Homology, Homotopy and Applications, vol. 21(2), 2019, pp.345–362

THE ADAMS–HILTON MODEL AND THE GROUP OF SELF-HOMOTOPY EQUIVALENCES OF A SIMPLY CONNECTED CW-COMPLEX

MAHMOUD BENKHALIFA

(*communicated by Alexander Mishchenko*)

Abstract

Let R be a principal ideal domain (PID). For a simply connected CW-complex X of dimension n, let Y be a space obtained by attaching cells of dimension q to X, $q > n$, and let $A(Y)$ denote an Adams–Hilton model of Y. If $\mathcal{E}(A(Y))$ denotes the group of homotopy self-equivalences of $A(Y)$ and $\mathcal{E}_*(A(Y))$ its subgroup formed of the elements inducing the identity on $H_*(Y, R)$, then we construct two short exact sequences:

$$\bigoplus_i H_q(\Omega X, R) \rightarrowtail \mathcal{E}(A(Y)) \twoheadrightarrow \Gamma_n^q,$$

$$\bigoplus_i H_q(\Omega X, R) \rightarrowtail \mathcal{E}_*(A(Y)) \twoheadrightarrow \Pi_n^q,$$

where $i = \operatorname{rank} H_q(Y, X; R)$, Π_n^q is a subgroup of $\mathcal{E}_*(A(X))$ and Γ_n^q is a subgroup of $\operatorname{aut}(\operatorname{Hom}(H_q(Y, X; R))) \times \mathcal{E}(A(X))$.

1. Introduction

Let R be a PID and let Y be a simply connected CW-complex. The Adams–Hilton model of Y is a chain algebra morphism

$$\Theta_Y : (\mathbb{T}(V), \partial) \to C_*(\Omega Y, R)$$

such that $H_*(\Theta_Y) : H_*(\mathbb{T}(V), \partial) \to H_*(\Omega Y, R)$ is an isomorphism of graded algebras and such that $H_*(V, d) \cong H_*(Y, R)$ as graded R-modules, where d denotes the linear part of the differential ∂ induced on the graded module of indecomposables V, where $C_*(\Omega Y, R)$ denotes the complex of non-degenerate cubic chains equipped with the multiplication induced by composition of loops and where $(\mathbb{T}(V), \partial)$ is the free chain R-algebra on the free graded R-module V. Let $A(Y)$ denote the Adams–Hilton model of the space Y.

As is well known, there is a reasonable concept of "homotopy" among chain algebra morphisms (see Section 3), analogous in many respects to the topological notion of homotopy. Consequently, let $\mathcal{E}(A(Y))$ denote the group of homotopy self-equivalences of the chain algebra $A(Y)$.

Received June 9, 2018, revised July 22, 2018, November 14, 2018; published on June 5, 2019.
2010 Mathematics Subject Classification: 55P10.
Key words and phrases: group of homotopy self-equivalences, Adams–Hilton model, Anick model, loop space.
Article available at http://dx.doi.org/10.4310/HHA.2019.v21.n2.a19

By the properties of the Adams–Hilton model, it is worth noting that if $\alpha\colon Y \to Y$ is a homotopy equivalence, then so is $A(\alpha)\colon A(Y) \to A(Y)$. Therefore there is a homomorphism of groups $\mathcal{E}(Y) \to \mathcal{E}(A(Y))$ sending $[\alpha]$ to $[A(\alpha)]$, where $\mathcal{E}(Y)$ is the group of homotopy self-equivalences of Y (see, for example, [**11**] for more details about this group).

The idea of inserting the group $\mathcal{E}(Y)$ in a short exact sequence of groups of the form $A \rightarrowtail \mathcal{E}(Y) \twoheadrightarrow B$ traces back to the first results on this group in the 1950s. Barcus–Barrett [**3**] gave an exact sequence describing the effect of a single cell attachment $Y = S^n \cup_\alpha S^{q-1}$ on the group $\mathcal{E}(Y)$. This basic result was refined and extended by later authors including Kahn [**13**], Oka–Sawashita–Sugawara [**14**], Benkhalifa–Smith[**11**] and Benkhalifa [**5, 7, 8**]. We refer the reader to [**15, 16**] for a comprehensive survey on these results including various exact sequences.

The aim of this paper is to study the effect of cell-attachment on the group $\mathcal{E}(A(X))$. More precisely let X be a simply connected CW-complex of dimension n and let

$$Y = X \cup_\alpha \Big(\bigcup_{i\in I} e_i^q\Big) \tag{1}$$

be the space obtained by attaching cells of dimension q to X by a map $\alpha\colon \underset{i\in I}{\vee} S^{q-1} \to X$. Let $\mathcal{E}_*(A(Y))$ denote the subgroup of $\mathcal{E}(A(Y))$ consisting of the elements inducing the identity on $H_*(Y,R)$. We prove:

Theorem 1. *For every n and for every $q > n$, there exist two short exact sequences of groups*

$$\underset{i}{\oplus} H_q(\Omega X, R) \rightarrowtail \mathcal{E}(A(Y)) \twoheadrightarrow \Gamma_n^q, \qquad \underset{i}{\oplus} H_q(\Omega X, R) \rightarrowtail \mathcal{E}_*(A(Y)) \twoheadrightarrow \Pi_n^q,$$

where $i = \operatorname{rank} H_q(Y,X;R)$, where Γ_n^q is a subgroup of $\operatorname{aut}(\operatorname{Hom}(H_q(Y,X;R))) \times \mathcal{E}(A(X))$ and Π_n^q is a subgroup of $\mathcal{E}_*(A(X))$ (see Definition 4.1)

An analogous problem was previously studied in [**6**], in terms of the Postnikov decomposition of the rational space Y and by the use of the Sullivan model in rational homotopy theory and it was shown that

Theorem 2 ([**6**], Corollary 3.3)**.** *There exist two exact short sequences:*

$$\operatorname{Hom}\big(\pi_q(Y); H^q(Y^{[n]})\big) \rightarrowtail \mathcal{E}(Y^{[n+1]}) \twoheadrightarrow D^n_{n-1},$$
$$\operatorname{Hom}\big(\pi_q(Y); H^q(Y^{[n]})\big) \rightarrowtail \mathcal{E}_\#(Y^{[n+1]}) \twoheadrightarrow \mathrm{G}^n_{n-1},$$

where D_n^q is a subgroup of $\operatorname{aut}\big(\operatorname{Hom}(\pi_q(Y), \mathbb{Q})\big) \times \mathcal{E}(Y^{[n]})$ and where G_n^q is a subgroup of $\mathcal{E}_\#(Y^{[n]})$. Here $Y^{[k]}$ denotes the k^{th} Postnikov section of Y and $\mathcal{E}_\#(Y^{[n]})$ denotes the subgroup of $\mathcal{E}(Y^{[n]})$ consisting of the elements inducing the identity on homotopy groups.

In particular, let Y be a simply connected CW-complex and ΣY the suspension of Y. A well-known theorem due to Bott–Samelson [**12**] asserts that, under the assumption that the homology $H_*(Y;R)$ is a free graded R-module, the chain algebra $\big(\mathbb{T}(H_*(Y;R)),0\big)$ (with the trivial differential) can be considered as an Adams–Hilton model for the space ΣY. Consequently, we prove that the group $\mathcal{E}(A(\Sigma Y))$ is simply identified with the group $\mathrm{aut}(A(\Sigma Y))$ of the chain algebra automorphisms of $A(\Sigma Y)$ and the subgroup $\mathcal{E}_*(A(\Sigma Y))$ is identified with the subgroup $\mathrm{aut}_*(A(\Sigma Y))$ of the chain algebra automorphisms inducing the identity on the graded module $H_*(Y,R)$. Moreover, applying Theorem 1, we obtain the following short exact sequences of groups

Theorem 3. *Let Y be a simply connected CW-complex and ΣY the suspension of Y. There exist two exact short sequences*

$$\begin{aligned}\mathrm{Hom}\Big(H_{q+1}(Y;R),\mathbb{T}_q(H_{<q}(Y;R))\Big) &\rightarrowtail \mathrm{aut}(A(\Sigma Y))\\ &\twoheadrightarrow \mathrm{aut}\big(H_{q+1}(Y;R)\big)\times \mathrm{aut}\big(A((\Sigma Y)^{q-1})\big),\\ \mathrm{Hom}\Big(H_{q+1}(Y;R),\mathbb{T}_q(H_{<q}(Y;R))\Big) &\rightarrowtail \mathrm{aut}_*(A(\Sigma Y))\twoheadrightarrow \mathrm{aut}_*\big(A((\Sigma Y)^{q-1})\big),\end{aligned}$$

where $(\Sigma Y)^{q-1}$ is the $(q-1)$-skeleton of the space ΣY.

For instance, using the above short exact sequence we can show the following results

$$\begin{aligned}\mathrm{aut}(A(\mathbb{S}^{n+1}\vee\mathbb{S}^{q+1}))&\cong\mathbb{Z}_2\times\mathbb{Z}_2,\\ \mathrm{aut}_*(A(\mathbb{S}^{n+1}\vee\mathbb{S}^{q+1}))&\cong\mathbb{Z}_2, &&\text{if } q\not\equiv 0 \pmod n,\\ \mathbb{Z}\rightarrowtail\mathrm{aut}(A(\mathbb{S}^{n+1}\vee\mathbb{S}^{q+1}))&\twoheadrightarrow\mathbb{Z}_2\times\mathbb{Z}_2,\\ \mathbb{Z}\rightarrowtail\mathrm{aut}_*(A(\mathbb{S}^{n+1}\vee\mathbb{S}^{q+1}))&\twoheadrightarrow\mathbb{Z}_2, &&\text{if } q\equiv \pmod n.\end{aligned}$$

Moreover, let $R\subseteq\mathbb{Q}$ be a subring with least non-invertible prime p. Using the Anick model theory [**1, 2**], if X is an r-connected CW-complex of dimension $n+1$ and $n<q\leqslant k$, where $k<min(r+2p-3,rp-1)$, then we prove

Theorem 4. *Let Y be the space in (1). The homomorphisms*

$$\mathcal{E}(Y_R)\to\mathcal{E}(A(Y_R)),\qquad \mathcal{E}_*(Y_R)\to\mathcal{E}_*(A(Y_R))$$

are injective, where Y_R denotes the R-localization of Y.

The paper is organized as follows. In Section 2 we recall briefly the notion of the Adams–Hilton model associated to a given simply connected space Y and the Bott–Samelson theorem concerning the Adams–Hilton model of the space ΣY. In Section 3 we establish Theorem 3, and in Section 4 we recall the notion of homotopy between chain algebra morphisms and we prove Theorem 2 and some of its corollaries.

2. Adams–Hilton model and Bott–Samelson theorem

Given a simply connected CW-complex Y. The Adams–Hilton model of Y is a chain algebra morphism

$$\Theta_Y\colon(\mathbb{T}(V),\partial)\to C_*(\Omega Y,R)$$

such that

$$H_*(\Theta_Y)\colon H_*(\mathbb{T}(V),\partial)\to H_*(\Omega Y,R)$$

is an isomorphism of graded algebras and such that

$$H_{i-1}(V,d)\cong H_i(Y,R),$$

as graded modules. Here $C_*(\Omega Y, R)$ denotes the complex of non-degenerate cubic chains equipped with the multiplication induced by composition of loops and $d\colon V \to V$ is the linear part of the differential ∂ defined by

$$\partial(v) - d(v) \in \mathbb{T}^{\geqslant 2}(V),$$

where $\mathbb{T}^{\geqslant 2}(V)$ is the graded R-module of decomposable elements, i.e., the elements of $\mathbb{T}(V)$ of length $\geqslant 2$. We denote by $A(Y)$ the chain algebra $(\mathbb{T}(V), \partial)$.

Let ΣY denote the suspension of Y. If the map $\sigma\colon Y \to \Omega\Sigma Y$ is the adjoint of $id_{\Sigma Y}$, then it induces a homomorphism of graded modules $\sigma_*\colon H_*(Y; R) \to H_*(\Omega\Sigma Y; R)$ which can be extended, by virtue of the universal property of the free chain algebra, to a homomorphism

$$\mathbb{T}(\sigma_*)\colon \mathbb{T}(H_*(Y; R)) \to H_*(\Omega\Sigma Y; R)$$

of graded algebras.

A theorem, due to Bott and Samelson [**12**], asserts that under the assumption that the homology $H_*(Y; R)$ is a free graded R-module, $\mathbb{T}(\sigma_*)$ is an isomorphism of R-algebras. Therefore the chain algebra $\big(\mathbb{T}(H_*(Y; R)), 0\big)$, with the trivial differential, can be considered as an Adams–Hilton model for the space ΣY, i.e.,

$$A(\Sigma Y) = \big(\mathbb{T}(H_*(Y; R)), 0\big). \tag{2}$$

By the properties of the Adams–Hilton model we derive

$$H_*(\Omega\Sigma Y; R) = H_*(A(\Sigma Y)) = \mathbb{T}(H_*(Y; R)). \tag{3}$$

Remark 2.1. It is important to mention here that as the graded R-module $H_*(Y; R)$ is assumed to be free, the two relations (2) and (3) imply that the Adams–Hilton model $(\mathbb{T}(V), 0)$ of ΣY satisfies

$$V_i \cong H_{i-1}(Y; R), \quad \forall i \geqslant 2. \tag{4}$$

3. The group of graded algebra automorphisms of the tensor algebra $\mathbb{T}(V)$

Let $\mathbb{T}(V_q \oplus V_{\leqslant n})$, where $q > n$, be a tensor algebra (considered as 1-connected chain algebra with trivial differential). Let us denote by $\operatorname{aut}(\mathbb{T}(V_q \oplus V_{\leqslant n}))$ the group of chain (graded) algebra automorphisms of $\mathbb{T}(V_q \oplus V_{\leqslant n})$.

If $\alpha \in \operatorname{aut}(\mathbb{T}(V_q \oplus V_{\leqslant n}))$, then it induces the following homomorphism

$$\alpha_k\colon V_k \to V_k \oplus \mathbb{T}_k(V_{\leqslant n}), \quad k \leqslant q$$

so define $\widetilde{\alpha}_k\colon V_k \to V_k$ such that $\alpha(v) - \widetilde{\alpha}_k(v) \in \mathbb{T}_k(V_{\leqslant n})$. Clearly $\widetilde{\alpha}_k$ is an automorphism of V_k. Hence denote by $\operatorname{aut}_*(\mathbb{T}(V_q \oplus V_{\leqslant n}))$ the subgroup of $\operatorname{aut}(\mathbb{T}(V_q \oplus V_{\leqslant n}))$ consisting of the elements α such that $\widetilde{\alpha}_k = id$ for all $k \leqslant q$.

The aim of this section is to establish the following theorem:

Theorem 3.1. *If $\mathbb{T}(V)$ is a 1-connected free graded tensor algebra, then we have the following two short exact sequences of groups*

$$\begin{aligned} \operatorname{Hom}(V_q, \mathbb{T}_q(V_{\leqslant n})) &\rightarrowtail \operatorname{aut}(\mathbb{T}(V_q \oplus V_{\leqslant n})) \twoheadrightarrow \operatorname{aut}(V_q) \times \operatorname{aut}(\mathbb{T}(V_{\leqslant n})), \\ \operatorname{Hom}(V_q, \mathbb{T}_q(V_{\leqslant n})) &\rightarrowtail \operatorname{aut}_*(\mathbb{T}(V_q \oplus V_{\leqslant n})) \twoheadrightarrow \operatorname{aut}_*(\mathbb{T}(V_{\leqslant n})). \end{aligned} \tag{5}$$

Proof. Let $(\mathbb{T}(V_q \oplus V_{\leqslant n}))$, where $q > n$, be a free graded tensor algebra. Define the map

$$\mathrm{g}\colon \operatorname{aut}(\mathbb{T}(V_q \oplus V_{\leqslant n})) \to \operatorname{aut}(V_q) \times \operatorname{aut}(\mathbb{T}(V_{\leqslant n}))$$

by setting:

$$\mathrm{g}(\alpha) = (\widetilde{\alpha}_q, \alpha_n), \tag{6}$$

where $\widetilde{\alpha}_q\colon V_q \to V_q$ is as above and where α_n is the restriction of α to $\mathbb{T}(V_{\leqslant n})$.

It is easy to see that g is a surjective morphism of groups. Indeed, let $(\xi, \gamma) \in \operatorname{aut}(V_q) \times \operatorname{aut}(\mathbb{T}(V_{\leqslant n}))$. Define $\alpha\colon \mathbb{T}(V_q \oplus V_{\leqslant n}) \to \mathbb{T}(V_q \oplus V_{\leqslant n})$ by setting:

$$\alpha(v) = \xi(v) \quad \text{and} \quad \alpha = \gamma \text{ on } V_{\leqslant n}.$$

Clearly we have $\widetilde{\alpha}_q = \xi$. Hence using (6) we derive $\mathrm{g}(\alpha) = (\xi, \gamma)$.

Finally, the following relations

$$\mathrm{g}(\alpha \circ \alpha') = (\widetilde{\alpha \circ \alpha'}_q, \alpha_n \circ \alpha'_n) = (\widetilde{\alpha}_q, \alpha_n) \circ (\widetilde{\alpha'}_q, \alpha'_n) = \mathrm{g}(\alpha) \circ \mathrm{g}(\alpha')$$

assure that g is a homomorphism of groups.

Consequently, we obtain the following short exact sequence of groups

$$\ker \mathrm{g} \rightarrowtail \operatorname{aut}(\mathbb{T}(V_q \oplus V_{\leqslant n}), 0) \overset{\mathrm{g}}{\twoheadrightarrow} \operatorname{aut}(V_q) \times \operatorname{aut}(\mathbb{T}(V_{\leqslant n})). \tag{7}$$

Next let us determine ker g. By (6) we can write:

$$\ker \mathrm{g} = \Big\{\alpha \in \operatorname{aut}(\mathbb{T}(V_q \oplus V_{\leqslant n})) \mid \widetilde{\alpha}_q = id_{V_q},\ \alpha_n = id_{\mathbb{T}(V_{\leqslant n})}\Big\}, \tag{8}$$

therefore for every $\alpha \in \ker \mathrm{g}$ we have:

$$\begin{aligned} \alpha(v) &= v + z_v, \qquad z_v \in \mathbb{T}_q(V_{\leqslant n}), \\ \alpha_n &= id_{\mathbb{T}(V_{\leqslant n})}. \end{aligned} \tag{9}$$

So define the map $\Psi\colon \ker \mathrm{g} \to \operatorname{Hom}(V_q, \mathbb{T}_q(V_{\leqslant n}))$ by setting

$$\Psi(\alpha)\colon V_q \to \mathbb{T}_q(V_{\leqslant n}), \qquad \Psi(\alpha)(v) = z_v. \tag{10}$$

On the one hand the relations (9) and (10) imply that

$$\alpha \circ \alpha'(v) = \alpha(v + z'_v) = v + z_v + z'_v$$

hence $\Psi(\alpha \circ \alpha')(v) = z_v + z'_v$. On the other hand we have

$$(\Psi(\alpha) + \Psi(\alpha'))(v) = \Psi(\alpha)(v) + \Psi(\alpha')(v) = z_v + z'_v.$$

Therefore $\Psi(\alpha \circ \alpha') = \Psi(\alpha) + \Psi(\alpha')$ implying that Ψ is a homomorphism of groups.

Now let $\alpha \in \ker \Psi$, then $\Psi(\alpha) = 0$ implying that $\Psi(\alpha)(v) = z_v = 0$, and according to (9), it follows that $\alpha = id$. Hence Ψ is injective. Finally, let $f \in \operatorname{Hom}(V_q, \mathbb{T}_q(V_{\leqslant n}))$, and define

$$\alpha(v) = v + f(v), \qquad \alpha_n = id_{\mathbb{T}(V_{\leqslant n})}.$$

By definition (10) we have $\Psi(\alpha)(v) = f(v)$, so $\Psi(\widetilde{\gamma}) = f$. It follows that Ψ is surjective, consequently Ψ is an isomorphism of groups.

Summarizing the short exact sequence (7) becomes

$$\mathrm{Hom}(V_q, \mathbb{T}_q(V_{\leqslant n})) \rightarrowtail \mathrm{aut}(\mathbb{T}(V_q \oplus V_{\leqslant n})) \overset{\mathrm{g}}{\twoheadrightarrow} \mathrm{aut}(V_q) \times \mathrm{aut}(\mathbb{T}(V_{\leqslant n})).$$

Next let $\hat{\mathrm{g}}$ denote the restriction of the homomorphism g to the subgroup $\mathrm{aut}_*(\mathbb{T}(V_q \oplus V_{\leqslant n}))$. As $\mathrm{aut}_*(\mathbb{T}(V_q \oplus V_{\leqslant n}))$ is formed by the elements α such that $\widetilde{\alpha}_k = id$ for all $k \leqslant q$, it follows that

$$\hat{\mathrm{g}}(\alpha) = (id_{V_q}, \alpha_n), \qquad \alpha_n \in \mathrm{aut}_*(\mathbb{T}(V_{\leqslant n})).$$

Hence we define $\widetilde{\mathrm{g}}\colon \mathrm{aut}_*(\mathbb{T}(V_q \oplus V_{\leqslant n})) \twoheadrightarrow \mathrm{aut}_*(\mathbb{T}(V_{\leqslant n}))$ by $\widetilde{\mathrm{g}}(\alpha) = \alpha_n$. The map $\widetilde{g}$ is a surjective homomorphism. Indeed, if $\gamma \in \mathrm{aut}_*(\mathbb{T}(V_{\leqslant n}))$, then $(id_{V_q}, \gamma) \in \mathrm{aut}(V_q) \times \mathrm{aut}(\mathbb{T}(V_{\leqslant n}))$. As the homomorphism g is surjective, there exists $\alpha \in \mathrm{aut}(\mathbb{T}(V_{\leqslant n}))$ such that $\mathrm{g}(\alpha) = (id_{V_q}, \gamma)$. Using (6) we deduce that

$$(id_{V_q}, \gamma) = (\widetilde{\alpha}_q, \alpha_n),$$

implying that $\alpha \in \mathrm{aut}_*(\mathbb{T}(V_{\leqslant n}))$ and $\widetilde{\mathrm{g}}(\alpha) = \gamma$.

Now from (8) we have ker g = ker $\widetilde{\mathrm{g}}$ and since ker g $\cong \mathrm{Hom}(V_q, \mathbb{T}_q(V_{\leqslant n}))$, we obtain the following short exact sequence

$$\mathrm{Hom}(V_q, \mathbb{T}_q(V_{\leqslant n})) \rightarrowtail \mathrm{aut}_*(\mathbb{T}(V_q \oplus V_{\leqslant n})) \twoheadrightarrow \mathrm{aut}_*(\mathbb{T}(V_{\leqslant n})). \qquad \square$$

Corollary 3.2. *If ΣY is a simply connected space of dimension $q+1$ and $X = (\Sigma Y)^q$ denotes the q skeleton of ΣY, then the following short sequence of groups is exact:*

$$\begin{aligned} \mathrm{Hom}\Big(H_{q+1}(Y;R), \mathbb{T}_q(H_{<q}(Y;R))\Big) &\rightarrowtail \mathrm{aut}(A(\Sigma Y)) \\ &\twoheadrightarrow \mathrm{aut}\big(H_{q+1}(Y;R)\big) \times \mathrm{aut}\big(A((\Sigma Y)^{q-1})\big), \\ \mathrm{Hom}\Big(H_{q+1}(Y;R), \mathbb{T}_q(H_{<q}(Y;R))\Big) &\rightarrowtail \mathrm{aut}_*(A(\Sigma Y)) \twoheadrightarrow \mathrm{aut}_*\big(A((\Sigma Y)^{q-1})\big). \end{aligned} \tag{11}$$

Proof. First the Adams–Hilton of the space ΣY is of the form $\mathbb{T}(V_q \oplus V_{\leqslant q-1})$ with trivial differential. Next we derive the two sequences in (11) by applying Theorem 3.1 and using the relations (2), (3) and (4). $\square$

Corollary 3.3. *Let $V_q = \{v_q\}$ be the free R-module of rank 1 and let $\mathbb{T}(V_q \oplus V_{\leqslant n})$, where $q > n$, be a free tensor algebra. Then the following short sequence of groups is exact:*

$$\begin{aligned} \mathbb{T}_q(V_{\leqslant n}) &\rightarrowtail \mathrm{aut}(\mathbb{T}(V_q \oplus V_{\leqslant n})) \twoheadrightarrow \mathrm{aut}(R) \times \mathrm{aut}(\mathbb{T}(V_{\leqslant n})), \\ \mathbb{T}_q(V_{\leqslant n}) &\rightarrowtail \mathrm{aut}_*(\mathbb{T}(V_q \oplus V_{\leqslant n})) \twoheadrightarrow \mathrm{aut}_*(\mathbb{T}(V_{\leqslant n})). \end{aligned} \tag{12}$$

Proof. The sequence (12) can be deduced from the exact sequence (5) by observing that $V_q = \{v_q\} \cong R$, so $\mathrm{Hom}(V_q, \mathbb{T}_q(V_{\leqslant n})) \cong \mathbb{T}_q(V_{\leqslant n})$ and $\mathrm{aut}(V_q) \cong \mathrm{aut}(R)$. $\square$

As an illustration of Corollary 3.3 we give the following example:

Example 3.4. Let $R = \mathbb{Z}$ and let $X = \mathbb{S}^{n+1}$ be the sphere of dimension $n+1$. Let $Y = \mathbb{S}^{n+1} \vee \mathbb{S}^{q+1}$, where $q > n$ and where the $q+1$-cell is trivially attached to $\mathbb{S}^{n+1}$.

Recall that $A(\Sigma Y)$ (respectively $A(\Sigma Y)$) denotes the Adams–Hilton model of the suspension of the space Y (respectively of X (see 2)) and let $\mathrm{aut}(A(\Sigma Y))$ (respectively $\mathrm{aut}(A(\Sigma Y)))$ denote the group of graded automorphisms of the free tensor algebra $A(\Sigma Y)$ (respectively $A(\Sigma X)$).

The Adams–Hilton models of $S^{n+1} = \Sigma S^n$ and $Y = \mathbb{S}^{n+1} \vee \mathbb{S}^{q+1}$ are, respectively,

$$A(S^{n+1}) = \mathbb{T}(H_*(S^n;\mathbb{Z})),$$
$$A(\mathbb{S}^{n+1} \vee \mathbb{S}^{q+1}) = A(\Sigma(\mathbb{S}^n \vee \mathbb{S}^q)) \cong \mathbb{T}(H_*(S^n \vee \mathbb{S}^q);\mathbb{Z}).$$

Recall that

$$H_q(S^n;\mathbb{Z}) = \begin{cases} 0, & \text{if } q \neq n, \\ \mathbb{Z}, & \text{if } q = n \end{cases}$$

and

$$H_*(S^n \vee \mathbb{S}^q;\mathbb{Z}) = H_*(S^n;\mathbb{Z}) \oplus H_*(\mathbb{S}^q;\mathbb{Z}).$$

Define the graded abelian group $V_q \oplus V_{\leqslant n}$ by

$$V_q \cong H_q(\mathbb{S}^q;\mathbb{Z}) \cong \mathbb{Z}, \qquad V_n \cong H_n(\mathbb{S}^n;\mathbb{Z}) \cong \mathbb{Z}, \qquad V_i = 0, \quad i \leqslant n-1.$$

Therefore we obtain

$$A(S^{n+1}) = \mathbb{T}(V_{\leqslant n}), \qquad A(\mathbb{S}^{n+1} \vee \mathbb{S}^{q+1}) = A(\Sigma(\mathbb{S}^n \vee \mathbb{S}^q)) \cong \mathbb{T}(V_q \oplus V_{\leqslant n};\mathbb{Z}).$$

Applying Corollary 3.3 we get

$$\mathbb{T}_q(H_*(S^n;\mathbb{Z})) \rightarrowtail \operatorname{aut}(A(\mathbb{S}^{n+1} \vee \mathbb{S}^{n+2})) \twoheadrightarrow \operatorname{aut}(\mathbb{Z}) \times \operatorname{aut}(\mathbb{T}(H_*(S^n;\mathbb{Z}))). \quad (13)$$

Let us compute $\operatorname{aut}(\mathbb{T}(H_*(S^n;\mathbb{Z})))$. Indeed, we have

$$\operatorname{aut}(\mathbb{T}(H_*(S^n;\mathbb{Z}))) = \operatorname{aut}(\mathbb{T}(V_{\leqslant n})) = \operatorname{aut}(\mathbb{T}(V_n \oplus V_{\leqslant n-1})).$$

Applying Corollary 3.3 again it follows that

$$\mathbb{T}_q(V_{\leqslant n-1}) \rightarrowtail \operatorname{aut}(\mathbb{T}(V_n \oplus V_{\leqslant n-1})) \twoheadrightarrow \operatorname{aut}(V_n) \times \operatorname{aut}(\mathbb{T}(H_*(V_{\leqslant n-1})))$$

and taking into account that $V_{\leqslant n-1} = 0$ we obtain

$$\operatorname{aut}(\mathbb{T}(V_n \oplus V_{\leqslant n-1})) \cong \operatorname{aut}(V_n).$$

Hence

$$\operatorname{aut}(\mathbb{T}(H_*(S^n;\mathbb{Z}))) \cong \operatorname{aut}(V_n) \cong \operatorname{aut}(\mathbb{Z}) = \mathbb{Z}_2.$$

Consequently, the exact sequence (13) becomes

$$\mathbb{T}_q(H_*(S^n;\mathbb{Z})) \rightarrowtail \operatorname{aut}(A(\mathbb{S}^{n+1} \vee \mathbb{S}^{q+1})) \twoheadrightarrow \mathbb{Z}_2 \times \mathbb{Z}_2.$$

But we have

$$\mathbb{T}_q(H_*(S^n;\mathbb{Z})) = \begin{cases} 0, & \text{if } q \not\equiv 0 \pmod{n}, \\ \mathbb{Z}, & \text{if } q \equiv 0 \pmod{n} \end{cases}$$

implying

$$\begin{aligned} \operatorname{aut}(A(\mathbb{S}^{n+1} \vee \mathbb{S}^{q+1})) &\cong \mathbb{Z}_2 \times \mathbb{Z}_2, && \text{if } q \not\equiv 0 \pmod{n}, \\ \mathbb{Z} \rightarrowtail \operatorname{aut}(A(\mathbb{S}^{n+1} \vee \mathbb{S}^{q+1})) &\twoheadrightarrow \mathbb{Z}_2 \times \mathbb{Z}_2, && \text{if } q \equiv 0 \pmod{n}. \end{aligned}$$

Finally, a similar computation shows that

$$\begin{aligned} \operatorname{aut}_*(A(\mathbb{S}^{n+1} \vee \mathbb{S}^{q+1})) &\cong \mathbb{Z}_2, && \text{if } q \not\equiv 0 \pmod{n}, \\ \mathbb{Z} \rightarrowtail \operatorname{aut}_*(A(\mathbb{S}^{n+1} \vee \mathbb{S}^{q+1})) &\twoheadrightarrow \mathbb{Z}_2, && \text{if } q \equiv 0 \pmod{n}. \end{aligned}$$

Notice that $\operatorname{aut}_*(A(\mathbb{S}^{n+1} \vee \mathbb{S}^{q+1}))$ is a normal subgroup of $\operatorname{aut}(A(\mathbb{S}^{n+1} \vee \mathbb{S}^{q+1}))$ and in the two cases the quotient group is

$$\frac{\operatorname{aut}(A(\mathbb{S}^{n+1}\vee\mathbb{S}^{q+1}))}{\operatorname{aut}_*(A(\mathbb{S}^{n+1}\vee\mathbb{S}^{q+1}))}\cong\mathbb{Z}_2.$$

In the second part of this paper we shall generalize the above results to the case when the differential given in $(\mathbb{T}(V),\partial)$ is not necessarily trivial. For this purpose we need the notion of homotopy between chain algebra morphisms which is analogous in many respects to the topological notion of homotopy.

4. The group of homotopy self-equivalences of chain algebra morphisms

4.1. Homotopy of chain algebra morphisms

(See [**4**] page 48 for more details). Let $(\mathbb{T}(V),\partial)$ be a 1-connected free chain algebra. Define the free algebra $\mathbb{T}(V'\oplus V''\oplus sV)$, where V',V'' are two isomorphic copies of V and sV is the (de)suspension of V. Then we define:

$$i',i''\colon \mathbb{T}(V)\to\mathbb{T}(V'\oplus V''\oplus sV),\qquad i'(v)=v',\qquad i''(v)=v'',$$

where $v'\in V',v''\in V$" are the two elements corresponding to $v\in V$. Now define $S\colon\mathbb{T}(V)\to\mathbb{T}(V'\oplus V''\oplus sV)$, of degree 1, as the unique graded module homomorphism which satisfies the following two conditions

$$S(v)=sv,\qquad S(x.y)=S(x).(i''(y)+(-1)^{|x|}i'(x))S(y),\quad \forall v\in V \text{ and } x,y\in\mathbb{T}(V).$$

Next we define the differential D on $\mathbb{T}(V'\oplus V''\oplus sV)$ by setting

$$D(sv)=v''-v'-S(\partial v),\qquad D(v')=i'(\partial v),\qquad D(v'')=i''(\partial v).\tag{14}$$

$(\mathbb{T}(V'\oplus V''\oplus sV),D)$ is called the cylinder chain algebra of $(\mathbb{T}(V),\partial)$.

Definition 4.1. A homotopy between two chain algebra morphisms $\alpha_1,\alpha_2\colon(\mathbb{T}(V),\partial)\to(\mathbb{T}(V),\partial)$ is a chain algebra morphism

$$F\colon(\mathbb{T}(V'\oplus V''\oplus sV),D)\to(\mathbb{T}(V),\partial)$$

such that $F\circ i'(v)=F(v')=\alpha_1(v)$ and $F\circ i''(v)=F(v'')=\alpha_2(v)$.

Definition 4.2. A chain algebra morphism $\alpha_1\colon(\mathbb{T}(V),\partial)\to(\mathbb{T}(V),\partial)$ is called a self-homotopy equivalence, if there exists a chain algebra morphism $\alpha_2\colon(\mathbb{T}(V),\partial)\to(\mathbb{T}(V),\partial)$ such that $\alpha_1\circ\alpha_2$ and $\alpha_2\circ\alpha_1$ are homotopic to the identity.

Definition 4.3. Let $\mathcal{E}(\mathbb{T}(V))$ denote the group of homotopy classes of self-homotopy equivalences of $(\mathbb{T}(V),\partial)$, under composition of chain algebra morphisms, and let $\mathcal{E}_*(\mathbb{T}(V))$ denote the subgroup consisting of those elements which induce the identity on the graded module of indecomposable V_*.

Thereafter we will need the following lemma:

Lemma 4.4. *Let $q>n$, let $V=V_q\oplus V_{\leqslant n}$ and let $\alpha_1,\alpha_2\colon(\mathbb{T}(V),\partial)\to(\mathbb{T}(V),\partial)$ be two chain algebra morphisms satisfying:*

$$\alpha_1(v)=v+z_1,\qquad \alpha_2(v)=v+z_2\quad \text{on } V_q\quad \text{and}\quad \alpha=\alpha'=\mathrm{id}\quad \text{on } V_{\leqslant n}.$$

Assume that $z_1-z_2=\partial(u)$, where $u\in\mathbb{T}_{q+1}(V)$. Then α_1 and α_2 are homotopic.

Proof. Define F by setting

$$\begin{array}{llllll} F(v') = v + z_1, & F(v'') = v + z_2 & \text{and} & F(sv) = u & \text{for} & v \in V^q, \\ F(v') = v, & F(v'') = v & \text{and} & F(sv) = 0 & \text{for} & v \in V^{\leqslant n}, \end{array}$$

then F is the required homotopy. □

We start with the following remarks.

Remark 4.5. If $(\mathbb{T}(V), 0)$ is a 1-connected free chain algebra with trivial differential, then the notion of homotopy is simply the equality. Indeed, let $\alpha_1, \alpha_2 \colon (\mathbb{T}(V), 0) \to (\mathbb{T}(V), 0)$ be two chain algebra morphisms and assume that they are homotopic. By Definition 4.1 there exists a chain algebra morphism

$$F \colon (\mathbb{T}(V' \oplus V'' \oplus sV), D) \to (\mathbb{T}(V), 0)$$

such that

$$F \circ i'(v) = F(v') = \alpha_1(v), \qquad F \circ i''(v) = F(v'') = \alpha_2(v). \tag{15}$$

As the differential ∂ is trivial and F is a chain algebra, it follows that

$$F \circ D = 0,$$

moreover, the relations (14) become

$$D(sv) = v'' - v', \qquad D(v') = 0, \qquad D(v'') = 0.$$

Therefore

$$0 = F \circ D(sv) = F(v'') - F(v').$$

Finally, according to (15) we deduce that $\alpha_1(v) = \alpha_2(v)$.

Remark 4.6. Let $(\mathbb{T}(V), 0)$ be a 1-connected free chain algebra with trivial differential. By virtue of Remark 4.5 we derive that the group $\mathcal{E}(\mathbb{T}(V))$ is identified with the group $\mathrm{aut}(\mathbb{T}(V))$ and $\mathcal{E}_*(\mathbb{T}(V))$ is identified with the subgroup $\mathrm{aut}_*(\mathbb{T}(V))$ introduced in the previous section.

4.2. The graded homomorphism b_* and the groups $\mathcal{D}_n^q$

Definition 4.7. Let $(\mathbb{T}(V_q \oplus V_{\leqslant n}), \partial)$ be a 1-connected chain algebra where $q > n$. We define the homomorphism $b_q \colon V_q \to H_{q-1}(\mathbb{T}(V_{\leqslant n}))$ by setting:

$$b_q(v) = [\partial(v)]. \tag{16}$$

Here $[\partial(v)]$ denotes the homology class of $\partial(v) \in \mathbb{T}_{q-1}(V_{\leqslant n})$.

For every 1-connected chain algebra $(\mathbb{T}(V_q \oplus V_{\leqslant n}), \partial)$, the homomorphism b_q is natural. Namely if $[\alpha] \in \mathcal{E}(\mathbb{T}(V_q \oplus V_{\leqslant n}))$, then the following diagram commutes:

$$\begin{array}{ccc} V_q & \xrightarrow{\widetilde{\alpha}_q} & V_q \\ {\scriptstyle b_q}\downarrow & & \downarrow{\scriptstyle b_q} \\ H_{q-1}(\mathbb{T}(V_{\leqslant n})) & \xrightarrow{H_{q-1}(\alpha_n)} & H_{q-1}(\mathbb{T}(V_{\leqslant n})) \end{array} \tag{17}$$

where

$$\widetilde{\alpha} \colon (V_q \oplus V_{\leqslant n}, d) \to (V_q \oplus V_{\leqslant n}, d) \tag{18}$$

is the graded homomorphism induced by α on the chain complex of indecomposables and where $\alpha_n\colon (\mathbb{T}(V_{\leqslant n}),\partial) \to (\mathbb{T}(V_{\leqslant n}),\partial)$ is the restriction of α. Here d denotes the linear part of the differential ∂ defined by the relation

$$\partial - d\colon V_{n+1} \to \mathbb{T}_n^{\geqslant 2}(V).$$

Definition 4.8. Given a 1-connected chain algebra $(\mathbb{T}(V_q \oplus V_{\leqslant n}),\partial)$ where $q > n$, set $V = V_q \oplus V_{\leqslant n}$. Let $\mathcal{D}_n^q$ be the subset of $\operatorname{aut}(V_q) \times \mathcal{E}(\mathbb{T}(V_{\leqslant n}))$ consisting of the couples $(\xi,[\alpha])$ making the following diagram commute

$$\begin{array}{ccc} V_q & \xrightarrow{\quad\xi\quad} & V_q \\ {\scriptstyle b_q}\downarrow & & \downarrow{\scriptstyle b_q} \\ H_{q-1}(\mathbb{T}(V_{\leqslant n})) & \xrightarrow{H_{q-1}(\alpha)} & H_{q-1}(\mathbb{T}(V_{\leqslant n})) \end{array} \tag{19}$$

Clearly $\mathcal{D}_n^q$ is a subgroup of $\operatorname{aut}(V_q) \times \mathcal{E}(\mathbb{T}(V_{\leqslant n}))$.

Remark 4.9. If $(\mathbb{T}(V),0)$ is a 1-connected free chain algebra with trivial differential, then according to the relation (16), the homomorphism b_q given in the diagram (19) is trivial. Moreover, we have

$$H_{q-1}(\mathbb{T}(V_{\leqslant n})) = \mathbb{T}_{q-1}(V_{\leqslant n}), \qquad H_{q-1}(\alpha) = \alpha_{q-1}.$$

As a result the group $\mathcal{D}_n^q$ consists of those pairs $(\xi,\alpha) \in \operatorname{aut}(V_q) \times \operatorname{aut}(\mathbb{T}(V_{\leqslant n}))$ making the following diagram commute (see Definition 4.8):

$$\begin{array}{ccc} V_q & \xrightarrow{\quad\xi\quad} & V_q \\ {\scriptstyle 0}\downarrow & & \downarrow{\scriptstyle 0} \\ \mathbb{T}_{q-1}(V_{\leqslant n}) & \xrightarrow{\gamma_{q-1}} & \mathbb{T}_{q-1}(V_{\leqslant n}) \end{array} \tag{20}$$

Therefore $\mathcal{D}_n^q$ is just the group $\operatorname{aut}(V_q) \times \operatorname{aut}(\mathbb{T}(V_{\leqslant n}))$ used in the previous section.

Proposition 4.10. *The map* $\mathrm{g}\colon \mathcal{E}(\mathbb{T}(V_q \oplus V_{\leqslant n})) \to \mathcal{D}_n^q$ *given by*

$$\mathrm{g}([\alpha]) = (\widetilde{\alpha}_q, [\alpha_n])$$

is a surjective homomorphism of groups.

Proof. First it is well known that if two chain morphisms are homotopic, then they induce the same graded linear maps on the chain complex of indecomposables, i.e., $\widetilde{\alpha} = \widetilde{\alpha'}$. Moreover, α_n, α'_n are homotopic and, using the diagram (17), we deduce that the map g is well defined.

Next let $(\xi,[\alpha_n]) \in \mathcal{D}_n^q$. Recall that, in the diagram (20), we have:

$$b_q(v) = \alpha_n \circ \partial(v) + \operatorname{Im}\partial_{\leqslant n},$$
$$b_q \circ \xi_q(v) = \partial \circ \xi(v) + \operatorname{Im}\partial_{\leqslant n},$$

where $\partial_{\leqslant n}\colon \mathbb{T}_q(V_{\leqslant n}) \to \mathbb{T}_{q-1}(V_{\leqslant n})$.

Since by Definition 4.8 this diagram commutes, the element $(\alpha_n \circ \partial - \partial \circ \xi)(v) \in$

$\operatorname{Im}\partial_{\leqslant n}$. As a consequence there exists $u_v \in \mathbb{T}_q(V^{\leqslant n})$ such that

$$(\alpha_n \circ \partial - \partial \circ \xi)(v) = \partial_{\leqslant n}(u_v). \tag{21}$$

Thus we define $\alpha\colon (\mathbb{T}(V_q \oplus V_{\leqslant n}), \partial) \to (\mathbb{T}(V_q \oplus V_{\leqslant n}), \partial)$ by setting

$$\alpha(v) = \xi(v) + u_v \quad \text{and} \quad \alpha = \alpha_n \text{ on } V_{\leqslant n}.$$

As $\partial(v) \in \mathbb{T}_{q-1}(V_{\leqslant n})$ then, by (21), we get

$$\partial \circ \alpha(v) = \partial(\xi(v)) + \partial_n(u_v) = \alpha_n \circ \partial(v) = \alpha \circ \partial(v).$$

So α is a chain algebra morphism. Now as $u_v \in \mathbb{T}_q(V_{\leqslant n})$ and $q > n$, the homomorphism $\widetilde{\alpha}_q\colon V_q \to V_q$ coincides with ξ.

Then it is well known (see [**1, 9**] and [**10**]) that any chain algebra morphism between two 1-connected chain algebras inducing a graded isomorphism on the homology of the chain complex of indecomposables (see 18) is a homotopy equivalence. Consequently, $[\alpha] \in \mathcal{E}(\mathbb{T}(V))$. Therefore g is onto.

Finally, the following relations

$$\begin{aligned} \mathrm{g}([\alpha][\alpha']) &= \mathrm{g}([\alpha \circ \alpha']) = (\widetilde{\alpha \circ \alpha'}_q, [\alpha_n \circ \alpha'_n]) \\ &= (\widetilde{\alpha}_q, [\alpha_n]) \circ (\widetilde{\alpha'}_q, [\alpha'_n]) = \mathrm{g}([\alpha]) \circ \mathrm{g}([\alpha']) \end{aligned}$$

assure that g is a homomorphism of groups. □

4.3. Characterization of ker g

Next by definition we have:

$$\ker \mathrm{g} = \Big\{ [\alpha] \in \mathcal{E}(\mathbb{T}(V_q \oplus V_{\leqslant n})) \mid \widetilde{\alpha}_q = id_{V_q},\ [\alpha_n] = [id_{\mathbb{T}(V_{\leqslant n})}] \Big\},$$

therefore for every $[\alpha] \in \ker \mathrm{g}$ we have:

$$\begin{aligned} \alpha(v) &= v + z, \quad z \in \mathbb{T}_q(V_{\leqslant n}), \\ \alpha_n &\simeq id_{\mathbb{T}(V_{\leqslant n})}. \end{aligned} \tag{22}$$

So define:

$$\theta_\alpha\colon V_q \to \mathbb{T}_q(V_{\leqslant n}) \quad \text{by } \theta_\alpha(v) = \alpha(v) - v. \tag{23}$$

Notice that the relations (22) and (23) imply that

$$\theta_{\alpha' \circ \alpha} = \theta_{\alpha'} + \theta_\alpha. \tag{24}$$

Remark 4.11. If the differential in the chain algebra $(\mathbb{T}(V_q \oplus V_{\leqslant n}))$ is trivial, then according to Remark 4.5 the formula (22) becomes

$$\begin{aligned} \alpha(v) &= v + z, \quad z \in \mathbb{T}_q(V_{\leqslant n}), \\ \alpha_n &= id_{\mathbb{T}(V_{\leqslant n})} \end{aligned}$$

implying that the element $\theta_\alpha(v) = z$ is a cycle in $\mathbb{T}_q(V_{\leqslant n})$. Notice that if the differential is not trivial, then $\theta_\alpha(v)$ need not be a cycle. However, we have the following crucial lemma:

Lemma 4.12. *Let* $[\alpha] \in \ker \mathrm{g}$. *Then there exists* $[\beta] \in \ker \mathrm{g}$ *satisfying:*

1. $\theta_\beta(v)$ *is a cycle in* $\mathbb{T}_q(V_{\leqslant n})$ *for every* $v \in V_q$

2. $\beta_n = id_{\mathbb{T}(V_{\leqslant n})}$
3. $[\beta] = [\alpha]$

Proof. Write $V = V_q \oplus V_{\leqslant n}$. Since, $[\alpha_n] = [id_{\mathbb{T}(V_{\leqslant n})}]$ there is a homotopy:

$$F\colon (\mathbb{T}(V'_{\leqslant n} \oplus V''_{\leqslant n} \oplus (sV)_{\leqslant n}), D) \to (\mathbb{T}(V_{\leqslant n}), \partial)$$

such that for every $x \in \mathbb{T}(V)$ we have

$$F \circ i'(x) = \alpha(x), \qquad F \circ i''(x) = x. \tag{25}$$

Thus we define β by setting:

$$\beta(v) = \begin{cases} \alpha(v) - F\big(S(\partial v)\big), & \text{for } v \in V_q; \\ v, & \text{for } v \in V_{\leqslant n}. \end{cases} \tag{26}$$

Notice that as $v \in V_q$, we deduce that $\partial v \in \mathbb{T}_{q-1}(V)$. It follows that $S(\partial v) \in \mathbb{T}_q(V'_{\leqslant n} \oplus V''_{\leqslant n} \oplus (sV)_{\leqslant n})$, so the element $F\big(S(\partial v)\big) \in \mathbb{T}_q(V_{\leqslant n})$.

Let us prove that β is a chain algebra morphism. Indeed first, for $v \in V_q$, using the relations (14), we deduce that

$$0 = D^2(sv) = D(v'' - v') - DS(\partial v). \tag{27}$$

Now by virtue of (25) and (26) we get

$$\begin{aligned}
\partial(\beta(v)) &= \partial \circ \alpha(v) - \partial \circ F\big(S(\partial v)\big) \\
&= \partial \circ \alpha(v) - F(DS(\partial v)) \\
&= \partial \circ \alpha(v) - F(D(v' - v'')) \\
&= \partial \circ \alpha(v) - F \circ i'(\partial v) + F \circ i''(\partial v) \\
&= \partial \circ \alpha(v) - \alpha \circ \partial(v) + \partial(v) \\
&= \partial(v) = \beta(\partial(v)).
\end{aligned}$$

Here we use (27) and the fact that $\partial(v) \in \mathbb{T}(V_{\leqslant n})$ and β is the identity on $V_{\leqslant n}$.

Consequently,

$$\partial(\theta_\beta(v)) = \partial(\beta(v) - v) = \partial(\beta(v)) - \partial(v) = \partial(v) - \partial(v) = 0.$$

Thus $\theta_\beta(v)$ is a cycle in $\mathbb{T}_q(V_{\leqslant n})$ and $\beta_n = id_{\mathbb{T}(V_{\leqslant n})}$. Next let us define

$$G\colon (\mathbb{T}(V' \oplus V'' \oplus (sV)), D) \to (\mathbb{T}(V), \partial)$$

by setting

$$\begin{aligned}
G(v') &= \alpha(v), && \text{on } V'_q, \\
G(v') &= \beta(v), && \text{on } V''_q, \\
G(sv) &= 0, && \text{on } (sV)_q, \\
G &= F, && \text{on } V'_{\leqslant n} \oplus V''_{\leqslant n} \oplus (sV)_{\leqslant n}.
\end{aligned} \tag{28}$$

Using (25) and (28), an easy computation shows

$$\begin{aligned}
\partial\circ G(v') &= \partial(\alpha(v)), & G\circ D(v') &= G\circ i'(\partial v) = F\circ i'(\partial v) = \alpha(\partial v),\\
\partial\circ G(v'') &= \partial(\beta(v)) = \partial v, & G\circ D(v'') &= G\circ i''(\partial v) = F\circ i''(\partial v) = \partial v,\\
\partial\circ G(sv) &= 0, & G\circ D(sv) &= G\circ(v'-v''-S(\partial v)) \qquad \text{by (14)}\\
&& &= G(v') - G(v'') - G(S(\partial v))\\
&& &= \alpha(v) - \beta(v) - G(S(\partial v)) \qquad \text{by (28)}\\
&& &= F\big(S(\partial v)\big) - G(S(\partial v))\\
&& &= 0.
\end{aligned}$$

Here we use the facts that $S(\partial v)\in\mathbb{T}_q(V'_{\leqslant n}\oplus V''_{\leqslant n}\oplus(sV)_{\leqslant n})$ and G and F coincide on $V'_{\leqslant n}\oplus V''_{\leqslant n}\oplus(sV)_{\leqslant n}$.

Finally, it easy to check (again by using (14)) that $G\circ i'(v)=\alpha(v)$ and $G\circ i''(v)=\beta(v)$ implying that $[\beta]=[\alpha]$. □

Thus Lemma 4.12 and the relation (23) allow us to define a map

$$\Phi\colon \ker \mathrm{g}\to \mathrm{Hom}\big(V_q, H_q(\mathbb{T}(V_{\leqslant n}))\big)$$

by setting $\Phi([\beta])(v)=\{\theta_\beta(v)\}$ for $v\in V_q$ where $[\beta]$ is chosen as in Lemma 4.12.

Proposition 4.13. *The map Φ is an isomorphism.*

Proof. Assume that $\Phi([\beta])(v)=\Phi([\beta'])(v)$ in $H_q(\mathbb{T}(V_{\leqslant n}))$, then $\theta_{\beta'}(v)-\theta_\beta(v)=\beta(v)-\beta'(v)$ is a boundary and Lemma 4.4 implies that $[\beta]=[\beta']$. Hence Φ is one to one.

Given a homomorphism $\chi\in\mathrm{Hom}\big(V_q,H_q(\mathbb{T}(V_{\leqslant n}))\big)$ and write $\chi(v)=\{\widetilde{\chi(v)}\}$, where $\widetilde{\chi(v)}$ is a cycle. We define $\beta\colon(\mathbb{T}(V),\partial)\to(\mathbb{T}(V),\partial)$ by:

$$\beta(v)=v+\widetilde{\chi(v)} \quad \text{for } v\in V_q \quad \text{and} \quad \beta=id \quad \text{on } V_{\leqslant n}.$$

Then β is a chain algebra morphism with $\Phi([\beta])=\chi$. Hence Φ is onto.

Finally, given $\beta,\beta'\in\ker\mathrm{g}$ as in Lemma 4.12. So $\beta(v)=v+\theta_\beta(v)$ and $\beta'(v)=v+\theta_{\beta'}(v)$ for $v\in V_q$. Therefore by (24) we get:

$$\beta'\circ\beta(v)=v+\theta_{\beta'}(v)+\theta_\beta(v)=v+\theta_{\beta'\circ\beta}(v).$$

Consequently, $\Phi([\beta'].[\beta])=\Phi([\beta'\circ\beta])=\theta_{\beta'\circ\beta}=\theta_{\beta'}+\theta_\beta=\Phi([\beta'])+\Phi([\beta])$. Thus Φ is a homomorphism of groups. □

Summarizing, we have proven:

Theorem 4.14. *Let $(\mathbb{T}(V_q\oplus V_{\leqslant n}),\partial)$ be a 1-connected chain algebra. Then there exists a short exact sequence of groups*

$$\mathrm{Hom}\big(V_q,H_q(\mathbb{T}(V_{\leqslant n}))\big)\rightarrowtail\mathcal{E}(\mathbb{T}(V_q\oplus V_{\leqslant n}))\twoheadrightarrow\mathcal{D}_n^q. \tag{29}$$

We now focus on the subgroup $\mathcal{E}_*(\mathbb{T}(V_q\oplus V_{\leqslant n}))$ of $\mathcal{E}(\mathbb{T}(V_q\oplus V_{\leqslant n}))$ formed of the elements inducing the identity on the graded homology module $H_*(V,d)$. Let us define $\mathcal{G}_n^q$ as the subgroup of $\mathcal{E}_*(\mathbb{T}(V_{\leqslant n}))$ consisting of the elements $[\alpha]$ satisfying $H_{q-1}(\alpha)\circ b_q=b_q$ where $b_q\colon V_q\to H_{q-1}(\mathbb{T}(V_{\leqslant n}))$ is as in (16).

Theorem 4.15. *Let $q>n$ and let $(\mathbb{T}(V_q\oplus V_{\leqslant n}),\partial)$ be a 1-connected chain algebra. Then there exists a short exact sequence of groups*

$$\mathrm{Hom}\big(V_q,H_q(\mathbb{T}(V_{\leqslant n}))\big)\rightarrowtail\mathcal{E}_*(\mathbb{T}(V_q\oplus V_{\leqslant n}))\twoheadrightarrow\mathcal{G}_n^q.$$

Proof. First let $[\beta] \in \ker \mathrm{g}$. Lemma 4.12 assures that $\widetilde{\alpha}_q = id_{V_q}$ and $\alpha_n = id_{\mathbb{T}(V_{\leqslant n})}$, therefore $\widetilde{\alpha} = id_V$. It follows that $\ker \mathrm{g} \subseteq \mathcal{E}_*(\mathbb{T}(V_q \oplus V_{\leqslant n}))$.

Next from (29) we obtain

$$\mathrm{g}\Big(\mathcal{E}_*(\mathbb{T}(V_q \oplus V_{\leqslant n}))\Big) = \Big\{\Psi([\alpha]) = (\widetilde{\alpha}_q, [\alpha_n]) \mid [\alpha] \in \mathcal{E}_*(\mathbb{T}(V_q \oplus V_{\leqslant n}))\Big\}.$$

As $[\alpha] \in \mathcal{E}_*(\mathbb{T}(V_q \oplus V_{\leqslant n}))$, the graded automorphism $H_*(\widetilde{\alpha})$ is the identity which, in turn, implies that $\widetilde{\alpha}_q = \mathrm{id}_{V^q}$ and, as the pair $(\mathrm{id}_{V^q}, [\alpha_n])$ makes the diagram (17) commute, we can identify $\mathrm{g}\big(\mathcal{E}_\sharp(\Lambda(V^q \oplus V^{\leqslant n}))\big)$ with the subgroup $\mathcal{G}^q_n$. □

Corollary 4.16. *Let $(\mathbb{T}(V_q \oplus V_{\leqslant n}), \partial)$ be a 1-connected chain algebra. If $\mathcal{E}_*(\mathbb{T}(V_{\leqslant n}))$ is trivial, then:*

$$\mathrm{Hom}\big(V_q, H_q(\mathbb{T}(V_{\leqslant n}))\big) \cong \mathcal{E}_*(\mathbb{T}(V_q \oplus V_{\leqslant n})).$$

Corollary 4.17. *Let $(\mathbb{T}(V_{2n} \oplus V_{\leqslant 2n-1}), \partial)$ be an n-connected chain algebra, i.e., $V_k = 0$ for $k < n$. Then*

$$\mathrm{Hom}\big(V_{2n}, V_n \otimes V_n\big) \cong \mathcal{E}_*(\mathbb{T}(V_{2n} \oplus V_{\leqslant 2n-1})). \tag{30}$$

Proof. First, as $(\mathbb{T}(V_{2n} \oplus V_{\leqslant 2n-1}), \partial)$ is n-connected, the group $\mathcal{E}_*(\mathbb{T}(V_{\leqslant 2n-1}))$ is trivial. Next clearly $H_{2n}(\mathbb{T}(V_{\leqslant 2n-1})) = V_n \otimes V_n$, hence (30) follows from Corollary 4.16. □

Corollary 4.18. *Let $(\mathbb{T}(V_q \oplus V_{\leqslant n}), \partial)$ be a 1-connected chain algebra. If the group $\mathcal{E}(\mathbb{T}(V_q \oplus V_{\leqslant n}))$ is finite, then the linear map b_q is injective.*

Proof. Assume that b_q is not injective and let $v_0 \neq 0 \in V_q$ be such that $b_q(v_0) = 0$. For every $a \neq 0 \in \mathbb{Q}$, we define $\xi_a \colon V_q \to V_q$ by

$$\xi(v_0) = av_0, \qquad \xi_a = id \quad \text{otherwise.}$$

Clearly the pair $(\xi_a, [id]) \in \mathrm{aut}(V_q) \times \mathcal{E}(\mathbb{T}(V_{\leqslant n}))$ for every $a \neq 0 \in \mathbb{Q}$ and makes following diagram commute

$$\begin{array}{ccc} V^q & \xrightarrow{\quad \xi^a \quad} & V^q \\ {\scriptstyle b_q}\downarrow & & \downarrow{\scriptstyle b_q} \\ H_{q-1}(\Lambda V^{\leqslant n}) & \xrightarrow{\quad id \quad} & H_{q-1}(\Lambda V^{\leqslant n}) \end{array}$$

Therefore $(\xi_a, [id]) \in \mathcal{D}^q_{n-1}$ for every $a \neq 0 \in \mathbb{Q}$ implying that the group $\mathcal{D}^q_n$ is infinite. Consequently, the group $\mathcal{E}(\mathbb{T}(V_q \oplus V_{\leqslant n}))$ is also infinite according the exact sequence (29). □

4.4. r-Mild differential graded Lie algebras

Let $R \subseteq \mathbb{Q}$ be a ring such that, for some prime p, R contains n^{-1} for $n < p$. A free differential graded Lie algebra $(\mathbb{L}(V), \partial)$ over R is called r-mild if

$$V_k = 0, \quad k \leqslant r-1, \quad k \geqslant pr-1.$$

Recall that, in [**1**], Anick defined a reasonable concept of "homotopy" among morphisms between free R-dgls, analogous in many respects to the topological notion of homotopy and proved the following result:

Theorem 4.19 ([**1**] proposition 3.3)**.** *Let $f, g\colon (\mathbb{L}(V), \partial) \to (\mathbb{L}(W), \partial)$ be two r-mild free differential graded Lie algebras. Then f, g are homotopic as dgl-morphisms if and only if Uf, Ug are homotopic as chain algebra morphisms.*

Proposition 4.20. *Let $(\mathbb{L}(V), \partial)$ be an r-mild free differential graded Lie algebra. For any $r \leqslant n < q < rp$, the homomorphism*

$$\phi_n^q\colon \mathcal{E}(\mathbb{L}(V_q \oplus V_{\leqslant n})) \to \mathcal{E}(\mathbb{T}(V_q \oplus V_{\leqslant n})), \quad \phi_n^q([\alpha]) = [U(\alpha)]$$

is injective.

Proof. First, it is well known that if α is equivalence of homotopy, then so is $U(\alpha)$, hence ϕ_n^q is well defined.

Next if $U(\alpha) \simeq id_{\mathbb{T}(V_q \oplus V_{\leqslant n})}$, then from Theorem 4.19 we deduce that $\alpha \simeq id_{\mathbb{L}(V_q \oplus V_{\leqslant n})}$. Therefore ϕ_n^q is injective. □

Using the same argument we can deduce

Corollary 4.21. *Let $(\mathbb{L}(V), \partial)$ be an r-mild free differential graded Lie algebra. For any $r \leqslant n < q < rp$, the homomorphism*

$$\psi_n^q\colon \mathcal{E}_*(\mathbb{L}(V_q \oplus V_{\leqslant n})) \to \mathcal{E}_*(\mathbb{T}(V_q \oplus V_{\leqslant n})), \quad \psi_n^q([\alpha]) = [U(\alpha)]$$

is injective.

5. Topological applications

Let X be a simply connected CW-complex of dimension $n+1$. For $q > n$ let

$$Y = X \cup_\alpha \Big(\bigcup_{i\in I} e_i^{q+1}\Big) \tag{31}$$

be the space obtained by attaching cells of dimension $q+1$ to X by a map $\alpha\colon \bigvee_{i\in I} \mathbb{S}^{q+1} \to X$.

Recall that the Adams–Hilton model of Y is a chain algebra morphism

$$\Theta_Y\colon (\mathbb{T}(V_q \oplus V_{\leqslant n}), \partial) \to C_*(\Omega Y, R)$$

such that

$$H_*(\Theta_Y)\colon H_*(\mathbb{T}(V_q \oplus V_{\leqslant n}), \partial) \to H_*(\Omega Y, R)$$

is an isomorphism of graded algebras and such that

$$H_{i-1}(V_q \oplus V_{\leqslant n}, d) \cong H_i(Y, R), \quad \text{as graded modules.}$$

Here $C_*(\Omega Y, R)$ denotes the complex of non-degenerate cubic chains equipped with the multiplication induced by composition of loops. We denote by $A(Y)$ the chain algebra $(\mathbb{T}(V_q \oplus V_{\leqslant n}), \partial)$.

Notice also that the free module V_i admits a basis consisting of the cells of dimension $i+1$ of Y and the differential ∂ is determined by the attaching maps of the cells (see for example [**1**] for more details). Consequently, we have

$$V_q \cong H_{q+1}(Y, X; R),$$

where $H_{q+1}(Y, X; R)$ denotes the free R-module of the homology of the pair (Y, X) in degree $q+1$.

By the properties of the Adams–Hilton model, the chain algebra $(\mathbb{T}(V_{\leqslant n}), \partial)$ may be considered as the Adams–Hilton model of X, i.e., $A(X) = (\mathbb{T}(V_{\leqslant n}), \partial)$. Moreover, if $[f] \in \mathcal{E}(A(X))$, then f induces the following commutative diagram

$$\begin{array}{ccc} H_{q-1}(\mathbb{T}(V_{\leqslant n})) & \xrightarrow{H_{q-1}(f)} & H_{q-1}(\mathbb{T}(V_{\leqslant n})) \\ \cong \downarrow H_{q-1}(\Theta_X) & & \cong \downarrow H_{q-1}(\Theta_X) \\ H_{q-1}(\Omega X, R) & \xrightarrow{H_{q-1}(\Theta_X)\circ H_{q-1}(f)\circ (H_{q-1}(\Theta_X))^{-1}} & H_{q-1}(\Omega X, R) \end{array}$$

Definition 5.1. Set $\widetilde{H}_{q-1}(f) = H_{q-1}(\Theta_X) \circ H_{q-1}(f) \circ (H_{q-1}(\Theta_X))^{-1}$. We define Γ_n^{q+1} to be the subset of $\operatorname{aut}\big(H_{q+1}(Y;R)\big) \times \mathcal{E}(A(X))$ consisting of the pairs $(\xi, [f])$ making the following diagram commute

$$\begin{array}{ccc} V_q \cong H_{q+1}(Y, X; R) & \xrightarrow{\xi} & H_{q+1}(Y, X; R) \cong V_q \\ b_q \downarrow & & \downarrow b_q \\ H_{q-1}(\Omega X, R) & \xrightarrow{\widetilde{H}_{q-1}(f)} & H_{q-1}(\Omega X, R) \end{array}$$

and

$$\Pi_n^{q+1} = \Big\{ [f] \in \mathcal{E}_*(A(X)) \mid \widetilde{H}_{q-1}(f) \circ b_q = b_q \Big\}.$$

Clearly Γ_n^{q+1} is a subgroup of $\operatorname{aut}\big(H_{q+1}(Y, X; R)\big) \times \mathcal{E}(A(X))$ and Π_n^{q+1} is a subgroup of $\mathcal{E}_*(A(X))$.

Remark 5.2. It is important to notice that if the homomorphism b_q is trivial, then

$$\Gamma_n^{q+1} = \operatorname{aut}\big(H_{q+1}(Y, X; R)\big) \times \mathcal{E}(A(X)), \qquad \Pi_n^{q+1} = \mathcal{E}_*(A(X))$$

and if b_q is an isomorphism, then from the commutative diagram (30) we deduce that $\xi = (b_q)^{-1} \circ H_{q-1}(f) \circ b_q$. Therefore the map

$$\mathcal{E}(A(X)) \to \Gamma_n^{q+1}, \qquad [f] \longmapsto \Big((b_q)^{-1} \circ H_{q-1}(f) \circ b_q, [\alpha] \Big)$$

is an isomorphism. In this case, if $[f] \in \Pi_n^{q+1}$, then $\widetilde{H}_{q-1}(f) \circ b_q = b_q$ and as b_q is an isomorphism it follows that $\widetilde{H}_{q-1}(f) = id$. Consequently,

$$\Pi_n^{q+1} = \Big\{ [f] \in \mathcal{E}_*(A(X)) \mid \widetilde{H}_{q-1}(f) = id \Big\}.$$

Theorem 5.3. *Let X be a simply connected CW-complex of dimension $n+1$ and let Y be as in* (31). Then there exist two short exact sequences of groups

$$\bigoplus_i H_q(\Omega X, R) \rightarrowtail \mathcal{E}(A(Y)) \twoheadrightarrow \Gamma_n^{q+1}, \qquad \bigoplus_i H_q(\Omega X, R) \rightarrowtail \mathcal{E}_*(A(Y)) \twoheadrightarrow \Pi_n^{q+1}. \quad (32)$$

Proof. The two sequences (32) follow from a mere transcription of Theorems 4.14 and 4.15 in the topological context using the properties of the Adams–Hilton model. Note that this model implies the identifications $\Gamma_n^{q+1} \cong \mathcal{D}_n^q$ and $\Pi_n^{q+1} \cong \mathcal{G}_n^q$. □

Combining Remark 5.2 and Theorem 5.3 we derive the following results:

Corollary 5.4. *Let X be a simply connected CW-complex of dimension $n+1$ and let Y be as in* (31). If the homomorphism $b_q\colon H_{q+1}(Y,X;R)\to H_{q-1}(\Omega X,R)$ is bijective, then there exist two short exact sequences of groups

$$\begin{aligned}&\bigoplus_i H_q(\Omega X,R)\rightarrowtail \mathcal{E}(A(Y))\twoheadrightarrow \mathcal{E}(A(X)),\\ &\bigoplus_i H_q(\Omega X,R)\rightarrowtail \mathcal{E}_*(A(Y))\twoheadrightarrow \Big\{[f]\in\mathcal{E}_*(A(X))\mid \widetilde{H}_{q-1}(f)=id\Big\}.\end{aligned}$$

Corollary 5.5. *Let X be a simply connected CW-complex of dimension $n+1$ and let Y be as in* (31). If the homomorphism $b_q\colon H_{q+1}(Y,X;R)\to H_{q-1}(\Omega X,R)$ is trivial, then there exist two short exact sequences of groups

$$\begin{aligned}&\bigoplus_i H_q(\Omega X,R)\rightarrowtail \mathcal{E}(A(Y))\twoheadrightarrow \operatorname{aut}\big(H_{q+1}(Y,X;R)\big)\times\mathcal{E}(A(X)),\\ &\bigoplus_i H_q(\Omega X,R)\rightarrowtail \mathcal{E}_*(A(Y))\twoheadrightarrow \mathcal{E}_*(A(X)).\end{aligned}$$

As a consequence of Corollaries 5.4 and 5.5 we derive

Corollary 5.6. *Let X be a simply connected CW-complex of dimension $n+1$ and let Y be as in* (31).

If the homomorphism b_q is trivial, then

$$\frac{\mathcal{E}(A(Y))}{\mathcal{E}_*(A(Y))}\cong \operatorname{aut}\big(H_{q+1}(Y,X;R)\big)\times\frac{\mathcal{E}(A(X))}{\mathcal{E}_*(A(X))}.$$

If b_q is an isomorphism, then

$$\frac{\mathcal{E}(A(Y))}{\mathcal{E}_*(A(Y))}\cong\frac{\mathcal{E}(A(X))}{\Big\{[f]\in\mathcal{E}_*(A(X))\mid \widetilde{H}_{q-1}(f)=id\Big\}}.$$

5.1. Anick model

We assume that $R\subseteq\mathbb{Q}$ is a ring with least non-invertible prime $p>2$. With R fixed, we take $1\leqslant r<k$ satisfying $k<min(r+2p-3,rp-1)$. When $R=\mathbb{Q}$ we assume $r=1$ and k is infinite.

Let X be an r-connected finite CW-complex of dimension $n+1\leqslant k$. For $k\geqslant q>n+1$, let

$$Y=X\cup_\alpha\Big(\bigcup_{i\in I}e_i^{q+1}\Big) \tag{33}$$

as in (31). Recall that the Anick model of Y (see [**1, 2**] for more details) is a free differential graded Lie algebra $(\mathbb{L}(V_q\oplus V_{\leqslant n}),\partial)$ over R such that

$$H_{*-1}(\mathbb{L}(V_q\oplus V_{\leqslant n}),\partial)\cong\pi_*(Y,)\otimes R,\qquad H_{*-1}(V_q\oplus V_{\leqslant n},d)\cong H_*(Y,R).$$

Moreover, by properties of this model we deduce

$$\mathcal{E}_*(\mathbb{L}(V_q\oplus V_{\leqslant n}))\cong\mathcal{E}_*(X_R),\qquad \mathcal{E}(\mathbb{L}(V_q\oplus V_{\leqslant n}))\cong\mathcal{E}(X_R).$$

Here X_R denotes the R-localization of X. From Proposition 4.20 and Corollary 4.21 we derive the following result

Theorem 5.7. *Let Y be the space in (33). The homomorphisms*

$$\mathcal{E}(Y_R) \to \mathcal{E}(A(Y_R)), \qquad \mathcal{E}_*(Y_R) \to \mathcal{E}_*(A(Y_R))$$

are injective.

References

[**1**] D.J. Anick, *Hopf algebras up to homotopy*, J. Amer. Math. Soc. (1989), 2(3): 417–452.

[**2**] D.J. Anick, *An R-local Milnor-Moore theorem*, Adv. Math. (1989), 77: 116–136.

[**3**] W. Barcus and M.G. Barratt, *On the homotopy classification of the extensions of a fixed map*, Trans. Amer. Math. Soc. (1958), 88: 57–74.

[**4**] H.J. Baues, *Homotopy type and homology*, Oxford Math. Monogr., Oxford University Press, New York, 1996.

[**5**] M. Benkhalifa, *On the group of self-homotopy equivalences of n-connected and $(3n+2)$-dimensional CW-Complex*, Topology Appl. (2018), 233: 1–15.

[**6**] M. Benkhalifa, *Postnikov decomposition and the group of self-equivalences of a rationalized space*, Homology Homotopy Appl. (2017), 19(1): 209–224.

[**7**] M. Benkhalifa, *Realizability of the group of rational self-homotopy equivalences*, J. Homotopy Relat. Struct. (2011), 5(1): 361–372.

[**8**] M. Benkhalifa, *Rational self-homotopy equivalences and Whitehead exact sequence*, J. Homotopy Relat. Struct. (2009), 4: 111–121.

[**9**] M. Benkhalifa, *Whitehead exact sequence and differential graded free Lie algebra*, J. Math. (2004), 10: 987–1005.

[**10**] M. Benkhalifa, *On the homotopy type of a chain algebra*, Homology Homotopy Appl. (2004), 6(1): 109–135.

[**11**] M. Benkhalifa and S.B. Smith, *The effect of cell attachment on the group of self-equivalences of an R-local space*, J. Homotopy Relat. Struct. (2015), 4(1): 135–144.

[**12**] R. Bott and H. Samelson, *On the Pontryagin product in spaces of paths*, Comment. Math. Helv. (1953), 27(1): 320–337.

[**13**] P.J. Kahn, *Self-equivalences of $(n-1)$-connected $2n$-manifolds*, Bull. Amer. Math. Soc. (1966), 72: 562–566.

[**14**] S. Oka, N. Sawashita and M. Sugawara, *On the group of self-equivalences of a mapping cone*, Hiroshima Math. J. (1974), 4: 9–28.

[**15**] J. Rutter, *The group of homotopy self-equivalence classes of CW-complexes*, Math. Proc. Cambridge Philos. Soc. (1983), 93: 275–293.

[**16**] J. Rutter, *Spaces of homotopy self-equivalences: A survey*, Lecture Notes in Math., vol. 162, Springer-Verlag, Berlin (1997).

Mahmoud Benkhalifa `mbenkhalifa@sharjah.ac.ae`

Department of Mathematics, Faculty of Sciences, University of Sharjah, Sharjah, United Arab Emirates