FIXED POINTS OF COISOTROPIC SUBGROUPS OF Γ_k ON DECOMPOSITION SPACES

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Abstract

We study the equivariant homotopy type of the poset $\mathcal{L}_{p^k}$ of orthogonal decompositions of $\mathbb{C}^{p^k}$. The fixed point space of the p-radical subgroup $\Gamma_k \subset U(p^k)$ acting on $\mathcal{L}_{p^k}$ is shown to be homeomorphic to a symplectic Tits building, a wedge of $(k-1)$-dimensional spheres. Our second result concerns $\Delta_k = (\mathbb{Z}/p)^k \subset U(p^k)$ acting on $\mathbb{C}^{p^k}$ by the regular representation. We identify a retract of the fixed point space of Δ_k acting on $\mathcal{L}_{p^k}$. This retract has the homotopy type of the unreduced suspension of the Tits building for $\mathrm{GL}_k(\mathbb{F}_p)$, also a wedge of $(k-1)$-dimensional spheres. As a consequence of these results, we find that the fixed point space of any coisotropic subgroup of Γ_k contains, as a retract, a wedge of $(k-1)$-dimensional spheres. We make a conjecture about the full homotopy type of the fixed point space of Δ_k acting on $\mathcal{L}_{p^k}$, based on a more general branching conjecture, and we show that the conjecture is consistent with our results.

1. Introduction

A *proper orthogonal decomposition* of $\mathbb{C}^n$ is an unordered collection of nontrivial, pairwise orthogonal, proper vector subspaces of $\mathbb{C}^n$ whose sum is $\mathbb{C}^n$. These decompositions have a partial ordering given by coarsening and accordingly form a topological poset category, denoted $\mathcal{L}_n$. The category $\mathcal{L}_n$ has a (topological) nerve, also denoted $\mathcal{L}_n$, and we trust that context will distinguish $\mathcal{L}_n$ the poset (a topological category) from $\mathcal{L}_n$ the nerve (a simplicial space). The action of $U(n)$ on $\mathbb{C}^n$ induces a natural action of $U(n)$ on $\mathcal{L}_n$, and we are interested in the fixed point spaces of the action of certain subgroups of $U(n)$ on $\mathcal{L}_n$.

The space $\mathcal{L}_n$ was introduced in [**Aro02**], in the context of the orthogonal calculus of M. Weiss. It plays an analogous role to that played in Goodwillie's homotopy calculus by the partition complex $\mathcal{P}_n$, the poset of proper nontrivial partitions of a set of n elements [**AM99**]. The space $\mathcal{L}_n$ made another, related appearance in [**AL07**], in the filtration quotients for a filtration of the spectrum bu that is analogous to the symmetric power filtration of the integral Eilenberg-Mac Lane spectrum. The

properties of $\mathcal{L}_n$ are particularly of interest in the context of the "bu-Whitehead Conjecture" [**AL10**, Conjecture 1.5].

The topology and some of the equivariant structure of $\mathcal{L}_n$ were studied in detail in [**BJL$^+$15**], and [**BJL$^+$**]. In particular, the goal of those papers was to determine, for a prime p and for all p-toral subgroups $H \subseteq U(n)$, whether $(\mathcal{L}_n)^H$ is contractible. This classification question is analogous to questions that had to be answered in [**ADL16**], in the course of calculating the Bredon homology of $\mathcal{P}_n$. In the case of $\mathcal{P}_n$, for coefficient functors that are Mackey functors taking values in $\mathbb{Z}_{(p)}$-modules, the p-subgroups of Σ_n with non-contractible fixed point spaces on $\mathcal{P}_n$ present obstructions to $\mathcal{P}_n$ having the same Bredon homology as a point. Fixed point spaces of subgroups of Σ_n acting on $\mathcal{P}_n$ were further studied in [**Aro15**] and [**AB18**]. In a different context, the spaces $\mathcal{L}_n$ were used in [**HL18**] to develop an obstruction theory for the existence of multiplicative complex orientations.

As was the case for $\mathcal{P}_n$, one expects that p-toral subgroups of $U(n)$ acting on $\mathcal{L}_n$ with non-contractible fixed point spaces will present obstructions to $\mathcal{L}_n$ having the same Bredon homology as a point, at least for coefficients that are Mackey functors taking values in $\mathbb{Z}_{(p)}$-modules. In this paper, we contribute to the understanding of these fixed point spaces by identifying two key cases of p-toral subgroups of $U(p^k)$ whose fixed point spaces on $\mathcal{L}_{p^k}$ are not only non-contractible, but actually have homology that is either free abelian or has a free abelian summand. When we put these together with a join formula from [**BJL$^+$**], we obtain a similar result for all coisotropic subgroups of Γ_k.

Our results have a similar flavor to results of [**AD01**] and [**ADL16**] in that they involve Tits buildings. We also show that the results obtained are consistent with a more general conjecture about the equivariant homotopy type of $\mathcal{L}_n$. The conjecture is analogous to the branching rule of [**Aro15**] and [**AB18**] for $\mathcal{P}_n$.

The results of the current work are used in [**BJL$^+$**] to give a complete classification of p-toral subgroups of $U(n)$ with contractible fixed point spaces on $\mathcal{L}_n$. Unlike the case for $\mathcal{P}_n$, where many elementary abelian p-subgroups of Σ_n have non-contractible fixed point spaces [**Aro15, AB18**], it turns out that the fixed point spaces of the actions of most p-toral subgroups of $U(n)$ on $\mathcal{L}_n$ are actually contractible. [**BJL$^+$**] shows that the only possible exceptions occur when $n = q^i p^j$, where q is a prime different from p. Theorems 1.1 and 1.2 below are used in [**BJL$^+$**] to settle these cases.

To state our results explicitly, we need some notation for the two p-toral subgroups that we study. First, let Δ_k denote the subgroup $(\mathbb{Z}/p)^k \subset U(p^k)$ where $(\mathbb{Z}/p)^k$ acts on $\mathbb{C}^{p^k}$ by the regular representation. Associated to Δ_k is the Tits building for $\mathrm{GL}_k(\mathbb{F}_p)$, denoted $T\,\mathrm{GL}_k(\mathbb{F}_p)$, which is the poset of proper, nontrivial subgroups of Δ_k, and which has the homotopy type of a wedge of spheres.

Second, let Γ_k be the irreducible projective elementary abelian p-subgroup of $U(p^k)$ (unique up to conjugacy), which is given by an extension

$$1 \longrightarrow S^1 \longrightarrow \Gamma_k \longrightarrow (\mathbb{Z}/p)^{2k} \longrightarrow 1. \qquad (1.1)$$

Here S^1 denotes the center of $U(p^k)$. (See Section 2 for a brief discussion of Γ_k, or [**Oli94**] or [**BJL$^+$**] for a detailed discussion from basic principles.) The extension (1.1) induces a symplectic form on $(\mathbb{Z}/p)^{2k}$ defined by lifting to Γ_k and taking the

commutator, which lies in S^1 and has order p. We define a coisotropic subgroup of Γ_k to be a subgroup that is the preimage in (1.1) of a coisotropic subspace of $(\mathbb{Z}/p)^{2k}$. (See Definition 2.7.) This allows us to associate to Γ_k the Tits building for the symplectic group, denoted $T\operatorname{Sp}_k(\mathbb{F}_p)$, which is the poset of proper coisotropic subgroups of $(\mathbb{Z}/p)^{2k}$, and like $T\operatorname{GL}_k(\mathbb{F}_p)$ has the homotopy type of a wedge of spheres.

We have two main results.

Theorem 1.1. *For $k \geqslant 1$, the fixed point space $\left(\mathcal{L}_{p^k}\right)^{\Gamma_k}$ is homeomorphic to $T\operatorname{Sp}_k(\mathbb{F}_p)$.*

By way of context, we point out that there is a more elementary, analogous result to Theorem 1.1 that establishes a homeomorphism between the fixed point space of the action of $\Delta_k \subset \Sigma_{p^k}$ on the partition complex $\mathcal{P}_{p^k}$, and the Tits building for $\operatorname{GL}_k(\mathbb{F}_p)$ [**ADL16**, Lemma 10.1]. The paper [**Aro02**] establishes a dictionary between the properties of the action of Σ_n on $\mathcal{P}_n$ and the action of $U(n)$ on $\mathcal{L}_n$. The dictionary translates $\Delta_k \subset \Sigma_{p^k}$ to $\Gamma_k \subset U(p^k)$, and translates the Weyl group of Δ_k in Σ_k, which is $\operatorname{GL}_k(\mathbb{F}_p)$, to the Weyl group of Γ_k in $U(p^k)$, which is $\operatorname{Sp}_k(\mathbb{F}_p)$. Therefore Theorem 1.1 is the result one would expect to get by taking the dictionary literally.

On the other hand, the dictionary does not give a prediction for $\left(\mathcal{L}_{p^k}\right)^{\Delta_k}$, although we explain later how the following theorem is consistent with a more general conjecture. Given a space X, let $X^\diamond$ denote the unreduced suspension of X.

Theorem 1.2. *For $k \geqslant 1$, the fixed point space $\left(\mathcal{L}_{p^k}\right)^{\Delta_k}$ has $T\operatorname{GL}_k(\mathbb{F}_p)^\diamond$ as a retract.*

We compute explicit examples for $k = 1$ in Examples 3.1 and 5.1.

Remark 1.3. As part of proving Theorem 1.2, we need to construct an inclusion $T\operatorname{GL}_k(\mathbb{F}_p)^\diamond \hookrightarrow \left(\mathcal{L}_{p^k}\right)^{\Delta_k}$, as well as a retraction. Constructing the inclusion is perhaps the sneakiest step in the paper. Contrary to what one might expect, the inclusion is not induced by a functor from the poset of subspaces of $\mathbb{F}_p^k$ to the poset of decompositions of $\mathbb{C}^{p^k}$. (Note that in any case, we want the suspension of the Tits building as the retract, and not the Tits building itself.) Rather, we need to use the *edgewise subdivision* of the poset of subspaces of $\mathbb{F}_p^k$ to model the space $T\operatorname{GL}_k(\mathbb{F}_p)^\diamond$. The edgewise subdivision is a poset whose objects are nested pairs $(H \subseteq K)$ of subgroups of Δ_k, and whose morphisms are twisted arrows. We construct a functor from the edgewise subdivision to the poset of decompositions of $\mathbb{C}^{p^k}$ using a mixture of the action of K on a basis of $\mathbb{C}^{p^k}$, and canonical decomposition of H-representations into H-isotypical summands. Details appear in the latter part of Section 4.

With Theorems 1.1 and 1.2 in place, we can use a join formula from [**BJL$^+$**] to identify a wedge of spheres as a retract of the fixed point space of any coisotropic subgroup of Γ_k acting on $\mathcal{L}_{p^k}$.

Corollary 1.4. *If $k \geqslant 1$ and $H \subseteq \Gamma_k$ is coisotropic, then $\left(\mathcal{L}_{p^k}\right)^H$ has a retract that is homotopy equivalent to a wedge of spheres of dimension $k - 1$.*

Proof. Because H is coisotropic, it has the form $\Gamma_s \times \Delta_t$ for some $s + t = k$ (Lemma 2.9). If $s = k$ or $t = k$, then the result is the same as Theorems 1.1 and 1.2, respectively, because $T\operatorname{Sp}_k(\mathbb{F}_p)$ and $T\operatorname{GL}_k(\mathbb{F}_p)^\diamond$ are both wedges of $(k-1)$-dimensional spheres. (See, for example, [**AB08**, Theorem 4.127].) By direct computation, this

statement includes the case $t = k = 1$, since $T \operatorname{GL}_1(\mathbb{F}_p)$ is empty and so its unreduced suspension is a 0-sphere, as required.

When s and t are both less than k, we apply [**BJL**$^+$, Theorem 9.2] to find that

$$\left(\mathcal{L}_{p^k}\right)^H \cong \left(\mathcal{L}_{p^t}\right)^{\Delta_t} * \left(\mathcal{L}_{p^s}\right)^{\Gamma_s}.$$

Hence by Theorems 1.1 and 1.2, $\left(\mathcal{L}_{p^k}\right)^H$ has $T \operatorname{GL}_t(\mathbb{F}_p)^{\diamond} * T \operatorname{Sp}_s(\mathbb{F}_p)$ as a retract. But s and t must both be at least 1, and as noted above, the spaces $T \operatorname{GL}_t(\mathbb{F}_p)^{\diamond}$ and $T \operatorname{Sp}_s(\mathbb{F}_p)$ each have the homotopy type of a wedge of spheres, of dimension $t - 1$ and $s - 1$, respectively. The corollary follows because $s + t = k$. □

Theorem 1.2 is good enough to complete the classification of [**BJL**$^+$]: all that is needed there is that the integral homology of $\left(\mathcal{L}_{p^k}\right)^{\Delta_k}$ has a summand that is a free abelian group. However, we actually have a conjectural description of the full homotopy type of the fixed point space $\left(\mathcal{L}_{p^k}\right)^{\Delta_k}$, based on a more general conjecture regarding the equivariant homotopy type of $\mathcal{L}_n$. We can embed $U(n-1) \subseteq U(n)$ (in a nonstandard way) as the symmetries of the orthogonal complement of the diagonal $\mathbb{C} \subset \mathbb{C}^n$, since that complement is an $(n-1)$-dimensional vector space over $\mathbb{C}$. Observe that the standard inclusion $\Sigma_n \hookrightarrow U(n)$ by permutation matrices actually factors through this inclusion $U(n-1) \subset U(n)$. Finally, let $S^{\bar{\rho}_n}$ denote the one-point compactification of the reduced standard representation of Σ_n on $\mathbb{R}^{n-1}$. The general conjecture is as follows.

Conjecture 1.5. *There is a $U(n-1)$-equivariant homotopy equivalence*

$$\mathcal{L}_n \simeq U(n-1)_+ \wedge_{\Sigma_n} \left(\mathcal{P}_n^{\diamond} \wedge S^{\bar{\rho}_n}\right).$$

Remark 1.6. Conjecture 1.5 is motivated by the role of $\mathcal{L}_n$ in orthogonal calculus. On the one hand, $\mathcal{L}_n$ is closely related to the n-th derivative of the functor $V \mapsto BU(V)$. This, together with the fibration sequence $S^1 \wedge S^V \to BU(V) \to BU(V \oplus \mathbb{C})$ implies that the restriction of $\mathcal{L}_n$ to $U(n-1)$ is closely related to the n-th derivative of the functor $V \mapsto S^1 \wedge S^V$. On the other hand, by connection with Goodwillie's homotopy calculus, the n-th derivative of this last functor is closely related to $\mathcal{P}_n^{\diamond} \wedge S^{\bar{\rho}_n}$. In fact, one can use this connection to prove that the equivalence in Conjecture 1.5 is true after taking suspension spectra and smashing with $EU(n)_+$. For more details see [**Aro02**], especially Theorem 3, which is equivalent to the assertion of the previous sentence, modulo standard manipulations involving Spanier-Whitehead duality.

In the final section of this paper, we show what the general statement in Conjecture 1.5 would imply about the actual homotopy type of $\left(\mathcal{L}_{p^k}\right)^{\Delta_k}$, and we check that implication against what we can prove beginning from Theorems 1.1 and 1.2. Let $C_G(H)$ denote the centralizer of a subgroup H in a group G. After some calculation, we find that Conjecture 1.5 implies the following conjecture.

Conjecture 1.7. *Let $\tilde{C} = C_{U(p^k)}(\Delta_k) / \left(\Delta_k \times S^1\right)$. There is a homotopy equivalence*

$$\left(\mathcal{L}_{p^k}\right)^{\Delta_k} \simeq \tilde{C}_+ \wedge T \operatorname{GL}_k(\mathbb{F}_p)^{\diamond}.$$

We observe that Conjecture 1.7 is consistent with Theorem 1.2, and this consistency can be regarded as evidence for the correctness of Conjecture 1.5.

Organization of the paper

In Section 2, we collect some background information about $\mathcal{L}_n$, the p-toral group Γ_k, and the symplectic Tits building. Section 3 proves Theorem 1.1 and computes an example. Section 4 proves Theorem 1.2, and lastly, in Section 5 we show how to deduce Conjecture 1.7 from Conjecture 1.5, and we compute another example.

Throughout the paper, we assume that we have fixed a prime p. By a subgroup of a Lie group, we always mean a closed subgroup. We write $N_G(H)$ and $C_G(H)$ for the normalizer and centralizer, respectively, of a subgroup H in a group G. The notation S^1 always means the center of the unitary group under discussion. We write ρ_n for the standard representation of Σ_n on $\mathbb{C}^n$, and we write $\bar{\rho}_n$ for the reduced standard representation (the quotient of the standard representation by the trivial representation).

2. Background on $\mathcal{L}_{p^k}$ and Γ_k

In this section, we give background results on the decomposition spaces $\mathcal{L}_n$, the group Γ_k, and the symplectic Tits building.

As explained in Section 1, $\mathcal{L}_n$ is a poset category internal to topological spaces: the objects and morphisms have an action of $U(n)$ and are topologized as disjoint unions of $U(n)$-orbits. If λ is an object of $\mathcal{L}_n$, then we write $\mathrm{cl}(\lambda)$ for the set of subspaces that make up λ, which are called the *classes* or *components* of λ. If a decomposition λ is stabilized by the action of a subgroup $H \subseteq U(n)$, then there is an action of H on $\mathrm{cl}(\lambda)$, which may be nontrivial.

In analyzing $(\mathcal{L}_n)^H$, there are two operations that are particularly helpful in constructing deformation retractions to subcategories.

Definition 2.1. Suppose that $H \subseteq U(n)$ is a closed subgroup, and λ is a decomposition in $(\mathcal{L}_n)^H$.

1. We define λ/H as the decomposition of $\mathbb{C}^n$ obtained by summing components of $\mathrm{cl}(\lambda)$ that are in the same orbit of the action of H on $\mathrm{cl}(\lambda)$.

2. If μ is a decomposition of $\mathbb{C}^n$ such that H acts trivially on $\mathrm{cl}(\mu)$ (i.e., every component of μ is a representation of H), then we define $\mu_{\mathrm{iso}(H)}$ as the refinement of μ obtained by taking the canonical decomposition of each component of μ into its H-isotypical summands.

Example 2.2. Let $\{e_1, e_2, e_3, e_4\}$ denote the standard basis for $\mathbb{C}^4$, and let $\Sigma_4 \subset U(4)$ act by permuting the basis vectors. Let ϵ denote the decomposition of $\mathbb{C}^4$ into the four lines determined by the standard basis. Let $H \cong \mathbb{Z}/2 \subset \Sigma_4$ be generated by $(1,2)(3,4)$. Then $\mu := \epsilon/H$ consists of two components $v_1 = \mathrm{Span}\{e_1, e_2\}$ and $v_2 = \mathrm{Span}\{e_3, e_4\}$.

Since each component of μ is a representation of H, we can refine μ as $(\epsilon/H)_{\mathrm{iso}(H)}$. Each of the components v_1 and v_2 decompose into one-dimensional eigenspaces for the action of H, one for the eigenvalue $+1$ and one for the eigenvalue -1. Hence $(\epsilon/H)_{\mathrm{iso}(H)}$ is a decomposition of $\mathbb{C}^4$ into four lines, each of which is fixed by H, where H acts on two of them by the identity and on the other two by multiplication by -1.

Since $\mathcal{L}_n$ has a topology, it is necessary that the operations of Definition 2.1 be continuous, which is proved in [**BJL**$^+$] using the following lemma, specialized from [**May99**, Lemma 1.1].

Lemma 2.3. *The path components of the object and morphism spaces of $(\mathcal{L}_n)^H$ are orbits of the identity component of the centralizer of H in $U(n)$.*

The proof of continuity of the operations of Definition 2.1 then goes by observing that the operations commute with the action of the centralizer of H in $U(n)$, which defines the topology of $(\mathcal{L}_n)^H$, since the orbits of $U(n)$ determine the topology of $\mathcal{L}_n$. See [**BJL**$^+$, Section 4].

Our next job is to identify a smaller subcomplex of $(\mathcal{L}_n)^H$ that is sometimes good enough to compute the homotopy type of $(\mathcal{L}_n)^H$.

Definition 2.4. Let $H \subseteq U(n)$ be a subgroup and suppose that λ is a decomposition in $(\mathcal{L}_n)^H$.

1. For $v \in \mathrm{cl}(\lambda)$, we define the *$H$-isotropy group of v*, denoted I_v, as $I_v = \{h \in H : hv = v\}$.

2. We say that λ has *uniform H-isotropy* if all elements of $\mathrm{cl}(\lambda)$ have the same H-isotropy group. In this case, we write I_λ for the H-isotropy group of any $v \in \mathrm{cl}(\lambda)$, provided that the group H is understood from context.

Example 2.5. Suppose that $\lambda \in \mathrm{Obj}\,(\mathcal{L}_n)^H$, and that H acts transitively on the set $\mathrm{cl}(\lambda)$. If there exists $v \in \mathrm{cl}(\lambda)$ such that $I_v \lhd H$, then λ necessarily has uniform H-isotropy. This is because the transitive action of H means that the H-isotropy groups of all components of λ are conjugate in H. Since I_v is normal, all the isotropy groups are actually the same.

More specifically, suppose that $H \subset U(n)$ has the property that $H/(H \cap S^1)$ is abelian (resp., elementary abelian), where S^1 denotes the center of $U(n)$. In this case we say that H is *projective abelian* (resp., *projective elementary abelian*). By the discussion above, if $\lambda \in \mathrm{Obj}\,(\mathcal{L}_n)^H$ has a transitive action of H on $\mathrm{cl}(\lambda)$, then λ has uniform H-isotropy, because every subgroup of H containing $H \cap S^1$ is normal.

For $H \subset U(n)$, let $\mathrm{Unif}(\mathcal{L}_n)^H$ denote the subposet of $(\mathcal{L}_n)^H$ consisting of objects with uniform H-isotropy. As in [**BJL**$^+$], we have the following lemma, stated slightly more generally here.

Lemma 2.6. *If $H \subset U(n)$ is a projective abelian subgroup, then the inclusion $\mathrm{Unif}(\mathcal{L}_n)^H \to (\mathcal{L}_n)^H$ induces a homotopy equivalence of nerves.*

Proof. Exactly the same proof as in [**BJL**$^+$] works here. Let λ be a decomposition in $(\mathcal{L}_n)^H$, with $\mathrm{cl}(\lambda) = \{v_1, \ldots, v_j\}$. Each I_{v_i} contains $H \cap S^1$, and so is normal in H because H is projective abelian. Let $J_\lambda = I_{v_1} \cdots I_{v_j}$, which is also a normal subgroup of H. We assert that λ/J_λ is a proper decomposition. If not, then J_λ (and hence also H) acts transitively on $\mathrm{cl}(\lambda)$. A transitive action of H on $\mathrm{cl}(\lambda)$ would tell us that $I_{v_1} = \cdots = I_{v_j} = J_\lambda$, and that I_{v_1}, for example, acts transitively on $\mathrm{cl}(\lambda)$. However, I_{v_1} fixes v_1, so λ would only have one component, a contradiction.

From this point, the proof is precisely as in [**BJL**$^+$], by doing the routine checks that $\lambda \mapsto \lambda/J_\lambda$ is a continuous deformation retraction from $(\mathcal{L}_n)^H$ to $\mathrm{Unif}(\mathcal{L}_n)^H$. $\square$

Our next order of business is to provide a little background on the groups whose fixed points we study in this paper. As in the introduction, we write Δ_k for the group $(\mathbb{Z}/p)^k \subset U(p^k)$ acting on the standard basis of $\mathbb{C}^{p^k}$ by the regular representation. One of the goals of this paper is to understand the fixed point space of Δ_k acting on $\mathcal{L}_{p^k}$ (Theorem 1.2 and Conjecture 1.7).

The other important group in our results is $\Gamma_k \subset U(p^k)$, which denotes a subgroup of $U(p^k)$ given by an extension

$$1 \longrightarrow S^1 \longrightarrow \Gamma_k \longrightarrow (\mathbb{Z}/p)^k \times (\mathbb{Z}/p)^k \longrightarrow 1,$$

and, of key importance, acts irreducibly on $\mathbb{C}^{p^k}$. The group Γ_k is discussed extensively and described explicitly in terms of matrices in [**Oli94**]. (See also [**BJL**$^+$] for a discussion from first principles.) Each factor of $(\mathbb{Z}/p)^k$ has a splitting back into Γ_k, though the splittings of the two factors do not commute in Γ_k. As a subgroup of $\Gamma_k \subseteq U(p^k)$, the image of the splitting of the first factor of $(\mathbb{Z}/p)^k$ can be regarded as Δ_k itself, acting on the standard basis of $\mathbb{C}^{p^k}$ by the regular representation. The image of the splitting of the second factor of $(\mathbb{Z}/p)^k$ acts via the regular representation on the p^k one-dimensional irreducible representations of Δ_k, which are pairwise nonisomorphic and span $\mathbb{C}^{p^k}$.

Moving on to Tits buildings, recall that a symplectic form on an $\mathbb{F}_p$-vector space is a nondegenerate alternating bilinear form. The vector space necessarily has even dimension. Lifting elements of Γ_k/S^1 to Γ_k and computing the commutator gives a well-defined symplectic form on $(\mathbb{Z}/p)^k \times (\mathbb{Z}/p)^k$. Oliver shows in [**Oli94**] that the Weyl group of Γ_k in $U(p^k)$ is the full group of automorphisms of this form, that is, the Weyl group of Γ_k in $U(p^k)$ is the symplectic group $\mathrm{Sp}_k(\mathbb{F}_p)$. Our next goal is to describe the symplectic Tits building, $T\,\mathrm{Sp}_k(\mathbb{F}_p)$.

Definition 2.7.

1. A subspace W of a symplectic vector space is called *coisotropic* if $W^\perp \subseteq W$.
2. We say that $J \subseteq \Gamma_k$ is a *coisotropic subgroup* if J is the inverse image of a coisotropic subspace of $(\mathbb{Z}/p)^{2k}$.
3. The *symplectic Tits building*, $T\,\mathrm{Sp}_k(\mathbb{F}_p)$, is the poset of proper coisotropic subgroups of Γ_k.

Example 2.8. To compute $T\,\mathrm{Sp}_1(\mathbb{F}_p)$, consider

$$1 \longrightarrow S^1 \longrightarrow \Gamma_1 \longrightarrow (\mathbb{Z}/p)^2 \longrightarrow 1.$$

Coisotropic subspaces have dimension at least half the dimension of the ambient vector space, so here a proper coisotropic subspace of $(\mathbb{Z}/p)^2$ has dimension one. Further, every one-dimensional subspace of a two-dimensional symplectic vector space is coisotropic. The vector space $(\mathbb{Z}/p)^2$ has $p+1$ one-dimensional subspaces. Since there are no possible inclusions between the subspaces, there are no morphisms in the poset, and therefore the nerve of $T\,\mathrm{Sp}_1(\mathbb{F}_p)$ consists of $p+1$ isolated points.

Remark. In the literature, the symplectic Tits building is usually defined in terms of *isotropic* subspaces. The poset of flags of isotropic subspaces is isomorphic to the poset of parabolic subgroups of the symplectic group $\mathrm{Sp}_k(\mathbb{F}_p)$, and this is why its geometric

realization is identified with the symplectic Tits building. In general, $T\operatorname{Sp}_k(\mathbb{F}_p)$ has the homotopy type of a wedge of spheres of dimension $k-1$. See [**AB08**, Section 6.6] for more details. Taking orthogonal complement defines a canonical (inclusion-reversing) bijection between isotropic and coisotropic subspaces, and for our purposes it is more natural to focus on the coisotropic subgroups.

Our final piece of background is some concrete information about coisotropic subgroups. Let $\mathbb{H}_s$ denote a $2s$-dimensional vector space over $\mathbb{Z}/p$ with a symplectic form, and let $\mathbb{T}_t$ denote a t-dimensional vector space with trivial form.

Lemma 2.9. *If $H \subseteq \Gamma_k$ is coisotropic, then H has the form $\Gamma_s \times \Delta_t$ where $s+t=k$.*

Proof. A coisotropic subspace of $(\mathbb{Z}/p)^{2k}$ has an alternating form isomorphic to $\mathbb{H}_s \oplus \mathbb{T}_t$ where $s+t=k$. Further, H is classified up to isomorphism by its commutator form, with $\mathbb{H}_s$ corresponding to Γ_s and $\mathbb{T}_t$ corresponding to Δ_t. (A proof is given in [**BJL$^+$**].) The result follows. $\qquad\square$

Lemma 2.10. *If $H \subseteq \Gamma_k$ is coisotropic, then H has irreducibles of dimension p^s $\iff$ $H \cong \Gamma_s \times \Delta_t$ where $s+t=k$.*

Proof. We already know from Lemma 2.9 that H is isomorphic to $H \cong \Gamma_s \times \Delta_t$ where $s+t=k$. The lemma follows from the fact that Γ_s is acting on $\mathbb{C}^{p^k}$ by a multiple of the standard representation, and the irreducible representations of $\Gamma_s \times \Delta_t$ are products of irreducible representations of Γ_s and (one-dimensional) irreducible representations of Δ_t. $\qquad\square$

3. Fixed points of Γ_k acting on $\mathcal{L}_{p^k}$

In this section, we prove the first theorem announced in the introduction.

Theorem 1.1. *For $k \geqslant 1$, the fixed point space $\left(\mathcal{L}_{p^k}\right)^{\Gamma_k}$ is homeomorphic to $T\operatorname{Sp}_k(\mathbb{F}_p)$.*

To get a feel for the result, we begin by computing the case $k=1$ of Theorem 1.1 directly.

Example 3.1. To compute $\left(\mathcal{L}_p\right)^{\Gamma_1}$, suppose that λ is a decomposition of $\mathbb{C}^p$ that is fixed by Γ_1. Because Γ_1 acts irreducibly on $\mathbb{C}^p$, the action of Γ_1 on $\operatorname{cl}(\lambda)$ is transitive, meaning that $\operatorname{cl}(\lambda)$ has one element, p^2 elements, or p elements. The first is impossible because λ is proper (must have more than one class), and the second is impossible because classes of λ must be nonzero (cannot have p^2 nonzero classes in a decomposition of $\mathbb{C}^p$). Hence λ is a decomposition of $\mathbb{C}^p$ into p lines. The kernel I_λ of the action map $\Gamma_1 \to \Sigma_{\operatorname{cl}(\lambda)}$ has the form $I_\lambda \cong S^1 \times \mathbb{Z}/p$. The decomposition λ is exactly the canonical decomposition of $\mathbb{C}^p$ into I_λ-isotypical representations. Hence there is a one-to-one correspondence between subgroups $I \cong S^1 \times \mathbb{Z}/p$ of Γ_1 and Γ_1-invariant decompositions λ of $\mathbb{C}^p$. There are $p+1$ subgroups I of the required form, and there are no possible inclusions, so $\left(\mathcal{L}_p\right)^{\Gamma_1}$ consists of $p+1$ points. Comparing to Example 2.8, we see that $T\operatorname{Sp}_1(\mathbb{F}_p)$ also consists of $p+1$ isolated points, as required by Theorem 1.1.

Example 3.1 brings up the point that while $T\mathrm{Sp}_k(\mathbb{F}_p)$ is a discrete poset, it is not initially clear that $(\mathcal{L}_{p^k})^{\Gamma_k}$ is discrete, because $\mathcal{L}_{p^k}$ itself is a topological poset. While it is not logically necessary to verify discreteness up front, we give a freestanding proof that $(\mathcal{L}_{p^k})^{\Gamma_k}$ is a discrete poset.

Lemma 3.2. *The object and morphism spaces of* $(\mathcal{L}_{p^k})^{\Gamma_k}$ *are discrete.*

Proof. By Lemma 2.3, the path components of $\mathrm{Obj}\,(\mathcal{L}_{p^k})^{\Gamma_k}$ are orbits of the centralizer of Γ_k in $U(p^k)$. However, Γ_k is centralized in $U(p^k)$ only by the center S^1 of $U(p^k)$ [**Oli94**, Prop. 4]. Since S^1 actually fixes every object of $\mathcal{L}_{p^k}$, the S^1-orbit of an object of $\mathcal{L}_{p^k}$ is just a point. Hence the path components of the object space of $(\mathcal{L}_{p^k})^{\Gamma_k}$ are single points, and the object space of $(\mathcal{L}_{p^k})^{\Gamma_k}$ is discrete. The same is then necessarily true of the morphism space, since there is at most one morphism between any two objects and the source and target maps are continuous on the morphism space.   □

The strategy for the proof of Theorem 1.1 is straightforward: to establish functors from $T\mathrm{Sp}_k(\mathbb{F}_p)$ to $(\mathcal{L}_{p^k})^{\Gamma_k}$ and back, and to show that their compositions are identity functors. Defining the functions on objects is not difficult. To show that the maps are functorial and compose to identity functors requires some representation theory.

We will define functions in both directions between the proper coisotropic subgroups of Γ_k and the objects of $(\mathcal{L}_{p^k})^{\Gamma_k}$. If H is a subgroup of Γ_k, let λ_H denote the canonical decomposition of $\mathbb{C}^{p^k}$ by H-isotypical summands. On the other hand, recall that if λ is an object of $(\mathcal{L}_{p^k})^{\Gamma_k}$, then λ necessarily has uniform Γ_k-isotropy (Example 2.5, because Γ_k acts irreducibly on $\mathbb{C}^{p^k}$). We denote this isotropy by $I_\lambda \subset U(p^k)$. We define the required correspondences between subgroups and decompositions as follows: if H is a coisotropic subgroup of Γ_k, then

$$F(H) = \lambda_H$$

and if λ is a decomposition in $(\mathcal{L}_{p^k})^{\Gamma_k}$, then

$$G(\lambda) = I_\lambda.$$

We need to check that the image of F consists of proper decompositions of $\mathbb{C}^{p^k}$, that the image of G consists of proper coisotropic subgroups, that F and G are functorial, and that F and G are inverses of each other when F is restricted to proper coisotropic groups.

To show that F and G are functors, we need a representation-theoretic lemma.

Lemma 3.3. *If H is a coisotropic subgroup of Γ_k, then the standard representation of Γ_k on $\mathbb{C}^{p^k}$ breaks into the sum of $[\Gamma_k : H]$ irreducible representations of H, all of equal dimension, and pairwise non-isomorphic.*

Proof. Direct computation of the character of Γ_k from the matrix representation in [**Oli94**] establishes that $\chi_{\Gamma_k}(x) = 0$ for $x \notin S^1$ and $\chi_{\Gamma_k}(s) = p^k s$ for $s \in S^1$, and hence the same is true for the character of H. By Lemma 2.9, we know $H \cong \Gamma_s \times \Delta_t$ with $s + t = k$. Computing the characters shows that the action of $H \cong \Gamma_s \times \Delta_t$ on $\mathbb{C}^{p^k} \cong$

$\mathbb{C}^{p^s} \otimes \mathbb{C}^{p^t}$ is conjugate to the action where Γ_s acts on the first factor by the standard representation and Δ_t acts on the second factor by the regular representation. Since H is a product, irreducible H-representations are obtained as tensor products of irreducible representations of Γ_s and of Δ_t. There are $p^t = [\Gamma_k : H]$ irreducibles of Δ_t acting on $\mathbb{C}^{p^t}$, all non-isomorphic, and the tensor products of these irreducibles with the standard representation of Γ_s are again irreducible, span $\mathbb{C}^{p^k}$, and are pairwise non-isomorphic (for example, since they have different characters). $\square$

We obtain the following corollary to Lemma 3.3.

Corollary 3.4. *If $J \subseteq \Gamma_k$ is coisotropic, then λ_J is the only J-isotypical decomposition of $\mathbb{C}^{p^k}$.*

Proof. A decomposition of $\mathbb{C}^{p^k}$ is J-isotypical if and only if each one of its components is an isotypical representation of J. Every J-isotypical decomposition of $\mathbb{C}^{p^k}$ is a refinement of λ_J. But by Lemma 3.3, each component of λ_J is irreducible. Hence λ_J has no J-isotypical refinements, and therefore it is the only J-isotypical decomposition of $\mathbb{C}^{p^k}$. $\square$

With Corollary 3.4 in hand, we can establish that F is functorial.

Proposition 3.5. *F is a functor from $T\operatorname{Sp}_k(\mathbb{F}_p)$ to $\left(\mathcal{L}_{p^k}\right)^{\Gamma_k}$.*

Proof. Suppose H is an object of $T\operatorname{Sp}_k(\mathbb{F}_p)$, that is, a proper coisotropic subgroup of Γ_k. Since $H \triangleleft \Gamma_k$, the action of Γ_k on $\mathbb{C}^{p^k}$ permutes the irreducible representations of H and hence stabilizes λ_H (while possibly permuting its components). Further, by Lemma 3.3, λ_H has $[\Gamma_k : H] > 1$ components, so λ_H is a proper decomposition of $\mathbb{C}^{p^k}$.

To establish naturality, suppose that $J \subseteq H$ are two coisotropic subgroups of Γ_k. Every component of λ_H is a representation of H, and hence also of J. Consider the decomposition $(\lambda_H)_{\operatorname{iso}(J)}$. It is J-isotypical, by definition, and so by Corollary 3.4, we know that $(\lambda_H)_{\operatorname{iso}(J)} = \lambda_J$. It follows that λ_J is a refinement of λ_H, so F is a functor on the poset of proper coisotropic subgroups of Γ_k. $\square$

Next we turn our attention to the function G from objects of $\left(\mathcal{L}_{p^k}\right)^{\Gamma_k}$ to subgroups of Γ_k. By way of preparation, we need a key representation-theoretic result similar to Lemma 3.3. Given an irreducible representation σ of a group G and another representation τ of G, let $[\tau : \sigma]$ denote the multiplicity of σ in τ.

Lemma 3.6. *Let λ be an object of $\left(\mathcal{L}_{p^k}\right)^{\Gamma_k}$, and let I_λ denote the (uniform) Γ_k-isotropy subgroup of its components. Then the representations of I_λ afforded by the components of λ are pairwise non-isomorphic irreducible representations of I_λ.*

Corollary 3.7. *If $\lambda \in \operatorname{Obj}\left(\mathcal{L}_{p^k}\right)^{\Gamma_k}$, then $FG(\lambda) = \lambda$.*

Proof. By definition, $G(\lambda) = I_\lambda$, so the question is to find the canonical isotypical decomposition of I_λ. Lemma 3.6 says that all components of λ are non-isomorphic irreducible representations of I_λ, so in fact, $F(I_\lambda) = \lambda$. $\square$

Proof of Lemma 3.6. Let ρ denote the standard representation of Γ_k on $\mathbb{C}^{p^k}$. The action of Γ_k/I_λ on $\mathrm{cl}(\lambda)$ is free and transitive (the latter because Γ_k acts irreducibly), so if we choose $v \in \mathrm{cl}(\lambda)$, then ρ is induced from the representation of I_λ given by v. We conclude that v is an irreducible representation of I_λ, since it induces the irreducible representation ρ. The same is true for every other component of λ, so the components of λ are a decomposition of $\mathbb{C}^{p^k}$ into I_λ-irreducibles.

We can apply Frobenius reciprocity (see, for example, [**Kna96**, Theorem 9.9]) to conclude that:

$$\left[\mathrm{Ind}_{I_\lambda}^{\Gamma_k}(v) : \rho\right] = [\rho|_{I_\lambda} : v].$$

Because $\mathrm{Ind}_{I_\lambda}^{\Gamma_k}(v) \cong \rho$, we conclude that $[\rho|_{I_\lambda} : v] = 1$. However, $\rho|_{I_\lambda}$ is a direct sum of the irreducible I_λ-modules given by the components of λ. If any other component of λ were isomorphic to v as a representation of I_λ, then we would have $[\rho|_{I_\lambda} : v] \geqslant 2$, contrary to the calculation above. $\qquad\square$

In addition to showing that F is a left inverse for G, Lemma 3.6 also allows us to check that subgroups in the image of G are actually proper coisotropic subgroups of Γ_k.

Lemma 3.8. *If λ is an object of $\left(\mathcal{L}_{p^k}\right)^{\Gamma_k}$, then I_λ is a proper coisotropic subgroup of Γ_k.*

Proof. We know that I_λ is strictly contained in Γ_k, because otherwise irreducibility of the action of Γ_k would imply that λ had only one component.

We have the following ladder of short exact sequences:

$$
\begin{array}{ccccccccc}
1 & \longrightarrow & S^1 & \longrightarrow & I_\lambda & \longrightarrow & W & \longrightarrow & 1 \\
& & \| & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & S^1 & \longrightarrow & \Gamma_k & \longrightarrow & (\mathbb{Z}/p)^{2k} & \longrightarrow & 1.
\end{array}
$$

We must show that if $z \in W^\perp \subseteq (\mathbb{Z}/p)^{2k}$, then in fact, $z \in W$. Recall that the symplectic form on $(\mathbb{Z}/p)^{2k}$ is given by the commutator pairing: if we denote lifts of z and w by $\tilde{z}$ and $\tilde{w}$, then the symplectic form evaluated on the pair (z,w) is given by the commutator $[\tilde{z}, \tilde{w}] \in S^1$. Hence if z pairs to 0 with all elements of W, it means that $\tilde{z}$ is actually in the centralizer of I_λ in Γ_k. Thus is it sufficient for us to show that if $\tilde{z} \in \Gamma_k$ centralizes I_λ, then $\tilde{z} \in I_\lambda$.

However, if $\tilde{z}$ centralizes I_λ and $v \in \mathrm{cl}(\lambda)$, then $\tilde{z}$ gives a nontrivial I_λ-equivariant map between the I_λ-representations v and $\tilde{z}v$. By Lemma 3.6, if $v \neq \tilde{z}v$, then v and $\tilde{z}v$ are non-isomorphic irreducible representations of I_λ, so Schur's Lemma tells us that there is no nontrivial I_λ-equivariant map. We conclude that $\tilde{z}v = v$, so $\tilde{z} \in I_\lambda$, as required. $\qquad\square$

Finally, the last step is to show that the functors F and G are inverses of each other.

Proof of Theorem 1.1. The functors $F\colon H \mapsto \lambda_H$ and $G\colon \lambda \mapsto I_\lambda$ induce the desired homeomorphism, once we show that they are inverses of each other. Corollary 3.7 already tells us that $FG(\lambda) = \lambda$. To finish the proof of the theorem, we must show if

H is proper and coisotropic, then $GF(H) = H$, that is, the Γ_k-isotropy subgroup of λ_H is H itself.

By definition of λ_H, the components of λ_H are H-representations, so certainly $H \subseteq I_{\lambda_H}$. Both H and I_{λ_H} are proper and coisotropic, by assumption and by Lemma 3.8, respectively. However, a coisotropic subgroup of Γ_k is determined up to isomorphism by the dimension of its irreducible summands in the standard representation of Γ_k (Lemma 2.10). Further, the components of λ_H are irreducible representations for both H (Lemma 3.3) and I_{λ_H} (Lemma 3.6). Hence the irreducible summands of H and I_{λ_H} are actually the same, and H and I_{λ_H} are isomorphic, and therefore equal.　□

4.　Fixed points of Δ_k acting on $\mathcal{L}_{p^k}$

Let $T\,\mathrm{GL}_k(\mathbb{F}_p)$ denote the Tits building for $\mathrm{GL}_k(\mathbb{F}_p)$, that is, the poset of proper nontrivial subgroups of Δ_k. In this section, we prove the following result.

Theorem 1.2. *For $k \geqslant 1$, the fixed point space $\left(\mathcal{L}_{p^k}\right)^{\Delta_k}$ has $T\,\mathrm{GL}_k(\mathbb{F}_p)^\diamond$ as a retract.*

To set up the proof, we follow a similar strategy to [**BJL**$^+$, Section 9]. Recall $\mathrm{Unif}\left(\mathcal{L}_{p^k}\right)^{\Delta_k}$ denotes the subposet of $\left(\mathcal{L}_{p^k}\right)^{\Delta_k}$ consisting of objects with uniform Δ_k-isotropy, and that $\mathrm{Unif}\left(\mathcal{L}_{p^k}\right)^{\Delta_k} \hookrightarrow \left(\mathcal{L}_{p^k}\right)^{\Delta_k}$ is a homotopy equivalence (Lemma 2.6). We analyze $\mathrm{Unif}\left(\mathcal{L}_{p^k}\right)^{\Delta_k}$ in terms of two subposets.

Definition 4.1.

1. Let $\left(\mathcal{L}_{p^k}\right)^{\Delta_k}_{\mathrm{Ntr}} \subseteq \mathrm{Unif}\left(\mathcal{L}_{p^k}\right)^{\Delta_k}$ consist of objects λ such that Δ_k does not act transitively on $\mathrm{cl}(\lambda)$.

2. Let $\left(\mathcal{L}_{p^k}\right)^{\Delta_k}_{\mathrm{move}} \subseteq \mathrm{Unif}\left(\mathcal{L}_{p^k}\right)^{\Delta_k}$ consist of objects λ such that Δ_k acts nontrivially on $\mathrm{cl}(\lambda)$.

Example 4.2. Choose an orthonormal basis E of $\mathbb{C}^{p^k}$ on which Δ_k acts freely and transitively. (Recall that Δ_k is acting on $\mathbb{C}^{p^k}$ by the regular representation.) Let ϵ be the corresponding decomposition of $\mathbb{C}^{p^k}$ into the lines, each line generated by an element of E. Then ϵ is an object of $\left(\mathcal{L}_{p^k}\right)^{\Delta_k}_{\mathrm{move}}$ but not of $\left(\mathcal{L}_{p^k}\right)^{\Delta_k}_{\mathrm{Ntr}}$, and the same is true for ϵ/K for any proper subgroup $K \subseteq \Delta_k$.

Conversely, let H be any nontrivial subgroup of Δ_k. Then λ_H is an element of $\left(\mathcal{L}_{p^k}\right)^{\Delta_k}_{\mathrm{Ntr}}$ but not of $\left(\mathcal{L}_{p^k}\right)^{\Delta_k}_{\mathrm{move}}$.

We observe that refinements of objects in $\left(\mathcal{L}_{p^k}\right)^{\Delta_k}_{\mathrm{Ntr}}$ are still in $\left(\mathcal{L}_{p^k}\right)^{\Delta_k}_{\mathrm{Ntr}}$, and refinements of objects in $\left(\mathcal{L}_{p^k}\right)^{\Delta_k}_{\mathrm{move}}$ are still in $\left(\mathcal{L}_{p^k}\right)^{\Delta_k}_{\mathrm{move}}$. Further, every object of $\mathrm{Unif}\left(\mathcal{L}_{p^k}\right)^{\Delta_k}$ is in one of these two subposets. Hence we have a pushout diagram of nerves

$$\begin{array}{ccc} \left(\mathcal{L}_{p^k}\right)^{\Delta_k}_{\mathrm{Ntr}} \cap \left(\mathcal{L}_{p^k}\right)^{\Delta_k}_{\mathrm{move}} & \longrightarrow & \left(\mathcal{L}_{p^k}\right)^{\Delta_k}_{\mathrm{Ntr}} \\ \downarrow & & \downarrow \\ \left(\mathcal{L}_{p^k}\right)^{\Delta_k}_{\mathrm{move}} & \longrightarrow & \mathrm{Unif}\left(\mathcal{L}_{p^k}\right)^{\Delta_k}. \end{array} \qquad (4.1)$$

We assert that this diagram is, in fact, a homotopy pushout: that the top row is a

Reedy cofibration, and the bottom left space is Reedy cofibrant. This is established by precisely the same argument as Proposition 9.11 of [**BJL**$^+$], with the identity component of the centralizer of Δ_k in $U(p^k)$ in place of the centralizers that are applicable in that work. Essentially, the point is that in each simplicial dimension, one is looking at an inclusion of a subset of path components.

To prove Theorem 1.2, we will use the expected steps to show that the nerve of $\mathrm{Unif}\left(\mathcal{L}_{p^k}\right)^{\Delta_k}$ has $T\,\mathrm{GL}_k(\mathbb{F}_p)^\diamond$ as a retract: finding a retraction map, exhibiting a corresponding inclusion, and showing that the inclusion and retraction compose to a self-equivalence of $T\,\mathrm{GL}_k(\mathbb{F}_p)^\diamond$. Our first step is to use diagram (4.1) to produce a map from the nerve of $\mathrm{Unif}\left(\mathcal{L}_{p^k}\right)^{\Delta_k}$ to the double cone on $T\,\mathrm{GL}_k(\mathbb{F}_p)$. Unlike the rest of the arguments in this paper, the map will not be realized on the categorical level, but only once we have passed to spaces by taking nerves. However, we begin on the categorical level. Define a function on object spaces,

$$G\colon \left(\mathcal{L}_{p^k}\right)^{\Delta_k}_{\mathrm{Ntr}} \cap \left(\mathcal{L}_{p^k}\right)^{\Delta_k}_{\mathrm{move}} \longrightarrow T\,\mathrm{GL}_k(\mathbb{F}_p)$$

by the formula $G(\lambda) = I_\lambda$.

Lemma 4.3. *The function G defines a continuous functor.*

Proof. First we need to check that $G(\lambda)$ is a proper, nontrivial subgroup of Δ_k. If λ is an object of $\left(\mathcal{L}_{p^k}\right)^{\Delta_k}_{\mathrm{move}}$, then I_λ is a proper subgroup of Δ_k. If I_λ were trivial, then Δ_k would act freely on $\mathrm{cl}(\lambda)$, implying that λ is a decomposition of $\mathbb{C}^{p^k}$ into p^k lines, freely permuted by Δ_k. But then the action of Δ_k on $\mathrm{cl}(\lambda)$ would be transitive, in contradiction of the assumption that $\lambda \in \left(\mathcal{L}_{p^k}\right)^{\Delta_k}_{\mathrm{Ntr}}$. Hence $G(\lambda)$ is a proper and nontrivial subgroup of Δ_k. To check that G defines a functor, we observe that if $\lambda \to \mu$ is a coarsening morphism in $\mathrm{Unif}\left(\mathcal{L}_{p^k}\right)^{\Delta_k}$, then $I_\lambda \subseteq I_\mu$.

The functor G is defined on a subcategory of $\mathrm{Unif}\left(\mathcal{L}_{p^k}\right)^{\Delta_k}$, and its target category is discrete. Continuity of G follows once we check that the assignment $\lambda \mapsto I_\lambda$ is constant on each path component of $\mathrm{Unif}\left(\mathcal{L}_{p^k}\right)^{\Delta_k}$. However, path components of $\mathrm{Unif}\left(\mathcal{L}_{p^k}\right)^{\Delta_k} \subseteq \left(\mathcal{L}_{p^k}\right)^{\Delta_k}$ are orbits of the centralizer of Δ_k. If c centralizes Δ_k, then $I_{c\lambda} = I_\lambda$. Hence the assignment $\lambda \mapsto I_\lambda$ is constant on path components of $\mathrm{Unif}\left(\mathcal{L}_{p^k}\right)^{\Delta_k}$, and G is therefore continuous. $\square$

Definition 4.4. The map from the nerve of $\mathrm{Unif}\left(\mathcal{L}_{p^k}\right)^{\Delta_k}$ to $T\,\mathrm{GL}_k(\mathbb{F}_p)^\diamond$ is defined as the map of homotopy colimits arising from the following map of diagrams induced by G in the upper left corner:

$$\left(\begin{array}{c} \left(\mathcal{L}_{p^k}\right)^{\Delta_k}_{\mathrm{Ntr}} \cap \left(\mathcal{L}_{p^k}\right)^{\Delta_k}_{\mathrm{move}} \longrightarrow \left(\mathcal{L}_{p^k}\right)^{\Delta_k}_{\mathrm{Ntr}} \\ \downarrow \\ \left(\mathcal{L}_{p^k}\right)^{\Delta_k}_{\mathrm{move}} \end{array} \right)$$

$$\Big\downarrow$$

$$\left(\begin{array}{c} T\,\mathrm{GL}_k(\mathbb{F}_p) \longrightarrow * \\ \downarrow \\ * \end{array} \right).$$

The next piece of the puzzle is to define a map from $T\,\mathrm{GL}_k(\mathbb{F}_p)^\diamond$ into $\mathrm{Unif}\left(\mathcal{L}_{p^k}\right)^{\Delta_k}$. This map will be defined on the categorical level, that is, by taking the nerve of a functor between two categories, but we need a different categorical model for $T\,\mathrm{GL}_k(\mathbb{F}_p)^\diamond$ in order to define the map. For this purpose, we recall some background on the edge subdivision of a category (also called a twisted arrow category). Suppose that $\mathcal{C}$ is a category; define the "edge subdivision" category $\mathrm{Sd}_e\,(\mathcal{C})$ of $\mathcal{C}$ as follows:

1. Objects of $\mathrm{Sd}_e\,(\mathcal{C})$ are morphisms $X \to Y$ of $\mathcal{C}$.

2. A morphism from $X \to Y$ to $C \to D$ is given by a *twisted arrow*, that is, a commuting diagram

$$
\begin{array}{ccc}
X & \longrightarrow & Y \\
\uparrow & & \downarrow \\
C & \longrightarrow & D.
\end{array}
$$

Note that if $\mathcal{C}$ is a poset, then $\mathrm{Sd}_e\,(\mathcal{C})$ is a poset as well.

Lemma 4.5 ([**Seg73**, Appendix 1]). *The geometric realizations of $\mathrm{Sd}_e\,(\mathcal{C})$ and $\mathcal{C}$ are naturally homeomorphic.*

Recall that $T\,\mathrm{GL}_k(\mathbb{F}_p)$ is the poset of proper, non-trivial subgroups of Δ_k. In what follows, let $\overline{T\,\mathrm{GL}_k(\mathbb{F}_p)}$ be the poset of *all* subgroups of Δ_k. Note that $\mathrm{Sd}_e\left(\overline{T\,\mathrm{GL}_k(\mathbb{F}_p)}\right)$ has a final object $\{e\} \to \Delta_k$, but no initial object.

Definition 4.6. Let $\mathcal{T}$ be the category $\mathrm{Sd}_e\,(T\,\mathrm{GL}_k(\mathbb{F}_p))$ and let $\mathcal{T}^\diamond$ be the category $\mathrm{Sd}_e\left(\overline{T\,\mathrm{GL}_k(\mathbb{F}_p)}\right)$ without the final object $\{e\} \to \Delta_k$. We will denote a generic object of $\mathrm{Sd}_e\left(\overline{T\,\mathrm{GL}_k(\mathbb{F}_p)}\right)$ by $H \subseteq K$.

To justify the notation $\mathcal{T}^\diamond$, we prove that the category $\mathcal{T}^\diamond$ does, in fact, give a model for the unreduced suspension of the Tits building.

Lemma 4.7. *The nerve of $\mathcal{T}^\diamond$ is homeomorphic to $|T\,\mathrm{GL}_k(\mathbb{F}_p)|^\diamond$.*

Proof. We define $\mathrm{Cone}^+(\mathcal{T})$ as the subposet of $\mathcal{T}^\diamond$ consisting of pairs $H \subseteq K$ where $H \neq \{e\}$. Likewise, we define $\mathrm{Cone}^-(\mathcal{T})$ as the subposet of $\mathcal{T}^\diamond$ consisting of pairs $H \subseteq K$ where $K \neq \Delta_k$.

A straightforward check shows that if $H \subseteq K$ is an object of $\mathrm{Cone}^+(\mathcal{T})$ (respectively, $\mathrm{Cone}^-(\mathcal{T})$), then $H \subseteq K$ can only be the target of morphisms from other objects in $\mathrm{Cone}^+(\mathcal{T})$ (respectively, $\mathrm{Cone}^-(\mathcal{T})$). We conclude that a sequence of composable morphism that ends in $\mathrm{Cone}^+(\mathcal{T})$ consists entirely of morphisms in $\mathrm{Cone}^+(\mathcal{T})$, and similarly for $\mathrm{Cone}^-(\mathcal{T})$. Therefore on the level of nerves, we have

$$
\mathrm{Cone}^+(\mathcal{T}) \cup \mathrm{Cone}^-(\mathcal{T}) = \mathcal{T}^\diamond.
$$

Since the intersection $\mathrm{Cone}^+(\mathcal{T}) \cap \mathrm{Cone}^-(\mathcal{T})$ is exactly $\mathcal{T}$, we have a pushout diagram of nerves

$$
\begin{array}{ccc}
\mathcal{T} & \longrightarrow & \mathrm{Cone}^+(\mathcal{T}) \\
\downarrow & & \downarrow \\
\mathrm{Cone}^-(\mathcal{T}) & \longrightarrow & \mathcal{T}^\diamond.
\end{array}
$$

Observe that $\mathrm{Cone}^+(\mathcal{T})$ is the edge subdivision of $T\,\mathrm{GL}_k(\mathbb{F}_p) \cup \{\Delta_k\}$ (adding in the

final object $\{\Delta_k\}$ to the category being subdivided) and similarly for $\mathrm{Cone}^-(\mathcal{T})$ (but by adding in the initial object $\{e\}$). Hence the nerves of $\mathrm{Cone}^+(\mathcal{T})$ and $\mathrm{Cone}^-(\mathcal{T})$ are each homeomorphic to a cone on the nerve of $\mathcal{T}$, and the result follows. $\qquad\square$

We will define a functor

$$F\colon \mathcal{T}^{\diamond} \longrightarrow \mathrm{Unif}\big(\mathcal{L}_{p^k}\big)^{\Delta_k}.$$

As in Example 4.2, we fix an orthonormal basis of $\mathbb{C}^{p^k}$ that is freely permuted by Δ_k, and let ϵ be the corresponding decomposition of $\mathbb{C}^{p^k}$ into lines. For an object $H \subseteq K$ of $T\,\mathrm{GL}_k(\mathbb{F}_p)^{\diamond}$, define F by

$$F(H \subseteq K) = (\epsilon/K)_{\mathrm{iso}(H)}.$$

Observe that this makes sense, because H acts trivially on the set of components of ϵ/K, so each component is a representation of H and can itself be refined into H-isotypical components.

A couple of routine checks are required.

Lemma 4.8. *The image $F(H \subseteq K)$ is an object of $\mathrm{Unif}\big(\mathcal{L}_{p^k}\big)^{\Delta_k}$.*

Proof. Since ϵ is stabilized by Δ_k and since H and K are normal in Δ_k, the operations of taking K-orbits and H-isotypical decomposition are stabilized by Δ_k. We also need to check that $F(H \subseteq K)$ is a proper decomposition. If K is a proper subgroup of Δ_k, then ϵ/K is proper, so certainly any refinement of it is proper. If $K = \Delta_k$, then ϵ/K has just one component, all of $\mathbb{C}^{p^k}$, but since H acts by copies of the regular representation, it acts non-isotypically. Hence $F(H \subseteq K)$ is a proper decomposition of $\mathbb{C}^{p^k}$.

To check whether $F(H \subseteq K)$ has uniform isotropy, first notice that since K centralizes H, an action of K on a subspace v fixes each of the canonical H-isotypical summands of v. Therefore K stabilizes each component of $(\epsilon/K)_{\mathrm{iso}(H)}$. But the action of Δ_k/K on ϵ/K is free, so the action of Δ_k/K on $(\epsilon/K)_{\mathrm{iso}(H)}$ is also free. Therefore $(\epsilon/K)_{\mathrm{iso}(H)}$ has K as the Δ_k-isotropy group of every component. $\qquad\square$

Lemma 4.9. *F is a functor.*

Proof. A morphism $(H_1 \subseteq K_1) \to (H_2 \subseteq K_2)$ of $\mathcal{T}^{\diamond}$ is given by a sequence of containments $H_2 \subseteq H_1 \subseteq K_1 \subseteq K_2$. We need to show that such a morphism gives rise to a coarsening morphism

$$(\epsilon/K_1)_{\mathrm{iso}(H_1)} \longrightarrow (\epsilon/K_2)_{\mathrm{iso}(H_2)}.$$

Certainly there is a coarsening morphism $\epsilon/K_1 \xrightarrow{c} \epsilon/K_2$, because $K_1 \subseteq K_2$. Components of both the source and the target of c are representations of H_1, since $H_1 \subseteq K_1 \subseteq K_2$, so we can take the isotypical refinement of c with respect to H_1 to obtain a morphism

$$(\epsilon/K_1)_{\mathrm{iso}(H_1)} \longrightarrow (\epsilon/K_2)_{\mathrm{iso}(H_1)}. \tag{4.2}$$

Following (4.2) with the morphism $(\epsilon/K_2)_{\mathrm{iso}(H_1)} \to (\epsilon/K_2)_{\mathrm{iso}(H_2)}$ gives the desired result. $\qquad\square$

Finally, we prove Theorem 1.2 by considering the compositions of the maps of diagrams induced by F and G.

Proof of Theorem 1.2. The three diagrams we need to consider are

$$\left(\begin{array}{c} \mathcal{T} \longrightarrow \operatorname{Cone}^+(\mathcal{T}) \\ \downarrow \\ \operatorname{Cone}^-(\mathcal{T}) \end{array} \right) \tag{4.3}$$

mapping on all three corners via $F \colon (H \subseteq K) \mapsto (\epsilon/K)_{\operatorname{iso}(H)}$ to

$$\left(\begin{array}{c} \left(\mathcal{L}_{p^k}\right)^{\Delta_k}_{\operatorname{Ntr}} \cap \left(\mathcal{L}_{p^k}\right)^{\Delta_k}_{\operatorname{move}} \longrightarrow \left(\mathcal{L}_{p^k}\right)^{\Delta_k}_{\operatorname{Ntr}} \\ \downarrow \\ \left(\mathcal{L}_{p^k}\right)^{\Delta_k}_{\operatorname{move}} \end{array} \right) \tag{4.4}$$

which then has a map of nerves induced by $G \colon \lambda \mapsto I_\lambda$ to

$$\left(\begin{array}{c} T\operatorname{GL}_k(\mathbb{F}_p) \longrightarrow * \\ \downarrow \\ * \end{array} \right). \tag{4.5}$$

We first need to check that the corners of diagram (4.3) map to the corners of diagram (4.4) as claimed. For the lower left-hand corner, notice that if $H \subseteq K \neq \Delta_k$ is an object of $\operatorname{Cone}^-(\mathcal{T})$, then there is a coarsening morphism

$$(\epsilon/K)_{\operatorname{iso}(H)} \longrightarrow \epsilon/K.$$

Since the set of components of ϵ/K has more than one element and a transitive (hence necessarily nontrivial) action of Δ_k, the action of Δ_k on the components of $(\epsilon/K)_{\operatorname{iso}(H)}$ is also nontrivial.

For the upper right-hand corner of diagram (4.4), if $\{e\} \neq H \subseteq K$ is an object of $\operatorname{Cone}^+(\mathcal{T})$, then we have a coarsening morphism

$$(\epsilon/K)_{\operatorname{iso}(H)} \longrightarrow (\epsilon/\Delta_k)_{\operatorname{iso}(H)} = \lambda_H.$$

However, λ_H has more than one component because H is nontrivial, and Δ_k acts trivially (hence nontransitively) on $\operatorname{cl}(\lambda_H)$ because H is central in Δ_k. Hence the action of Δ_k on the components of $(\epsilon/K)_{\operatorname{iso}(H)}$ cannot be transitive either.

The maps given between diagrams (4.3), (4.4), and (4.5) give maps on homotopy pushouts:

$$\mathcal{T}^\diamond \longrightarrow \operatorname{Unif}\left(\mathcal{L}_{p^k}\right)^{\Delta_k} \longrightarrow T\operatorname{GL}_k(\mathbb{F}_p)^\diamond.$$

To prove the theorem, it is sufficient to show that the composition of diagrams (4.3), (4.4), and (4.5) gives a homotopy equivalence of nerves on the upper left-hand corner,

$$\mathcal{T} \longrightarrow \left(\mathcal{L}_{p^k}\right)^{\Delta_k}_{\operatorname{Ntr}} \cap \left(\mathcal{L}_{p^k}\right)^{\Delta_k}_{\operatorname{move}} \longrightarrow T\operatorname{GL}_k(\mathbb{F}_p).$$

However, the composition takes an object $H \subseteq K$ of $\mathcal{T}$ to the isotropy subgroup of $(\epsilon/K)_{\operatorname{iso}(H)}$, which is K itself, as in the proof of Lemma 4.8. Hence the composition $\mathcal{T} \to T\operatorname{GL}_k(\mathbb{F}_p)$ maps $(H \subseteq K)$ to K, which induces an equivalence of nerves by [**Qui73**, p. 94]. □

5. Conjectures

In the introduction, we presented a general conjecture regarding the $U(n-1)$-equivariant homotopy type of $\mathcal{L}_n$. Recall that $\mathcal{P}_n$ denotes the poset of proper non-trivial partitions of a set of n elements and $\mathcal{P}_n^\diamond$ denotes its unreduced suspension. The group Σ_n is embedded in $U(n)$ via the standard (permutation) representation, and $S^{\bar{\rho}_n}$ denotes the representation sphere of the reduced standard representation of Σ_n on $\mathbb{C}^n$.

Conjecture 1.5. *There is a $U(n-1)$-equivariant homotopy equivalence*
$$\mathcal{L}_n \simeq U(n-1)_+ \wedge_{\Sigma_n} \left(\mathcal{P}_n^\diamond \wedge S^{\bar{\rho}_n} \right).$$

In this section, we show that the following conjecture follows from Conjecture 1.5.

Conjecture 1.7. *Let $\tilde{C} = C_{U(p^k)}(\Delta_k) / (\Delta_k \times S^1)$. There is a homotopy equivalence*
$$\left(\mathcal{L}_{p^k} \right)^{\Delta_k} \simeq \tilde{C}_+ \wedge T\,\mathrm{GL}_k(\mathbb{F}_p)^\diamond.$$

The case $k = 1$ is computed explicitly in Example 5.1.

Dividing $C_{U(p^k)}(\Delta_k) \cong (U(1))^{p^k}$ by the subgroup $\Delta_k \times S^1$ still leaves us with a torus, so we have a homeomorphism $\tilde{C} \cong \left(S^1 \right)^{p^k - 1}$. Recall that $T\,\mathrm{GL}_k(\mathbb{F}_p)^\diamond$ is a wedge of spheres of dimension $k - 1$. Conjecture 1.7 would tell us that for $k > 1$, the fixed point space $\left(\mathcal{L}_{p^k} \right)^{\Delta_k}$ is a wedge of spheres of varying dimensions. Further, by the join formula from [**BJL**$^+$], we have
$$\left(\mathcal{L}_{p^{s+t}} \right)^{\Gamma_s \times \Delta_t} \simeq (\mathcal{L}_{p^s})^{\Gamma_s} * (\mathcal{L}_{p^t})^{\Delta_t},$$
which would also be a wedge of spheres (of varying dimensions for $t > 0$) provided that either $s > 0$ or $t > 1$.

Recall that we are considering $U(p^k - 1) \subset U(p^k)$ as the symmetries of the orthogonal complement of the diagonal $\mathbb{C} \subset \mathbb{C}^{p^k}$. The subgroup $\Delta_k \subset \Sigma_{p^k}$ is a subgroup of $U(p^k - 1)$ with this embedding. To show that Conjecture 1.7 follows from Conjecture 1.5, we need to calculate the fixed points of $\Delta_k \subset \Sigma_{p^k}$ acting on
$$U(p^k - 1)_+ \wedge_{\Sigma_{p^k}} \left(\mathcal{P}_{p^k}^\diamond \wedge S^{\bar{\rho}_{p^k}} \right). \tag{5.1}$$
In general, the fixed points of $D \subseteq G$ on a space with an action of $H \subseteq G$ induced up to G is
$$(G \times_H X)^D = \bigcup_{[g] \in N(D;H)/H} \{g\} \times X^{g^{-1}Dg}, \tag{5.2}$$
where $N_G(D;H) = \{g \in G : g^{-1}Dg \subseteq H\}$. Thus we need $N_{U(p^k-1)}(\Delta_k; \Sigma_{p^k})$.

To calculate $N_{U(p^k)}(\Delta_k; \Sigma_{p^k})$, suppose that $u \in U(p^k)$ satisfies $u^{-1}\Delta_k u \subset \Sigma_{p^k} \subset U(p^k)$, which means that all elements of $u^{-1}\Delta_k u$ are permutation matrices. The character of $u^{-1}\Delta_k u$ is the same as that of Δ_k, i.e., zero on all nonidentity elements, which tells us that $u^{-1}\Delta_k u$ acts freely and hence transitively on $\{1, \ldots, p^k\}$. But then Δ_k and $u^{-1}\Delta_k u$ are both transitive elementary abelian p-subgroups of Σ_{p^k}, which means that they are conjugate inside of Σ_{p^k} itself. So there exists $\sigma \in \Sigma_{p^k}$ such that $\sigma^{-1}\Delta_k \sigma = u^{-1}\Delta_k u \subset \Sigma_{p^k}$.

However, all automorphisms of Δ_k are realized by the action of its normalizer in Σ_{p^k}. By changing the choice of σ if necessary, we can actually make the stronger assertion that σ and u induce the same automorphism of Δ_k, i.e. $\sigma^{-1}d\sigma = u^{-1}du$ for all $d \in \Delta_k$. Thus $u\sigma^{-1}$ centralizes every $d \in \Delta_k$, and u is in the coset $C_{U(p^k)}(\Delta_k)\,\sigma$. We conclude that

$$N_{U(p^k)}\left(\Delta_k; \Sigma_{p^k}\right) = \bigcup_{\sigma \in \Sigma_{p^k}} C_{U(p^k)}\left(\Delta_k\right)\sigma.$$

Since the centralizer of Δ_k in Σ_{p^k} is Δ_k itself, $C_{U(p^k)}(\Delta_k) \cap \Sigma_{p^k} = \Delta_k$. It follows that the formula for $N_{U(p^k)}\left(\Delta_k; \Sigma_{p^k}\right)$ can be rewritten as

$$N_{U(p^k)}\left(\Delta_k; \Sigma_{p^k}\right) = C_{U(p^k)}\left(\Delta_k\right) \times_{\Delta_k} \Sigma_{p^k}.$$

Next we restrict to $U\left(p^k - 1\right) \subset U\left(p^k\right)$, and observe that

$$N_{U(p^k-1)}\left(\Delta_k; \Sigma_{p^k}\right) = N_{U(p^k)}\left(\Delta_k; \Sigma_{p^k}\right) \cap U\left(p^k - 1\right).$$

We have already found that $N_{U(p^k)}\left(\Delta_k; \Sigma_{p^k}\right)$ is a union of cosets $C_{U(p^k)}\left(\Delta_k\right)\sigma$, and $\sigma \in \Sigma_{p^k} \subset U\left(p^k - 1\right)$, so we need only compute the intersection of $C_{U(p^k)}\left(\Delta_k\right)$ with $U\left(p^k - 1\right)$. Recall that $C_{U(p^k)}\left(\Delta_k\right) = (U(1))^{p^k}$, where each copy of $U(1)$ acts on a different irreducible representation of Δ_k on $\mathbb{C}^{p^k}$. However, $U\left(p^k - 1\right)$ is the symmetry group of the orthogonal complement of the diagonal $\mathbb{C} \subset \mathbb{C}^{p^k}$, and the diagonal is actually the trivial representation of Δ_k, so we find

$$C_{U(p^k)}\left(\Delta_k\right) \cap U\left(p^k - 1\right) = (U(1))^{p^k - 1},$$

where each $U(1)$ acts on a different nontrivial irreducible representation of Δ_k, and

$$N_{U(p^k-1)}\left(\Delta_k; \Sigma_{p^k}\right) = \bigcup_{\sigma \in \Sigma_{p^k}} (U(1))^{p^k-1}\,\sigma = (U(1))^{p^k-1} \times_{\Delta_k} \Sigma_{p^k}.$$

Taking the quotient by Σ_k, we find that the indexing set in (5.2) applied to (5.1) is

$$N_{U(p^k-1)}(\Delta_k; \Sigma_{p^k})/\Sigma_{p^k} = (U(1))^{p^k-1}/\Delta_k.$$

To finish the calculation, we note that $\left(S^{\bar{\rho}_{p^k}}\right)^{\Delta_k} \cong S^0$ and we recall that by [**ADL16**, Lemma 10.1], $\left(\mathcal{P}_{p^k}^\diamond\right)^{\Delta_k}$ is equivalent to $T\,\mathrm{GL}_k(\mathbb{F}_p)^\diamond$. Assembling all the pieces,

$$\left[U\left(p^k - 1\right)_+ \wedge_{\Sigma_{p^k}} \left(\mathcal{P}_{p^k}^\diamond \wedge S^{\bar{\rho}_{p^k}}\right)\right]^{\Delta_k} = \bigcup_{[g] \in N_{U(p^k-1)}\left(\Delta_k; \Sigma_{p^k}\right)/\Sigma_{p^k}} \{[g]\}_+ \wedge \left(\mathcal{P}_{p^k}^\diamond \wedge S^{\bar{\rho}_{p^k}}\right)^{\Delta_k}$$

$$\cong \left(U(1)^{p^k-1}\right)/\Delta_{k+} \wedge \left(\mathcal{P}_{p^k}^\diamond \wedge S^{\bar{\rho}_{p^k}}\right)^{\Delta_k}$$

$$\cong C_{U(p^k)}\left(\Delta_k\right)/\left(\Delta_k \times S^1\right)_+ \wedge T\,\mathrm{GL}_k(\mathbb{F}_p)^\diamond,$$

where the S^1 in the last line is the center of $U\left(p^k\right)$.

We conclude that Conjecture 1.7 follows from Conjecture 1.5.

Example 5.1. We can compute $(\mathcal{L}_p)^{\Delta_1}$ explicitly. (In fact, this is done via completely

elementary manipulations in [**BJL**$^+$**15**] for $p = 2$.) There are two types of decompositions λ in $(\mathcal{L}_p)^{\Delta_1}$:

(i) Δ_1 acts freely on $\text{cl}(\lambda)$, in which case λ has p components, each of which is a line;

(ii) Δ_1 acts trivially on $\text{cl}(\lambda)$, in which case each component of λ is a representation of Δ_1.

In the first situation, the decompositions of $\mathbb{C}^p$ into lines that are freely (and therefore transitively) permuted by Δ_1 have no refinements, and also no coarsenings that are stabilized by Δ_1. We assert that they are all in a single orbit of $C_{U(p)}(\Delta_1) \cong (U(1))^p$. For suppose that λ and μ are such decompositions, with $\text{cl}(\lambda) = \{v_1, \ldots, v_p\}$ and $\text{cl}(\mu) = \{w_1, \ldots, w_p\}$. Choose an isomorphism f from v_1 and w_1, and consider the unique extension of f to a Δ_1-equivariant map $u \in U(p)$. Then $u\lambda = \mu$, and u centralizes Δ_1 by construction. Some linear algebra allows us to show that if $u \in C_{U(p)}(\Delta_1) \cong (U(1))^p$ stabilizes λ, then $u \in S^1 \times \Delta_1$, so this component of the object space is homeomorphic to $C_{U(p)}(\Delta_1)/\left(S^1 \times \Delta_1\right)$.

On the other hand, the decompositions of $\mathbb{C}^p$ whose components are each stabilized by Δ_1 are sums of the p distinct one-dimensional representations of Δ_1 in its regular representation on $\mathbb{C}^p$. There are coarsening morphisms between such decompositions, but there are no morphisms from such decompositions to those of the paragraph above. There is an initial object in the subcategory of objects λ in $(\mathcal{L}_p)^{\Delta_1}$ with trivial action on $\text{cl}(\lambda)$, namely the canonical decomposition of $\mathbb{C}^p$ into the lines that are the irreducible representations of Δ_1.

Hence we can actually deduce that

$$(\mathcal{L}_p)^{\Delta_1} \cong \text{Cone}\,(\mathcal{P}_p) \sqcup C_{U(p)}(\Delta_1)/\left(S^1 \times \Delta_1\right)$$
$$\simeq C_{U(p)}(\Delta_1)/\left(S^1 \times \Delta_1\right)_+ \wedge T\,\text{GL}_1(\mathbb{F}_p)^\diamond$$

because $T\,\text{GL}_1(\mathbb{F}_p) = \emptyset$. The result is in conformity with Conjecture 1.7, and is also in agreement with the calculation for $p = 2$ in [**BJL**$^+$**15**], where it was found that $(\mathcal{L}_2)^{\mathbb{Z}/2} \cong * \sqcup S^1$.

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