definition 3.4. Suppose that $\mathcal{C}$ is an A_∞-category and $f \in Hom_\mathcal{C}(Y_0, Y_1)$ is a cycle, so that $m_1(f) = 0$, then an object $X \in Ob(\mathcal{C})$ *is a cone of f in $\mathcal{C}$* when there is an isomorphism

$$\mathcal{Y}_X \cong \mathcal{C}(f)$$

in the homotopy category $H^0(\mathcal{C}\text{-mod})$ of $\mathcal{C}$-modules. In particular, *f has a cone $C(f)$ in $\mathcal{C}$* when there is such a cone object X in $\mathcal{C}$. Since the Yoneda embedding is full and faithful up to homotopy, any two cones of a single f must be isomorphic in $H^0(\mathcal{C})$.

Use of the Yoneda embedding to characterize the cone construction in dg and A_∞-categories appeared in [**3, 18**].

3.1. A relative perspective on triangles

In what follows we review a different perspective of triangulated categories often attributed to Kontsevich, see [**7**], [**17**, I, (3g)].

If $f \in Hom_\mathcal{C}(X, Y)$ is a cycle of degree $|f| = 0$ and $Z \in Ob(\mathcal{C})$ is the cone on f, as above, then there are cycles $g\colon Y \to Z$ and $h\colon Z \to X$ of degrees $|g| = 0$ and $|h| = 1$ respectively which is summarized by

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1].$$

While any such collection of maps within an A_∞-category $\mathcal{C}$ could be called a triangle, a distinguished triangle must satisfy the additional properties:

1. In the homotopy category, composing any two adjacent maps is zero

$$m_2(g, f) = 0, \quad m_2(h, g) = 0 \quad \text{and} \quad m_2(f, h) = 0. \tag{6}$$

2. The Massey product of three consecutive maps is identity

$$m_3(h, g, f) = 1_X, \quad m_3(f, h, g) = 1_Y \quad \text{and} \quad m_3(g, f, h) = 1_Z. \tag{7}$$

It happens that these conditions suffice to distinguish distinguished triangles. In particular, $Z \simeq C(f)$ when conditions (1) and (2) hold. There is a category Δ which packages the information above in such a way that A_∞-functors $t\colon \Delta \to \mathcal{C}$ from Δ to $\mathcal{C}$ correspond to distinguished triangles in $\mathcal{C}$.

Definition 3.5. There is an A_∞-category Δ which encodes the constraints satisfied by a distinguished triangle. The objects are given by the set $Ob(\Delta) = \{A, B, C\}$

and maps are given by identity maps $1_A, 1_B$ and 1_C together with maps $\alpha\colon A \to B$, $\beta\colon B \to C$ and $\gamma\colon C \to A$ of degrees $|\alpha| = 0$, $|\beta| = 0$ and $|\gamma| = 1$, respectively.
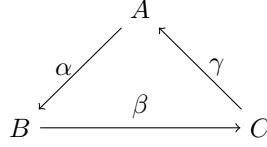
The A_∞-structure is determined by the requirements of strict unitality, as in Definition 2.1, and equations (6) and (7) above. This is the partially wrapped Fukaya category of the disk with three marked points on the boundary, see Definition 4.15 [5, 9, 14].

Theorem 3.6. *A triangle $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$ in $H^0(\mathcal{C})$ is distinguished if and only if there is an A_∞-functor $t\colon \Delta \to \mathcal{C}$ such that*

1. *$t(A) = X$, $t(B) = Y$ and $t(C) = Z$,*
2. *$[t](\alpha) = f$, $[t](\beta) = g$ and $[t](\gamma) = h$,*

where $[t] = H^0(t)\colon H^0(\Delta) \to H^0(\mathcal{C})$.

Informally, the theorem above says that A_∞-functors correspond to distinguished triangles in a given A_∞-category $\mathcal{C}$. This theorem will be used to rephrase the condition that an A_∞-category $\mathcal{C}$ contains a cone object $C(f)$ for every cycle $f \in Hom_{\mathcal{C}}(X, Y)$.

3.2. A_2-representations as a subcategory

Since the third object Z in a distinguished triangle is determined up to isomorphism by its realization as the cone on the map $f\colon X \to Y$ between the other two, the category Δ is Morita equivalent to the subcategory consisting of two objects and a map between them

$$A_2 := [X \xrightarrow{f} Y].$$

This is called the A_2-quiver. A *module* or *representation* M of A_2 consists of two k-vector spaces V and W assigned to the objects X and Y

$$M(X) = V \quad \text{and} \quad M(Y) = W$$

together with a linear map $M(f)\colon V \to W$. In other words, a functor $A_2 \to Vect_k$. A map $q\colon M \to N$ between two representations is a natural transformation. If M and N are two representations then there is a sum $M \oplus N$ determined by the assignments $(M \oplus N)(X) = M(X) \oplus N(X)$, $(M \oplus N)(Y) = M(Y) \oplus N(Y)$ and $(M \oplus N)(f) = M(f) \oplus N(f)$.

The representations of A_2 form the objects of an abelian category A_2-mod containing precisely three indecomposable objects: P, S_1 and S_2 determined by the table of functors:

1. $P(X) = k$, $P(Y) = k$ and $P(f) = 1_k$.
2. $S_1(X) = k$, $S_1(Y) = 0$ and $S_1(f) = 0$.
3. $S_2(X) = 0$, $S_2(Y) = k$ and $S_2(f) = 0$.

These modules form a short exact sequence

$$0 \to S_2 \to P \to S_1 \to 0, \tag{8}$$

which is universal in the sense that the structure of the category A_2-mod is determined by (8) and the axioms of abelian categories. The space $Ext^1(S_1, S_2)$ is spanned by this extension. All of the other Ext-groups vanish because S_2 and P are projective. From equation (8), we see that the chain complex $Q = [S_2 \to P]$ is a projective resolution of S_1. Theorem 2.7 can be used compute the A_∞-structure of the Ext-algebra:

$$End^*(Q \oplus P \oplus S_2) \xrightarrow{\sim} Ext^*(S_1 \oplus P \oplus S_2)$$

as in Example 2.4. The A_∞-structure on the right-hand side of this equation is well-known, see [**12**, App. B. 2], up to sign conventions, it is identical to the category Δ

$$\Delta \cong Ext^*(S_1 \oplus P \oplus S_2).$$

It is in this way that the A_∞-category Δ, the principal datum of a triangulated category and the definition of algebraic cone stem directly from the representation theory of the A_2-quiver.

3.3. Cones as completions

The discussion above leads us to a generalization of the requirement that an A_∞-category have cones corresponding to each cycle.

Definition 3.7. Suppose that $\iota \colon A \subset \bar{A}$ is a pair of A_∞-categories. Then an A_∞-category $\mathcal{C}$ is ι-complete when the pullback functor: $\iota^* \colon \mathbb{R}Hom(\bar{A}, \mathcal{C}) \to \mathbb{R}Hom(A, \mathcal{C})$ is quasi-essentially surjective (i.e. $H^0(\iota^*)$ is essentially surjective).

If $\mathcal{C}$ is ι-complete then, up to homotopy, every functor $F \colon A \to \mathcal{C}$ from A into $\mathcal{C}$ lifts to a functor $\tilde{F} \colon \bar{A} \to \mathcal{C}$ in such a way that the diagram below commutes.

$$
\begin{array}{ccc}
A & \xrightarrow{\ F\ } & \mathcal{C} \\
{\scriptstyle \iota}\big\downarrow & \nearrow & \\
\bar{A} & {\scriptstyle \tilde{F}} &
\end{array}
$$

Example 3.8. Suppose that A is a diagram category consisting of a collection of disjoint points and $\bar{A} = CA$ is the cone category obtained by adding an initial object. Then a category $\mathcal{C}$ containing A-limits will be ι-complete (where $\iota \colon A \to CA$).

In examples of most interest the functor ι^* is a quasi-equivalence; both quasi-essentially surjective and quasi-fully faithful.

The definition below introduces some terminology to clarify the generalization we are making.

Definition 3.9. Suppose $\iota \colon A \to \bar{A}$ is a pair as in Definition 3.7 above. A functor $F \colon A \to \mathcal{C}$ is a *morphism* or *ι-morphism*. A lift $\tilde{F} \colon \bar{A} \to \mathcal{C}$ is a *ι-distinguished triangle* associated F. For any morphism F, the additional information needed to define a lift $\tilde{F}$ is a *cone* or *ι-cone* on F.

Example 3.10. Suppose B is a finite dimensional algebra over k and $\mathcal{S}$ is the collection of finite dimensional simple B-modules $\mathcal{S} \subset B$-mod. Set $S = \oplus_{M \in \mathcal{S}} M$. Then there are A_∞-categories

$$A_B = End^*(S) \quad \text{and} \quad \bar{A}_B = End^*(S \oplus B).$$

In the second term, B is used to denote the algebra B viewed as a left B-module over itself. After applying Theorem 2.7 to both sides, the inclusion $A_B \hookrightarrow \bar{A}_B$ of dg algebras determines an A_∞-functor $\iota\colon A_B \to \bar{A}_B$. In this sense, there is a notion of ι-completeness associated to every finite dimensional algebra B.

When $B = A_2$, Theorem 3.6 combines with the discussion in Section 3.2 to show that an A_∞-category $\mathcal{C}$ contains a cone $C(f)$ for every cycle $f \in \mathcal{C}$ if and only if $\mathcal{C}$ is ι-complete as in the example above.

Since there are many choices of ι, it is important to limit investigation to interesting choices. In the next section we will investigate this condition for the preprojective algebra of the A_2-quiver. For this category, it might make sense to call the ι-distinguished triangles, distinguished pyramids.

4. Preprojective cones

In this section we introduce the preprojective algebras ΠQ and compute the A_∞-category Π associated to the derived endomorphisms of indecomposable modules over ΠA_2. The category Π constitutes our generalization of the category Δ which was seen to describe distinguished triangles in Theorem 3.6.

A *quiver* is a finite directed graph $Q = (Q_0, Q_1)$ consisting of *vertices* Q_0 and *edges* Q_1. Each edge $f \in Q_1$ has a *start* $s(f) \in Q_0$ and a *tail* $t(f) \in Q_0$. For example, A_2 in Section 3.2 is a quiver of the form $A_2 = (\{X, Y\}, \{X \xrightarrow{f} Y\})$, $t(f) = Y$ and $s(f) = X$.

Associated to any quiver Q, is the *path algebra* kQ consisting of k-linear combinations of paths between the vertices in Q. Any two such paths $a, b \in kQ$ multiply by concatenation ab when the vertex at which a ends agrees with the vertex at which b begins; the product is defined to be zero otherwise. The category of left modules over the path algebra kQ is equivalent to the category of functors $Q\text{-mod} = Hom(Q, Vect_k)$ appearing in Section 3.2

$$kQ\text{-mod} \cong Q\text{-mod}.$$

Given a quiver Q, one may form another quiver $\overline{Q}$ which is obtained by adding a formal inverse $f^*\colon t(f) \to s(f)$ to each arrow $f\colon s(f) \to t(f)$. If the set of these arrows is denoted by Q_1^* then $\overline{Q} = (Q_0, Q_1 \cup Q_1^*)$. Let ρ be the element of the path algebra $k\overline{Q}$ given by the sum

$$\rho = \sum_{f \in Q_1} (ff^* - f^*f).$$

The *preprojective algebra* is the quotient of the path algebra of $\overline{Q}$ by the ideal generated by ρ:

$$\Pi Q = k\overline{Q}/(\rho).$$

Remark 4.1. The category ΠQ-mod is a kind of next-simplest most-interesting replacement for the category Q-mod, see [4] or [16, Thm. C].

Example 4.2. When $Q = A_n$ is the graph consisting of n vertices $\{1, \ldots, n\}$ with one directed edge $(i, i+1): i \to i+1$ for $1 \leqslant i < n$. Then the graph $\overline{A}_n$ is formed by adding the inverses $(i, i+1)^* = (i+1, i): i+1 \to i$ pictured below.

$$1 \rightleftarrows \cdots \rightleftarrows i-1 \rightleftarrows i \rightleftarrows i+1 \rightleftarrows \cdots \rightleftarrows n$$
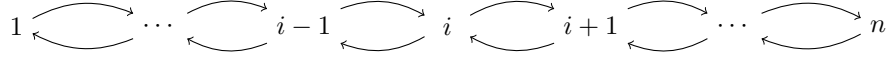
Quotienting the path algebra $k\overline{A}_n$ by the ideal (ρ), described above, implies the relations of the preprojective algebra

$$(i, i-1)(i-1, i) = (i, i+1)(i+1, i) \quad \text{for} \quad i = 2, \ldots, n-1,$$
$$(1, 2)(2, 1) = 0 \quad \text{and} \quad (n, n-1)(n-1, n) = 0.$$

These preprojective algebras are closely related to the algebras studied by Khovanov-Seidel: if $A_n^!$ denotes the Koszul dual of the Khovanov-Seidel algebra then the preprojective algebra is obtained by adding one relation

$$\Pi A_n = A_n^! / \langle (n, n-1)(n-1, n) \rangle,$$

see [11] and [15, §4].

4.1. The algebra ΠA_2

In this section we will discuss the algebra ΠA_2 and its representation theory in more detail.

Definition 4.3. The algebra ΠA_2 may be thought of as a category with two objects $Ob(\Pi A_2) = \{1, 2\}$, each with its own identity map 1_1 or 1_2, and two maps $(12): 1 \to 2$ and $(21): 2 \to 1$ which satisfy two relations:

$$(12)(21) = 0 \quad \text{and} \quad (21)(12) = 0.$$

The quiver underlying this construction is pictured below.

$$1 \underset{(21)}{\overset{(12)}{\rightleftarrows}} 2$$
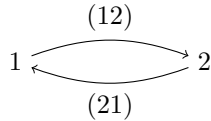
The representation theory of this algebra is well-known. The abelian category ΠA_2-mod of finitely generated representations has four indecomposable modules: S_1, S_2, P_1 and P_2 [8, §8]. The first two modules S_1 and S_2 are 1-dimensional simple modules associated to the vertices 1 and 2,

$$S_1 = k(1) \quad \text{and} \quad S_2 = k(2),$$

where (i) acts as the identity on S_i and all other basis elements of ΠA_2 act trivially. The second two modules are projective modules spanned by the set of paths which begin at their respective vertices

$$P_1 = \Pi A_2(1) \quad \text{and} \quad P_2 = \Pi A_2(2).$$

While there are no maps of degree zero between simple modules, the arrows (12) and (21) in the definition above induce maps $(12): P_1 \to P_2$ and $(21): P_2 \to P_1$ between

projective modules. Each of the two maps, indicated by bold arrows in the diagram below, has a kernel and image, which give the two maps between projectives and simples also pictured in the diagrams below.
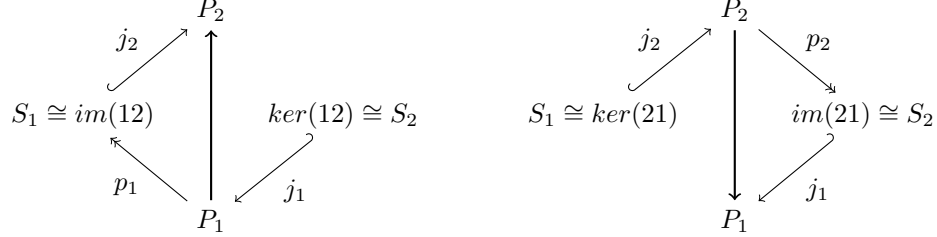
There is an obvious symmetry implicit in the discussion above which we next record.

Proposition 4.4. *There is an involution* $\kappa\colon \Pi A_2\text{-mod} \to \Pi A_2\text{-mod}$ *induced by exchanging the indecomposable modules*

$$P_1 \leftrightarrow P_2 \quad \text{and} \quad S_1 \leftrightarrow S_2.$$

The two short exact sequences

$$0 \to S_1 \xrightarrow{j_2} P_2 \xrightarrow{p_2} S_2 \to 0 \quad \text{and} \quad 0 \to S_2 \xrightarrow{j_1} P_1 \xrightarrow{p_1} S_1 \to 0$$

correspond to two maps $\alpha\colon S_2 \to S_1$ and $\beta\colon S_1 \to S_2$ which span the groups $Ext^1(S_2, S_1)$ and $Ext^1(S_1, S_2)$, respectively.

There are projective resolutions Q_i of simple modules S_i by the projective modules P_1 and P_2

$$Q_i = [\cdots \to P_1 \xrightarrow{(12)} P_2 \xrightarrow{(21)} P_1 \to \cdots \to P_i] \xrightarrow{p_i} S_i \to 0.$$

The structure exhibited among the indecomposables of ΠA_2 is a kind of double of the structure discussed in Section 3.2. In this paper we seek to understand what the ΠA_2 analogue of Δ corepresents: a preprojective analogue of distinguished triangles and algebraic cones. In order to answer this question we construct an A_∞-category analogue Π of Δ.

In more detail, we wish to understand the Ext-algebra of $M = S_1 \oplus P_1 \oplus P_2 \oplus S_2$. After replacing each simple S_i with the projective resolution Q_i, and setting $\tilde{M} = Q_1 \oplus P_1 \oplus P_2 \oplus Q_2$, Theorem 2.7, allows us to compute an A_∞-structure on the Ext-algebra $Ext^*(M)$ so that the dg category $End^*(\tilde{M})$ and $Ext^*(M)$ are A_∞-equivalent.

$$End^*(\tilde{M}) \xrightarrow{\sim} Ext^*(M).$$

The A_∞-category $\Pi = Ext^*(M)$ will serve as our replacement for Δ in what follows.

4.2. The category Π

As an A_∞-category Π has $m_1 = 0$. The generating maps of the category Π are pictured as follows:
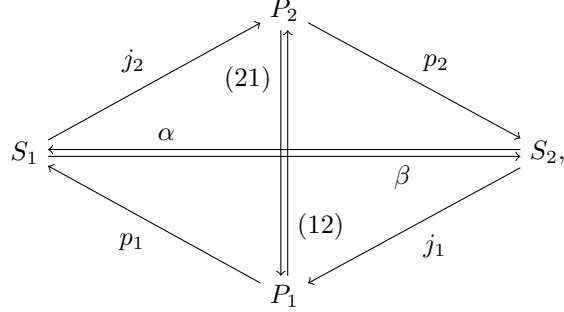
where $|\alpha| = 1$, $|\beta| = 1$ and all other maps have degree 0.

Remark 4.5. All of the tables in this section are written so that the left column is equivalent to κ of the right column; κ is defined in Proposition 4.4.

Let $u_1 := \alpha\beta$ and $u_2 := \beta\alpha$ denote the degree 2 endomorphisms of S_1 and S_2 respectively. Then the composition

$$m_2 \colon Hom(X_1, X_2) \otimes Hom(X_0, X_1) \to Hom(X_0, X_2), \qquad f \otimes g \mapsto f \circ g$$

is determined by the requirements of the identity maps and the table:

$$
\begin{aligned}
m_2(j_1, p_2) \quad &= (21), & m_2(j_2, p_1) \quad &= (12), \\
m_2(u_1^n, u_1^m) \quad &= u_1^{n+m}, & m_2(u_2^n, u_2^m) \quad &= u_2^{n+m}, \\
m_2(\beta u_1^m, u_1^n) \quad &= \beta u_1^{n+m}, & m_2(\alpha u_2^m, u_2^n) \quad &= \alpha u_2^{n+m}, \\
m_2(u_2^m, \beta u_1^n) \quad &= \beta u_1^{n+m}, & m_2(u_1^m, \alpha u_2^n) \quad &= \alpha u_2^{n+m}, \\
m_2(\alpha u_2^m, \beta u_1^n) \quad &= u_1^{n+m+1}, & m_2(\beta u_1^m, \alpha u_2^n) \quad &= u_2^{n+m+1}.
\end{aligned}
$$

In other words, all of the compositions are zero besides those involving identity maps, $j_1 p_2 = (21)$, $j_2 p_1 = (12)$ and the maps α and β which generate a free subalgebra. Together with maps $u_1 = \alpha\beta$ and $u_2 = \beta\alpha$ there are relations

$$u_2\beta = \beta u_1 \qquad \text{and} \qquad u_1\alpha = \alpha u_2.$$

The generating set pictured above determines the basis for each Hom-space:

$$
\begin{aligned}
Hom(P_1, S_1) &= p_1, & Hom(P_2, S_2) &= p_2, \\
Hom(S_1, P_2) &= j_2, & Hom(S_2, P_1) &= j_1, \\
Hom(P_1, P_2) &= (12), & Hom(P_2, P_1) &= (21), \\
Hom(P_1, P_1) &= 1_{P_1}, & Hom(P_2, P_1) &= 1_{P_2}, \\
Hom(S_1, S_1) &= u_1^n, & Hom(S_2, S_2) &= u_2^n, \\
Hom(S_1, S_2) &= \beta u_1^n, & Hom(S_2, S_1) &= \alpha u_2^n,
\end{aligned}
$$

where $n \geqslant 0$.

When $d = 3$, the A_∞-multiplication map

$$m_3 \colon Hom(X_2, X_3) \otimes Hom(X_1, X_2) \otimes Hom(X_0, X_1) \to Hom(X_0, X_3)[-1]$$

is determined by the constraints of strict unitality (see (2)) and the behavior of certain

triangles within the graph above under action of m_2. There are two basic triangles:

$$(A) \quad m_3(\alpha, p_2, j_2) = 1_{S_1}, \qquad\qquad m_3(\beta, p_1, j_1) = 1_{S_2},$$
$$(B) \quad m_3(p_2, j_2, \alpha) = 1_{S_2}, \qquad\qquad m_3(p_1, j_1, \beta) = 1_{S_1}, \qquad (9)$$
$$m_3(j_2, \alpha, p_2) = 1_{P_2}, \qquad\qquad m_3(j_1, \beta, p_1) = 1_{P_1}$$

consisting of rotations of the upper and lower faces of the tetrahedron pictured above.

There is a trick to finding several other non-trivial m_3-products in Π. They are implied by the A_∞-relations and the basic triangles above. Ignoring signs for a moment, the first A_∞-relation, $d = 4$ in equation (1), to incorporate the m_3-operation is written in long form as follows:

$$m_3(m_2(h, f_3), f_2, f_1) + m_3(h, m_2(f_3, f_2), f_1) + m_3(h, f_3, m_2(f_2, f_1))$$
$$+ m_2(m_3(h, f_3, f_2), f_1) + m_2(h, m_3(f_3, f_2, f_1)) = 0. \qquad (10)$$

So when all but the first and last terms in the sum vanish, each face $m_3(f_3, f_2, f_1) = g$ of the tetrahedron above gives rise to a number of other m_3-operations. These can be constructed by using non-trivial m_2-compositions on either the left

$$m_3(m_2(h, f_3), f_2, f_1) = m_2(h, m_3(f_3, f_2, f_1)) = m_2(h, g),$$

or, by symmetry, on the right

$$m_3(f_3, f_2, m_2(f_1, h)) = m_2(m_3(f_3, f_2, f_1), h) = m_2(g, h).$$

Using this trick, each case (A) and (B) above gives the three additional compositions:

$$(A) \quad m_3(\alpha, p_2, (12)) \quad = p_1, \qquad\qquad m_3(\beta, p_1, (21)) \quad = p_2,$$
$$m_3(u_2^n, p_2, j_2) \quad = \beta u_1^{n-1}, \qquad\qquad m_3(u_1^n, p_1, j_1) \quad = \alpha u_2^{n-1},$$
$$m_3(\alpha u_2^n, p_2, j_2) \quad = u_1^n, \qquad\qquad m_3(\beta u_1^n, p_1, j_1) \quad = u_2^n,$$
$$(B) \quad m_3((21), j_2, \alpha) \quad = j_1, \qquad\qquad m_3((12), j_1, \beta) \quad = j_2,$$
$$m_3(p_2, j_2, u_1^n) \quad = \beta u_1^{n-1}, \qquad\qquad m_3(p_1, j_1, u_2^n) \quad = \alpha u_2^{n-1},$$
$$m_3(p_2, j_2, \alpha u_2^n) \quad = u_2^n, \qquad\qquad m_3(p_1, j_1, \beta u_1^n) \quad = u_1^n.$$

When $d = 4$, the A_∞-multiplication map

$$m_4 \colon Hom(X_3, X_4) \otimes Hom(X_2, X_3) \otimes Hom(X_1, X_2) \otimes Hom(X_0, X_1)$$
$$\to Hom(X_0, X_4)[-2]$$

is determined by the constraints of strict unitality (see equation (2)) and the behavior of certain triangles within the graph above under action of m_2. The two basic operations below correspond to the left and right faces of the tetrahedron on the page pictured above:

$$(A) \quad m_4(p_1, (21), j_2, u_1) = 1_{S_1}, \qquad\qquad m_4(p_2, (12), j_1, u_2) = 1_{S_2},$$
$$(B) \quad m_4(u_1, p_1, (21), j_2) = 1_{S_1}, \qquad\qquad m_4(u_2, p_2, (12), j_1) = 1_{S_2},$$
$$m_4(j_2, u_1, p_1, (21)) = 1_{P_2}, \qquad\qquad m_4(j_1, u_2, p_2, (12)) = 1_{P_1}, \qquad (11)$$
$$m_4((21), j_2, u_1, p_1) = 1_{P_1}, \qquad\qquad m_4((12), j_1, u_2, p_2) = 1_{P_2}.$$

As explained for the m_3-operations above, due to the vanishing of some terms in the A_∞-relation for the m_4-operation, we can act with the m_2-operation on either the

left or the right of the basic triangles to obtain a few more m_4-operations. Each case, (A) and (B), determines three more m_4-operations:

(A)
$$m_4(p_1,(21),j_2,u_1^{n+1}) = u_1^n, \qquad m_4(p_2,(12),j_1,u_2^{n+1}) = u_2^n,$$
$$m_4(p_1,(21),j_2,\alpha u_2^{n+1}) = \alpha u_2^n, \qquad m_4(p_2,(12),j_1,\beta u_1^{n+1}) = \beta u_1^n,$$
$$m_4((12),(21),j_2,u_1) = j_2, \qquad m_4((21),(12),j_1,u_2) = j_1,$$

(B)
$$m_4(u_1^{n+1},p_1,(21),j_2) = u_1^n, \qquad m_4(u_2^{n+1},p_2,(12),j_1) = u_2^n,$$
$$m_4(u_2^{n+1}\beta,p_1,(21),j_2) = u_2^n\beta, \qquad m_4(u_1^{n+1}\alpha,p_2,(12),j_1) = u_1^n\alpha,$$
$$m_4(u_1,p_1,(21),(12)) = p_1, \qquad m_4(u_2,p_2,(12),(21)) = p_2.$$

In order to facilitate our description of the rest of the A_∞-structure, we will use the following notation

$$(212) = (21) \otimes (12) \quad \text{and} \quad (121) = (12) \otimes (21).$$

For instance,

$$(212)^n = (21) \otimes (12) \otimes (21) \otimes (12) \otimes \cdots \otimes (21) \otimes (12) \qquad n\text{-times.}$$

The classification of higher homotopies (see (13) later on) shows that the only non-zero higher operations must contain an expression of the form:

$$p_2 \otimes (121)^n \otimes j_2, \qquad p_1 \otimes (212)^n \otimes j_1,$$
$$p_2 \otimes (121)^{n+1}, \qquad p_1 \otimes (212)^{n+1},$$
$$p_2 \otimes (121)^n \otimes (12), \qquad p_1 \otimes (212)^n \otimes (21),$$
$$p_2 \otimes (121)^n \otimes (12) \otimes j_1, \qquad p_1 \otimes (212)^n \otimes (21) \otimes j_2.$$

Remark 4.6. Intuitively speaking, if we view the input of an A_∞-operation $m_n(f_n, \ldots \ldots, f_1)$ as a path $f_1, \ldots, f_n$ in the graph featured at the beginning of Section 4.2 then the higher A_∞-operations that we find can be seen as extensions of the lower order operations discussed above. Each of these extensions is formed by adding a loop of the form (121) or (212) to the path while balancing the grading by adding a loop of the form $u_1 = \alpha\beta$ or $u_2 = \beta\alpha$. The equations above list the ways in which the loops (121) or (212) can be added. See Remark 4.8.

This classification result is accomplished by Theorem 4.7, which is a computation using homotopy perturbation theory (Theorem 2.7). We need to introduce a few more preliminaries before proceeding to the theorem.

Our goal in Theorem 4.7 is to compute all of the compositions of maps corresponding to the decorated binary trees discussed in Section 2.2. The key is to construct homotopies "h-maps" for compositions which are nullhomotopic and study compositions of these homotopies. We now define certain important maps in $End^*(\tilde{M})$. In what follows below, $i \in \mathbb{Z}/2$.

Below is the map $h_{S_iS_i}^{(n)}: S_i \to S_i[-2n]$.

$$
\begin{array}{ccccccccc}
\cdots & \longrightarrow & P_i & \longrightarrow & P_{i+1} & \longrightarrow & P_i & & \\
& & \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle 1} & & \\
\cdots & \longrightarrow & P_i & \longrightarrow & P_{i+1} & \longrightarrow & P_i & \longrightarrow & P_{i+1} & \longrightarrow & \cdots & \longrightarrow & P_{i+1} & \longrightarrow & P_i
\end{array}
$$

Below is the map $h^{(n)}_{S_i S_{i+1}}: S_{i+1} \to S_i[-2n-1]$.

$$\cdots \longrightarrow P_{i+1} \longrightarrow P_i \longrightarrow P_{i+1}$$
$$\downarrow{\scriptstyle 1} \qquad \downarrow{\scriptstyle 1} \qquad \downarrow{\scriptstyle 1}$$
$$\cdots \longrightarrow P_{i+1} \longrightarrow P_i \longrightarrow P_{i+1} \longrightarrow P_i \longrightarrow \cdots \longrightarrow P_{i+1} \longrightarrow P_i$$

Below is the map $h^{(n)}_{S_i P_i}: P_i \to S_i[-2n]$.

$$P_i$$
$$\downarrow{\scriptstyle 1}$$
$$\cdots \longrightarrow P_{i+1} \longrightarrow P_i \longrightarrow P_{i+1} \longrightarrow P_i \longrightarrow \cdots \longrightarrow P_i$$

Below is the map $h^{(n)}_{S_i P_{i+1}}: P_{i+1} \to S_i[-2n-1]$.

$$P_{i+1}$$
$$\downarrow{\scriptstyle 1}$$
$$\cdots \longrightarrow P_i \longrightarrow P_{i+1} \longrightarrow P_i \longrightarrow P_{i+1} \longrightarrow \cdots \longrightarrow P_i$$

Below is the map $h^{(n)}_{P_i S_i}: S_i \to P_i[2n]$.

$$\cdots \longrightarrow P_i \longrightarrow P_{i+1} \longrightarrow P_i \longrightarrow P_{i+1} \longrightarrow \cdots \longrightarrow P_i$$
$$\downarrow{\scriptstyle 1}$$
$$P_{i+1}$$

Below is the map $h^{(n)}_{P_{i+1} S_i}: S_i \to P_{i+1}[2n+1]$.

$$\cdots \longrightarrow P_i \longrightarrow P_{i+1} \longrightarrow P_i \longrightarrow P_{i+1} \longrightarrow \cdots \longrightarrow P_i$$
$$\downarrow{\scriptstyle 1}$$
$$P_i$$

The trees appearing in (4) propagate from leaves to root. The homotopies appear in order determined by distance from the leaves. The initial homotopies H arise as follows:

$$
\begin{aligned}
H(p_1 j_1) &= h^{(0)}_{S_1 S_2}, & H(p_2 j_2) &= h^{(0)}_{S_2 S_1}, \\
H(j_1(\beta u_1^n)) &= h^{(n)}_{P_1 S_1}, & H(j_2(\alpha u_2^n)) &= h^{(n)}_{P_2 S_2}, \\
H(p_1(21)) &= h^{(0)}_{S_1 P_2}, & H(p_2(12)) &= h^{(0)}_{S_2 P_1}, \\
H(j_2 u_1^n) &= h^{(n-1)}_{P_2 S_1}, \quad n > 0, & H(j_1 u_2^n) &= h^{(n-1)}_{P_1 S_2}, \quad n > 0.
\end{aligned}
\tag{12}
$$

"Higher" h-maps arise by applying H to these initial h maps as follows:

$$
\begin{aligned}
H(h_{S_1P_2}^{(n)} \circ j_2) = h_{S_1S_1}^{(n+1)}, \quad n \geqslant 0, \qquad && H(h_{S_2P_1}^{(n)} \circ j_1) = h_{S_2S_2}^{(n+1)}, \quad n \geqslant 0, \\
H(h_{S_1P_2}^{(n)} \circ (12)) = h_{S_1P_1}^{(n+1)}, \quad n \geqslant 0, \qquad && H(h_{S_2P_1}^{(n)} \circ (21)) = h_{S_2P_2}^{(n+1)}, \quad n \geqslant 0, \\
H(h_{S_1P_1}^{(n)} \circ j_1) = h_{S_1S_2}^{(n)}, \quad n \geqslant 1, \qquad && H(h_{S_2P_2}^{(n)} \circ j_2) = h_{S_2S_1}^{(n)}, \quad n \geqslant 1, \\
H(h_{S_1P_1}^{(n)} \circ (21)) = h_{S_1P_2}^{(n)}, \quad n \geqslant 1, \qquad && H(h_{S_2P_2}^{(n)} \circ (12)) = h_{S_2P_1}^{(n)}, \quad n \geqslant 1, \\
H((12) \circ h_{P_1S_1}^{(n)}) = h_{P_2S_1}^{(n-1)}, \quad n \geqslant 1, \qquad && H((21) \circ h_{P_2S_2}^{(n)}) = h_{P_1S_2}^{(n-1)}, \quad n \geqslant 1, \\
H((21) \circ h_{P_2S_1}^{(n)}) = h_{P_1S_1}^{(n)}, \quad n \geqslant 0, \qquad && H((12) \circ h_{P_1S_2}^{(n)}) = h_{P_2S_2}^{(n)}, \quad n \geqslant 0.
\end{aligned}
\tag{13}
$$

Compositions of h's with elements in the Ext-algebra and other h's produce the elements in the Ext-algebra listed below. The map H is zero on any element f representing a cycle in the Ext-algebra:

$$
\begin{aligned}
u_1^n \circ h_{S_1S_2}^{(m)} = u_1^{n-m-1}\alpha, \qquad && u_2^n \circ h_{S_2S_1}^{(m)} = u_2^{n-m-1}\beta, \\
u_2^n \beta \circ h_{S_1S_2}^{(m)} = u_2^{n-m}, \qquad && u_1^n \alpha \circ h_{S_2S_1}^{(m)} = u_1^{n-m}, \\
h_{P_1S_1}^{(0)} \circ p_1 = 1_{P_1}, \qquad && h_{P_2S_2}^{(0)} \circ p_2 = 1_{P_2}, \\
(12) \circ h_{P_1S_1}^{(0)} = j_2, \qquad && (21) \circ h_{P_2S_2}^{(0)} = j_1, \\
u_2^m \beta \circ h_{S_1P_2}^{(n)} = \delta_{m,n} p_2, \qquad && u_1^m \alpha \circ h_{S_2P_1}^{(n)} = \delta_{m,n} p_1, \\
u_1^n \circ h_{S_1S_1}^{(m)} = u_1^{n-m}, \qquad && u_2^n \circ h_{S_2S_2}^{(m)} = u_2^{n-m}, \\
u_1^n \circ h_{S_1P_1}^{(m)} = \delta_{n,m} p_1, \qquad && u_2^n \circ h_{S_2P_2}^{(m)} = \delta_{n,m} p_2, \\
u_2^k \beta \circ h_{S_1S_1}^{(n)} = u_2^{k-n}\beta, \qquad && u_1^k \alpha \circ h_{S_2S_2}^{(n)} = u_1^{k-n}\alpha, \\
h_{P_2S_1}^{(m)} \circ h_{S_1P_2}^{(n)} = \delta_{m,n} 1_{P_2}, \qquad && h_{P_1S_2}^{(m)} \circ h_{S_2P_1}^{(n)} = \delta_{m,n} 1_{P_1}, \\
h_{P_1S_1}^{(m)} \circ h_{S_1P_1}^{(n)} = \delta_{m,n} 1_{P_1}, \qquad && h_{P_2S_2}^{(m)} \circ h_{S_2P_2}^{(n)} = \delta_{m,n} 1_{P_2},
\end{aligned}
\tag{14}
$$

where $\delta_{m,n}$ is the Kronecker delta. We also have the following list of "partial" terminating operations. Let $\gamma_1 = \alpha$, $\gamma_2 = \beta$ and let $i \in \mathbb{Z}/2$. These compositions are not cycles, so the map H is defined to be zero on them.

Below is the map $h_{S_iS_i}^{(n)} \circ u_i^k$.

$$
\begin{array}{ccccccc}
\cdots \longrightarrow & P_i & \longrightarrow & P_{i+1} & \longrightarrow & P_i & \longrightarrow \cdots \longrightarrow P_i \\
& \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle 1} & \\
\cdots \longrightarrow & P_i & \longrightarrow & P_{i+1} & \longrightarrow & P_i & \longrightarrow P_{i+1} \longrightarrow \cdots \longrightarrow P_{i+1} \longrightarrow P_i
\end{array}
$$

Below is the map $h_{S_iS_i}^{(n)} \circ u_i^k \gamma_i$.

$$
\begin{array}{ccccccc}
\cdots \longrightarrow & P_i & \longrightarrow & P_{i+1} & \longrightarrow & P_i & \longrightarrow \cdots \longrightarrow P_{i+1} \\
& \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle 1} & \\
\cdots \longrightarrow & P_i & \longrightarrow & P_{i+1} & \longrightarrow & P_i & \longrightarrow P_{i+1} \longrightarrow \cdots \longrightarrow P_{i+1} \longrightarrow P_i
\end{array}
$$

Below is the map $h_{S_iS_{i+1}}^{(n)} \circ u_{i+1}^k$.

$$
\begin{array}{ccccccc}
\cdots \longrightarrow & P_{i+1} & \longrightarrow & P_i & \longrightarrow & P_{i+1} & \longrightarrow \cdots \longrightarrow P_{i+1} \\
& \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle 1} & \\
\cdots \longrightarrow & P_{i+1} & \longrightarrow & P_i & \longrightarrow & P_{i+1} & \longrightarrow P_i \longrightarrow \cdots \longrightarrow P_{i+1} \longrightarrow P_i
\end{array}
$$

Below is the map $h^{(n)}_{S_i S_{i+1}} \circ u^k_{i+1} \gamma_{i+1}$.

$$
\begin{array}{ccccccccccc}
\cdots & \longrightarrow & P_{i+1} & \longrightarrow & P_i & \longrightarrow & P_{i+1} & \longrightarrow & \cdots & \longrightarrow & P_i \\
& & \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle 1} & & & & \\
\cdots & \longrightarrow & P_{i+1} & \longrightarrow & P_i & \longrightarrow & P_{i+1} & \longrightarrow P_i \longrightarrow \cdots \longrightarrow P_i & \longrightarrow & P_i
\end{array}
$$

Below is the map $h^{(n)}_{S_i P_{i+1}} \circ h^{(k)}_{P_{i+1} S_i}$.

$$
\begin{array}{ccccccccc}
\cdots & \longrightarrow & P_{i+1} & \longrightarrow & P_i & \longrightarrow & P_{i+1} & \longrightarrow \cdots \longrightarrow & P_i \\
& & & & & & \downarrow{\scriptstyle 1} & & \\
\cdots & \longrightarrow & P_{i+1} & \longrightarrow & P_i & \longrightarrow & P_{i+1} & \longrightarrow P_i \longrightarrow \cdots \longrightarrow P_{i+1} \longrightarrow & P_i
\end{array}
$$

Below is the map $h^{(n)}_{S_i P_i} \circ h^{(k)}_{P_i S_i}$.

$$
\begin{array}{ccccccccc}
\cdots & \longrightarrow & P_i & \longrightarrow & P_{i+1} & \longrightarrow & P_i & \longrightarrow \cdots \longrightarrow & P_i \\
& & & & & & \downarrow{\scriptstyle 1} & & \\
\cdots & \longrightarrow & P_i & \longrightarrow & P_{i+1} & \longrightarrow & P_i & \longrightarrow P_{i+1} \longrightarrow \cdots \longrightarrow P_{i+1} \longrightarrow & P_i
\end{array}
$$

Below is the map $p_i \circ h^{(k)}_{P_i S_i}$.

$$
\begin{array}{ccccccccc}
\cdots & \longrightarrow & P_i & \longrightarrow & P_{i+1} & \longrightarrow & P_i & \longrightarrow \cdots \longrightarrow & P_i \\
& & & & & & \downarrow{\scriptstyle 1} & & \\
\cdots & \longrightarrow & P_i & \longrightarrow & P_{i+1} & \longrightarrow & P_i & &
\end{array}
$$

Below is the map $h^{(n)}_{S_{i+1} P_{i+1}} \circ h^{(k)}_{P_{i+1} S_i}$.

$$
\begin{array}{ccccccccc}
\cdots & \longrightarrow & P_{i+1} & \longrightarrow & P_i & \longrightarrow & P_{i+1} & \longrightarrow \cdots \longrightarrow & P_i \\
& & & & & & \downarrow{\scriptstyle 1} & & \\
\cdots & \longrightarrow & P_{i+1} & \longrightarrow & P_i & \longrightarrow & P_{i+1} & \longrightarrow P_i \longrightarrow \cdots \longrightarrow P_i \longrightarrow & P_{i+1}
\end{array}
$$

Below is the map $p_{i+1} \circ h^{(k)}_{P_{i+1} S_i}$.

$$
\begin{array}{ccccccccc}
\cdots & \longrightarrow & P_{i+1} & \longrightarrow & P_i & \longrightarrow & P_{i+1} & \longrightarrow \cdots \longrightarrow & P_i \\
& & & & & & \downarrow{\scriptstyle 1} & & \\
\cdots & \longrightarrow & P_{i+1} & \longrightarrow & P_i & \longrightarrow & P_{i+1} & &
\end{array}
$$

Below is the map $h^{(n)}_{S_i P_{i+1}} \circ h^{(k)}_{P_{i+1} S_{i+1}}$.

$$
\begin{array}{ccccccccc}
\cdots & \longrightarrow & P_{i+1} & \longrightarrow & P_i & \longrightarrow & P_{i+1} & \longrightarrow \cdots \longrightarrow & P_{i+1} \\
& & & & & & \downarrow{\scriptstyle 1} & & \\
\cdots & \longrightarrow & P_{i+1} & \longrightarrow & P_i & \longrightarrow & P_{i+1} & \longrightarrow P_{i+1} \longrightarrow \cdots \longrightarrow P_{i+1} \longrightarrow & P_i
\end{array}
$$

Theorem 4.7. *All of the non-trivial operations for* Π *are given by the list below once combined with the corresponding list obtained by applying the automorphism* κ *(Proposition 4.4) which exchanges the nodes* 1 *and* 2 *in the underlying quiver.*

This list contains the m_2 *operations:*

$$m_2(j_1, p_2) = (21),$$
$$m_2(u_1^n, u_1^m) = u_1^{n+m},$$
$$m_2(\beta u_1^m, u_1^n) = \beta u_1^{n+m},$$
$$m_2(u_2^m, \beta u_1^n) = \beta u_1^{n+m},$$
$$m_2(\alpha u_2^m, \beta u_1^n) = u_1^{n+m+1}.$$

This list contains the higher operations:

$$m_{2n+2k+5}((12), (212)^k, j_1, \beta u_1^{n+k+1}, p_1, (212)^n, (21)) = 1_{P_2},$$
$$m_{2n+3}((12), (212)^n, j_1, \beta u_1^n) = j_2,$$
$$m_{2n+3}((212)^k, j_1, \beta u_1^n, p_1, (212)^{n-k}) = 1_{P_1},$$
$$m_{2n+3}(u_1^k, p_1, (212)^n, j_1) = u_1^{k-n-1}\alpha,$$
$$m_{2n+3}(u_2^k\beta, p_1, (212)^n, j_1) = u_2^{k-n},$$
$$m_{2n+4}(u_1^k, p_1, (212)^n, (21), j_2) = u_1^{k-n-1},$$
$$m_{2n+4}(p_1, (212)^n, (21), j_2, u_1^k) = u_1^{k-n-1},$$
$$m_{2n+4}(u_2^k\beta, p_1, (212)^n, (21), j_2) = u_2^{k-n-1}\beta,$$
$$m_{2n+3}(u_2^n\beta, p_1, (212)^n, (21)) = p_2,$$
$$m_{2n+2}(u_1^n, p_1, (212)^n) = p_1,$$
$$m_{2k+2n+4}((121)^k, j_2, u_1^{n+k+1}, p_1, (212)^n, (21)) = 1_{P_2},$$
$$m_{2n+4}((121)^{n+1}, j_2, u_1^{n+1}) = j_2,$$
$$m_{2n+4}((212)^k, (21), j_2, u_1^{n+1}, p_1, (212)^{n-k}) = 1_{P_1},$$
$$m_{2m+3}(p_2, (121)^m, j_2, \alpha u_2^n) = u_2^{n-m},$$
$$m_{2m+3}(p_2, (121)^m, j_2, u_1^n) = \beta u_1^{n-(m+1)},$$
$$m_{2m+4}(p_1, (21), (121)^m, j_2, \alpha u_2^{n+1}) = \alpha u_2^{n-m}.$$

Proof. The A_∞-structure on Π is determined by the dg structure on $End^*(\tilde{M})$ and Theorem 2.7. The dg category $End^*(\tilde{M})$ only has only non-trivial first and second multiplications (the derivation and the natural algebra multiplication). See Example 2.4 for more details.

Since all of the higher multiplications in $End^*(\tilde{M})$ are trivial, in order to compute m_n we must determine all possible binary trees with n input edges satisfying certain properties. Let $f_n, \ldots, f_1 \in \Pi$ such that the composition $f_{i+1} \circ f_i$ makes sense for $i = 1, \ldots, n-1$. The input edges (read from left to right) are labeled $f_n, \ldots, f_1$. First one includes each $f_i \in \Pi$ into $End^*(\tilde{M})$. The internal edges are labeled by H.

From a calculation we see that $H(f_{i+1} \circ f_i)$ for $f_{i+1}, f_i \in \Pi$ is non-zero only in the cases listed in (12).

From a calculation we see that higher homotopy maps are produced only when H is applied to a product of the form hf or fh where h is some higher homotopy map and f is an element in $\Pi = Ext^*(M)$. The possibilities are listed in (13). In particular, the only non-zero way to grow a binary tree labeled as in Section 2.2 is illustrated in the remark below.

Finally, to produce an element in Π, one must apply the projection map p described in (3) to certain products of two elements of $End^*(\tilde{M})$ listed in (14).

Any other composition that we need to consider produces a non-cycle and so H of it is set to zero. These cases are enumerated above as partial terminating operations. $\square$

Remark 4.8. The A_∞-maps m_2 are determined by the composition in the *Ext*-algebra. The maps m_3 and m_4 can be done by hand. The only way to inductively evolve a binary tree to give non-zero higher operation is illustrated below, see Remark 4.6.
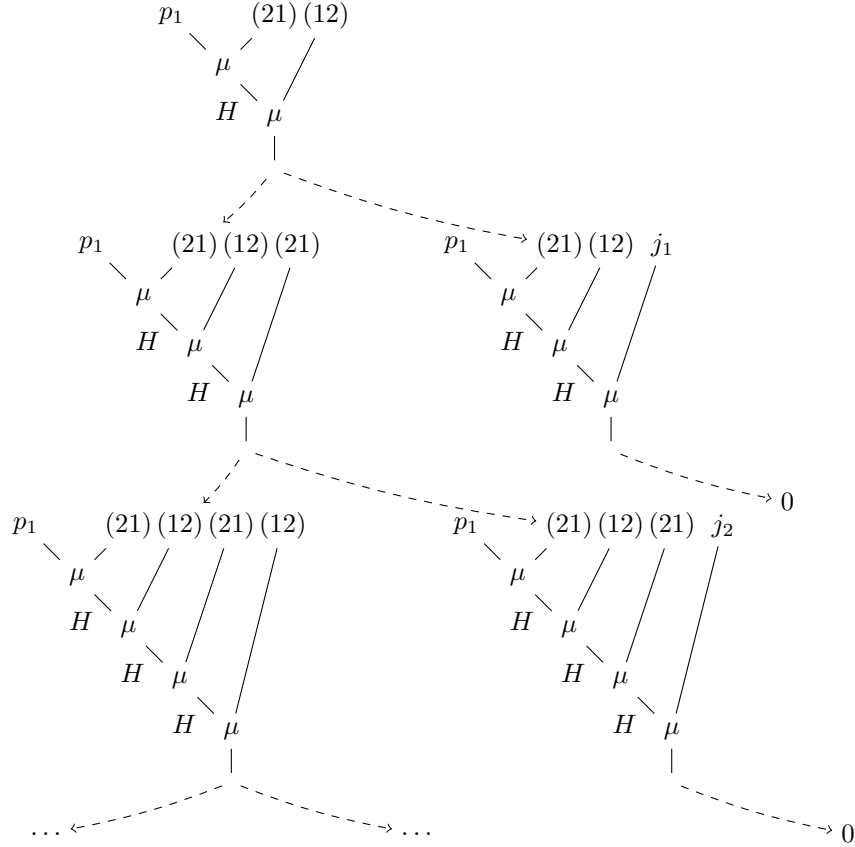
4.3. The preprojective cone

Recall from Definition 3.7 that our abstract cones are determined by a certain lifting problem. This section combines all of the bits and pieces from previous sections and provides some explanation as to what we have computed.

Before proceeding, it is useful to observe the following remark.

Remark 4.9. If $F\colon \mathcal{C} \to \mathcal{D}$ is an A_∞-functor such that $F^d = 0$ for $d \geqslant 2$, then the A_∞-relation in Definition 2.5 becomes

$$m_d^{\mathcal{D}}(F^1(f_d), \ldots, F^1(f_1)) = F^1(m_d^{\mathcal{C}}(f_d, \ldots, f_1)). \tag{15}$$

The proposition below is a detailed version of the comments at the end of Example 3.10 in the preprojective setting.

Proposition 4.10. *The subcategory π associated to the two simple modules $S_1, S_2 \in \Pi A_2$ -mod*

$$S_1 \overset{\alpha}{\underset{\beta}{\rightleftarrows}} S_2$$

is formal; the higher A_∞-structure $m_d^\pi = 0$ for $d > 2$. The inclusion $\iota \colon \pi \hookrightarrow \Pi$ is an A_∞-functor $\iota = \{\iota^d\}$ which is determined by the assignments: $\iota(S_i) := S_i$ on objects, $\iota^1(\alpha) := \alpha$, $\iota^1(\beta) := \beta$ on maps and $\iota^d := 0$ for $d \geqslant 2$.

Proof. The category π is formal by Theorem 4.7 since there are no non-trivial higher operations in the list involving only entries of the form α and β. Since π is formal, we need only check that ι satisfies equation (15). This again follows from the observation that there are no relations among α and β in Π and there are no higher A_∞-relations, $m_d^\Pi|_\pi \equiv 0$ for $d > 2$, by Theorem 4.7. $\square$

In the notation introduced by the proposition, the lifting problem in Definition 3.7 can be restated by the commutative diagram

$$
\begin{array}{ccc}
\pi & \xrightarrow{\ F\ } & \mathcal{D}. \\
{\scriptstyle \iota}\Big\uparrow\Big\downarrow & \nearrow\!\!\!{\scriptstyle \tilde{F}} & \\
\Pi & &
\end{array}
$$

So the initial data is an A_∞-functor $F\colon \pi \to \mathcal{D}$ and a cone on F is determined by a lift along ι, i.e. an A_∞-functor $\tilde{F}\colon \Pi \to \mathcal{D}$ for which $\tilde{F} \circ \iota \simeq F$ in the category $\mathcal{A}_\infty(\pi, \mathcal{D})$.

The theorem below shows that the upper and lower parts of the Π-diagram at the beginning of Section 4.2 are triangles in the sense of Theorem 3.6. It follows that, up to homotopy, the portions of the category in the completion, $\tilde{F}(P_1)$ and $\tilde{F}(P_2)$, are classical cones on the maps $F^1(\beta)$ and $F^1(\alpha)$, respectively.

Theorem 4.11. *If $F\colon \Pi \to \mathcal{D}$ is an A_∞-functor from the preprojective category to an A_∞-category $\mathcal{D}$ then the objects associated to P_i in $\mathcal{D}$ are homotopy equivalent to cones on the maps α and β. More precisely,*

$$F(P_1) \simeq C(F^1(\beta)) \qquad \text{and} \qquad F(P_2) \simeq C(F^1(\alpha)). \tag{16}$$

Proof. The category Δ is pictured in Definition 3.5. There are three objects A,B and C together with maps $\alpha\colon A \to B$, $\beta\colon B \to C$ and $\gamma\colon C \to A$ with degrees $|\alpha| = 0$, $|\beta| = 0$ and $|\gamma| = 1$.

For $i = 1, 2$, there are assignments $\iota_i \colon Ob(\Delta) \hookrightarrow Ob(\Pi)$ and $\iota_i^1 \colon Hom_\Delta(X_1, X_0) \to Hom_\Pi(\iota_i(X_1), \iota_i(X_0))$ given by

	A	B	C
ι_1	S_2	P_1	S_1
ι_2	S_1	P_2	S_2

and

	α	β	γ
ι_1^1	j_1	p_1	β
ι_2^1	j_2	p_2	α

Set $\iota_i^d := 0$ for $d \geqslant 2$. In order to see that the assignments ι_i are A_∞-functors, we must check that the A_∞-relation equation (15) above is satisfied. The cases ι_1 and ι_2

are symmetric. So consider ι_1, then equation (15) holds because the only non-identity compositions are zero

$$j_1 \circ \beta = 0 \qquad p_1 \circ j_1 = 0 \qquad \beta \circ p_1 = 0$$

and the only higher A_∞-operation supported by the morphisms $\{p_1, j_1, \beta\}$ in Π are given by the m_3 in equation (9); these agree with equation (7).

Thus the restrictions formed by the compositions $F \circ \iota_i \colon \Delta \to \mathcal{D}$ are A_∞-functors. By Theorem 3.6, each of the two restrictions determines a triangle in $\mathcal{D}$. In particular, the image of each vertex of $\Delta \subset \Pi$ must be homotopic to the cone on the morphism in subcategory complementary to the vertex. A special case of this is equation (16). $\square$

Example 4.12. Suppose we have an A_∞-functor $F \colon \pi \to \mathcal{D}$. The structure maps $F \colon Ob(\pi) \to Ob(\mathcal{D})$ and $F^1_{X,Y} \colon Hom_\pi(X,Y) \to Hom_\mathcal{D}(F(X), F(Y))$ determine two cycles

$$S_1' \underset{\beta'}{\overset{\alpha'}{\rightleftarrows}} S_2',$$

where $S_1' = F(S_1)$, $S_2' = F(S_2)$, $\alpha' = F^1(\alpha)$ and $\beta' = F^1(\beta)$. If $\mathcal{D}$ is triangulated then there are objects $C(\alpha')$ and $C(\beta')$ in $\mathcal{D}$ which are cones in the sense of Definition 3.4. By the theorem above, a lift $\tilde{F} \colon \Pi \to \mathcal{D}$ of F along ι must associate to P_1 and P_2 objects which are homotopy equivalent to $C(\beta')$ and $C(\alpha')$ respectively. When $\mathcal{D}$ is a dg category of complexes, this can be made very explicit. The diagram for Π at the beginning of Section 4.2 becomes

$$(S_1' \oplus S_2[1]', d_{C(\alpha')})$$

$$(1,0)^t \qquad\qquad (0,1)$$

$$S_1' \underset{\beta'}{\overset{\alpha'}{\rightleftarrows}} S_2'.$$

$$(1,0) \qquad\qquad (0,1)^t$$

$$(S_1[1]' \oplus S_2', d_{C(\beta')})$$

Since $(21) = j_1 \circ p_2$ and $(12) = j_2 \circ p_1$, the matrices associated to these maps are

$$(21) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad (12) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

and we see directly the relations $(12)(21) = 0$ and $(21)(12) = 0$.

4.4.   Coda

The remainder of this paper establishes some context which pertains to future work.

Remark 4.13. Just as Proposition 4.10 shows that the subcategory π determined by simples embeds in Π, the quiver presentation for ΠA_2 in Definition 4.3 embeds in Π

by the Yoneda embedding. So for Π, there are two notions of map which yield the same notion of a ι-distinguished triangle. One comes from the α and β maps between simple modules, and another comes from the (21) and (12) maps between projective modules. These two are dual in the sense that some of the data arising from a ι-distinguished triangle of one ι-cone construction is the same as the initial data for the other. Since exchanging simples and projectives results only in a rotation of the triangle in the construction of the A_2-cone, this shows that the preprojective cones exhibit some new behavior.

Remark 4.14. The simplicity of Theorem 4.11 seems incongruous with the complexity of the A_∞-structure found in Theorem 4.7. The A_∞-structure is complicated in part because of the trick in equation (10) and in part because it is recording higher Postnikov systems among compositions of α and β maps. We will give an informal explanation of the first non-trivial example.

Recall from Section 3.1 that functors $\Delta \to \mathcal{D}$ correspond to distinguished triangles in $\mathcal{D}$ and that Δ agrees with the partially wrapped Fukaya category $\mathcal{F}_3$ of the disk with three marked points. There is an extension of these statements.

Definition 4.15. The partially wrapped Fukaya category $\mathcal{F}_n$ of the disk D^2 with n marked points along the boundary is the A_∞-category with n-objects $Ob(\mathcal{F}_n) = \{X_i\}_{i \in \mathbb{Z}/n}$ and maps $1_{X_i} \colon X_i \to X_i$ and $f_i \colon X_i \to X_{i+1}$. The gradings are chosen to satisfy the constraint $\sum_i |f_i| = n - 2$. The only non-trivial A_∞-operations are compositions with identity and cyclic permutations of

$$m_n(f_n, \ldots, f_1) = 1_{X_1}.$$

See [**9**, above Eqn. (3.21)], [**14**, **5**] or other references [**17**, I (3g) Rmk. 3.11]. For $n \geqslant 3$, functors from the partially wrapped Fukaya category $\mathcal{F}_n \to \mathcal{D}$ correspond to n-fold extensions among objects in $\mathcal{D}$. Since there is a Morita equivalence

$$\mathcal{F}_n \sqcup_{i,j} \mathcal{F}_m \xrightarrow{\sim} \mathcal{F}_{n+m-2}$$

associated to the gluing of the ith boundary object in (D^2, n) to the jth boundary object in (D^2, m) [**9**, §3.6], functors $\mathcal{F}_n \to \mathcal{D}$ correspond to n-fold extensions among objects in $\mathcal{D}$ since every disk with n marked points can be subdivided into a gluing of disks with 3 marked points. On the other hand, precisely the same logic as was used in Theorem 4.11 to establish the existence of functors (for $i = 1, 2$) $\iota_i \colon \Delta \to \Pi$ for the triangles formed by the maps $\{j_1, p_1, \beta\}$ and $\{j_2, p_2, \alpha\}$ corresponding to A_∞-operations in equation (9) can be used to establish the existence of A_∞-functors $\kappa_i \colon \mathcal{F}_4 \to \Pi$ for the quadrilaterals corresponding to the maps $\{p_1, (21), j_2, u_1\}$ and $\{p_2, (12), j_1, u_2\}$ appearing in equation (11). So in addition to determining two distinguished triangles:

$$\cdots \to S_1' \xrightarrow{\beta'} S_2' \to C(\beta') \to S_1'[1] \to \cdots \qquad \text{and}$$

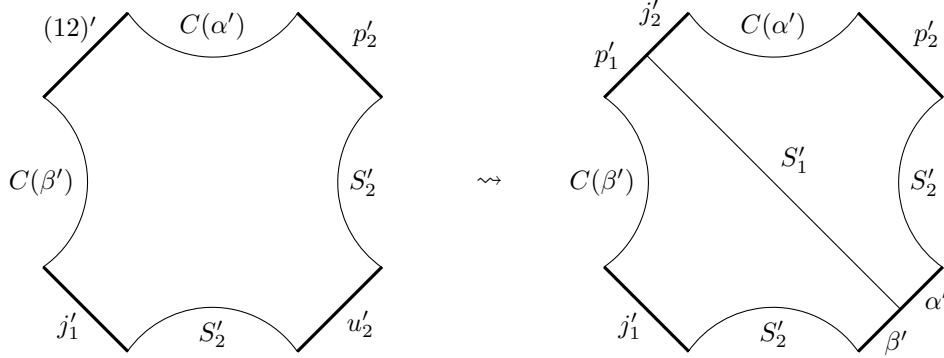$$\cdots \to S_2' \xrightarrow{\alpha'} S_1' \to C(\alpha') \to S_2'[1] \to \cdots$$

as in Theorem 4.11 earlier, an A_∞-functor $\tilde{F} \colon \Pi \to \mathcal{D}$ corresponding to a cone on $F \colon \pi \to \mathcal{D}$ determines 4-fold Postnikov systems via restrictions $\tilde{F}\kappa_i \colon \mathcal{F}_4 \to \mathcal{D}$. For

the maps $\{p_2, (12), j_1, u_2\}$ this corresponds to

$$\cdots \to C(\beta') \xrightarrow{(12)'} C(\alpha') \xrightarrow{p_2'} S_2' \xrightarrow{u_2'} S_2'[2] \to \cdots,$$

where $(12)' = \tilde{F}^1(12)$, $p_2' = \tilde{F}^1(p_2)$ and $u_2' = \tilde{F}^1(u_2)$. Topologically the quadrilateral formed by the gluing of the two triangles below along the object S_1'.

We conclude with a connection between the A_∞-category Π and an important category coming from symplectic topology. The wrapped Fukaya category $D^\pi \mathcal{W}(P)$ of the pair of pants $P = S^2 \backslash \{D_1^2, D_2^2, D_3^2\}$ can be generated by three Lagrangians X_0, X_1 and X_2. The A_∞-subcategory A determined by these objects was studied in relation to the homological mirror symmetry conjecture [**2**]. Proposition 4.19 constructs a functor from this category to a version of the preprojective category Π'.

Definition 4.16 ([**2**])**.** The generating category A has objects $Ob(A) = \{X_0, X_1, X_2\}$ and morphisms

$$Hom_A(X_i, X_j) = \begin{cases} k[x_i, y_i]/(x_i y_i) & i = j, \\ k[x_{i+1}]u_{i,i+1} = u_{i,i+1}k[y_i] & j = i+1, \\ k[y_{i-1}]v_{i,i-1} = v_{i,i-1}k[x_i] & j = i-1, \\ 0 & \text{otherwise.} \end{cases}$$

These maps compose according to

$$(x_i^k u_{i-1,i}) \circ (v_{i,i-1} x_i^l) = x_i^{k+l+1} \qquad \text{and} \qquad (v_{i,i-1} x_i^l) \circ (x_i^k u_{i-1,i}) = y_{i-1}^{k+l+1} \qquad (17)$$

and are graded by setting $|u_{0,1}| = 1$, $|v_{1,0}| = 1$ and $|u_{i,i+1}| = 0$, $|v_{i,i-1}| = 0$ in all other cases. This is pictured on the left-hand side of (19). The higher A_∞-operations are determined by

$$\begin{array}{ll} m_3(u_{2,0}, u_{1,2}, u_{0,1}) = 1_{X_0}, & m_3(v_{1,0}, v_{2,1}, v_{0,2}) = 1_{X_0}, \\ m_3(u_{0,1}, u_{2,0}, u_{1,2}) = 1_{X_1}, & m_3(v_{2,1}, v_{0,2}, v_{1,0}) = 1_{X_1}, \\ m_3(u_{1,2}, u_{0,1}, u_{2,0}) = 1_{X_2}, & m_3(v_{0,2}, v_{1,0}, v_{2,1}) = 1_{X_2}. \end{array} \qquad (18)$$

In order to make Π look like A in Definition 4.16, we need to combine the two projective objects P_1 and P_2 into one by forming the direct sum $P := P_1 \oplus P_2$. The following definition recalls how direct sums are formed. This is a special case of additivization [**17**, I (3k)].

Definition 4.17. If $X, Y \in Ob(\mathcal{A})$ are objects in an A_∞-category $\mathcal{A}$ then the direct sum $X \oplus Y \in D^\pi(\mathcal{A})$, [**17**, I (4b)], satisfies

$$Hom(X \oplus Y, Z) = Hom(X, Z) \oplus Hom(Y, Z),$$
$$Hom(W, X \oplus Y) = Hom(W, X) \oplus Hom(W, Y),$$

and the A_∞-operations extend additively

$$m_k(a_k, \ldots, (a_i, a_i'), \ldots, a_1) = m_k(a_k, \ldots, a_i, \ldots, a_1) + m_k(a_k, \ldots, a_i', \ldots, a_1).$$

This allows one to define an A_∞-category $Mat(\mathcal{A})$ of direct sums of objects of $\mathcal{A}$. The new category Π' is formed by combining the two projectives.

Definition 4.18. The category Π' is the full A_∞-subcategory of $Mat(\Pi)$ formed by the objects $\{S_1, S_2, P\}$ where $P := P_1 \oplus P_2$. The A_∞-structure of Π' is determined by Theorem 4.7 and Definition 4.17 above. The objects and morphisms in Π' are pictured on the right-hand side of (19).

$$(19)$$

Proposition 4.19. *There is a canonical A_∞-functor $G\colon A \to \Pi'$.*

Proof. The A_∞-functor $G = \{G^d\}$ is defined by mapping the left-hand side of the diagram above to the right-hand side of the diagram. In more detail, define $G\colon Ob(A) \to Ob(\Pi')$ by setting $G(X_0) := S_1$, $G(X_1) := S_2$ and $G(X_2) := P$. The map $G^1\colon Hom_A(X_i, X_j) \to Hom_{\Pi'}(G(X_i), G(X_j))$ is determined by setting

$$
\begin{array}{ll}
G^1(u_{0,1}) := \beta, & G^1(v_{1,0}) := \alpha, \\
G^1(u_{1,2}) := j_1, & G^1(v_{2,1}) := p_2, \\
G^1(u_{2,0}) := p_1, & G^1(v_{0,2}) := j_2,
\end{array}
\tag{20}
$$

and the observation that equation (17) implies that the maps $u_{i,i+1}$ and $v_{i,i-1}$ for $i \in \mathbb{Z}/3$ generate A.

Since the higher maps $G^d := 0$ vanish for $d \geqslant 2$ then we need only check equation (15). The only non-trivial A_∞-operations m_n for $n > 2$ in A are determined by equation (18), however, equation (20) can be used to translate back and forth between equations (18) and (9). This shows that G satisfies equation (15) for all of the A_∞-operations m_n for $n \geqslant 3$.

When $n = 2$, equation (15) means that G^1 is a homomorphism. The only relation in A is $x_i y_i = 0$ in $End(X_i)$ for $i = 0, 1, 2$. Equation (17), shows that

$$x_i = u_{i-1,i} v_{i,i-1} \quad \text{and} \quad y_i = v_{i+1,i} u_{i,i+1}$$

in A. Combining this with equation (20) gives

$$x_0 = u_{2,0}v_{0,2} \mapsto p_1 j_2 = 0, \qquad y_0 = v_{1,0}u_{0,1} \mapsto \alpha\beta = u_1,$$
$$x_1 = u_{0,1}v_{1,0} \mapsto \beta\alpha = u_2, \qquad y_1 = v_{2,1}u_{1,2} \mapsto p_2 j_1 = 0,$$
$$x_2 = u_{1,2}v_{2,1} \mapsto j_1 p_2 = (21), \qquad y_2 = v_{0,2}u_{2,0} \mapsto j_2 p_1 = (12).$$

Finally, using this calculation we check that $x_i y_i = y_i x_i = 0$:

$$x_0 y_0 \mapsto 0u_1 = 0, \qquad y_0 x_0 \mapsto u_1 0 = 0,$$
$$x_1 y_1 \mapsto u_2 0 = 0, \qquad y_1 x_1 \mapsto 0 u_2 = 0,$$
$$x_2 y_2 \mapsto (21)(12) = 0, \qquad y_2 x_2 \mapsto (12)(21) = 0.$$

$\square$

References

[1] D. Auroux *A beginner's introduction to Fukaya categories*, Contact and symplectic topology, Bolyai Soc. Math. Stud., 26, János Bolyai Math. Soc., Budapest, 85–136, 2014.

[2] M. Abouzaid, D. Auroux, A.I. Efimov, L. Katzarkov and D. Orlov, *Homological mirror symmetry for punctured spheres*, J. Amer. Math. Soc., 26, 1051–1083, 2013.

[3] A.I. Bondal and M.M. Kapranov, *Enhanced Triangulated Categories*, Math. USSR Sb., 70, 1, 93–107, 1991.

[4] W. Crawley-Boevey, *Geometry of the moment map for representations of quivers*, Compos. Math., 126, 257–293, 2001.

[5] T. Dyckerhoff, M. Kapranov, *Triangulated surfaces in triangulated categories*, arXiv:1306.2545.

[6] G. Faonte, *A-infinity functors and homotopy theory of dg-categories*, arXiv: 1412.1255v1.

[7] K. Lefèvre-Hasegawa, *Sur les A-infini catégories*, Univ. Paris 7, Thesis, arXiv:math/0310337.

[8] C. Geiss, B. Leclerc and J. Schröer, *Rigid modules over preprojective algebras*, Invent. Math., 165, no 3, 589–632, 2006.

[9] F. Haiden, L. Katzarkov and M. Kontsevich, *Flat surfaces and stability structures*, Publ. Math. Inst. Hautes Études Sci., 126, 1, 247–318, 2017.

[10] B. Keller, *A-infinity algebras, modules and functor categories*, Homology Homotopy Appl., 3, 1, 1–35, 2001.

[11] M. Khovanov and P. Seidel, *Quivers, Floer cohomology and braid group actions*, J. Amer. Math. Soc., 15, 1, 203–271, 2002.

[12] D. Madsen, *Homological aspects in representation theory*, Thesis, Norwegian University of Science and Technology, 2002.

[13] M. Markl, *Transferring A_∞ (strongly homotopy associative) structures*, Rend. Circ. Mat. Palermo (2), 79, 139–151, 2006.

[14] D. Nadler, *Cyclic symmetries of A_n-quiver representations*, Adv. Math., 269, 346–363, 2015.

[15] Y. Qi and J. Sussan, *A Categorification of the Burau representation at prime roots of unity*, Selecta Math. (N.S.), 22, 3, 1157–1193, 2016.

[**16**] C.M. Ringel, *The preprojective algebra of a quiver*, Algebras and modules II, CMS Conf. Proc., 24, Amer. Math. Soc., Providence, RI, 467–480, 1998.

[**17**] P. Seidel, *Fukaya categories and Picard-Lefschetz theory*, Zur. Lect. Adv. Math., EMS, 2008.

[**18**] Y. Soibelman, *Non-commutative geometry and deformations of A_∞-algebras and A_∞-categories*, Arbeitstagung 2003.

[**19**] B. Toën, *The homotopy theory of dg-categories and derived Morita theory*, Invent. Math., 167, 3, 615–667, 2007.

Benjamin Cooper   ben-cooper@uiowa.edu

University of Iowa, Department of Mathematics, 14 MacLean Hall, Iowa City, IA 52242-1419, USA

Joshua Sussan   jsussan@mec.cuny.edu

CUNY Medgar Evers, Department of Mathematics, 1650 Bedford Ave, Brooklyn, NY 11225, USA