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A HIGHER WHITEHEAD THEOREM AND THE EMBEDDING OF QUASICATEGORIES IN PREDERIVATORS

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Abstract

We prove a categorified Whitehead theorem showing that the 2-functor HO associating a prederivator to a quasicategory reflects equivalences. The question of whether HO is bicategorically fully faithful (that is, whether morphisms and 2-morphisms can be uniquely lifted from prederivators to quasicategories) is more subtle. We can show that small quasicategories embed fully faithfully, both bicategorically and with respect to a certain simplicial enrichment, into prederivators defined on arbitrary small categories. When the quasicategories are not necessarily small, or when the prederivators are defined only on homotopically finite categories, the 2-categorical argument breaks down, although the simplicial version continues to go through. We give a conjectural counterexample to bicategorical full faithfulness in general.

1. Introduction

Every notion of an abstract homotopy theory $\mathcal{C}$, whether an ∞-category or a model category, admits the common underlying structure the homotopy category $\mathrm{Ho}(\mathcal{C})$. Moreover, for every category J there exists a homotopy theory $\mathcal{C}^J$ of J-shaped diagrams in $\mathcal{C}$, which thus has its own homotopy category $\mathrm{Ho}(\mathcal{C}^J)$. Indeed, each homotopy theory $\mathcal{C}$ gives rise to a 2-functor $\mathrm{Ho}(\mathcal{C}^{(-)})$ sending categories to categories. This is known as the "prederivator" of $\mathcal{C}$. (Pre)derivators were, in fact, axiomatized independently by Grothendieck [**Gro90**], Heller [**Hel88**] and Franke [**Fra96**], before the modern development of flexible models of ∞-categories.

Prederivators are thus often treated in the literature as a notion of abstract homotopy theory, but this intuition has not always been referred to mathematical fact. One might naturally ask for an embedding of quasicategories in prederivators, in any of various homotopical senses. It is clear that not every prederivator is levelwise equivalent to the prederivator associated to a quasicategory. Fuentes-Keuthan, Kędziorek, and Rovelli have recently described the image up to isomorphism of prederivators [**DFKR18**], but there is as yet no proposed description of the image up to equivalence. We view the latter problem as the key remaining question in this area, which will be investigated in future work on 2-categorical Brown representability.

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The question of embedding quasicategories into prederivators bifurcates into two approaches, as quasicategories naturally form a simplicially enriched category $\mathbf{QCat}_\bullet$, while prederivators naturally form only a 2-category $\underline{\mathbf{PDer}}$. We compare both $\mathbf{QCat}_\bullet$ to the simplicially enriched category $\mathbf{PDer}_\bullet$ of [MR] and $\underline{\mathbf{PDer}}$ to the 2-category $\underline{\mathbf{QCat}}$ of quasicategories first studied by Joyal.

The reason for investigating the 2-categorical as well as the simplicial comparison is that the 2-category $\underline{\mathbf{PDer}}$ is to understand to what extent the theory of quasicategories can be reduced to ordinary category theory: $\underline{\mathbf{PDer}}$ is an ordinary 2-category of 2-functors into $\underline{\mathbf{Cat}}$.

Summary of results

We will denote henceforth by $\underline{\mathbf{QCAT}}$ the 2-category of quasicategories. A prederivator is a 2-functor $\mathscr{D}\colon \underline{\mathbf{Dia}}^{\mathrm{op}} \to \underline{\mathbf{CAT}}$, where the indexing category $\underline{\mathbf{Dia}}$ is generally either the 2-category $\underline{\mathbf{Cat}}$ of small categories or that $\underline{\mathbf{HFin}}$ of homotopy finite categories. Denoting the 2-category of prederivators by $\underline{\mathbf{PDER}}$, we can construct 2-functors $\mathrm{HO}\colon \underline{\mathbf{QCAT}} \to \underline{\mathbf{PDER}}$ for each $\underline{\mathbf{Dia}}$.

Our results are as follows.

First result: In Theorem 4.1, we show that the 1-category $\mathbf{QCAT}$ of quasicategories embeds fully faithfully in any category $\mathbf{PDER}^{\mathrm{str}}$ of prederivators with strictly 2-natural morphisms. This extends to an embedding $\mathbf{QCAT}_\bullet \to \mathbf{PDER}_\bullet$ of simplicial categories, where the domain has the usual simplicial enrichment. Thus, quasicategories and their mapping spaces can be recovered *up to isomorphism* from their prederivators and strict maps. The on-the-nose quality of this statement reflects the use of strict transformations of prederivators, as opposed to the pseudo-natural transformations that appear in the later results.

Second result: In Theorem 5.1, we show that $\mathrm{HO}\colon \underline{\mathbf{QCAT}} \to \underline{\mathbf{PDER}}$ is bicategorically fully faithful when restricted to small quasicategories, as long as $\underline{\mathbf{Dia}}$ contains all of $\underline{\mathbf{Cat}}$. The main tool is Joyal's delocalization theorem, as published by Stevenson [Ste16], which we recall as Theorem 5.4. This allows us to write every quasicategory as a localization of a 1-category.

This result is similar to the theorem of Renaudin [Ren09] that a certain 2-category of combinatorial model categories embeds bicategorically fully faithfully in $\underline{\mathbf{PDER}}_!$, the 2-category of cocomplete prederivators with cocontinuous morphisms, insofar as combinatorial model categories model locally presentable quasicategories, cocontinuous maps out of which are determined by restriction to small dense subcategories.

Third result: Our final result, applicable in more generality, is Theorem 6.4, which shows that every version of HO is bicategorically *conservative*, in that equivalences of quasicategories are reflected by HO. In other words, the prederivator is enough to distinguish equivalence classes of abstract homotopy theories, no matter which size choices we make. The proof is unrelated to that of Theorem 5.1, and relies on the author's Whitehead theorem for the 2-category of unpointed spaces [Arl18].

Conventions: We will denote the category, the 2-category, and the simplicial category of foos respectively by

$$\mathbf{foo}, \quad \underline{\mathbf{foo}}, \quad \mathbf{foo}_\bullet$$

Furthermore, when applicable, the above will designate the category of *small* foos while

$$\mathbf{FOO}, \quad \underline{\mathbf{FOO}}, \quad \mathbf{FOO}_\bullet$$

will refer to *large* ones. We operationalize the term *large* to mean "small with respect to the second-smallest Grothendieck universe."

For us 2-categories are strict: they have strictly associative composition and strict units preserved on the nose by 2-functors. We denote the horizontal composition of 2-morphisms by $*$, so that if $\alpha\colon f \Rightarrow g\colon x \to y$ and $\beta\colon h \Rightarrow k\colon y \to z$, we have $\beta * \alpha\colon h \circ f \Rightarrow k \circ g$. If $\mathcal{C}$ is a category (or a 2-category, simplicially enriched category, etc.) with objects c_1 and c_2, we denote the set (or category, simplicial set, etc.) of morphisms by $\mathcal{C}(c_1, c_2)$.

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2. Background on 2-categories and prederivators

Below we recall the various 2-categorical definitions we will require.

Definition 2.1. Morphisms between 2-functors will be either 2-natural or pseudonatural transformations depending on context. Let us recall that, if $\mathcal{K}, \mathcal{L}$ are 2-categories and $F, G\colon \mathcal{K} \to \mathcal{L}$ are 2-functors, a pseudonatural transformation $\Lambda\colon F \Rightarrow G$ consists of

- Morphisms $\Lambda_x\colon F(x) \to G(x)$ associated to every object $x \in \mathcal{K}$
- 2-morphisms $\Lambda_f\colon \Lambda_y \circ F(f) \Rightarrow G(f) \circ \Lambda_x$ for every morphism $f\colon x \to y$ in $\mathcal{K}$

satisfying the coherence conditions:

- (Pseudonaturality) Λ_f is an isomorphism, for every f.
- (Coherence) Λ is a functor from the underlying 1-category of $\mathcal{K}$ to the category of pseudo-commutative squares in $\mathcal{L}$, that is, squares commuting up to a chosen isomorphism, where composition is by pasting.
- (Respect for 2-morphisms) For every 2-morphism $\alpha\colon f \Rightarrow g\colon x \to y$ in $\mathcal{K}$, we have the equality of 2-morphisms
$$\Lambda_g \circ (\Lambda_y * F(\alpha)) = (G(\alpha) * \Lambda_x) \circ \Lambda_f\colon \Lambda_y \circ F(f) \Rightarrow G(g) \circ \Lambda_x.$$

In case all the Λ_f are identities, we say that Λ is strictly 2-natural, in which case the axiom of coherence is redundant, and that of respect for 2-morphisms becomes simply $\Lambda_y * F(\alpha) = G(\alpha) * \Lambda_x$.

Definition 2.2. The morphisms between pseudonatural transformations, are called *modifications.* A modification $\Xi\colon \Lambda \Rrightarrow \Gamma\colon F \Rightarrow G\colon \mathcal{K} \to \mathcal{L}$ consists of 2-morphisms $\Xi_x\colon \Lambda_x \to \Gamma_x$ for each object $x \in \mathcal{K}$, subject to the condition

$$(G(f) * \Xi_x) \circ \Lambda_f = \Gamma_f \circ (\Xi_y * F(f))\colon \Lambda_y \circ F(f) \Rightarrow G(f) \circ \Gamma_x,$$

for any morphism $f\colon x \to y$ in $\mathcal{K}$. When F and G are strict, this simplifies to

$$G(f) * \Xi_x = \Xi_y * F(f).$$

An *equivalence* between the objects $x, y \in \mathcal{K}$ consists of two morphisms $f\colon x \leftrightarrow y\colon g$ together with invertible 2-morphisms $\alpha\colon g \circ f \cong \mathrm{id}_x$ and $\beta\colon f \circ g \cong \mathrm{id}_y$.

If $F\colon \mathcal{K} \to \mathcal{L}$ is a 2-functor between 2-categories, then in general we say F is "locally φ" if φ is a predicate applicable to functors between 1-categories which holds of each functor $\mathcal{K}(x, y) \to \mathcal{L}(F(x), F(y))$ induced by F. For instance, we can in this way ask that F be locally essentially surjective, locally fully faithful, or locally an equivalence.

We shall use the phrase "bicategorically φ" for global properties of F that categorify the property φ as applied to a single functor of categories. For instance, we shall use "bicategorically fully faithful" as a synonym for the potentially misleading term "local equivalence". Finally, F will be said to be "bicategorically conservative" if it reflects equivalences: whenever we have $f\colon x \to y$ in $\mathcal{K}$ such that $F(f)$ is an equivalence in $\mathcal{L}$, we can conclude f is an equivalence in $\mathcal{K}$.

Remark 2.3. Observe that any bicategorically fully faithful 2-functor $F\colon \mathcal{K} \to \mathcal{L}$ is bicategorically conservative. Given an equivalence $H(f)\colon H(x) \leftrightarrows H(y)\colon g$, since quasi-inverses are closed under isomorphism and H is locally essentially surjective, we may assume that $g = H(g')$ for some $g'\colon y \to x$. Now we have only to note that $g'f$ and fg' are isomorphic to their respective identities since $H(g')H(f)$ and $H(f)H(g')$ are.

We now recall the definitions relevant to the theory of derivators.

Definition 2.4. Denote by $\underline{\mathbf{HFin}}$ the 2-category of *homotopy finite* categories. A category is homotopy finite, often (confusingly) called finite direct, if its nerve has finitely many nondegenerate simplices; equivalently, if it is finite, skeletal, and admits no nontrivial endomorphisms.

Definition 2.5. A *prederivator* is a 2-functor $\mathscr{D}\colon \underline{\mathbf{Dia}}^{\mathrm{op}} \to \underline{\mathbf{CAT}}$ into the 2-category $\underline{\mathbf{CAT}}$ of large categories. The 2-category $\underline{\mathbf{Dia}}$ will be, for us, either the 2-category of small categories $\underline{\mathbf{Cat}}$ or the 2-category $\underline{\mathbf{HFin}}$ of homotopy finite categories.

We will often denote $\mathscr{D}(u)$ by u^*, for $u\colon J \to K$ a functor in $\underline{\mathbf{Dia}}$, and similarly for a 2-morphism α in $\underline{\mathbf{Dia}}$.

For categories $J, K \in \underline{\mathbf{Dia}}$, we have a functor $\mathrm{dia}_J^K\colon \mathscr{D}(J \times K) \to \mathscr{D}(J)^K$ induced by the action of $\mathscr{D}$ on the functors and natural transformations from $[0]$ to K. We refer to dia_J^K as a "partial underlying diagram functor," and when $J = [0]$ simply as the "underlying diagram functor," denoted dia^K. See [**Gro13**, Section 1] for more on the associated diagram.

Below are those axioms of derivators that are relevant to this paper. We stick with the traditional numbering due to Maltsiniotis [**Mal05**] but leave out the axioms we shall not consider. The 2-functor $\mathscr{D}$ is a *semiderivator* if it satisfies the first two of the following axioms, and *strong* if it satisfies (Der5). We introduce here a variant (Der5′) of the fifth axiom, prederivators satisfying which will be called *smothering*. Let us remark that, in the presence of Axiom (Der2), (Der5′) requires exactly that $\mathrm{dia}_J^{[1]}$ be smothering in the sense of [**RV15a**], which explains the nomenclature.

(Der1) Let $(J_i)_{i\in I}$ be a family of objects of $\underline{\mathbf{Dia}}$ such that $\coprod_I J_i \in \underline{\mathbf{Dia}}$. Then the canonical map $\mathscr{D}(\coprod_I J_i) \to \prod_I \mathscr{D}(J_i)$ is an equivalence.

(Der2) For every $J \in \underline{\mathbf{Dia}}$, the underlying diagram functor $\mathrm{dia}^J\colon \mathscr{D}(J) \to \mathscr{D}([0])^J$ is conservative, i.e., reflects equivalences.

(Der5) For every $J \in \underline{\mathbf{Dia}}$, the partial underlying diagram functor $\mathrm{dia}_J^{[1]}\colon \mathscr{D}(J \times [1]) \to \mathscr{D}(J)^{[1]}$ is full and essentially surjective on objects.

(Der5′) For every $J \in \underline{\mathbf{Dia}}$, the partial underlying diagram functor $\mathrm{dia}_J^{[1]}\colon \mathscr{D}(J \times [1]) \to \mathscr{D}(J)^{[1]}$ is full and surjective on objects.

A morphism of prederivators is a pseudonatural transformation, and a 2-morphism is a modification (see Definition 2.1). Altogether, we get the 2-category $\underline{\mathbf{PDER}}_{\underline{\mathbf{Dia}}}$ of prederivators defined on $\underline{\mathbf{Dia}}$. We shall make use of the shorthand $\underline{\mathbf{PDER}}$ to represent a 2-category of prederivators defined on an arbitrary $\underline{\mathbf{Dia}}$. When we insist on strictly 2-natural transformations, we get the sub-2-category $\underline{\mathbf{PDER}}^{\mathrm{str}}$, of which we will primarily use the underlying category, $\mathbf{PDER}^{\mathrm{str}}$. Our capitalization convention will reserve the notation $\underline{\mathbf{PDer}}$ for a 2-category of "small" prederivators defined as those with values in the 2-category $\underline{\mathbf{Cat}}$ of small categories.

3. The basic construction

In this section, we will define the homotopy prederivator associated to a quasicategory as the value of a functor, a 2-functor, and a simplicial functor.

The prederivator associated to a quasicategory

We denote the category associated to the poset $0 < 1 < \cdots < n$ by $[n]$, so that $[0]$ is the terminal category. The simplex category Δ is the full subcategory of $\mathbf{Cat}$ on the categories $[n]$.

If S is a simplicial set, that is, a functor $\Delta^{\mathrm{op}} \to \mathbf{Set}$, then we denote its set of n-simplices by $S([n]) = S_n$. The face map $S_n \to S_{n-1}$ which forgets the i^{th} vertex will be denoted d_i^n or just d_i. We denote by Δ^n the simplicial set represented by $[n] \in \Delta$. Equivalently, $\Delta^n = N([n])$, where we recall that the nerve $N(J)$ of a category J is the simplicial set defined by the formula $N(J)_n = \mathbf{Cat}([n], J)$. The natural extension of N to a functor is a fully faithful embedding of categories in simplicial sets. See [**Joy08**, Proposition B.0.13].

We recall that a *quasicategory* [**Joy08**], called an ∞-category in [**Lur09**], is a simplicial set Q in which every inner horn has a filler. That is, every map $\Lambda_i^n \to Q$ extends to an n-simplex $\Delta^n \to Q$ when $0 < i < n$, where $\Lambda_i^n \subseteq \Delta^n$ is the simplicial subset generated by all faces $d_j\Delta^n$ with $j \neq i$. For instance, when $n = 2$, the only inner horn is Λ_1^2, and then the filler condition simply says we may compose "arrows" (that is, 1-simplices) in Q, though not uniquely. Morphisms of quasicategories are simply morphisms of simplicial sets. The quasicategories in which every inner horn has a *unique* filler are, up to isomorphism, the nerves of categories; in particular, the nerve functor $N\colon \mathbf{CAT} \to \mathbf{SSET}$ factors through the subcategory of quasicategories, $\mathbf{QCAT}$.

Every quasicategory Q has a homotopy category $\mathrm{Ho}(Q)$, the 1-category defined as follows. The objects of $\mathrm{Ho}(Q)$ are simply the 0-simplices of Q. For two 0-simplices q_1, q_2, temporarily define $Q_{q_1,q_2} \subseteq Q_1$ to be the set of 1-simplices f with initial vertex q_1 and final vertex q_2. Then the hom-set $\mathrm{Ho}(Q)(q_1, q_2)$ is the quotient of Q_{q_1,q_2} which identifies *homotopic* 1-simplices. Here two 1-simplices $f_1, f_2 \in Q_{q_1,q_2}$ are said to be homotopic if f_1, f_2 are two faces of some 2-simplex in which the third face is both outer and degenerate. We have a functor $\mathrm{Ho}\colon \mathbf{QCAT} \to \mathbf{CAT}$ from quasicategories to categories, left adjoint to the nerve $N\colon \mathbf{CAT} \to \mathbf{QCAT}$. This follows from the fact that a morphism $f\colon Q \to R$ of quasicategories preserves the homotopy relation between 1-simplices, so that it descends to a well defined functor $\mathrm{Ho}(f)\colon \mathrm{Ho}(Q) \to \mathrm{Ho}(R)$. In fact, $\mathrm{Ho}\colon \mathbf{QCat} \to \mathbf{Cat}$ admits an extension, sometimes denoted τ_1, to all of $\mathbf{SSet}$, which is still left adjoint to N. But it is not amenable to computation.

The fact that the Joyal model structure is Cartesian and has the quasicategories as its the fibrant objects implies (see [**RV15a**, 2.2.8]) that Q^S is a quasicategory for every simplicial set S and quasicategory Q. In particular, the category of quasicategories is enriched over itself via the usual simplicial exponential

$$(R^Q)_n = \mathbf{SSET}(Q \times \Delta^n, R).$$

It is immediately checked that the homotopy category functor Ho preserves finite products, so that by change of enrichment we get, finally, *the 2-category of quasicategories*, $\underline{\mathbf{QCAT}}$. Its objects are quasicategories, and for quasicategories Q, R, the hom-category $\underline{\mathbf{QCAT}}(Q, R)$ is simply the homotopy category $\mathrm{Ho}(R^Q)$ of the hom-quasicategory R^Q. This permits the following tautological definition of equivalence of quasicategories.

Definition 3.1. An equivalence of quasicategories is an equivalence in $\underline{\mathbf{QCAT}}$.

Remark 3.2. Thus an equivalence of quasicategories is a pair of maps $f\colon Q \leftrightarrows R\colon g$ together with two homotopy classes $a = [\alpha], b = [\beta]$ of morphisms $\alpha\colon Q \to Q^{\Delta^1}$, $\beta\colon R \to R^{\Delta^1}$, with endpoints gf and id_Q, respectively, fg and id_R, such that a is an isomorphism in $\mathrm{Ho}(Q^Q)$, as is b in $\mathrm{Ho}(R^R)$. We can make the definition yet more explicit by noting that, for each $q \in Q_0$, the map α sends q to some $\alpha(q) \in Q_1$, and recalling that the invertibility of a is equivalent to that of each homotopy class $[\alpha(q)]$, as explicated for instance in the statement below:

Lemma 3.3 ([**RV15a**, 2.3.10])**.** *The equivalence class* $[\alpha]$ *of a map* $\alpha\colon Q \to R^{[1]}$ *is an isomorphism in the homotopy category* $\mathrm{Ho}(R^Q)$ *if and only if, for every vertex* $q \in Q_0$ *of* Q*, the equivalence class* $[\alpha(q)]$ *is an isomorphism in* $\mathrm{Ho}(R)$*.*

We now construct the 2-functor $\mathrm{HO}\colon \underline{\mathbf{QCAT}} \to \underline{\mathbf{PDer}}$ (with respect to an arbitrary $\underline{\mathbf{Dia}}$). Restricting to $\underline{\mathbf{QCat}}$ gives us all the forms of HO of interest to us.

We first extend Ho to a 2-functor of the same name, $\mathrm{Ho}\colon \underline{\mathbf{QCAT}} \to \underline{\mathbf{CAT}}$. This still sends a quasicategory to its homotopy category; we must define the action on morphism categories. This will be for each R and Q a functor

$$\mathrm{Ho}_{Q,R}\colon \underline{\mathbf{QCAT}}(Q,R) = \mathrm{Ho}(R^Q) \to \mathrm{Ho}(R)^{\mathrm{Ho}(Q)} = \underline{\mathbf{CAT}}(\mathrm{Ho}(Q),\mathrm{Ho}(R)).$$

The functor $\mathrm{Ho}_{Q,R}$ is defined as the transpose of the composition

$$\mathrm{Ho}(R^Q) \times \mathrm{Ho}(Q) \cong \mathrm{Ho}(R^Q \times Q) \xrightarrow{\mathrm{Ho}(\mathrm{ev})} \mathrm{Ho}(R)$$

across the product-hom adjunction in the 1-category **Cat**. For this isomorphism we have used again the preservation of finite products by Ho. The morphism $\mathrm{ev}\colon R^Q \times Q \to R$ is evaluation, the counit of the adjunction $(-) \times Q \dashv (-)^Q$ between endofunctors of **QCAT**.

We also need a 2-functor $N\colon \underline{\mathbf{CAT}} \to \underline{\mathbf{QCAT}}$ sending a category $J \in \underline{\mathbf{CAT}}$ to $N(J)$. The map on hom-categories is the composition $J^K \cong \mathrm{Ho}(N(J^K)) \cong \mathrm{Ho}(N(J)^{N(K)})$. The first isomorphism is the inverse of the counit of the adjunction $\mathrm{Ho} \dashv N$, which is an isomorphism by full faithfulness of the nerve. The second uses the fact that N preserves exponentials, see [**Joy08**, Proposition B.0.16].

Finally, we require the following fact: a monoidal functor $F\colon \mathcal{V} \to \mathcal{W}$ induces a 2-functor $F_*(-)\colon \mathcal{V} - \underline{\mathbf{Cat}} \to \mathcal{W} - \underline{\mathbf{Cat}}$ between 2-categories of $\mathcal{V}$- and $\mathcal{W}$-enriched categories. The fully general version of this claim was apparently not published until recently; it comprises Chapter 4 of [**Cru08**]. In our case, the functor Ho is monoidal insofar as it preserves products and thus it induces the 2-functor $\mathrm{Ho}_*(-)$ sending simplicially enriched categories, simplicial functors, and simplicial natural transformations to 2-categories, 2-functors, and 2-natural transformations.

Now we define the homotopy prederivator.

Definition 3.4. Let Q be a quasicategory. Then the *homotopy prederivator* $\mathrm{HO}(Q)\colon \underline{\mathbf{Dia}}^{\mathrm{op}} \to \underline{\mathbf{CAT}}$ is given as the composition

$$\underline{\mathbf{Dia}}^{\mathrm{op}} \xrightarrow{N^{\mathrm{op}}} \underline{\mathbf{QCAT}}^{\mathrm{op}} \xrightarrow{Q^{(-)}} \underline{\mathbf{QCAT}} \xrightarrow{\mathrm{Ho}} \underline{\mathbf{CAT}}.$$

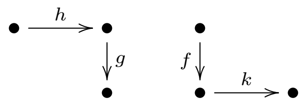
In particular, $\mathrm{HO}(Q)$ maps a category J to the homotopy category of J-shaped diagrams in Q, that is, to $\mathrm{Ho}(Q^{N(J)})$. Given a morphism of quasicategories $f\colon Q \to R$, we have a strictly 2-natural morphism of prederivators (see Definition 2.1) $\mathrm{HO}(f)\colon \mathrm{HO}(Q) \to \mathrm{HO}(R)$ given as the analogous composition $\mathrm{HO}(f) = \mathrm{Ho} \circ f^{(-)} \circ N$, so that for each category J the functor $\mathrm{HO}(f)_J$ is given by post-composition with f, that is, by $\mathrm{Ho}(f^{N(J)})\colon \mathrm{Ho}(Q^{N(J)}) \to \mathrm{Ho}(R^{N(J)})$.

We now record the axioms which are satisfied by the homotopy prederivator of any quasicategory. First, a quasicategorical lemma:

Lemma 3.5. *Let Q be a quasicategory, and $X\colon [1] \times [1] \to \mathrm{Ho}(Q)$ a commutative square in its homotopy category. Suppose we have chosen $f, g \in Q_1$ representing the vertical edges of X, so that $[f] = X|_{\{0\}\times[1]}$ and $[g] = X|_{\{1\}\times[1]}$. Then there exists $\widehat{X}\colon f \to g$ in $\mathrm{Ho}(Q^{\Delta^1})$ lifting X, in the sense that $0^*\widehat{X} = X|_{[1]\times\{0\}}$ and $1^*\widehat{X} = X|_{[1]\times\{1\}}$.*

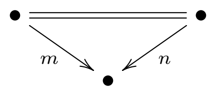
Proof. We must show that any homotopy-commutative square $X\colon [1] \times [1] \to \mathrm{Ho}(Q)$ with chosen lifts $f, g \in (Q)_1$ of its left and right edges underlies a morphism $\widehat{X}\colon f \to g$

in $\mathrm{Ho}(Q^{\Delta^1})$. For this we first lift the top and bottom edges of X to some h and k in Q_1 and choose 2-simplices a, b filling the horns



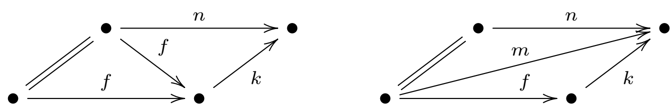
respectively. Let $m = d_1 a$ and $n = d_1 b$ be the inner faces of a and b.

Since X was homotopy commutative, we know $[g] \circ [h] = [k] \circ [f]$ in $\mathrm{Ho}(Q)$, that is, $[m] = [n]$. So there exists a 2-simplex c with boundary



giving a homotopy between $d_1 a$ and $d_1 b$.

Now we define a map $H \colon \Lambda^3_1 \to Q$ with $d_0 H = b, d_2 H = c$, and $d_3 H$ degenerate on f, as below:



Filling H to a 3-simplex $\hat{H}$, we get the desired square by juxtaposing $d_1 \hat{H}$ and a. □

Proposition 3.6. *For any quasicategory Q, the homotopy prederivator* $\mathrm{HO}(Q)$ *satisfies the axioms (Der1), (Der2), and (Der5′).*

Proof. The axiom (Der1) follows from the fact that $Q \mapsto Q^J$ preserves coproducts in J, and that Ho preserves all products. (Der2) is an application of Lemma 3.3, with Q specialized to $N(J)$ for some $J \in \underline{\mathbf{Dia}}$. For (Der5′), surjectivity of $\mathrm{dia}_J^{[1]}$ follows immediately from the definition of the homotopy category. Fullness is exactly Lemma 3.5. □

It may be worth noting that, while it is possible to define a 2-category $\underline{\mathbf{SSet}}$ of simplicial sets using τ_1 and extend HO to $\underline{\mathbf{SSet}}$, the prederivator associated to an arbitrary simplicial set S will not, in general, satisfy any of the three axioms. It is straightforward to see that $\mathrm{HO}(S)$ need not satisfy (Der2) or (Der5), while the reason (Der1) may fail is that τ_1, unlike Ho, need not preserve infinite products.

The simplicial enrichment of prederivators

The 2-functor $\mathrm{HO} \colon \underline{\mathbf{QCAT}} \to \underline{\mathbf{PDer}}$ factors through the subcategory $\underline{\mathbf{PDer}}^{\mathrm{str}}$ in which the morphisms are required to be strictly 2-natural. Its underlying category $\mathbf{PDer}^{\mathrm{str}}$ admits a simplicial enrichment $\mathbf{PDer}_\bullet$, as we now recall.

Muro and Raptis showed how to define the simplicially enriched category $\mathbf{PDer}_\bullet$ in [MR]. First, note that for any prederivator $\mathscr{D}$ and each category $J \in \underline{\mathbf{Dia}}$ we have a shifted prederivator $\mathscr{D}^J = \mathscr{D} \circ (J \times -)$. This shift is a special case of the cartesian closed structure on $\underline{\mathbf{PDer}}$ discussed in [Hel97, Section 4]. Explicitly, given two prederivators $\mathscr{D}_1, \mathscr{D}_2$, and denoting by $\widehat{J}$ the prederivator represented by a small category J, the exponential is defined by $\mathscr{D}_2^{\mathscr{D}_1}(J) = \underline{\mathbf{PDer}}(\widehat{J} \times \mathscr{D}_1, \mathscr{D}_2)$. Then the 2-categorical Yoneda lemma implies that the shifted prederivator $\mathscr{D}^J$ is canonically isomorphic to the prederivator exponential $\mathscr{D}^{\widehat{J}}$. This allows us to interpret expressions such as $\mathscr{D}^\alpha \colon \mathscr{D}^u \Rightarrow \mathscr{D}^v \colon \mathscr{D}^K \to \mathscr{D}^J$, when $\alpha\colon u \Rightarrow v\colon J \to K$ is a natural transformation, by using the internal hom 2-functor.

Remark 3.7. For a natural transformation $\alpha\colon u \Rightarrow v\colon J \to K$ between functors in **Cat**, the preceding definition of $\mathscr{D}^\alpha$ gives only a shadow of the full action of α on $\mathscr{D}$. The natural transformation α corresponds naturally to a functor $\bar{\alpha}\colon J \times [1] \to K$, associated to which we have a prederivator morphism $\mathscr{D}^{\bar{\alpha}}\colon \mathscr{D}^K \to \mathscr{D}^{J\times[1]}$, that is, a family of functors $\mathscr{D}(K \times I) \to \mathscr{D}(J \times I \times [1])$. This is strictly more information, as composing with the underlying diagram functor $\mathrm{dia}^{[1]}_{J\times I}\colon \mathscr{D}(J \times I \times [1]) \to \mathscr{D}(J \times I)^{[1]}$ recovers our original $\mathscr{D}^\alpha$. What is happening here is that the entity $\mathscr{D}^{(-)}$ is more than a 2-functor $\underline{\mathbf{Cat}}^{\mathrm{op}} \to \underline{\mathbf{PDer}}$: it is a simplicial functor $N_*\underline{\mathbf{Cat}}^{\mathrm{op}} \to \mathbf{PDer}_\bullet$ from the simplicial category of nerves of categories to the simplicial category of prederivators, which we must now define.

For each category J let $\mathrm{diag}_J\colon J \to J \times J$ be the diagonal functor.

Definition 3.8. We define $\mathbf{PDer}_\bullet$ as a simplicially enriched category whose objects are the prederivators. The mapping simplicial sets have n-simplices as follows: $\mathbf{PDer}_n(\mathscr{D}_1, \mathscr{D}_2) = \mathbf{PDer}^{\mathrm{str}}(\mathscr{D}_1, \mathscr{D}_2^{[n]})$. For $(f, g) \in \mathbf{PDer}_n(\mathscr{D}_2, \mathscr{D}_3) \times \mathbf{PDer}_n(\mathscr{D}_1, \mathscr{D}_2)$, the composition $f * g\colon \mathscr{D}_1 \to \mathscr{D}_3^{[n]}$ is given by the formula below, in which we repeatedly apply the internal hom 2-functor discussed above Remark 3.7.

$$\mathscr{D}_1 \xrightarrow{g} \mathscr{D}_2^{[n]} \xrightarrow{f^{[n]}} \left(\mathscr{D}_3^{[n]}\right)^{[n]} \cong \mathscr{D}_3^{[n]\times[n]} \xrightarrow{\mathscr{D}_3^{\mathrm{diag}_{[n]}}} \mathscr{D}_3^{[n]}.$$

We can now extend HO to a simplicially enriched functor. The definition follows formally from the following interpretation of the simplicial enrichments on **QCAT** and $\mathbf{PDer}^{\mathrm{str}}$. Each category has a given cosimplicial object, respectively given by the representable simplicial sets $\Delta^\bullet$ and the representable prederivators $\widehat{[\bullet]}$. We have a natural isomorphism $\widehat{[\bullet]} \cong \mathrm{HO}(\Delta^\bullet)$ following from the full faithfulness of the nerve. This shows that for any quasicategory R, the simplicial prederivator $\mathrm{HO}(R)^{[\bullet]}$ is isomorphic to $\mathrm{HO}(R^{\Delta^\bullet})$. In particular, we can define $\mathrm{HO}\colon \mathbf{QCAT}_\bullet(Q, R) \to \mathbf{PDer}_\bullet(\mathrm{HO}(Q), \mathrm{HO}(R))$ on n-simplices by the composition

$$\mathbf{QCAT}(Q, R^{\Delta^n}) \to \mathbf{PDer}^{\mathrm{str}}(\mathrm{HO}(Q), \mathrm{HO}(R^{\Delta^n})) \cong \mathbf{PDer}^{\mathrm{str}}(\mathrm{HO}(Q), \mathrm{HO}(R)^{[n]}).$$

We also have canonical cosimplicial objects in **QCAT** and $\mathbf{PDer}^{\mathrm{str}}$, given by $\Delta^\bullet \times \Delta^\bullet$ and $\widehat{[\bullet]} \times \widehat{[\bullet]}$ respectively, which are mapped isomorphically into each other by HO. Furthermore, the isomorphisms $\mathrm{HO}(\Delta^\bullet) \cong \widehat{[\bullet]}$ and $\mathrm{HO}(\Delta^\bullet \times \Delta^\bullet) \cong \widehat{[\bullet]} \times \widehat{[\bullet]}$ commute with the diagonal and projection morphisms used in the definition of the simplicial compositions in $\mathbf{QCAT}_\bullet$ and $\mathbf{PDer}_\bullet$, so that the suggested definition HO respects the simplicial compositions.

In [**MR**] a restriction of this enrichment, which we now recall, was of primary interest. Each prederivator $\mathscr{D}$ has an "essentially constant" shift by a small category J denoted $\mathscr{D}^J_{\mathrm{eq}}$. This is defined as follows: $\mathscr{D}^J_{\mathrm{eq}}(K) \subseteq \mathscr{D}(J \times K)$ is the full subcategory on those objects $X \in \mathscr{D}(J \times K)$ such that in the partial underlying diagram $\mathrm{dia}^J_K(X) \in$

$\mathscr{D}(K)^J$, the image of every morphism of J is an isomorphism in $\mathscr{D}(K)$. We shall only need $J = [n]$, when an object of $\mathscr{D}^{[n]}_{\mathrm{eq}}(K)$ has as its partial underlying diagram a chain of n isomorphisms in $\mathscr{D}(K)$.

Then we get another simplicial enrichment:

Definition 3.9. The simplicial category $\mathbf{PDer}^{\mathrm{eq}}_{\bullet}$ is the sub-simplicial category of $\mathbf{PDer}_{\bullet}$ with

$$\mathbf{PDer}^{\mathrm{eq}}_n(\mathscr{D}_1, \mathscr{D}_2) = \mathbf{PDer}^{\mathrm{str}}(\mathscr{D}_1, \mathscr{D}^{[n]}_{2,\mathrm{eq}}).$$

This leads to the notion of equivalence of prederivators under which Muro and Raptis showed Waldhausen K-theory is invariant. We say *coherent* below where Muro and Raptis use *strong*, to avoid ambiguity.

Definition 3.10. A coherent isomorphism between strict 2-natural transformations $F\colon \mathscr{D} \to \mathscr{E}, G\colon \mathscr{E} \to \mathscr{D}$ between prederivators is given by 2-morphisms $\alpha\colon \mathscr{D} \to \mathscr{D}^{[1]}_{\mathrm{eq}}$ and $\beta\colon \mathscr{E} \to \mathscr{E}^{[1]}_{\mathrm{eq}}$ such that the vertices of α are GF and $\mathrm{id}_{\mathscr{D}}$, and similarly for β.

The 2-natural transformations F and G comprise a *coherent equivalence* of prederivators if there exist zigzags of coherent isomorphisms connecting GF to $\mathrm{id}_{\mathscr{D}_1}$ and FG to $\mathrm{id}_{\mathscr{D}_2}$.

By [**Joy08**, Proposition B.0.15], the extension $\tau_1\colon \mathbf{SSet} \to \mathbf{Cat}$ of the homotopy category functor Ho to the entirety of $\mathbf{SSet}$ preserves finite products. Thus the simplicial categories $\mathbf{PDer}_{\bullet}$ and $\mathbf{PDer}^{\mathrm{eq}}_{\bullet}$ give rise to 2-categories $\tau_{1*}\underline{\mathbf{PDer}}$ and $\tau_{1*}\underline{\mathbf{PDer}}^{\mathrm{eq}}$ by applying τ_1 to each hom-simplicial set.

Remark 3.11. We are now provided with an abundance of notions of equivalences of prederivators. To wit, we have:

(1) The equivalences in the 2-category $\tau_{1*}\mathbf{PDer}^{\mathrm{eq}}_{\bullet}$.
(2) The equivalences in the 2-category $\tau_{1*}\mathbf{PDer}_{\bullet}$.
(3) The coherent equivalences as defined above.
(4) The equivalences in the 2-category $\underline{\mathbf{PDer}}^{\mathrm{str}}$.
(5) The morphisms in $\underline{\mathbf{PDer}}^{\mathrm{str}}$ which induce levelwise equivalences of categories.
(6) The equivalences in the 2-category $\underline{\mathbf{PDer}}$.
(7) The morphisms in $\underline{\mathbf{PDer}}$ which induce levelwise equivalences of categories.

Proposition 3.12. *The following implications hold among the above classes of equivalences.*

$$\begin{array}{ccccccccc} (1) & \Longrightarrow & (2) & & & & & & \\ \Downarrow & & \Downarrow & & & & & & \\ (3) & \Longrightarrow & (4) & \Longrightarrow & (5) & \Longrightarrow & (6) & \Longleftrightarrow & (7) \end{array}$$

Proof. The implications (4) $\implies$ (5) and (6) $\implies$ (7) are immediate, since evaluation at any J gives a 2-functor $\underline{\mathbf{PDer}} \to \underline{\mathbf{Cat}}$. Since (5) $\implies$ (7) trivially, to prove (5) $\implies$ (6) it is enough to show (7) $\implies$ (6). Given a pseudonatural transformation $F\colon \mathscr{D}_1 \to \mathscr{D}_2$ inducing equivalences of categories $F_J\colon \mathscr{D}_1(J) \to \mathscr{D}_2(J)$ for all $J \in \underline{\mathbf{Dia}}$, arbitrary choices of quasi-inverses $G_J\colon \mathscr{D}_2(J) \to \mathscr{D}_1(J)$ to the F_J can be compiled into a quasi-inverse for F in $\underline{\mathbf{PDer}}$, as can easily be checked directly. More abstractly, this is a case of a basic result in 2-dimensional universal algebra, since $\underline{\mathbf{PDer}}^{\mathrm{str}}$ is the 2-category of strict algebras for a 2-monad on the 2-category of $\underline{\mathbf{Dia}}$-indexed families of categories. See [**Lac07**, Proposition 4.10].

It thus remains to handle the implications involving the simplicial enrichments. Recall that if $\mathcal{K}_\bullet$ is a simplicial category, then the 2-category $\tau_{1*}\mathcal{K}$ has as hom-categories $\mathcal{K}_{\tau_1}(x,y) = \tau_1(\mathcal{K}(x,y)$. Thus an equivalence in $\tau_{1*}\mathcal{K}$ is given by morphisms $f\colon x \leftrightarrows y\colon g$ together with four strings of edges in hom-simplicial sets of $\mathcal{K}_\bullet$, linking id_x and id_y to and from gf and fg, respectively, which become mutually inverse in $\tau_{1*}\mathcal{K}(x,x)$, respectively $\tau_{1*}\mathcal{K}(y,y)$.

This description yields the implications (1) $\implies$ (2), since an edge in $\mathbf{PDer}^{\mathrm{eq}}_\bullet$ is an edge in $\mathbf{PDer}_\bullet$, and (1) $\implies$ (3), since a string of edges in $\mathbf{PDer}^{\mathrm{eq}}_\bullet$ is, in particular, a zigzag. For (3) $\implies$ (4), consider a coherent equivalence $F\colon \mathscr{D}_1 \leftrightarrows \mathscr{D}_2\colon G$. Then by assumption, we may choose a zigzag

$$\mathrm{id}_{\mathscr{D}_1} \to T_1 \leftarrow T_2 \to \cdots \to T_n \leftarrow GF$$

in which the arrows are 1-simplices in $\mathbf{PDer}^{\mathrm{eq}}_\bullet(\mathscr{D}_1, \mathscr{D}_1)$. By definition of $\mathbf{PDer}^{\mathrm{eq}}_\bullet$, each such 1-simplex induces an invertible modification in $\underline{\mathbf{PDer}}^{\mathrm{str}}(\mathscr{D}_1, \mathscr{D}_1)$. Thus we may compose along the zigzag, inverting as necessary, to get a modification $\mathrm{id}_{\mathscr{D}_1} \cong GF$. Similarly, we get $FG \cong \mathrm{id}_{\mathscr{D}_2}$, so F and G are mutually quasi-inverse in $\underline{\mathbf{PDer}}^{\mathrm{str}}$. For (2) $\implies$ (4), the natural morphism of simplicial sets $\mathbf{PDer}_\bullet(\mathscr{D}_1, \mathscr{D}_1)$ $\to N\underline{\mathbf{PDer}}^{\mathrm{str}}(\mathscr{D}_1, \mathscr{D}_1)$ induces an identity-on-objects functor $\tau_{1*}\mathbf{PDer}_\bullet(\mathscr{D}_1, \mathscr{D}_1)$ $\to \underline{\mathbf{PDer}}^{\mathrm{str}}(\mathscr{D}_1, \mathscr{D}_1)$. Thus if $\mathrm{id}_{\mathscr{D}_1}$ and GF are isomorphic in $\tau_{1*}\mathbf{PDer}_\bullet(\mathscr{D}_1, \mathscr{D}_1)$, they are isomorphic in $\underline{\mathbf{PDer}}^{\mathrm{str}}(\mathscr{D}_1, \mathscr{D}_1)$. □

Remark 3.13. In general, no implications between the various classes of equivalences hold other than those listed above. We forebear to give explicit counterexamples, but suggest the nature of each obstruction here. To get (6) $\implies$ (5) or (5) $\implies$ (4) would require a strictifiability theorem for pseudonatural transformations, which does not generally hold. The implications (4) $\implies$ (3) and (4) $\implies$ (2) are versions of Axiom (Der5), weakened to allow lifting into strings or zigzags of coherent morphisms but strengthened to apply to endomorphisms of $\mathscr{D}$, rather than just to objects in the values of $\mathscr{D}$. The easiest way to get (3) $\implies$ (1) is for $\mathbf{PDer}_\bullet(\mathscr{D}_i, \mathscr{D}_i)$ to admit composition of edges, while the easiest way to get (2) $\implies$ (1) is for $\mathbf{PDer}^{\mathrm{eq}}_\bullet(\mathscr{D}_i, \mathscr{D}_i)$ to admit inversion of 1-arrows.

As the above remarks suggest, many of these implications collapse when the $\mathscr{D}_i$ are associated to quasicategories. By Theorem 4.1 below, any *strict* 2-natural transformation $F\colon \mathrm{HO}(Q) \to \mathrm{HO}(R)$ arises from a morphism $f\colon Q \to R$ of quasicategories. By Theorem 6.4, a map $f\colon Q \to R$ of quasicategories is an equivalence if and only if $\mathrm{HO}(f)$ is an equivalence in $\underline{\mathbf{PDer}}$. Furthermore, the simplicial clause of Theorem 4.1 shows that an equivalence of quasicategories induces an equivalence in $\tau_{1*}\mathbf{PDer}^{\mathrm{eq}}_\bullet$. Thus, for prederivators associated to quasicategories, all of (1), (2), (3), (4), and (5) are equivalent.

When the quasicategories Q and R are small and $\underline{\mathbf{Dia}} = \underline{\mathbf{Cat}}$, Theorem 5.1 implies that even if $F\colon \mathrm{HO}(Q) \to \mathrm{HO}(R)$ is merely pseudonatural, if it is an equivalence in $\underline{\mathbf{PDer}}$ then it arises from an equivalence of quasicategories. Thus in that case, all seven notions of equivalences are equivalent. However, in Conjecture 6.1 we suggest a pseudonatural equivalence between **$\underline{\mathbf{HFin}}$**-indexed prederivators of small quasicategories which does not arise from an equivalence, so we do not expect a full collapse of equivalence notions in general, even for quasicategories.

4. The simplicial embedding $\mathbf{QCAT}_\bullet \to \mathbf{PDer}_\bullet$

In this section, we prove that categories of arbitrarily large quasicategories embed fully faithfully in any category of prederivators and strict morphisms. We extend this result to a fully faithful embedding of simplicial categories, as well as of categories enriched in Kan complexes.

Theorem 4.1. *The ordinary functor* $\mathrm{HO}\colon \mathbf{QCAT} \to \mathbf{PDer}^{\mathrm{str}}$ *is fully faithful.*

We first give some corollaries.

Corollary 4.2. *The simplicial functor* $\mathrm{HO}\colon \mathbf{QCAT}_\bullet \to \mathbf{PDer}_\bullet$ *is simplicially fully faithful.*

Proof. The action of HO on n-simplices was defined as map of $\mathbf{QCAT}(Q, R^{\Delta^n}) \to \mathbf{PDer}^{\mathrm{str}}(\mathrm{HO}(Q), \mathrm{HO}(R^{\Delta^n}))$ induced by HO followed by the canonical isomorphism

$$\mathbf{PDer}^{\mathrm{str}}(\mathrm{HO}(Q), \mathrm{HO}(R^{\Delta^n})) \cong \mathbf{PDer}^{\mathrm{str}}(\mathrm{HO}(Q), \mathrm{HO}(R)^{[n]}).$$

Thus, HO induces an isomorphism on n-simplices for every n and is simplicially fully faithful. □

Define, for the moment, $\mathbf{QPDer}_\bullet \subseteq \mathbf{PDer}_\bullet$ to be the image of quasicategories in prederivators, so that Corollary 4.2 gives an isomorphism of simplicial categories $\mathbf{QCAT}_\bullet \cong \mathbf{QPDer}_\bullet$. In particular, $\mathbf{QPDer}_\bullet$ is not merely a simplicial category, but actually a category enriched in quasicategories.

Recall that the inclusion of Kan complexes into quasicategories has a right adjoint ι, which we will call the Kan core. For a quasicategory Q, the core ιQ is the subsimplicial set such that an n-simplex $x \in Q_n$ is in $(\iota Q)_n$ if and only if every 1-simplex of x is an isomorphism in $\mathrm{Ho}(Q)$. See [**Joy02**, Section 1].

As a right adjoint, ι preserves products, so that for any quasicategorically enriched category $\mathcal{C}$ we have an associated Kan complex-enriched category $\iota_*\mathcal{C}$, given by taking the core homwise. (This change of enrichment does not exist on a point-set level for general simplicially enriched categories, which explains our inelegant introduction of $\mathbf{QPDer}_\bullet$.)

Corollary 4.3. *The associated prederivator functor* $\mathrm{HO}\colon \mathbf{QCAT}_\bullet \to \mathbf{PDer}_\bullet$ *induces an isomorphism of* **Kan**-*enriched categories* $\iota_*\mathrm{HO}\colon \iota_*\mathbf{QCAT}_\bullet \to \iota_*\mathbf{QPDer}_\bullet$.

Proof. The given **Kan**-enriched functor exists by the argument of [**Cru08**] described above. It is defined predictably, in the manner of Equation 1 below. We just have to show that $\iota_*\mathrm{HO}$ induces isomorphisms on hom-objects, since $\iota_*\mathrm{HO}$ is bijective on objects by definition. Given the isomorphism $\mathrm{HO}_{Q,R}\colon \mathbf{QCAT}_\bullet(Q,R) \cong \mathbf{PDer}_\bullet(\mathrm{HO}(Q), \mathrm{HO}(R))$ of Theorem 4.1, we get isomorphisms

$$\iota(\mathrm{HO}_{Q,R})\colon \iota(\mathbf{QCAT}_\bullet(Q,R)) \cong \iota(\mathbf{PDer}_\bullet(\mathrm{HO}(Q), \mathrm{HO}(R))) \tag{1}$$

as desired. □

Remark 4.4. The **Kan**-enriched category $\iota_*\mathbf{QCAT}_\bullet$ is a model of the homotopy theory of homotopy theories, which thus embeds into prederivators. In particular, the homotopy category of homotopy theories embeds in the simplicial homotopy category of $\mathbf{PDer}^{\mathrm{eq}}_\bullet$.

In Section 5, we improve this to show that *the homotopy 2-category* in the sense of [RV15b] embeds in the 2-category $\underline{\mathbf{PDer}}_{\mathbf{Cat}}$, a much more concrete object, under certain size assumptions. The word *the* is partially justified here by work of Low [Low13] indicating that the 2-category $\underline{\mathbf{QCat}}$ has a universal role analogous to that of "*the* homotopy category", namely, the homotopy category of spaces.

We turn to the proof of Theorem 4.1. We must show that the ordinary functor HO gives an isomorphism between the sets $\mathbf{QCAT}(Q, R)$ and $\mathbf{PDer}^{\mathrm{str}}(\mathrm{HO}(Q), \mathrm{HO}(R))$. This is Proposition 4.8, whose proof has the following outline:

(1) Eliminate most of the data of a prederivator map by showing strict maps $\mathrm{HO}(Q) \to \mathrm{HO}(R)$ are determined by their restriction to natural transformations between ordinary functors $\mathbf{Cat}^{\mathrm{op}} \to \mathbf{Set}$. This is Lemma 4.6.
(2) Show that $\mathrm{HO}(Q)$ and $\mathrm{HO}(R)$ recover Q and R upon restricting the domain to Δ^{op} and the codomain to **Set**, and that natural transformations as in the previous step are in bijection with maps $Q \to R$. This is Lemma 4.7.
(3) Show that $\mathrm{HO}(f)$ restricts back to f for a map $f\colon Q \to R$, which implies that HO is faithful, and that a map $F\colon \mathrm{HO}(Q) \to \mathrm{HO}(R)$ is exactly HO applied to its restriction, which implies that HO is full. This constitutes the proof of Proposition 4.8 proper.

Let us begin with step (1).

Definition 4.5. A **Dia**-set is a large presheaf on **Dia** that is, an ordinary functor $\mathbf{Dia}^{\mathrm{op}} \to \mathbf{SET}$.

Given a prederivator $\mathscr{D}$, let $\mathscr{D}^{\mathrm{ob}}\colon \mathbf{Dia}^{\mathrm{op}} \to \mathbf{SET}$ be its underlying **Dia**-set, so that $\mathscr{D}^{\mathrm{ob}}$ sends a small category J to the set of objects $\mathrm{ob}(\mathscr{D}(J))$ and a functor $u\colon I \to J$ to the action of $\mathscr{D}(u)$ on objects.

Recall that where (Der5) requires that $\mathrm{dia}\colon \mathscr{D}(J \times [1]) \to \mathscr{D}(J)^{[1]}$ be (full and) essentially surjective, (Der5′) insists on actual surjectivity on objects. The following lemma shows that under this assumption most of the apparent structure of a strict prederivator map is redundant.

Lemma 4.6. *A strict morphism $F\colon \mathscr{D}_1 \to \mathscr{D}_2$ between prederivators satisfying (Der5′) is determined by its restriction to the underlying* **Dia***-sets $\mathscr{D}_1^{\mathrm{ob}}, \mathscr{D}_2^{\mathrm{ob}}$. That is, the restriction functor from prederivators satisfying (Der5′) to* **Dia***-sets is faithful.*

Proof. The data of a strict morphism $F\colon \mathscr{D}_1 \to \mathscr{D}_2$ is that of a functor $F_J\colon \mathscr{D}_1(J) \to \mathscr{D}_2(J)$ for every J.[1]

The induced map $F^{\mathrm{ob}}\colon \mathscr{D}_1^{\mathrm{ob}} \to \mathscr{D}_2^{\mathrm{ob}}$ is given by the action of F on objects. So to show faithfulness it is enough to show that, given a family of functions

[1] Note the simplification here over pseudonatural transformations, which require also a natural transformation associated to every functor and do not induce maps of **Dia**-sets. That is the fundamental difficulty leading to the dramatically different techniques of the next sections.

$r_J\colon \mathrm{ob}(\mathscr{D}_1(J)) \to \mathrm{ob}(\mathscr{D}_2(J))$, that is, the data required in a natural transformation between **Dia**-sets, there is at most one 2-natural transformation with components $F_J\colon \mathscr{D}_1(J) \to \mathscr{D}_2(J)$ and object parts $\mathrm{ob}(F_J) = r_J$.

Indeed, suppose F is given with object parts $r_J = \mathrm{ob}(F_J)$ and let $f\colon X \to Y$ be a morphism in $\mathscr{D}_1(J)$. Then by Axiom (Der5′), f is the underlying diagram of some $\widehat{f} \in \mathscr{D}_1(J \times [1])$. By 2-naturality, the following square must commute:

$$\begin{array}{ccc} \mathscr{D}_1(J \times [1]) & \xrightarrow{F_{J\times[1]}} & \mathscr{D}_2(J \times [1]) \\ {\scriptstyle \mathrm{dia}_J^{[1]}}\downarrow & & \downarrow{\scriptstyle \mathrm{dia}_J^{[1]}} \\ \mathscr{D}_1(J)^{[1]} & \xrightarrow{F_J} & \mathscr{D}_2(J)^{[1]}. \end{array}$$

Indeed, $\mathrm{dia}_J^{[1]}$ is the action of a prederivator on the unique natural transformation between the two functors $0, 1\colon [0] \to [1]$ from the terminal category to the arrow category, as is described in full detail below [**Gro13**, Proposition 1.7]. Thus the square above is an instance of the axiom of respect for 2-morphisms. It follows that we must have $F_J(f) = F_J(\mathrm{dia}_J^{[1]}\widehat{f}) = \mathrm{dia}_J^{[1]}(r_{J\times[1]}(\widehat{f}))$.

Thus if F, G are two strict morphisms $\mathscr{D}_1 \to \mathscr{D}_2$ with the same restrictions to the underlying **Dia**-sets, they must coincide, as claimed. □

Note the above does not claim that the restriction functor is full: the structure of a strict prederivator map is determined by the action on objects of each $\mathscr{D}_1(J), \mathscr{D}_2(J)$, but it is not generally true that an arbitrary map of **Dia**-sets will admit a well defined extension to morphisms.

We proceed to step (2) of the proof.

Let us recall the theory of pointwise Kan extensions for 1-categories. Let $F\colon \mathcal{C} \to \mathcal{D}$ and $G\colon \mathcal{C} \to \mathcal{E}$ be functors. At least if $\mathcal{C}$ and $\mathcal{D}$ are small and $\mathcal{E}$ is complete, then we always have a right Kan extension $F_*G\colon \mathcal{D} \to \mathcal{E}$ characterized by the adjunction formula $\mathcal{E}^{\mathcal{D}}(H, F_*G) \cong \mathcal{E}^{\mathcal{C}}(H \circ F, G)$ and computed on objects by

$$F_*G(d) = \lim_{d\downarrow F} G \circ q.$$

Here $d \downarrow F$ is the comma category with objects $(c, f\colon d \to F(c))$ and morphisms the maps in $\mathcal{C}$ making the appropriate triangle commute, and $q\colon d \downarrow F \to \mathcal{C}$ is the projection.

Lemma 4.7. *Let $j\colon \Delta^{\mathrm{op}} \to \mathbf{Dia}^{\mathrm{op}}$ be the inclusion. Then for any quasicategory R, the **Dia**-set $\mathrm{HO}(R)^{\mathrm{ob}}$ underlying $\mathrm{HO}(R)$ is the right Kan extension of R along j.*

Proof. For any small category J, the **Dia**-set $\mathrm{HO}(R)^{\mathrm{ob}}$ takes J to the set of simplicial set maps from J to R:

$$\mathrm{HO}(R)^{\mathrm{ob}}(J) = \mathrm{ob}(\mathrm{Ho}(R^{N(J)})) = \mathbf{SSET}(N(J), R).$$

We shall show that the latter is the value required of j_*R at J, which exists and is calculated via Equation 5.1 since **SET** is complete (in the sense of a universe in which its objects constitute the small sets).

First, one of the basic properties of presheaf categories implies that $N(J)$ is a colimit over its category of simplices. That is, $N(J) = \operatorname{colim}_{\Delta\downarrow NJ} y \circ q$, where $q\colon \Delta \downarrow NJ \to \Delta$ is the projection and $y\colon \Delta \to \mathbf{SSet}$ is the Yoneda embedding.

Then we can rewrite the values of $\mathrm{HO}(R)^{\mathrm{ob}}$ as follows:

$$\begin{aligned}\mathrm{HO}(R)^{\mathrm{ob}}(J) = \mathbf{SSET}(N(J), R) &= \mathbf{SSET}(\operatorname*{colim}_{\Delta\downarrow NJ} y\circ q, R)\\ &\cong \lim_{(\Delta\downarrow NJ)^{\mathrm{op}}} \mathbf{SSET}(y\circ q, R) \cong \lim_{(\Delta\downarrow NJ)^{\mathrm{op}}} R\circ q^{\mathrm{op}}.\end{aligned}$$

The last isomorphism follows from the Yoneda lemma.

The indexing category $(\Delta\downarrow N(J))^{\mathrm{op}}$ has as objects pairs $(n, f\colon \Delta^n \to N(J))$ and as morphisms $\bar{a}\colon (n,f)\to(m,g)$, the maps $a\colon \Delta^m\to\Delta^n$ such that $f\circ a = g$. That is, $(\Delta\downarrow N(J))^{\mathrm{op}} \cong N(J)\downarrow \Delta^{\mathrm{op}}$, where on the right-hand side $N(J)$ is viewed as an object of $\mathbf{SSET}^{\mathrm{op}}$. Using the full faithfulness of the nerve functor N, we see $(\Delta\downarrow N(J))^{\mathrm{op}} \cong J\downarrow\Delta^{\mathrm{op}}$, where again $J\in\mathbf{Dia}^{\mathrm{op}}$.

Thus, if q^{op} serves also to name the projection $J/\Delta^{\mathrm{op}}\to\Delta^{\mathrm{op}}$, we may continue the computation above with

$$\mathrm{HO}(R)^{\mathrm{ob}}(N(J)) \cong \lim_{N(J)\downarrow\Delta^{\mathrm{op}}} R\circ q^{\mathrm{op}}.$$

This is exactly the formula for $j_*R(J)$ recalled above. The isomorphism thus constructed is certainly natural with respect to the action on maps of the Kan extension, so the lemma is established. □

We arrive at step (3).

Proposition 4.8. *The homotopy prederivator functor* $\mathrm{HO}\colon \mathbf{QCAT}\to\mathbf{PDer}^{\mathrm{str}}$ *is a fully faithful embedding of 1-categories.*

Proof. Note that, by Lemma 4.7, the restriction of $\mathrm{HO}(Q)^{\mathrm{ob}}$ to a functor $\Delta^{\mathrm{op}}\to\mathbf{SET}$ is canonically isomorphic to Q, since Kan extensions along fully faithful functors are splittings of restriction. Thus a map $F\colon \mathrm{HO}(Q)\to\mathrm{HO}(R)$ restricts to a map $\rho(F)\colon Q\to R$. In fact, we have a natural isomorphism $\rho\circ\mathrm{HO}\cong \mathrm{id}_{\mathbf{QCAT}}$, so that $\rho\circ\mathrm{HO}(f)$ is again f, up to this isomorphism. Indeed, given $f\colon Q\to R$, we already know how to compute $\mathrm{HO}(f)$ as $\mathrm{Ho}\circ\left(f^{N(-)}\right)$. Then the restriction $\rho(\mathrm{HO}(f))\colon Q\to R$, which we are to show coincides with f, is given by $\rho(\mathrm{HO}(f))_n = \mathrm{ob}\circ\mathrm{Ho}\circ f^{\Delta^n}$. That is, $\rho(\mathrm{HO}(f))$ acts by the action of f on the objects of the homotopy categories of Q^{Δ^n} and R^{Δ^n}. In other words, it acts by the action of f on the sets $\mathbf{SSET}(\Delta^n, Q)$ and $\mathbf{SSET}(\Delta^n, R)$; via Yoneda, $\rho(\mathrm{HO}(f))$ acts by f itself.

It remains to show that $\mathrm{HO}(\rho(F)) = F$ for any $F\colon \mathrm{HO}(Q)\to\mathrm{HO}(R)$. By Lemma 4.6 it suffices to show that the restrictions of $\mathrm{HO}(\rho(F))$ and F to the underlying $\mathbf{Dia}$-sets coincide. Using Lemma 4.7 and the adjunction characterizing the Kan extension, we have

$$\begin{aligned}\mathbf{SET}^{\mathbf{Dia}^{\mathrm{op}}}(\mathrm{HO}(Q)^{\mathrm{ob}}, \mathrm{HO}(R)^{\mathrm{ob}}) = \mathbf{SET}^{\mathbf{Dia}^{\mathrm{op}}}(j_*Q, j_*R) &\cong \mathbf{SSET}(j^*j_*Q, R)\\ &\cong \mathbf{SSET}(Q,R).\end{aligned}$$

In particular, maps between $\mathrm{HO}(Q)^{\mathrm{ob}}$ and $\mathrm{HO}(R)^{\mathrm{ob}}$ agree when their restrictions to Q and R do. Thus we are left to show that $\rho(\mathrm{HO}(\rho(F))) = \rho(F)$. But as we showed above, $\rho\circ\mathrm{HO}$ is the identity map on $\mathbf{SSET}(Q,R)$, so the proof is complete. □

5. The embedding $\underline{\mathbf{QCat}}\to\underline{\mathbf{PDer}}$ of 2-categories

We shall now prove an analogous embedding theorem in the 2-categorical setting.

Theorem 5.1. *Let* $\underline{\mathbf{QCat}}$ *denote the 2-category of small quasicategories. Then the 2-functor* HO: $\underline{\mathbf{QCat}} \to \underline{\mathbf{PDer}_{\mathbf{Cat}}}$ *is bicategorically fully faithful; that is, it induces equivalences of hom-categories* $\underline{\mathbf{QCat}}(Q,R) \simeq \underline{\mathbf{PDer}_{\mathbf{Cat}}}(\mathrm{HO}(Q), \mathrm{HO}(R))$ *for any quasicategories* Q *and* R.

We get a Whitehead theorem for quasicategories as a corollary, following Remark 2.3. However, note that the following is implied by Theorem 6.4, whereas neither one of Theorem 5.1 and Theorem 6.4 implies the other.

Corollary 5.2. *The 2-functor* HO: $\underline{\mathbf{QCat}} \to \underline{\mathbf{PDer}_{\mathbf{Cat}}}$ *is bicategorically conservative; that is, if* $f\colon Q \to R$ *is a morphism of small quasicategories such that* $\mathrm{HO}(f)$ *is an equivalence in* $\underline{\mathbf{PDer}_{\mathbf{Cat}}}$, *then* f *is an equivalence in* $\underline{\mathbf{QCat}}$.

The core tool for the proof is Theorem 5.4 below, which says that every quasicategory is a localization of a category. It is due to Joyal but was first published by Stevenson in [**Ste16**].

First we recall the notion of ∞-localization, often just "localization," for simplicial sets and quasicategories.

Definition 5.3. Let $f\colon S \to T$ be a map of simplicial sets and $\mathcal{W} \subseteq S_1$ a set of edges. For any quasicategory Q, let $Q^S_{\mathcal{W}}$ be the full sub-quasicategory of Q^S on those maps $g\colon S \to Q$ such that $g(w)$ is an equivalence in Q for every edge $w \in \mathcal{W}$.

Then we say f exhibits T as an ∞-localization of S at $\mathcal{W}$ if, for every quasicategory Q, the morphism $f^*\colon Q^T \to Q^S$ factors via an equivalence $f^*\colon Q^T \to Q^S_{\mathcal{W}}$ of quasicategories.

In particular, if $f\colon S \to T$ is a localization at $\mathcal{W}$ then for any quasicategory Q, the pullback $f^*\colon \mathrm{Ho}(Q^T) \to \mathrm{Ho}(Q^S)$ is fully faithful, as we will use repeatedly below. Specifically, f^* is an equivalence onto the full subcategory $\mathrm{Ho}(Q^S_{\mathcal{W}}) \subseteq \mathrm{Ho}(Q^S)$, since the 2-functor Ho preserves equivalences.

Let $\Delta \downarrow S$ be the category of simplices of a simplicial set S, and let $p_S\colon N(\Delta \downarrow S) \to S$ be the natural extension of the projection $(f\colon \Delta^m \to S) \mapsto f(m)$. Finally, let $\mathcal{L}_S$ be the class of arrows $a\colon (f\colon \Delta^m \to S) \to (g\colon \Delta^n \to S)$ in $\Delta \downarrow S$ such that $a(n) = m$, that is, the last-vertex maps. Then we have the following theorem:

Theorem 5.4 ([**Ste16**])**.** *For any quasicategory* Q, *the last-vertex projection* p_Q *exhibits* Q *as an* ∞*-localization of the nerve* $N(\Delta \downarrow Q)$ *at the class* $\mathcal{L}_Q$.

Thus every quasicategory Q is canonically a localization of its category $\Delta \downarrow Q$ of simplices.

Remark 5.5. Observe that $N(\Delta \downarrow (-))$ constitutes an endofunctor of simplicial sets and that $p\colon N(\Delta \downarrow (-)) \to \mathrm{id}_{\mathbf{SSet}}$ is a natural transformation.

We turn to the proof.

Proof of Theorem 5.1. First, we must show that if $F\colon \mathrm{HO}(Q) \to \mathrm{HO}(R)$ is a pseudonatural transformation, then there exists $h\colon Q \to R$ and an isomorphism $\Lambda\colon \mathrm{HO}(h) \cong F$. Observe that, since Q is small, $\Delta \downarrow Q$ is in $\underline{\mathbf{Cat}}$. Now we claim that $F_{\Delta\downarrow Q}(p_Q)\colon \Delta \downarrow Q \to R$ sends the class $\mathcal{L}_Q$ of last-vertex maps into equivalences in R. Indeed, if $\ell\colon \Delta^1 \to$

$\Delta \downarrow Q$ is in $\mathcal{L}_Q$, then we have, using F's respect for 2-morphisms and the structure isomorphism F_ℓ,

$$F_{[0]}(\mathrm{dia}(\ell^* p_Q)) = \mathrm{dia}(F_{[1]}(\ell^* p_Q)) \cong \mathrm{dia}(\ell^* F_{\Delta\downarrow J}(p_Q)).$$

Thus $\mathrm{dia}(\ell^* F_{\Delta\downarrow J}(p_Q))$ is an isomorphism in $\mathrm{Ho}(R)$, since $\mathrm{dia}(\ell^* p_Q)$ is an isomorphism in $\mathrm{Ho}(Q)$. Then using the delocalization theorem, we can define $h\colon Q \to R$ as any map admitting an isomorphism $\sigma\colon h \circ p_Q \cong F_{\Delta\downarrow Q}(p_Q)$. From σ, we get an invertible modification

$$\mathrm{HO}(\sigma)\colon \mathrm{HO}(h \circ p_Q) \Rightarrow \mathrm{HO}(F_{\Delta\downarrow Q}(p_Q))\colon \mathrm{HO}(\Delta \downarrow Q) \to \mathrm{HO}(R).$$

We now construct an invertible modification $\Lambda\colon \mathrm{HO}(h) \Rightarrow F\colon \mathrm{HO}(Q) \to \mathrm{HO}(R)$. Fixing $J \in \underline{\mathbf{Cat}}$ and $X\colon N(J) \to Q$, since $p_J^*\colon \mathrm{HO}(R)(J) \to \mathrm{HO}(R)(\Delta \downarrow J)$ is fully faithful we can uniquely define $\Lambda_{X,J}\colon \mathrm{HO}(h)_J(X) \cong F_J(X)$ by giving $p_J^*(\Lambda_{X,J})$. To wit, we require $p_J^*(\Lambda_{X,J})$ to be the composition

$$\begin{aligned} p_J^*\mathrm{HO}(h)_J(X) &= h \circ X \circ p_J = h \circ p_Q \circ \Delta \downarrow X \\ &\cong F_{\Delta\downarrow Q}(p_Q) \circ \Delta \downarrow X \cong F_{\Delta\downarrow J}(p_Q \circ \Delta \downarrow X) = F_{\Delta\downarrow J}(X \circ p_J) \\ &\cong F_J(X) \circ p_J. \end{aligned}$$

The first isomorphism is a component of $\mathrm{HO}(\sigma)$, while the latter two are components of F. The naturality of $\Lambda_{J,X}$ in X thus follows from the facts that F is pseudonatural and that $\mathrm{HO}(\sigma)$ is a modification. So we have natural isomorphisms $\Lambda_J\colon \mathrm{HO}(h)_J \Rightarrow F_J$ for each J. To verify that the Λ_J assemble into a modification, consider any $u\colon K \to J$. Then we must show that, for any $X\colon J \to Q$, the diagram

$$\begin{array}{ccc} \mathrm{HO}(h)_J(X) \circ u & \xrightarrow{\Lambda_{J,X} * u} & F_J(X) \circ u \\ \| & & \downarrow{\scriptstyle F_u} \\ \mathrm{HO}(h)_K(X \circ u) & \xrightarrow{\Lambda_{K,X\circ u}} & F_K(X \circ u) \end{array}$$

commutes. Using, as always, full faithfulness of the pullback along a localization, we may precompose with p_K. Then the modification axiom is verified by the commutativity of the following diagram:

$$\begin{array}{ccccc} hXup_K & \xrightarrow{\Lambda_{J,X} * up_K} & F_J(X)up_K & & \\ \| & & \| & & \\ hXp_J\Delta \downarrow u & \xrightarrow{\Lambda_{J,X} * p_J\Delta\downarrow u} & F_J(X)p_J\Delta \downarrow u & = & F_J(X)up_K \\ \| & & \uparrow{\scriptstyle F_{p_J} * \Delta\downarrow u} & & \uparrow{\scriptstyle F_u * p_K} \\ hp_Q\Delta \downarrow Xu & & F_{\Delta\downarrow J}(Xp_J)\Delta \downarrow u & & F_K(Xu)p_K \\ \| & & \| & & \uparrow{\scriptstyle F_{p_K}} \\ F_{\Delta\downarrow Q}(p_Q)\Delta \downarrow Xu & \xrightarrow{F^{-1}_{\Delta\downarrow X} * \Delta\downarrow u} & F_{\Delta\downarrow J}(p_Q\Delta \downarrow X)\Delta \downarrow u & & F_{\Delta\downarrow K}(Xup_K) \\ & \searrow{\scriptstyle F^{-1}_{\Delta\downarrow Xu}} & \uparrow{\scriptstyle F_{\Delta\downarrow u}} & // & \\ & & F_{\Delta\downarrow K}(p_Q\Delta \downarrow Xu). & & \end{array}$$

The upper left square commutes since $up_K = p_J\Delta \downarrow u$. The left central hexagon commutes by definition of $\Lambda_{J,X}$, and the lower left triangle and right-hand heptagon commute by functoriality of the pseudonaturality isomorphisms of F. Meanwhile, the outer route around the diagram from $hXup_K$ to $F_J(X)up_K$ is $F_u\Lambda_{K,Xu}$, while the inner route is $\Lambda_{J,X} * up_K$. So Λ is an invertible modification $\mathrm{HO}(h) \cong F$, as desired.

We have shown that HO induces an essentially surjective functor $\underline{\mathbf{QCat}}(Q,R) \to \underline{\mathbf{PDer}}(\mathrm{HO}(Q),\mathrm{HO}(R))$. We next consider full faithfulness. So, assume given a modification

$$\Xi\colon \mathrm{HO}(f) \Rightarrow \mathrm{HO}(g)\colon \mathrm{HO}(Q) \to \mathrm{HO}(R).$$

We must show there exists a unique $\xi\colon f \Rightarrow g$ with $\mathrm{HO}(\xi) = \Xi$. First, we consider

$$\Xi_{p_Q}\colon f \circ p_Q \to g \circ p_Q,$$

which is a morphism in $\mathrm{HO}(R)(\Delta \downarrow Q)$. According to (Der5′), we can lift this to a map $\widehat{\Xi}_{p_Q}\colon \Delta \downarrow Q \to R^{\Delta^1}$ with $\mathrm{dia}(\widehat{\Xi}_{p_Q}) = \Xi_{p_Q}$.

Since the domain and codomain $f \circ p_Q$ and $g \circ p_Q$ of $\widehat{\Xi}_{p_Q}$ invert the last-vertex maps $\mathcal{L}_Q$, by (Der2) so does $\widehat{\Xi}_{p_Q}$ itself. Thus by the delocalization theorem we get $\widehat{\Xi}'\colon Q \to R^{\Delta^1}$ with an isomorphism

$$a\colon \widehat{\Xi}' \circ p_Q \cong \widehat{\Xi}_{p_Q}.$$

The domain and codomain

$$0^*a\colon 0^*(\widehat{\Xi}' \circ p_Q) \cong fp_Q \text{ and } 1^*a\colon 1^*(\widehat{\Xi}' \circ p_Q) \cong gp_Q$$

give rise to unique isomorphisms

$$i\colon 0^*\widehat{\Xi}' \cong f \text{ and } j\colon 1^*\widehat{\Xi}' \cong g.$$

Now we can construct $\widehat{\Xi}\colon Q \to R^{\Delta^1}$ as a lift of the composite

$$f \xrightarrow{i^{-1}} 0^*\widehat{\Xi}' \xrightarrow{\mathrm{dia}(\widehat{\Xi}')} 1^*\widehat{\Xi}' \xrightarrow{j} g$$

in $\mathrm{Ho}(R^Q)$. Using the fullness clause of (Der5′), we can choose an isomorphism $b\colon \widehat{\Xi} \cong \widehat{\Xi}'$ in $\mathrm{Ho}((R^{\Delta^1})^Q)$ lifting $(i^{-1}, j^{-1})\colon \mathrm{dia}(\widehat{\Xi}) \to \mathrm{dia}(\widehat{\Xi}')$.

Then $a \circ (b * p_Q)\colon \widehat{\Xi} \circ p_Q \to \widehat{\Xi}_{p_Q}$ is an isomorphism with endpoints fixed, insofar as $0^*(b * p_Q) = i^{-1} * p_Q = 0^*a^{-1}$ and similarly $1^*(b * p_Q) = 1^*a^{-1}$. Thus $\mathrm{dia}(\widehat{\Xi} \circ p_Q) = \mathrm{dia}(\widehat{\Xi}_{p_Q}) = \Xi_{p_Q}$ in $\mathrm{Ho}(R^{\Delta\downarrow Q})$.

Notice that if $\widehat{\Xi}_2\colon Q \to R^{\Delta^1}$ is any other morphism satisfying $\mathrm{dia}(\widehat{\Xi}_2 \circ p_Q) = \Xi_{p_Q}$, then $\mathrm{dia}(\widehat{\Xi}_2) = \mathrm{dia}(\widehat{\Xi})$, since pullback along p_Q is faithful. So we have a unique candidate $\xi := \mathrm{dia}(\widehat{\Xi})\colon f \Rightarrow g$; it remains to show that $\mathrm{HO}(\xi) = \Xi$.

To that end, we claim that for every $X\colon J \to Q$, we have $\mathrm{HO}(\xi)_X = \xi * X = \Xi_X$. As above, it suffices to precompose X with p_J, and then we have

$$\begin{aligned} \xi * X * p_J = \mathrm{dia}(\widehat{\Xi}) * p_Q * \Delta \downarrow X &= \mathrm{dia}(\widehat{\Xi} \circ p_Q) \circ \Delta \downarrow X \\ &= \Xi_{p_Q} * \Delta \downarrow X = \Xi_{p_Q \circ \Delta\downarrow X} = \Xi_{X \circ p_J} = \Xi_X * p_J \end{aligned}$$

as desired. In the equations above we have used the 2-functoriality of $\mathrm{HO}(R)$, naturality of p, and the modification property of Ξ. So $\mathrm{HO}(\xi) = \Xi$, as was to be shown. □

6. Whitehead's theorem for quasicategories

As discussed above, Theorem 5.1 says that HO is bicategorically fully faithful, and in particular, bicategorically conservative, when the domain is small quasicategories and $\underline{\mathbf{Dia}} = \underline{\mathbf{Cat}}$. In this section, we show that HO is at least bicategorically conservative no matter what assumptions are placed on the domain and codomain. We first give informal arguments that this is the best one should hope for.

Evidence against a stronger result

We do not hope to prove all of Theorem 5.1 for arbitrary choices of domain and codomain, as is most intuitive to see in the case of $\mathrm{HO}\colon \underline{\mathbf{QCat}} \to \underline{\mathbf{PDer}_{\mathbf{HFin}}}$. Since the 2-category $\underline{\mathbf{HFin}}$ of homotopy finite categories is small, prederivators of that domain form a "concrete 2-category" in the strongest possible sense. That is, we have a 2-functor $U\colon \underline{\mathbf{PDer}_{\mathbf{HFin}}} \to \underline{\mathbf{Cat}}$, faithful on 1- and 2-morphisms, given by

$$U(\mathscr{D}) = \prod_{J \in \underline{\mathbf{HFin}}} \mathscr{D}(J) \times \prod_{u\colon K \to J} \mathscr{D}(J)^{[1]}.$$

For $F\colon \mathscr{D}_1 \to \mathscr{D}_2$, we have $U(F) = ((F_J), (F_u\colon \mathscr{D}(J)^{[1]} \to \mathscr{D}(K)^{[1]}))$, where the functor F_u sends $f\colon X \to Y$ to the arrow $u^*F(X) \to F(u^*Y)$ which can be defined in two equivalent ways using the pseudonaturality isomorphisms of F. Similarly, for $\Xi\colon F_1 \Rightarrow F_2$, we have $U(\Xi) = ((\Xi_J), (\Xi_u\colon F_u \Rightarrow G_u))$, where the components of Ξ_u are $u * \Xi_X$ and Ξ_{u^*Y}. It is straightforward to check that these objects are, respectively, a functor and a natural transformation.[2]

Since $\mathrm{HO}\colon \underline{\mathbf{QCat}} \to \underline{\mathbf{PDer}_{\mathbf{HFin}}}$ is faithful on 1-morphisms, if it were also faithful on 2-morphisms then $\underline{\mathbf{QCat}}$ would be a strictly concrete 2-category in the same sense as $\underline{\mathbf{PDer}_{\mathbf{HFin}}}$. In perhaps more familiar terms, there would be no "phantom homotopies" between maps of quasicategories. That this should be the case strains credulity, given the famous theorem of Freyd [**Fre04**] that the category of spaces **Hot** is not concrete (i.e. admits no faithful functor to **Set**,) though we propose no specific counterexample.

We do conjecture a specific counterexample to the claim that $\mathrm{HO}\colon \underline{\mathbf{QCat}} \to \underline{\mathbf{PDer}_{\mathbf{HFin}}}$ is locally essentially surjective. That is, we suggest a pseudonatural transformation $F\colon \mathrm{HO}(Q) \to \mathrm{HO}(R)$ not isomorphic to the image of any quasicategory morphism $f\colon Q \to R$. A mapping telescope argument rules out any countable Q, so we turn to the simplest possible uncountable example.

Denote then, as usual, the least uncountable ordinal by ω_1. We construct a "partially coherent" form of ω_1 as follows. We have a functor $D\colon \omega_1 \to \mathbf{SSet}$ that sends each countable ordinal $\alpha \in \omega_1$ to its nerve and each map $\alpha < \beta$ in ω_1 to the standard inclusion $N\alpha \rightarrowtail N\beta$. Since D is projectively cofibrant, the homotopy colimit of D in $\mathbf{SSet}_{\mathrm{Joyal}}$ is equivalent both to $N\omega_1$ and to the homotopy colimit of the simplicial set $n \mapsto \coprod_{\alpha_0 < \cdots < \alpha_n} N(\alpha_n)$. We define $\omega_1^{A_2}$ as the homotopy colimit in $\mathbf{SSet}_{\mathrm{Joyal}}$ of the restriction of this simplicial object to $\Delta^{\mathrm{op}}_{\leqslant 2}$, the full subcategory of Δ^{op} on the objects $[0], [1], [2]$.

[2]The reason for the unfamiliar u terms in the definition of C is that a pseudonatural transformation is not determined by its action on objects.

Thus, concretely, $\omega_1^{A_2}$ is given by first taking the disjoint union of the nerves of the countable ordinals α, then gluing in an isomorphism from α to its image in β whenever $\alpha < \beta$, and finally, gluing in a commutative triangle between these isomorphisms for each triple $\alpha < \beta < \gamma$. A map from $\omega_1^{A_2}$ into a quasicategory thus corresponds to maps out of each countable ordinal which are homotopic via homotopies which commute, but which need not satisfy any higher coherences. This explains the notation $\omega_1^{A_2}$ as an analogy between a (fully coherent) homotopy colimit and an A_∞-algebra.

There is a canonical pseudonatural transformation $F\colon \mathrm{HO}(\omega_1) \to \mathrm{HO}(\omega_1^{A_2})$ defined as follows. Given any homotopically finite category J and a map $X\colon N(J) \to N(\omega_1)$, let $\alpha \in \omega_1$ be the least ordinal through which X factors. Then we define $F(X)\colon N(J) \to \omega_1^{A_2}$ as the composition of X with the canonical inclusion of α in $\omega_1^{A_2}$. The pseudonaturality constraints of F are constructed from the canonical isomorphisms in $\omega_1^{A_2}$.

Conjecture 6.1. *The pseudonatural transformation F defined above is not isomorphic to a strict transformation, and is thus not in the essential image of* HO*.*

To prove the conjecture, it would suffice to show that every morphism $N(\omega_1) \to \omega_1^{A_2}$ is bounded in terms of the countable ordinals α whose image in $\omega_1^{A_2}$ it intersects. This boundedness should follow from analysis of the Kan complex under $\omega_1^{A_2}$ produced by collapsing the image of each α to a point.

Proof of Whitehead's theorem for quasicategories

We will use the main theorem of [**Arl18**], which says that the 2-category $\underline{\mathbf{KAN}} \subseteq \underline{\mathbf{QCAT}}$ of Kan complexes is strongly generated by the tori $(S^1)^n$, in the sense that a morphism $f\colon X \to Y$ of Kan complexes is a homotopy equivalence if and only if, for each n, the functor $\underline{\mathbf{KAN}}((S^1)^n, f)$ is an equivalence of groupoids.[3] We rephrase this in a form more convenient for our purposes:

Theorem 6.2. *The restriction of* $\mathrm{HO}\colon \underline{\mathbf{QCAT}} \to \underline{\mathbf{PDer}_{\mathbf{HFin}}}$ *to the 2-category* $\underline{\mathbf{KAN}}$ *reflects equivalences.*

Recall that equivalences in $\underline{\mathbf{PDer}_{\mathbf{HFin}}}$ in the abstract 2-categorical sense coincide with pseudonatural transformations which induce equivalences of categories levelwise.

Proof. Given $f\colon X \to Y$ in $\underline{\mathbf{KAN}}$, the image $\mathrm{HO}(f)$ is an equivalence in $\underline{\mathbf{PDer}_{\mathbf{HFin}}}$ if and only if, for every homotopically finite category J, the induced functor $\mathrm{Ho}(f^{N(J)})\colon \mathrm{Ho}(X^{N(J)}) \to \mathrm{Ho}(Y^{N(J)})$ is an equivalence. Since the classical model structure on simplicial sets is also Cartesian, we have equivalences $\mathrm{Ho}(X^{N(J)}) \simeq \mathrm{Ho}(X^{\mathrm{Ex}^\infty(N(J))})$, and similarly for Y, where Ex^∞ is Kan's fibrant replacement functor. Now, by Thomason's theorem [**Tho80**], as J varies, $\mathrm{Ex}^\infty(N(J))$ runs through all finite homotopy types. In particular, if $\mathrm{HO}(f)$ is an equivalence in $\underline{\mathbf{PDer}}$, then f induces equivalences $\mathrm{Ho}(X^{(S^1)^n}) \to \mathrm{Ho}(Y(S^1)^n)$ for every n, which is to say, $\underline{\mathbf{KAN}}((S^1)^n, f)$ is an equivalence. Thus f must be an equivalence, by [**Arl18**]. □

To make use of the above result to prove results on the relationship between quasicategories and their prederivators, we first recall what Rezk has described as the fundamental theorem of quasicategory theory. First, a quasicategory Q has mapping

[3] We write S^1 for any Kan complex of the homotopy type of the circle.

spaces $Q(x,y)$ for each $x,y \in Q$, which can be given various models. We shall use the balanced model in which we have $Q(x,y) = \{(x,y)\} \times_{Q\times Q} Q^{\Delta^1}$, so that an n-simplex of $Q(x,y)$ is a prism $\Delta^n \times \Delta^1$ in Q which is degenerate on x and y at its respective endpoints.

We say that a map $f\colon Q \to R$ of quasicategories is *fully faithful* if it induces an equivalence of Kan complexes $Q(x,y) \to R(f(x),f(y))$ for every $x,y \in Q$. It is essentially surjective if, for every $z \in R$, there exists $x \in Q$ and an edge $a\colon f(x) \to z$ which becomes an isomorphism in $\mathrm{Ho}(R)$. Then we have

Theorem 6.3 (Joyal)**.** *A map $f\colon Q \to R$ of quasicategories is an equivalence in the sense of Definition 3.1 if and only if it is fully faithful and essentially surjective.*

Now we can prove our Whitehead theorem for quasicategories.

Theorem 6.4. *Let $f\colon Q \to R$ be a map of quasicategories, and suppose that $\mathrm{HO}(f)$ is an equivalence in $\underline{\mathbf{PDer}}$. Then f is an equivalence of quasicategories.*

Proof. Since $\mathrm{Ho}(f)$ is an equivalence by assumption, f is essentially surjective. Thus we have only to show f is fully faithful. By Theorem 6.2, it suffices to show that $\mathrm{HO}(f)$ induces an equivalence $\mathrm{HO}(Q(x,y)) \cong \mathrm{HO}(R(f(x),f(y)))$ in $\underline{\mathbf{PDer}}$ for every x and y in Q. What is more, since for any J we have $Q(x,y)^{NJ} \cong Q^{NJ}(p_J^* x, p_J^* y)$, it suffices at last to show that f induces equivalences $f_{x,y}\colon \mathrm{Ho}(Q(x,y)) \to \mathrm{Ho}(R(f(x),f(y)))$ on the homotopy categories of mapping spaces.

Essential surjectivity is proved via an argument that also appeared in the construction of $\widehat{\Xi}$ in the proof of Theorem 5.1. Namely, from essential surjectivity of $\mathrm{HO}(f)$, given any $X \in \mathrm{Ho}(R(f(x),f(y)))$ and any $Y \in \mathrm{HO}(Q)([1])$ with an isomorphism $s\colon \mathrm{HO}(f)(Y) \cong X$ in $\mathrm{HO}(R)([1])$, we see by conservativity and fullness of $\mathrm{HO}(f)$ that we have isomorphisms $0^*Y \cong x \in \mathrm{HO}(Q)(J)$ and, similarly, $1^*Y \cong y$. Composing these isomorphisms and $\mathrm{dia}Y$ in $\mathrm{Ho}(Q)$ gives a morphism $x \to y$ in $\mathrm{Ho}(Q)$ isomorphic to $\mathrm{dia}Y$ in $\mathrm{Ho}(Q)^{[1]}$. By (Der5′) and (Der2) we can lift this to an isomorphism $r\colon Y' \cong Y$ in $\mathrm{HO}(Q)([1])$ such that $0^*(s \circ \mathrm{HO}(f)(r)) = \mathrm{id}_x$ and $1^*(s \circ \mathrm{HO}(f)(r)) = \mathrm{id}_y$. This implies that $s \circ \mathrm{HO}(f)(r)$ may be lifted to an isomorphism $\mathrm{HO}(f)(Y') \cong X$ in $\mathrm{Ho}(R(f(x),f(y))$. Thus $f_{x,y}$ is essentially surjective.

For fullness, we observe that if $a\colon Y_1 \to Y_2 \in \mathrm{HO}(Q)([1])$ verifies $Y_1, Y_2\colon x \to y$, $0^*\mathrm{HO}(f)(a) = \mathrm{id}_{f(x)}$, and $1^*\mathrm{HO}(f)(a) = \mathrm{id}_{f(y)}$, then we have also $0^*(a) = \mathrm{id}_x$ and $1^*a = \mathrm{id}_y$, since $\mathrm{HO}(f)$ is faithful. This implies that a can be lifted to a morphism $a'\colon Y_1 \to Y_2$ in $\mathrm{Ho}(Q(x,y))$ with $f_{x,y}(a) = \mathrm{HO}(f)(a)$. And since $\mathrm{HO}(f)$ is full, every morphism $\mathrm{HO}(f)(Y_1) \to \mathrm{HO}(f)(Y_2)$ in $\mathrm{Ho}(R(f(x),f(y)))$ is equal to $\mathrm{HO}(f)(a)$ in $\mathrm{HO}(R)([1])$, for some a.

Finally, we turn to faithfulness. Suppose we have morphisms $a,b\colon Y_1 \to Y_2$ in $\mathrm{Ho}(Q(x,y))$ with $f_{x,y}(a) = f_{x,y}(b)$ in $\mathrm{Ho}(R(f(x),f(y)))$. We wish to show $a = b$. First, we may represent a and b by $\hat{a}, \hat{b} \in \mathrm{HO}(Q)([1]\times[1])$, each with boundary

$$\begin{array}{ccc} x & \xrightarrow{Y_1} & y \\ \| & & \| \\ x & \xrightarrow{Y_2} & y. \end{array}$$

Let $\partial[2]$ denote the category on objects $0,1,2$ freely generated by *three* arrows $0 \to 1, 1 \to 2, 0 \to 2$, so that $N\partial[2]$ is Joyal equivalent to $\partial\Delta^2$. The lifts $\hat{a}$ and $\hat{b}$

fit together in a diagram $W \in \mathrm{HO}(Q)([1]\times\partial[2])$ with $(01)^*W = q^*Y_1$, $(02)^*W = \hat{a}$, and $(12)^*W = \hat{b}$, where $q\colon [1]\times[1]\to[1]$ projects out the last coordinate. The significance of W is that we have $a = b$ if and only if W admits an extension Z to $\mathrm{HO}(Q)([1]\times[2])$ such that $Z|_{\{0\}\times[2]} = p^*_{[2]}x$ and $Z|_{\{1\}\times[2]} = p^*_{[2]}y$. It suffices to exhibit $W' \in \mathrm{HO}(Q)([1]\times\partial[2])$ with $W'|_{0\times\partial[2]} = p^*_{\partial[2]}x$ and $W'|_{1\times\partial[2]} = p^*_{\partial[2]}y$ admitting such an extension Z', together with an isomorphism $t\colon W \to W'$ in $\mathrm{HO}(Q)([1]\times\partial[2])$ such that $t|_{0\times\partial[2]} = \mathrm{id}_{p^*_{\partial[2]}x}$ and $t|_{1\times\partial[2]} = \mathrm{id}_{p^*_{\partial[2]}y}$. Indeed, in this situation W and W' both represent maps from S^1 to the Kan complex $Q(x,y)$, Z and Z' represent putative extensions to Δ^2, and t represents a homotopy between them.

In particular, since by assumption $\mathrm{HO}(f)(a) = \mathrm{HO}(f)(b)$ in $\mathrm{Ho}(R(f(x),f(y)))$, there exists an extension T of $\mathrm{HO}(f)(W)$ to $\mathrm{HO}(R)([1]\times[2])$ with trivial endpoints, as above. Now take $\hat{T} \in \mathrm{HO}(Q)([1]\times[2])$ with an isomorphism $s\colon \mathrm{HO}(f)(\hat{T}) \cong T$. In particular, this gives isomorphisms $\mathrm{HO}(f)(\hat{T})|_{\{0\}\times[2]} \cong p^*_{[2]}f(x)$ and $\mathrm{HO}(f)(\hat{T})|_{\{1\}\times[2]} \cong p^*_{[2]}f(y)$ in $\mathrm{HO}(R)([2])$, which lift uniquely to isomorphisms $\hat{T}|_{\{0\}\times[2]} \cong p^*_{[2]}x$ and $\hat{T}|_{\{1\}\times[2]} \cong p^*_{[2]}y$ in $\mathrm{HO}(Q)([2])$. Composing these isomorphisms with $\mathrm{dia}\hat{T}$ and lifting into $\mathrm{HO}(Q)([1]\times[2])$ gives $Z' \in \mathrm{HO}(Q)([1]\times[2])$ with $Z'|_{\{0\}\times[2]} = p^*_{[2]}x$ and $Z'|_{\{1\}\times[2]} = p^*_{[2]}y$, together with an isomorphism $t'\colon \mathrm{HO}(f)(Z') \cong T$ in $\mathrm{HO}(R)([1]\times[2])$ inducing the identity on $p^*_{[2]}f(x)$ and $p^*_{[2]}f(y)$, respectively. Restricting t' to $[1]\times\partial[2]$ and lifting to $\mathrm{HO}(Q)([1]\times\partial[2])$ specifies an isomorphism $t\colon Z'|_{[1]\times\partial[2]} \cong W$ such that $t|_{0\times\partial[2]} = \mathrm{id}_{p^*_{\partial[2]}x}$ and $t|_{1\times\partial[2]} = \mathrm{id}_{p^*_{\partial[2]}y}$. As we saw above, this suffices to guarantee that W admits an extension Z as desired. □

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