

BOX-HOMOTOPY AND THE REDUCTION OF pro^* - $HTop$ CATEGORY

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Abstract

A new generalized definition of homotopy is proposed, including a new class of continuous mappings that we call box-homotopies. It turns out that to be box-homotopic is an equivalence relation on $Top(X, Y)$ and $HTop(X, Y)$, and that it is well adjusted with composition, which allows us to introduce a new category $H_{\square}Top$, the corresponding quotient category of the category $HTop$, and consequently, pro - $H_{\square}Top$. We propose a new functor \tilde{R} from pro^* - $HTop$ to pro - $H_{\square}Top$, which represents morphisms in pro^* -category as morphisms in pro -category between more complex objects.

1. Introduction

In this article we observe a reduced power of a topological space, an object that can be found in studies of Rudin, Kunen and Bankston (see, for example, [6, 3] or [1]). It is a product of topological space X by itself countably many times, given the box topology, and reduced to a quotient space by an equivalence relation saying that two sequences of elements in X are related if they differ at most on a finite number of coordinates. This quotient space will be denoted by \tilde{X} or $\prod_{\mathbb{N}} X$.

We list some of its properties. For instance, it preserves separation properties from the original space. Also, in \tilde{X} , intersections of countably many open sets, in other words G_{δ} sets, are open. In the literature such a space is called a P -space in the sense of Gillman–Henriksen.

In the sequel we propose a generalization of the notion of homotopy, a relation that we call box-homotopy and denote by \sim_{\square} . Naturally, all homotopic maps are box-homotopic, and we provide an example showing that the converse does not hold. Then we construct a new category, $H_{\square}Top$, with topological spaces as objects and box-homotopy classes of continuous maps as morphisms, as it proves out that box-homotopy is an equivalence relation on $Top(X, Y)$, and that it is well adjusted with the composition.

Finally we propose a new functor \tilde{R} from pro^* - $HTop$ to pro - $H_{\square}Top$, which represents morphisms of pro^* -category as morphisms of pro -category between more complex objects. For instance, inverse systems in Example 7.2 of [2] are not isomorphic

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in $pro\text{-}HTop$, but they are isomorphic in $pro^*\text{-}HTop$, and consequently, their images under \tilde{R} are isomorphic in $pro\text{-}H\Box Top$.

2. Preliminaries

Let us provide a brief survey of the required foreknowledge (see [2]).

Let $\mathbf{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda)$ and $\mathbf{Y} = (Y_\mu, q_{\mu\mu'}, M)$ be two inverse systems in some category \mathcal{C} and $f: M \rightarrow \Lambda$ a function called the index function. A morphism $(f, f_\mu): \mathbf{X} \rightarrow \mathbf{Y}$ consists of the function f and a set of morphisms $f_\mu: X_{f(\mu)} \rightarrow Y_\mu$, one for each $\mu \in M$, in \mathcal{C} , such that, for every pair $\mu, \mu' \in M$, $\mu \leq \mu'$, there exists a $\lambda \in \Lambda$, $\lambda \geq f(\mu)$, $f(\mu')$ such that

$$f_\mu p_{f(\mu)\lambda} = q_{\mu\mu'} f_{\mu'} p_{f(\mu')\lambda}.$$

The category $inv\text{-}\mathcal{C}$ has all inverse systems for objects, and all morphisms (f, f_μ) between inverse systems as morphisms.

A morphism $(f, f_\mu): \mathbf{X} \rightarrow \mathbf{Y}$ is equivalent to a morphism $(f', f'_\mu): \mathbf{X} \rightarrow \mathbf{Y}$, denoted by $(f, f_\mu) \sim (f', f'_\mu)$, if for every $\mu \in M$ there exists a $\lambda \in \Lambda$, $\lambda \geq f(\mu)$, $f'(\mu)$ such that $f_\mu p_{f(\mu)\lambda} = f'_\mu p_{f'(\mu)\lambda}$. The category $pro\text{-}\mathcal{C}$ has all inverse systems for objects, and all equivalence classes $\mathbf{f} = [(f, f_\mu)]$ of all morphisms (f, f_μ) between inverse systems as morphisms.

There are another two categories with the same objects, but with richer morphism classes, containing the previous two as subcategories and we shall need one of them. Let $\mathbf{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda)$ and $\mathbf{Y} = (Y_\mu, q_{\mu\mu'}, M)$ be two inverse systems in some category \mathcal{C} and $f: M \rightarrow \Lambda$ a function called the index function. A morphism $(f, f_\mu^n): \mathbf{X} \rightarrow \mathbf{Y}$ consists of the function f and a set of morphisms $f_\mu^n: X_{f(\mu)} \rightarrow Y_\mu$, $n \in \mathbb{N}$, $\mu \in M$, in \mathcal{C} , such that, for every pair $\mu, \mu' \in M$, $\mu \leq \mu'$, there exists $\lambda \in \Lambda$, $\lambda \geq f(\mu)$, $f(\mu')$ and $n \in \mathbb{N}$ such that for every $n' \geq n$,

$$f_\mu^{n'} p_{f(\mu)\lambda} = q_{\mu\mu'} f_{\mu'}^{n'} p_{f(\mu')\lambda}.$$

The category $inv^*\text{-}\mathcal{C}$ has all inverse systems for objects, and all morphisms (f, f_μ^n) between inverse systems as morphisms.

A morphism $(f, f_\mu^n): \mathbf{X} \rightarrow \mathbf{Y}$ is equivalent to a morphism $(f', f_{\mu'}^{n'}): \mathbf{X} \rightarrow \mathbf{Y}$, denoted by $(f, f_\mu^n) \sim (f', f_{\mu'}^{n'})$, if for every $\mu \in M$ there exists a $\lambda \in \Lambda$, $\lambda \geq f(\mu)$, $f'(\mu)$ and $n \in \mathbb{N}$ such that $f_\mu^{n'} p_{f(\mu)\lambda} = f_{\mu'}^{n'} p_{f'(\mu)\lambda}$, for every $n' \geq n$. The category $pro^*\text{-}\mathcal{C}$ has all inverse systems for objects, and all equivalence classes $\mathbf{f}^* = [(f, f_\mu^n)]$ of all morphisms (f, f_μ^n) between inverse systems as morphisms.

By a mapping that holds inverse systems fixed, and that sends a morphism $\mathbf{f} = [(f, f_\mu)]$ in $pro\text{-}\mathcal{C}$ to a morphism $\mathbf{f}^* = [(f, f_\mu^n)]$ in $pro^*\text{-}\mathcal{C}$, where $f_\mu^n = f_\mu$ for every $n \in \mathbb{N}$, the functor $\underline{J}_{\mathcal{C}}: pro\text{-}\mathcal{C} \rightarrow pro^*\text{-}\mathcal{C}$ is given.

For a given pair of categories $(\mathcal{C}, \mathcal{D})$, where \mathcal{D} is dense in \mathcal{C} , the shape category $Sh_{(\mathcal{C}, \mathcal{D})}$ has objects of \mathcal{C} as objects, and its morphisms are $F = \langle \mathbf{f} \rangle = \langle [(f, f_\mu)] \rangle: X \rightarrow Y$ for $\mathbf{f} = [(f, f_\mu)]: \mathbf{X} \rightarrow \mathbf{Y}$, where \mathbf{X} and \mathbf{Y} are systems associated with X and Y (respectively) by \mathcal{D} -expansions. Here, $\mathbf{f} \sim \mathbf{f}'$ means that, if $\mathbf{i}: \mathbf{X} \rightarrow \mathbf{X}'$ and $\mathbf{j}: \mathbf{Y} \rightarrow \mathbf{Y}'$ are canonical isomorphisms between expansions of the same object, $\mathbf{f}'\mathbf{i} = \mathbf{j}\mathbf{f}$ in $pro\text{-}\mathcal{D}$. Obviously, the realizing category for the category $Sh_{(\mathcal{C}, \mathcal{D})}$ is the category $pro\text{-}\mathcal{D}$. Specially, the category $HPol$, the homotopy category of polyhedra, is dense in

the category $HTop$, and the realizing category for the category $Sh_{(HTop,HPol)} \equiv Sh$ is the category $pro\text{-}HPol$.

For every $f: X \rightarrow Y$ of \mathcal{C} and every pair of expansions $p: X \rightarrow \mathbf{X}$ and $q: Y \rightarrow \mathbf{Y}$ there exists a unique $\mathbf{f}: \mathbf{X} \rightarrow \mathbf{Y}$ of $pro\text{-}\mathcal{D}$ such that

$$\begin{array}{ccc} \mathbf{X} & \xleftarrow{p} & X \\ \mathbf{f} \downarrow & & \downarrow f \\ \mathbf{Y} & \xleftarrow{q} & Y \end{array}$$

commutes in $pro\text{-}\mathcal{C}$. The shape functor $S: \mathcal{C} \rightarrow Sh_{(\mathcal{C},\mathcal{D})}$ is defined by $S(X) = X$, $X \in Ob(\mathcal{C})$, and $S(f) = F = \langle \mathbf{f} \rangle$, $f \in Mor(\mathcal{C})$.

3. Reduced power of topological space

Let \mathcal{F} be a countable filter on \mathbb{N} that consists of all $A \subseteq \mathbb{N}$ such that $\mathbb{N} \setminus A$ is finite. Let X be a set. We will denote by \tilde{X} the quotient set $\prod_{\mathbb{N}} X / \sim$, where $(x_n) \sim (x'_n)$ for $(x_n), (x'_n) \in \prod_{\mathbb{N}} X$ if $\{n \mid x_n = x'_n\} \in \mathcal{F}$.

We want to define a topology on \tilde{X} when (X, \mathcal{T}) is a topological space. It is natural to take a quotient topology which we get from $\prod_{\mathbb{N}} X$ by projection. Let us show that the box topology on the cartesian product $\prod_{\mathbb{N}} X$ is, for our purpose, a better choice to introduce a topology on \tilde{X} than the commonly used product topology.

Lemma 3.1. *Let (X, \mathcal{T}) be a topological space, and let $\prod_{\mathbb{N}} X$ have the product topology \mathcal{T}_p . Then the quotient topology on \tilde{X} is indiscrete.*

Proof. A collection $\mathcal{B}_p = \left\{ \prod_{\mathbb{N}} U_n \mid U_n \in \mathcal{T}, U_n \neq X \text{ for only finitely many } n \right\}$ is a basis for \mathcal{T}_p . Let $V \in \mathcal{B}_p$. Hence, $V = \prod_{\mathbb{N}} U_n$ where $U_n \in \mathcal{T}$ and $\{n \mid U_n = X\} \in \mathcal{F}$. Also, let $p: \prod_{\mathbb{N}} X \rightarrow \tilde{X}$ be a projection. Then

$$\begin{aligned} p(V) &= p\left(\prod_{\mathbb{N}} U_n\right) = \{[(x_n)] \in \tilde{X} \mid \{n \mid x_n \in U_n\} \in \mathcal{F}\} \\ &= \{[(x_n)] \in \tilde{X} \mid \{n \mid x_n \in X\} \in \mathcal{F}\} = \tilde{X}. \end{aligned}$$

If $U \subseteq \tilde{X}$ is open in \tilde{X} , $U \neq \emptyset$, then $p^{-1}(U)$ is open in $\prod_{\mathbb{N}} X$. Therefore, $p^{-1}(U) = \bigcup_{V \in \mathcal{S}} V$, $\mathcal{S} \subseteq \mathcal{B}_p$. We have

$$p(p^{-1}(U)) = p\left(\bigcup_{V \in \mathcal{S}} V\right) = \bigcup_{V \in \mathcal{S}} p(V) = \tilde{X}.$$

Since $p(p^{-1}(U)) \subseteq U$, we get $\tilde{X} \subseteq U$, and consequently, $\tilde{X} = U$. \square

We will write $\nabla_{\mathbb{N}} U_n \equiv p\left(\prod_{\mathbb{N}} U_n\right) = \{[(x_n)] \in \tilde{X} \mid \{n \mid x_n \in U_n\} \in \mathcal{F}\}$ for $U_n \subseteq X$, $n \in \mathbb{N}$.

Lemma 3.2. *If the cartesian product $\prod_{\mathbb{N}} X$ is provided with the box topology \mathcal{T}_b , the collection*

$$\tilde{\mathcal{B}} = \left\{ \nabla_{\mathbb{N}} U_n \mid U_n \in \mathcal{T}, \text{ for every } n \in \mathbb{N} \right\}$$

is a basis for the quotient topology on \tilde{X} .

Proof. Let us first show that $\tilde{\mathcal{B}}$ is indeed a topology basis. Let $[(x_n)] \in \tilde{X}$. Since \mathcal{T} is a cover of X , for every $n \in \mathbb{N}$ there is $U_n \in \mathcal{T}$ such that $x_n \in U_n$. Thus, $[(x_n)] \in \nabla_{\mathbb{N}} U_n$, so $\tilde{\mathcal{B}}$ is a cover of \tilde{X} .

Let $\nabla_{\mathbb{N}} U_n, \nabla_{\mathbb{N}} V_n \in \tilde{\mathcal{B}}$. Then,

$$\begin{aligned} \nabla_{\mathbb{N}} U_n \cap \nabla_{\mathbb{N}} V_n &= \{[(x_n)] \in \tilde{X} \mid \{n \mid x_n \in U_n\} \in \mathcal{F}\} \cap \{[(x_n)] \in \tilde{X} \mid \{n \mid x_n \in V_n\} \in \mathcal{F}\} \\ &= \{[(x_n)] \in \tilde{X} \mid \{n \mid x_n \in U_n \cap V_n\} \in \mathcal{F}\} \\ &= \nabla_{\mathbb{N}} (U_n \cap V_n), \end{aligned}$$

implying that the intersection of any two elements of $\tilde{\mathcal{B}}$ is an element of $\tilde{\mathcal{B}}$.

Let us prove that $\tilde{\mathcal{B}}$ generates the quotient topology

$$\tilde{\mathcal{T}} = \left\{ U \subseteq \tilde{X} \mid p^{-1}(U) \subseteq \prod_{\mathbb{N}} X \text{ is open in } \mathcal{T}_b \right\}.$$

Let $U \in \tilde{\mathcal{T}}$. That means, if $\mathcal{T} = \{U_\lambda, \lambda \in \Lambda\}$, that $p^{-1}(U) = \bigcup_{\alpha \in A} \prod_{\mathbb{N}} U_{\alpha_n}$ for some $A \subseteq \Lambda^{\mathbb{N}}$ and $U_{\alpha_n} \in \mathcal{T}$ for every $\alpha \in A$. Since p is a surjection, we have

$$U = p(p^{-1}(U)) = p \left(\bigcup_{\alpha \in A} \prod_{\mathbb{N}} U_{\alpha_n} \right) = \bigcup_{\alpha \in A} p \left(\prod_{\mathbb{N}} U_{\alpha_n} \right) = \bigcup_{\alpha \in A} \nabla_{\mathbb{N}} U_{\alpha_n}. \quad \square$$

Remark 3.3. Usually, when a product space $\prod_{\mathbb{N}} X$ is provided with the box topology, it is denoted by $\square X$, and when it is provided with the product topology, it is denoted by X^ω .

The space $(\tilde{X}, \tilde{\mathcal{T}})$, abbreviated \tilde{X} , can be found in studies of Rudin, Kunen and Bankston (for example in [6, 3] or [1]), and is referred to as a reduced box product or a reduced power. In the sequel we will list some properties of this space, and we will include some simple concise proofs with new denotation.

Theorem 3.4. *If X is a discrete space, then \tilde{X} is a discrete space, too.*

Proof. The statement is obvious. □

Lemma 3.5. *If (F_n) is a sequence of sets such that $F_n \subseteq X$ is closed in X for every $n \in \mathbb{N}$, then $\nabla_{\mathbb{N}} F_n$ is closed in \tilde{X} .*

Proof. The complement of $\nabla_{\mathbb{N}} F_n$ in \tilde{X} is given with

$$\left(\nabla_{\mathbb{N}} F_n\right)^c = \{[(x_n)] \mid \{n \mid x_n \in F_n\} \notin \mathcal{F}\} = \{[(x_n)] \mid \{n \mid x_n \in F_n^c\} \text{ is cofinal in } \mathbb{N}\}.$$

Therefore, its preimage by the projection p is

$$p^{-1}\left(\left(\nabla_{\mathbb{N}} F_n\right)^c\right) = \bigcup_{\substack{K \subseteq \mathbb{N} \\ \text{cofinal in } \mathbb{N}}} \prod_{\mathbb{N}} W_{Kn}, \quad \text{where } W_{Kn} = \begin{cases} F_n^c, & n \in K \\ X, & n \in \mathbb{N} \setminus K \end{cases},$$

and that is an open set in $\square_{\mathbb{N}} X$. \square

Theorem 3.6. *If X is a T_i space for $i = 0, 1, 2$ or 3 , then \tilde{X} is also a T_i space.*

Proof. We will provide the proof of the statement for T_1 and T_3 spaces. The rest is similar.

Let (X, \mathcal{T}) be a T_1 space and let $[(x_n)], [(y_n)] \in \tilde{X}$, $[(x_n)] \neq [(y_n)]$. That means there is an infinite set $A \subseteq \mathbb{N}$ such that $x_n \neq y_n$ for every $n \in A$. X is a T_1 space, so, for every $n \in A$ there is an open neighborhood $U_n \subseteq X$ of x_n which does not contain y_n . For $n \in \mathbb{N} \setminus A$ let $U_n = X$. In this way we get a neighborhood $\nabla_{\mathbb{N}} U_n$ of $[(x_n)]$ that does not contain $[(y_n)]$, implying that $(\tilde{X}, \tilde{\mathcal{T}})$ is a T_1 space.

Now, let (X, \mathcal{T}) be a regular space. According to the previous, \tilde{X} is a T_1 space. For arbitrarily chosen $[(x_n)] \in \tilde{X}$ and $F \subseteq \tilde{X}$ closed in \tilde{X} such that $[(x_n)] \notin F$, there is a basic set $\nabla_{\mathbb{N}} U_n$ such that $[(x_n)] \in \nabla_{\mathbb{N}} U_n$, and $F \subseteq \tilde{X} \setminus \nabla_{\mathbb{N}} U_n$. Then $J := \{n \mid x_n \in U_n\} \in \mathcal{F}$. Since X is regular, for every $n \in J$ we can choose an open set $V_n \subseteq X$ such that

$$x_n \in V_n \subseteq \bar{V}_n \subseteq U_n.$$

For $n \notin J$ let us put $V_n = X$. We have

$$[(x_n)] \in \nabla_{\mathbb{N}} V_n \subseteq \nabla_{\mathbb{N}} \bar{V}_n \subseteq \nabla_{\mathbb{N}} U_n.$$

According to 3.5, the set $\nabla_{\mathbb{N}} \bar{V}_n$ is closed in \tilde{X} . So, $\tilde{X} \setminus \nabla_{\mathbb{N}} \bar{V}_n$ is an open set in \tilde{X} containing F , and $\nabla_{\mathbb{N}} V_n \cap \left(\tilde{X} \setminus \nabla_{\mathbb{N}} \bar{V}_n\right) = \emptyset$. \square

The next statement can be found in [6] and [3] in a more general form.

Theorem 3.7. *\tilde{X} is a P -space.*

Proof. We need to prove that an intersection of a countable collection of open sets in \tilde{X} is an open set. It is sufficient to prove the statement for basic sets.

Let $[(x_n)] \in \bigcap_{k \in \mathbb{N}} \nabla_{\mathbb{N}} U_n^k$, where $U_n^k \in \mathcal{T}$ for every $k \in \mathbb{N}$ and for every $n \in \mathbb{N}$. Let $F(n) = \{k \in \mathbb{N} \mid k \leq n, x_n \in U_n^k\}$. The set $F(n)$ is finite, and therefore

$$V_n = \bigcap_{k \in F(n)} U_n^k$$

is open in X . Notice that $F(n)$ can be empty for a finite number of $n \in \mathbb{N}$. In that case, let V_n be the whole space X . We claim that $[(x_n)] \in \nabla_{\mathbb{N}} V_n \subseteq \bigcap_{k \in \mathbb{N}} \nabla_{\mathbb{N}} U_n^k$. If we fix

some $k \in \mathbb{N}$, we have

$$\{n \mid k \in F(n)\} = \{n \mid k \leq n, x_n \in U_n^k\} \in \mathcal{F}.$$

Also,

$$\{n \mid k \in F(n)\} \subseteq \{n \mid V_n \subseteq U_n^k\} \in \mathcal{F},$$

which implies

$$[(x_n)] \in \nabla_{\mathbb{N}} V_n \subseteq \nabla_{\mathbb{N}} U_n^k,$$

for arbitrary $k \in \mathbb{N}$. Therefore,

$$[(x_n)] \in \nabla_{\mathbb{N}} V_n \subseteq \bigcap_{k \in \mathbb{N}} \nabla_{\mathbb{N}} U_n^k,$$

and we conclude that $\bigcap_{k \in \mathbb{N}} \nabla_{\mathbb{N}} U_n^k$ is open in \tilde{X} . □

Remark 3.8. There are several definitions of a P -space in the literature. We use the definition in the sense of Gillman–Henriksen.

According to [5], in a P -space, regularity and complete regularity are equivalent. It can be easily seen from the fact that in a regular space X for every $x \in X$ and $F \subseteq X$ closed in X not containing x we can construct a sequence (V_n) of open sets in X such that

$$x \in V_n \subseteq X \setminus F, \quad \text{for every } n \in \mathbb{N}$$

and

$$\overline{V_{n+1}} \subseteq V_n, \quad \text{for every } n \in \mathbb{N}.$$

Then, $x \in \bigcap_{n \in \mathbb{N}} V_n$, and if X is a P -space, the set $\bigcap_{n \in \mathbb{N}} V_n$ is open in X . It is also closed since $\bigcap_{n \in \mathbb{N}} V_n = \bigcap_{n \in \mathbb{N}} \overline{V_n}$, and it is disjoint with F , so the map $f: X \rightarrow [0, 1]$ defined by

$$f(x) = \begin{cases} 1, & x \in \bigcap_{n \in \mathbb{N}} V_n, \\ 0, & x \in X \setminus \bigcap_{n \in \mathbb{N}} V_n, \end{cases}$$

is continuous. Obviously, every regular P -space has a clopen base. Consequently, every regular P -space is totally disconnected (every point in such a space can be separated from every other point by a clopen neighborhood).

Hence, if X is a completely regular space, \tilde{X} is a completely regular space, too, and it is totally disconnected.

Also, if X is zero-dimensional with respect to the small inductive dimension, or totally disconnected, so is \tilde{X} (see [1]).

Example 3.9. The space $\square_{\mathbb{N}} \{0, 1\} / \sim$ is discrete and it is an example of a reduced power of a compact space that is not compact.

Example 3.10. An example of a P -space that is not discrete is a reduced power of Sierpiński space. If $X = \{0, 1\}$ has a topology $\mathcal{T} = \{\emptyset, \{1\}, \{0, 1\}\}$, then $\{[(x_n)]\}$ is not open in \tilde{X} whenever $\{n \in \mathbb{N} \mid x_n = 1\} \notin \mathcal{F}$.

Example 3.11. The space $\tilde{\mathbb{R}}$ is completely regular since \mathbb{R} is, and it is zero-dimensional and totally disconnected.

More properties of \tilde{X} can be found in [5] and [6].

4. Box-homotopy

To clear the matter of continuity of mappings that we use in the sequel, we begin with simple lemmata.

Lemma 4.1. *The projection $p_n: \prod_{\mathbb{N}} X \rightarrow X$ to the n -th coordinate space is a continuous mapping, for every $n \in \mathbb{N}$.*

Proof. The statement is obviously true. □

Remark 4.2. The similar statement for the product topology is an elementary fact.

Lemma 4.3. *Let X_n and Y_n be topological spaces and $f_n: X_n \rightarrow Y_n$ a mapping, for every $n \in \mathbb{N}$. The mapping $\prod_{\mathbb{N}} f_n: \prod_{\mathbb{N}} X_n \rightarrow \prod_{\mathbb{N}} Y_n$ between products, both with either product topologies or box topologies, defined by*

$$\prod_{\mathbb{N}} f_n((x_n)) = (f_n(x_n))$$

is continuous if and only if f_n is continuous for every n .

Proof. Let us prove the statement in the case of box topology on products. The product topology case is an elementary fact.

Let $\prod_{\mathbb{N}} V_n$ be a basic set in $\prod_{\mathbb{N}} Y_n$. That means that V_n is open in Y_n for every $n \in \mathbb{N}$. We have

$$\left(\prod_{\mathbb{N}} f_n \right)^{-1} \left(\prod_{\mathbb{N}} V_n \right) = \prod_{\mathbb{N}} \left((f_n)^{-1}(V_n) \right).$$

If f_n is continuous for every $n \in \mathbb{N}$, $(f_n)^{-1}(V_n)$ is open in X_n for every $n \in \mathbb{N}$, hence $\prod_{\mathbb{N}} \left((f_n)^{-1}(V_n) \right)$ is open in $\prod_{\mathbb{N}} X_n$, implying that $\prod_{\mathbb{N}} f_n$ is continuous.

To prove the converse, let us fix any $k \in \mathbb{N}$, let $V_k \subset Y_k$ be an open set in Y_k and let $V_n = Y_n$ for $n \neq k$. Then,

$$\left(\prod_{\mathbb{N}} f_n \right)^{-1} \left(\prod_{\mathbb{N}} V_n \right) = \prod_{\mathbb{N}} W_n,$$

where $W_n = X_n$ for $n \neq k$ and $W_k = f_k^{-1}(V_k)$. If $\prod_{\mathbb{N}} f_n$ is continuous, then $\prod_{\mathbb{N}} W_n$ is open in $\prod_{\mathbb{N}} X_n$. Hence, W_k is open in X_k , implying that f_k is continuous. Since k was arbitrarily chosen, we conclude that f_n is continuous for every $n \in \mathbb{N}$. □

Remark 4.4. The statement of the previous lemma is obviously valid if we replace sequences with finite (of course, equal) number of spaces and mappings.

Definition 4.5. Let X, Y be topological spaces, and $f, g: X \rightarrow Y$ continuous maps, and let $n \in \mathbb{N} \cup \{\omega\}$. We will say that f is n -homotopic to g , $f \underset{n}{\sim} g$, if there is a continuous mapping $H: X \times I^n \rightarrow Y$, where I^n has the product topology, satisfying

$$\begin{aligned} H(x, 0, 0, 0, \dots, 0) &= f(x), \\ H(x, 1, 1, 1, \dots, 1) &= g(x), \end{aligned}$$

when $n \in \mathbb{N}$, and

$$\begin{aligned} H(x, 0, 0, 0, \dots) &= f(x), \\ H(x, 1, 1, 1, \dots) &= g(x), \end{aligned}$$

for $n = \omega$, for every $x \in X$.

Proposition 4.6. Let $n \in \mathbb{N} \cup \{\omega\}$ and let $f, g \in \text{Top}(X, Y)$. Then f is n -homotopic to g if and only if f is homotopic to g .

Proof. The statement of the theorem is obvious if $n = 1$ ($\underset{1}{\sim} = \simeq$).

Let us fix an $n \in \mathbb{N} \cup \{\omega\}$. If $H: X \times I \rightarrow Y$ is a homotopy from f to g , then the mapping $H_n: X \times I^n \rightarrow Y$ defined by $H_n = H \circ (\text{id}_X \times p_1)$, where $p_1: I^n \rightarrow I$ is a projection to the first coordinate, is an n -homotopy from f to g . It is continuous because projection is continuous.

Likewise, if $H_n: X \times I^n \rightarrow Y$ is an n -homotopy from f to g , then $H: X \times I \rightarrow Y$ defined by $H = H_n \circ (\text{id}_X \times \Delta_I)$, where $\Delta_I: I \rightarrow I^n$ is defined by $\Delta_I(t) = (t, t, \dots)$, is a homotopy from f to g . It is continuous for every $n \in \mathbb{N} \cup \{\omega\}$ because of the universal property for the product topology saying that a mapping to the product of an arbitrary collection of topological spaces (with the product topology) is continuous if and only if all its coordinate mappings are continuous. \square

To generalize the ω notion of homotopy, we give the following definition.

Definition 4.7. Let X, Y be topological spaces, and $f, g: X \rightarrow Y$ continuous mappings. We will say that f and g are box-homotopic, $f \underset{\square}{\sim} g$, if there is a continuous mapping $H: X \times \square_{\mathbb{N}} I \rightarrow Y$ satisfying

$$\begin{aligned} H(x, 0, 0, 0, \dots) &= f(x), \\ H(x, 1, 1, 1, \dots) &= g(x), \end{aligned}$$

for every $x \in X$.

To prove that the box-homotopy is an equivalence relation on $\text{Top}(X, Y)$, we will need the following lemma.

Lemma 4.8. $\Delta = \{(x, x, x, \dots) \mid x \in I\} \subset \square_{\mathbb{N}} I$ is a retract of the space $\square_{\mathbb{N}} I$.

Proof. Let

$$\begin{aligned} C &= \left\{ (x_n) \in \square_{\mathbb{N}} I \mid (x_n) \text{ converges in } I \right\} \subset \square_{\mathbb{N}} I, \\ D &= \left\{ (x_n) \in \square_{\mathbb{N}} I \mid (x_n) \text{ diverges in } I \right\} \subset \square_{\mathbb{N}} I. \end{aligned}$$

C is open since every $(x_n) \in C$ has an open neighborhood

$$\square_{\mathbb{N}} \left(\left\langle x_n - \frac{1}{2^n}, x_n + \frac{1}{2^n} \right\rangle \cap I \right) \subset C.$$

D is also open since every $(x_n) \in D$ has an open neighborhood

$$\square_{\mathbb{N}} \left(\left\langle x_n - \frac{1}{2^n}, x_n + \frac{1}{2^n} \right\rangle \cap I \right) \subset D.$$

Therefore, $\{C, D\}$ is a separation of $\square_{\mathbb{N}}I$.

We claim that a mapping $f: C \rightarrow \square_{\mathbb{N}}I$ defined by $f((x_n)) = (x, x, \dots)$, where $x = \lim_{n \rightarrow \infty} (x_n)$, is continuous. Let $\square_{\mathbb{N}}V_n \subset \square_{\mathbb{N}}I$ be an arbitrary basic set. If $f^{-1}(\square_{\mathbb{N}}V_n) \neq \emptyset$, and $(x_n) \in f^{-1}(\square_{\mathbb{N}}V_n)$ with $x = \lim_{n \rightarrow \infty} (x_n)$, then

$$f \left(\left(\square_{\mathbb{N}} \left\langle x_n - \frac{1}{2^n}, x_n + \frac{1}{2^n} \right\rangle \right) \cap \square_{\mathbb{N}}I \right) = \{(x, x, x, \dots)\} \subset \square_{\mathbb{N}}V_n.$$

Therefore, f is continuous.

Now, we can define $r: \square_{\mathbb{N}}I \rightarrow \Delta$ by

$$r((x_n)) = \begin{cases} (\frac{1}{2}, \frac{1}{2}, \dots) & (x_n) \in D \\ f((x_n)) & (x_n) \in C \end{cases}.$$

Since $\{C, D\}$ is a separation of $\square_{\mathbb{N}}I$, r is continuous. Also, $r|_{\Delta} = \text{id}_{\Delta}$. Therefore, r is a retraction from $\square_{\mathbb{N}}I$ to its subspace Δ . \square

Theorem 4.9. *To be box-homotopic is an equivalence relation on $Top(X, Y)$.*

Proof. Reflexivity. The mapping $H: X \times \square_{\mathbb{N}}I \rightarrow Y$ defined by $H = f \circ p_1$ is continuous according to 4.1, and it is a box-homotopy from f to f .

Symmetry. If $H: X \times \square_{\mathbb{N}}I \rightarrow Y$ is a box-homotopy from f to g , then the map $G: X \times \square_{\mathbb{N}}I \rightarrow Y$, $G = H \circ (\text{id}_X \times (1 - \text{id}_I) \times (1 - \text{id}_I) \times \dots)$, is a box-homotopy from g to f . According to 4.3 it is continuous.

Transitivity. Suppose $H_1: X \times \square_{\mathbb{N}}I \rightarrow Y$ is a box-homotopy from f to g , and $H_2: X \times \square_{\mathbb{N}}I \rightarrow Y$ is a box homotopy from g to h . The mapping $H: X \times \left(\prod_{\mathbb{N}} [0, \frac{1}{2}] \cup \prod_{\mathbb{N}} [\frac{1}{2}, 1] \right) \rightarrow Y$, where $\prod_{\mathbb{N}} [0, \frac{1}{2}] \cup \prod_{\mathbb{N}} [\frac{1}{2}, 1] \subset \square_{\mathbb{N}}I$ has the subspace topology, defined by

$$\begin{aligned} H|_{X \times \prod_{\mathbb{N}} [0, \frac{1}{2}]} &= H_1 \circ \left(\text{id}_X \times (2 \cdot \text{id}_{[0, \frac{1}{2}]}) \times (2 \cdot \text{id}_{[0, \frac{1}{2}]}) \times \dots \right), \\ H|_{X \times \prod_{\mathbb{N}} [\frac{1}{2}, 1]} &= H_2 \circ \left(\text{id}_X \times (2 \cdot \text{id}_{[\frac{1}{2}, 1]} - 1) \times (2 \cdot \text{id}_{[\frac{1}{2}, 1]} - 1) \times \dots \right) \end{aligned}$$

is continuous according to 4.3 and the pasting lemma because $X \times \prod_{\mathbb{N}} [0, \frac{1}{2}]$ and $X \times \prod_{\mathbb{N}} [\frac{1}{2}, 1]$ are closed in $X \times \left(\prod_{\mathbb{N}} [0, \frac{1}{2}] \cup \prod_{\mathbb{N}} [\frac{1}{2}, 1] \right)$, and $H|_{X \times \prod_{\mathbb{N}} [0, \frac{1}{2}]}$ and $H|_{X \times \prod_{\mathbb{N}} [\frac{1}{2}, 1]}$

coincide on their intersection $X \times \{(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \dots)\}$. Also,

$$\begin{aligned} H(x, 0, 0, \dots) &= H_1(x, 0, 0, \dots) = f(x), \\ H(x, 1, 1, \dots) &= H_2(x, 1, 1, \dots) = h(x). \end{aligned}$$

Hence, if r is a retraction from $\square_{\mathbb{N}}I$ to its subspace Δ , the mapping $H \circ (\text{id}_X \times r)$ is a box-homotopy from f to h . \square

Theorem 4.10. *Let $f, f': X \rightarrow Y$ and $g, g': Y \rightarrow Z$ be continuous maps, and let $g \circ f, g' \circ f': X \rightarrow Z$ be the respective composite maps. If $f \underset{\square}{\sim} f'$ and $g \underset{\square}{\sim} g'$, then $g \circ f \underset{\square}{\sim} g' \circ f'$.*

Proof. Let $H_1: X \times \square_{\mathbb{N}}I \rightarrow Y$ be a box-homotopy from f to f' , and $H_2: Y \times \square_{\mathbb{N}}I \rightarrow Z$ a box homotopy from g to g' . The map $G_1 = g \circ H_1: X \times \square_{\mathbb{N}}I \rightarrow Z$ is a box-homotopy from $g \circ f$ to $g \circ f'$, and the map $G_2: X \times \square_{\mathbb{N}}I \rightarrow Z$ defined by $G_2 = H_2 \circ ((f' \circ \text{id}_X) \times \text{id}_I \times \text{id}_I \times \dots)$ is a box-homotopy from $g \circ f'$ to $g' \circ f'$. Now, using transitivity of the relation $\underset{\square}{\sim}$, we conclude that $g \circ f \underset{\square}{\sim} g' \circ f'$. \square

Theorem 4.11. *If two mappings are homotopic, then they are box-homotopic.*

Proof. Let $H: X \times I \rightarrow Y$ be a homotopy from f to g . We can define a box-homotopy $G: X \times \square_{\mathbb{N}}I \rightarrow Y$ from f to g by $G = H \circ (\text{id}_X \times p_1)$. \square

Remark 4.12. The converse does not hold due to the fact that the universal property of product topology does not have its equivalent for the box topology. We will provide a counterexample in the sequel.

The relation $\underset{\square}{\sim}$ divides the set $Mor(Top)$ into equivalence classes, and it is a congruence (it is well adjusted with the composition). Therefore, there exists the quotient category $Top/\underset{\square}{\sim}$. We will denote it by $H_{\square}Top$, and the box-homotopy class of a morphism f will be denoted by $[f]_{\square}$.

5. The reduction of pro^* - $HTop$ category

If a morphism $[(f^n)]: (X) \rightarrow (Y)$ between rudimentary systems in pro^* - Top is given, we will denote by $\prod_{\mathbb{N}} f^n$ the mapping from (\tilde{X}) to (\tilde{Y}) induced by $\prod_{\mathbb{N}} f^n: \square_{\mathbb{N}}X \rightarrow \square_{\mathbb{N}}Y$, $\prod_{\mathbb{N}} f^n((x_n)) = (f^n(x_n))$, for $(x_n) \in \square_{\mathbb{N}}X$.

Theorem 5.1. *The mapping $\prod_{\mathbb{N}} f^n$ is a well defined morphism of pro - Top .*

Proof. Let $(f^n) \in inv^*Top((X), (Y))$. The map $\prod_{\mathbb{N}} f^n: \square_{\mathbb{N}}X \rightarrow \square_{\mathbb{N}}Y$ is continuous according to 4.3. Suppose $p: \square_{\mathbb{N}}X \rightarrow \tilde{X}$ and $q: \square_{\mathbb{N}}Y \rightarrow \tilde{Y}$ are quotient maps. Then,

$q\left(\prod_{\mathbb{N}} f^n\right): \square_{\mathbb{N}}X \rightarrow \tilde{Y}$ is continuous and, for $(x_n), (x'_n) \in \square_{\mathbb{N}}X$ such that $(x_n) \sim (x'_n)$,

$$\left(q\left(\prod_{\mathbb{N}} f^n\right)\right)((x_n)) = \left(q\left(\prod_{\mathbb{N}} f^n\right)\right)((x'_n)).$$

Thus there is a unique continuous map $g: \tilde{X} \rightarrow \tilde{Y}$ such that

$$g \circ p = q \circ \left(\prod_{\mathbb{N}} f^n\right).$$

Obviously, $g = \nabla_{\mathbb{N}} f^n$. Let $(f^n), (f'^n) \in inv^* - Top((X), (Y))$ such that $f^n = f'^n$ for all but finitely many $n \in \mathbb{N}$. Then $\left(\prod_{\mathbb{N}} f^n\right)((x_n)) \sim \left(\prod_{\mathbb{N}} f'^n\right)((x_n))$ in $\square_{\mathbb{N}}Y$ for $(x_n) \in \square_{\mathbb{N}}X$. Hence, $\prod_{\mathbb{N}} f^n$ and $\prod_{\mathbb{N}} f'^n$ induce the same map $\nabla_{\mathbb{N}} f^n$. \square

Lemma 5.2. *Let X and Y be two topological spaces and let $p_n: \square_{\mathbb{N}}X \rightarrow X$ and $q_n: \square_{\mathbb{N}}Y \rightarrow Y$ be projections to the n -th coordinate space, for every $n \in \mathbb{N}$. Then, the mapping*

$$i = (p_1 \times q_1, p_2 \times q_2, \dots): \left(\square_{\mathbb{N}}X\right) \times \left(\square_{\mathbb{N}}Y\right) \rightarrow \square_{\mathbb{N}}(X \times Y)$$

is a homeomorphism.

Proof. The mapping i is obviously a bijection. A basic open set in $\square_{\mathbb{N}}(X \times Y)$ is of the form $\square_{\mathbb{N}}(U_n \times V_n)$, where $U_n \subset X$ is open in X and $V_n \subset Y$ is open in Y , for every $n \in \mathbb{N}$. Its preimage under i is

$$\begin{aligned} & ((p_1 \times q_1)^{-1}(U_1 \times V_1)) \cap ((p_2 \times q_2)^{-1}(U_2 \times V_2)) \cap \dots \\ &= ((p_1)^{-1}(U_1) \times (q_1)^{-1}(V_1)) \cap ((p_2)^{-1}(U_2) \times (q_2)^{-1}(V_2)) \cap \dots \\ &= \left(\bigcap_{\mathbb{N}} (p_n)^{-1}(U_n)\right) \times \left(\bigcap_{\mathbb{N}} (q_n)^{-1}(V_n)\right) \\ &= \left(\square_{\mathbb{N}}U_n\right) \times \left(\square_{\mathbb{N}}V_n\right), \end{aligned}$$

and that is an open set in $\left(\square_{\mathbb{N}}X\right) \times \left(\square_{\mathbb{N}}Y\right)$. Therefore, i is continuous. Moreover, since

$$i\left(\left(\square_{\mathbb{N}}U_n\right) \times \left(\square_{\mathbb{N}}V_n\right)\right) = \square_{\mathbb{N}}(U_n \times V_n),$$

i^{-1} is continuous as well. \square

Theorem 5.3. *Let $(f^n), (f'^n) \in inv^* - Top((X), (Y))$ such that f^n is homotopic to f'^n for every $n \in \mathbb{N}$. Then $\nabla_{\mathbb{N}} f^n \sim_{\square_{\mathbb{N}}} \nabla_{\mathbb{N}} f'^n$.*

Proof. Let, for every $n \in \mathbb{N}$, a map $H_n: X \times I \rightarrow Y$ be a homotopy from f^n to f'^n . Let i be the homeomorphism from Lemma 5.2. The mapping $H: \square_{\mathbb{N}}X \times \square_{\mathbb{N}}I \rightarrow \square_{\mathbb{N}}Y$

defined by $H = \left(\prod_{\mathbb{N}} H_n\right) \circ i$ is continuous by Lemmas 4.3 and 5.2, and it induces

a continuous map $\tilde{H}: \tilde{X} \times_{\mathbb{N}} \square I \rightarrow \tilde{Y}$. Indeed, for any fixed $(t_n) \in \square I$, if $(x_n) \sim (x'_n)$, then

$$\{n \mid H_n(x_n, t_n) = H_n(x'_n, t_n)\} \in \mathcal{F}.$$

Thus $(H_n(x_n, t_n)) \sim (H_n(x'_n, t_n))$. It is easy to verify that $\tilde{H}/_{\tilde{X} \times \{(0,0,\dots)\}} = \nabla_{\mathbb{N}} f^n$ and $\tilde{H}/_{\tilde{X} \times \{(1,1,\dots)\}} = \nabla_{\mathbb{N}} f'^n$. \square

Lemma 5.4. *The mapping $j: \square I \rightarrow \square_{\mathbb{N}}(\square I)$ defined by*

$$j((x_n)) = ((y_n^1), (y_n^2), \dots),$$

where $y_n^m = x_{2^{m-1}(2n-1)}$ for every $m, n \in \mathbb{N}$, is a homeomorphism.

Proof. Notice that the mapping $c: \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$, $c(m, n) = 2^{m-1}(2n-1)$ is a bijection commonly used for proving that the Cartesian product of two countable sets is countable. Therefore, j is a bijection.

Let $\square_{m \in \mathbb{N}} \left(\square_{n \in \mathbb{N}} V_n^m \right)$ be a basic open set in $\square_{\mathbb{N}}(\square X)$.

$$j_X^{-1} \left(\square_{m \in \mathbb{N}} \left(\square_{n \in \mathbb{N}} V_n^m \right) \right) = j_X^{-1} \left(\square_{n \in \mathbb{N}} V_n^1 \times \square_{n \in \mathbb{N}} V_n^2 \times \dots \right) = \square_{k \in \mathbb{N}} U_k,$$

where $U_k = V_n^m$ for $k = c(m, n)$. Therefore, j_X is continuous.

The openness of j_X is proved analogously. \square

Remark 5.5. The similar statement is valid if I is replaced by X , for every $X \in \text{Ob}(\text{Top})$.

Theorem 5.6. *Let $(f^n), (f'^n) \in \text{inv}^*\text{-Top}((X), (Y))$ such that f^n is box-homotopic to f'^n for every $n \in \mathbb{N}$. Then $\nabla_{\mathbb{N}} f^n \sim_{\square_{\mathbb{N}}} \nabla_{\mathbb{N}} f'^n$.*

Proof. Let, for every $n \in \mathbb{N}$, a map $H_n: X \times_{\mathbb{N}} \square I \rightarrow Y$ be a box-homotopy from f^n to f'^n . Let i and j_I be the mappings from Lemmas 5.2 and 5.4. The mapping $H: \square X \times_{\mathbb{N}} \square I \rightarrow \square Y$ defined by

$$H = \left(\prod_{\mathbb{N}} H_n \right) \circ i \circ (\text{id}_{\square X} \times j_I)$$

is continuous by Lemmas 4.3, 5.2 and 5.4, therefore it is a box-homotopy between $\prod_{\mathbb{N}} f^n$ and $\prod_{\mathbb{N}} f'^n$, and it induces a continuous map $\tilde{H}: \tilde{X} \times_{\mathbb{N}} \square I \rightarrow \tilde{Y}$, a box-homotopy between $\nabla_{\mathbb{N}} f^n$ and $\nabla_{\mathbb{N}} f'^n$. \square

The following simple example shows that box-homotopic maps need not be homotopic.

Example 5.7. Let $(f^n), (g^n) \in \text{inv}^*\text{-Top}((I), (I))$ such that $f^n = 0$ and $g^n = 1$ for every $n \in \mathbb{N}$. The maps f^n and g^n are homotopic for every $n \in \mathbb{N}$, and, as we have seen, $\nabla_{\mathbb{N}} f^n, \nabla_{\mathbb{N}} g^n: \tilde{I} \rightarrow \tilde{I}$ are box-homotopic by Theorem 5.3. But, they are not homotopic.

Indeed, if $H: \tilde{I} \times I \rightarrow \tilde{I}$ is a homotopy between $\nabla_{\mathbb{N}} f^n$ and $\nabla_{\mathbb{N}} g^n$, then $H([(x_n)], 0) = [(0, 0, \dots)]$, and $H([(x_n)], 1) = [(1, 1, \dots)]$ for every $[(x_n)] \in \tilde{I}$. Then for every $[(x_n)]$, a map $I \rightarrow \tilde{I}$ defined by $t \mapsto H([(x_n)], t)$ is a path from $[(0, 0, \dots)]$ to $[(1, 1, \dots)]$ in \tilde{I} , and that is a contradiction because \tilde{I} is totally disconnected.

Now, let $[(f^n)] \in pro^*HTop((X), (Y))$. Recall, $([f^n])$ and $([f'^n])$ are equivalent if there is an $n_0 \in \mathbb{N}$ such that $[f^n] = [f'^n]$ for every $n \geq n_0$.

We claim that $([f^n])$ and $([f'^n])$ induce the same box-homotopy class $\left[\nabla_{\mathbb{N}} f^n \right]_{\square}$. Indeed, let (f^n) and (f'^n) be two sequences of continuous maps between spaces X and Y such that $f^n \sim f'^n$ by a homotopy H_n for every $n \geq n_0$. We can define a mapping $H: \square_{\mathbb{N}}(X \times I) \rightarrow \square_{\mathbb{N}}Y$ by

$$q_n \circ H = \begin{cases} H_n, & n \geq n_0, \\ H_{n_0}, & n \leq n_0, \end{cases}$$

where $q_n: \square_{\mathbb{N}}Y \rightarrow Y$ is a projection. H is continuous, and $H \circ i$, where i is the mapping from Lemma 5.2, induces a box homotopy $\tilde{H}: \tilde{X} \times \square_{\mathbb{N}}I \rightarrow \tilde{Y}$, $\tilde{H}([(x_n)], (t_n)) = [H((x_n, t_n))]$, between $\nabla_{\mathbb{N}} f^n$ and $\nabla_{\mathbb{N}} f'^n$. Indeed,

$$\begin{aligned} \tilde{H}([(x_n)], (0, 0, \dots)) &= [(f^n(x_n))], \\ \tilde{H}([(x_n)], (1, 1, \dots)) &= [(f'^n(x_n))]. \end{aligned}$$

Therefore, by the correspondence $pro^*HTop \rightarrow pro-H_{\square}Top$ defined by

$$\tilde{R}((X_{\lambda}, [p_{\lambda\lambda'}], \Lambda)) = \left(\tilde{X}_{\lambda}, \left[\nabla_{\mathbb{N}} p_{\lambda\lambda'} \right]_{\square}, \Lambda \right),$$

for $(X_{\lambda}, [p_{\lambda\lambda'}], \Lambda) \in Ob(pro^*HTop)$, and

$$\tilde{R}([(f, [f_{\mu}^n])]) = \left[\left(f, \left[\nabla_{\mathbb{N}} f_{\mu}^n \right]_{\square} \right) \right],$$

for $[(f, [f_{\mu}^n])] \in Mor(pro^*HTop)$, the function \tilde{R} is well defined. By chasing an appropriate diagram one can easily check that $\left[\left(f, \left[\nabla_{\mathbb{N}} f^n \right]_{\square} \right) \right]$ is indeed a morphism of $pro-H_{\square}Top$ and that functorial properties are valid.

The functor \tilde{R} is neither faithful nor full. For instance, morphisms $\mathbf{f} = [(f^n)]$, $\mathbf{g} = [(g^n)] \in pro^*HTop((X), (Y))$ such that f^n and g^n are box-homotopic but not homotopic for every $n \in \mathbb{N}$, are different, and, by 5.6, $\tilde{R}(\mathbf{f}) = \tilde{R}(\mathbf{g})$. Also, morphisms in $pro-H_{\square}Top$ between inverse systems of non- P -spaces do not come from pro^*HTop under \tilde{R} .

In this way we get a functor that reduces the category pro^*HTop to the category $pro-H_{\square}Top$, making the objects more complex, but the number of morphisms is significantly reduced, which makes analyzing and constructing isomorphisms easier.

We can go further on and in a similar manner define a certain correspondence $\tilde{\tilde{R}}: pro^*H_{\square}Top \rightarrow pro-H_{\square}Top$ by

$$\tilde{\tilde{R}}((X_{\lambda}, [p_{\lambda\lambda'}]_{\square}, \Lambda)) = \left(\tilde{X}_{\lambda}, \left[\nabla_{\mathbb{N}} p_{\lambda\lambda'} \right]_{\square}, \Lambda \right),$$

for $(X_\lambda, [p_{\lambda\lambda'}]_\square, \Lambda) \in Ob(pro^*-H_\square Top)$, and

$$\tilde{R}\left(\left(f, [f_\mu^n]_\square\right)\right) = \left[\left(f, \left[\nabla_{\mathbb{N}} f_\mu^n\right]_\square\right)\right],$$

for $\left(f, [f_\mu^n]_\square\right) \in Mor(pro^*-H_\square Top)$. It proves out that \tilde{R} is also a well defined functor.

Now, if we apply the known coarse shape theory techniques, recalling that every topological space admits a polyhedral resolution, the sequence of functors

$$\begin{aligned} HTop &\xrightarrow{\underline{S}} pro\text{-}HPol \xrightarrow{\underline{J}} pro^*\text{-}HPol \xrightarrow{\tilde{R}} pro\text{-}H_\square Top \xrightarrow{\underline{J}} \\ &pro^*\text{-}H_\square Top \xrightarrow{\tilde{R}} pro\text{-}H_\square Top \xrightarrow{\underline{J}} \\ &pro^*\text{-}H_\square Top \xrightarrow{\tilde{R}} pro\text{-}H_\square Top \xrightarrow{\underline{J}} \dots \end{aligned}$$

induces a sequence of classifications coarser than the coarse shape type. Here, \underline{S} acts as a shape functor by fixing an expansion of a topological space.

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