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ON SOME SEQUENCES OF MODULES OVER THE MOD p STEENROD ALGEBRA

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Abstract

In [**3**] Toda conjectured the exactness of some sequences of modules over the Steenrod algebra mod 2. One of the conjectures was proved by Wall [**1**]. In this note we show that in general the analogue of the Toda conjecture for $p > 2$ does not hold. We give an upper bound on dimensions in which exactness holds. Also we consider examples of exact sequences.

1. Introduction

For the Steenrod algebra $\mathcal{A}_2$ consider the subalgebra $\mathcal{S}_k$ generated by the elements $Sq^1, Sq^2, Sq^4, \ldots, Sq^{2^k}$. In the paper [**3**] Toda conjectured exactness of the sequence

$$\mathcal{A}_2 \xrightarrow{\varphi_r} \mathcal{A}_2/\mathcal{A}_2\mathcal{S}_{r-2} \xrightarrow{\varphi_r} \mathcal{A}_2/\mathcal{A}_2\mathcal{S}_{r-1}, \tag{1}$$

where $\varphi_r(x) = x \cdot Sq^{2^r}$. The conjecture was proved by Wall in [**1**].

In this paper we discuss the generalization of the conjecture to the case $p > 2$. Our main results are Theorems 1.1, 1.2 and 1.3 about exactness or non-exactness of sequences of type (1). We describe their statements as well as the structure of the paper at the end of the Introduction.

Fix a prime $p > 2$. The mod p Steenrod algebra $\mathcal{A}_p$ contains the subalgebra $\overline{\mathcal{A}}_p$ (with 1) generated over $\mathbb{Z}/p$ by the elements P^i, $i \geqslant 1$, subject to the Adem relations

$$\text{for } a < pb, \qquad P^a P^b = \sum_{j=0}^{[a/p]} (-1)^{a+j} \binom{(p-1)(b-j)-1}{a-pj} P^{a+b-j} P^j.$$

Here $P^0 = 1$ and $\deg P^i = 2(p-1)i$.

Let $\mathcal{S}_r$ be the subalgebra of $\overline{\mathcal{A}}_p$ generated by P^{p^j}, where $0 \leqslant j \leqslant r$, and let $\mathcal{I}_j$ be the left ideal $\mathcal{S}_r\mathcal{S}_j$ of $\mathcal{S}_r$ generated by $\mathcal{S}_j$. Note that $\mathcal{I}_0 \subset \mathcal{I}_1 \subset \mathcal{I}_2 \subset \cdots$, and if some element is equal to zero modulo $\mathcal{I}_r$, then it belongs to $\mathcal{I}_{r-1}$.

Consider the two maps

$$\alpha_1 \text{ and } \alpha_2 \colon \overline{\mathcal{A}}_p \to \overline{\mathcal{A}}_p/\overline{\mathcal{A}}_p\mathcal{S}_{r-2},$$

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defined by the formulas

$$\alpha_1(x) = xP^{ap^r} \bmod \overline{\mathcal{A}}_p\mathcal{S}_{r-2}, \qquad \alpha_2(x) = x(P^{p^r})^a \bmod \overline{\mathcal{A}}_p\mathcal{S}_{r-2},$$

for some fixed $1 \leqslant a \leqslant p-1$. Also consider two further maps

$$\beta_1 \text{ and } \beta_2 \colon \overline{\mathcal{A}}_p/\overline{\mathcal{A}}_p\mathcal{S}_{r-2} \to \overline{\mathcal{A}}_p/\overline{\mathcal{A}}_p\mathcal{S}_{r-1},$$

defined by

$$\beta_1(x) = xP^{bp^r} \bmod \overline{\mathcal{A}}_p\mathcal{S}_{r-1}, \qquad \beta_2(x) = x(P^{p^r})^b \bmod \overline{\mathcal{A}}_p\mathcal{S}_{r-1},$$

for some fixed $1 \leqslant b \leqslant p-1$. We prove that β_1 and β_2 are well defined (Lemma 2.4).

For the sequence (1) the equality $\varphi_r \circ \varphi_r = 0$ follows from the fact that $(Sq^{2^r})^2 \in \mathcal{S}_{r-1}$. The corresponding statement for $p > 2$ is $(P^{p^r})^p \in \mathcal{S}_{r-1}$. Hence it is natural to consider the sequence

$$\overline{\mathcal{A}}_p \xrightarrow{\alpha_i} \overline{\mathcal{A}}_p/\overline{\mathcal{A}}_p\mathcal{S}_{r-2} \xrightarrow{\beta_j} \overline{\mathcal{A}}_p/\overline{\mathcal{A}}_p\mathcal{S}_{r-1}, \tag{2}$$

where the maps α_i and β_j are chosen such that $a + b = p$. In this case $\beta_j \circ \alpha_i = 0$ for any $i, j = 1, 2$. The generalization of the Toda conjecture for $p > 2$ is the statement that $\ker \beta_j = \operatorname{im} \alpha_i$ for suitable i, j.

Before we state the main results, note that $\overline{\mathcal{A}}_p$ is a free module over $\overline{\mathcal{S}_r}$, where $\overline{\mathcal{S}_r}$ is the subalgebra $\mathcal{S}_r$ together with the identity element (see Lemma 6). Hence, exactness of the sequence (2) is equivalent to exactness of the sequence

$$\overline{\mathcal{S}}_r \xrightarrow{\alpha_i} \mathcal{S}_r/\mathcal{I}_{r-2} \xrightarrow{\beta_j} \mathcal{S}_r/\mathcal{I}_{r-1}. \tag{3}$$

In what follows all the modules under consideration are trivial in degrees different from $2(p-1)k$, where k is an integer. Hence, for simplicity, for a given module M, we denote by $M^{(k)}$ the submodule of all elements of degree $2(p-1)k$, in particular, $M = \bigoplus_k M^{(k)}$. We refer to n as the grading of the elements of $M^{(n)}$.

The sequence (3) splits into the direct sum over all k of the sequences

$$\overline{\mathcal{S}_r}^{(k)} \xrightarrow{\alpha_i} (\mathcal{S}_r/\mathcal{I}_{r-2})^{(ap^r+k)} \xrightarrow{\beta_j} (\mathcal{S}_r/\mathcal{I}_{r-1})^{(p^{r+1}+k)}. \tag{4}$$

By Lemma 2.3 the homomorphisms β_1 and β_2 coincide up to nonzero scalar multiple, and therefore they have the same kernel. Hence, we can consider only $\beta = \beta_1$ or $\beta = \beta_2$.

Our main results split into three cases $r = 0$, $r = 1$, and $r \geqslant 2$.

Case $r = 0$**.** Note that $(P^1)^q = q!P^q$ for $1 \leqslant q \leqslant p-1$, therefore it is enough to consider the maps $\alpha(x) = \alpha_2(x) = x(P^1)^a$ and $\beta(x) = \beta_2(x) = x(P^1)^b$. Both ideals $\mathcal{I}_{r-1}$ and $\mathcal{I}_{r-2}$ are trivial.

Theorem 1.1. *The sequence*

$$\overline{\mathcal{A}}_p \xrightarrow{\alpha} \overline{\mathcal{A}}_p \xrightarrow{\beta} \overline{\mathcal{A}}_p$$

is exact.

Case $r = 1$**.** Consider $\alpha(x) = xP^p$ and $\beta(x) = xP^{(p-1)p}$, that is $a = 1$ and $b = p-1$.

Theorem 1.2. *For the sequence*

$$\overline{\mathcal{S}}_1^{(k)} \xrightarrow{\alpha} \mathcal{S}_1^{(p+k)} \xrightarrow{\beta} (\mathcal{S}_1/\mathcal{I}_0)^{(p^2+k)},$$

the following statements hold:

1. *The sequence is exact for $k < p$.*
2. *For $k = p$ the quotient $(\ker\beta/\operatorname{im}\alpha)^{(p+k)}$ has dimension 1. Moreover, the element $P^1P^p(P^1)^{p-1}$ belongs to $\ker\beta$ and is not contained in the image of α.*

In other words, for $r = 1$ and for our choice of α and β the sequence (4) is exact up to a certain 'critical' grading and at the 'critical' grading $\dim\ker\beta/\operatorname{im}\alpha = 1$. The same statement holds for $r > 1$, but with a different 'critical' grading.

Case $r \geqslant 2$. Consider $\beta(x) = x \cdot P^{(p-1)p^r}$ and $\alpha(x) = x \cdot P^{p^r}$, that is $a = 1$ and $b = p - 1$.

Theorem 1.3. *For the sequence*

$$\overline{\mathcal{S}_r}^{(k)} \xrightarrow{\alpha} (\mathcal{S}_r/\mathcal{I}_{r-2})^{(p^r+k)} \xrightarrow{\beta} (\mathcal{S}_r/\mathcal{I}_{r-1})^{(p^{r+1}+k)},$$

the following statements hold:

1. *The sequence is exact for $k < 2p^{r-1} + p^{r-2}$.*
2. *For $k = 2p^{r-1} + p^{r-2}$ the quotient $\ker\beta/\operatorname{im}\alpha$ has dimension one, moreover, the element $2P^{p^{r-1}}P^{p^r}P^{p^{r-2}}P^{p^{r-1}} - P^{p^{r-2}}P^{p^{r-1}}P^{p^r}P^{p^{r-1}}$ belongs to $\ker\beta$, but is not contained in the image of α.*

In Corollaries 6.1, 8.2 we show that for any other choice of α_i with $1 < a < p$ there is no exactness for $r \geqslant 1$. This explains why we consider only case $a = 1$, $b = p - 1$ in Theorems 1.2, 1.3.

In Section 2 we discuss some preliminaries concerning the subalgebra $\overline{\mathcal{A}}_p$ and the main tool in our proofs—the Z-basis of $\overline{\mathcal{A}}_p$, developed in our paper [**6**]. Some important relations in the subalgebra $\mathcal{S}_r$ modulo the ideals $\mathcal{I}_{r-1}$ and $\mathcal{I}_{r-2}$ are proved in Section 3. In Section 4 we prove Theorem 1.1 and consider another analogue of the sequence $\mathcal{A}_2 \xrightarrow{\varphi_0} \mathcal{A}_2 \xrightarrow{\varphi_0} \mathcal{A}_2, \varphi_0(x) = xSq^1$, namely, the sequence $\mathcal{A}_p \xrightarrow{\psi_j} \mathcal{A}_p \xrightarrow{\psi_j} \mathcal{A}_p, \psi_j(x) = xQ_j$. The elements Q_j form an exterior subalgebra of $\mathcal{A}_p$, for details see [**2**]. In Sections 5 and 6 we prove Theorem 1.2. The parts (1) and (2) are separated for the convenience of the reader. Similarly, in Sections 7 and 8 we prove Theorem 1.3. Finally, in Section 9 we make some remarks, including a discussion of the case of the (whole) Steenrod algebra $\mathcal{A}_p$.

2. Preliminaries

Recall the following property of mod p binomial coefficients which is frequently used throughout the paper:

Lemma 2.1 ([**5**, 2.6], or Lucas' Lemma)**.** *Let p be a prime number. Suppose $a = a_kp^k + \cdots + a_1p + a_0$ and $b = b_kp^k + \cdots + b_1p + b_0$, $0 \leqslant a_j, b_j < p$ are the p-adic expansions of nonnegative integers a and b. Then*

$$\binom{b}{a} = \binom{b_k}{a_k} \cdots \binom{b_1}{a_1}\binom{b_0}{a_0} \bmod p.$$

Lemma 2.2. *In $\mathcal{S}_r$ we have:*

1. $P^{p^r}P^{ap^r} \equiv (a+1)\,P^{(a+1)\,p^r} \mod \mathcal{I}_{r-1}$ *for* $a < p$.
2. $(P^{p^r})^a \equiv a!\,P^{ap^r} \mod \mathcal{I}_{r-1}$ *for* $1 \leqslant a \leqslant p$.
3. $P^{ap^r}P^{bp^r} \equiv (-1)^a\binom{p-1-b}{a}P^{(a+b)p^r} \mod \mathcal{I}_{r-1}$ *for integers* $a, b \in [1, p-1]$.

Proof. 1. Applying the Adem relations to the product $P^{p^r}P^{ap^r}$ we obtain

$$P^{p^r}P^{ap^r} \equiv (-1)^{p^r}\binom{ap^r(p-1)-1}{p^r}P^{(a+1)\,p^r} \mod \mathcal{I}_{r-1}.$$

One has $ap^r(p-1)-1 = (a-1)p^{r+1}+(p-a)p^r-1$ hence the binomial coefficient in the last equality is equal to $p-a-1 \mod p$.

Statement 2 follows from 1 by induction.

3. From the Adem relations we obtain $P^{ap^r}P^{bp^r} \equiv (-1)^a\,c\,P^{(a+b)p^r} \mod \mathcal{I}_{r-1}$, where the binomial coefficient c is equal to

$$c = \binom{bp^r(p-1)-1}{ap^r} = \binom{(b-1)p^{r+1}+(p-b)p^r-1}{ap^r} = \binom{p-b-1}{a} \mod p. \qquad \square$$

Consider two maps $\hat{\beta}_j$, $j = 1, 2$:

$$\hat{\beta}_j : \overline{\mathcal{A}}_p \to \overline{\mathcal{A}}_p/\overline{\mathcal{A}}_p\mathcal{S}_{r-1},$$

where $\hat{\beta}_1(x) = xP^{bp^r}$ and $\hat{\beta}_2(x) = x(P^{p^r})^b$, $1 \leqslant b \leqslant p-1$.

The following statement is a direct consequence of Lemma 2.2.

Lemma 2.3. $\hat{\beta}_2 = b!\,\hat{\beta}_1$. *In particular,* $\ker\hat{\beta}_1 = \ker\hat{\beta}_2$ *for* $1 \leqslant b \leqslant p-1$.

Lemma 2.4. *Assume* $r \geqslant 2$. *Then for* $i = 1, 2$ *and* $1 \leqslant b \leqslant p-1$ *we have* $\hat{\beta}_i(x) \in \mathcal{I}_{r-1}$ *for any* $x \in \mathcal{I}_{r-2}$. *In particular, the maps* $\hat{\beta}_i$ *induce well defined maps* $\beta_i : \overline{\mathcal{A}}_p/\overline{\mathcal{A}}_p\mathcal{S}_{r-2} \to \overline{\mathcal{A}}_p/\overline{\mathcal{A}}_p\mathcal{S}_{r-1}$.

Proof. It is enough to check that $P^{p^j}P^{bp^r} \in \mathcal{I}_{r-1}$ for $j \leqslant r-2$. Obviously we have

$$P^{p^j}P^{bp^r} \equiv (-1)^{p^j}\binom{bp^r(p-1)-1}{p^j}P^{bp^r+p^j} \mod \mathcal{I}_{r-1}.$$

Calculating the binomial coefficient we obtain

$$P^{p^j}P^{bp^r} \equiv P^{bp^r+p^j} \mod \mathcal{I}_{r-1}. \tag{5}$$

Since $P^{p^r-p^{j+1}+p^j} \in \mathcal{S}_{r-1}$, the product $P^{(b-1)p^r+p^{j+1}}P^{p^r-p^{j+1}+p^j}$ belongs to $\mathcal{I}_{r-1}$. Applying the Adem relations (which is possible as $r \geqslant j+2$) we obtain

$$P^{(b-1)p^r+p^{j+1}}P^{p^r-p^{j+1}+p^j} \equiv (-1)^b\,c\,P^{bp^r+p^j} \mod \mathcal{I}_{r-1}, \tag{6}$$

where the binomial coefficient

$$c = \binom{(p^r-p^{j+1}+p^j)(p-1)-1}{p^{j+1}+(b-1)p^r} = \underbrace{\binom{p-2}{b-1}}_{r\text{-th}}\binom{*}{0}\cdots\binom{*}{0}\underbrace{\binom{1}{1}}_{j+1\text{-th}}\binom{*}{0}\cdots\binom{*}{0}$$

is not zero.

The equalities (5) and (6) imply $P^{p^j}P^{bp^r} \in \mathcal{I}_{r-1}$. $\square$

From Lemmas 2.3 and 2.4 it follows that it is enough to consider only $\beta = \beta_1$.

Lemma 2.5. *For $a + b = p$ and $i = 1, 2$ we have $\beta \circ \alpha_i = 0$ in the sequence*

$$\overline{\mathcal{A}}_p \xrightarrow{\alpha_i} \overline{\mathcal{A}}_p/\overline{\mathcal{A}}_p\mathcal{S}_{r-2} \xrightarrow{\beta} \overline{\mathcal{A}}_p/\overline{\mathcal{A}}_p\mathcal{S}_{r-1}. \tag{7}$$

Proof. Remind that $\beta(x) = xP^{bp^r}$. Then for $\alpha_1(x) = xP^{ap^r}$ the statement follows immediately from Lemma 2.2(3).

For $\alpha_1(x) = x(P^{p^r})^a$ proceed by induction on a applying Lemma 2.2(1). □

The natural generalization of the Toda conjecture is the exactness of the sequence (7).

Remark 2.6. Symbolic calculations show that α_1 and α_2 give different statements about exactness of (7) for $a > 1$.

Now we are going to describe the main tool of our work, namely the Z-basis constructed in our paper [**6**]. By $<_L$ we denote the left lexicographical ordering of finite sequences of integers.

Definition 2.7. Let $Z_k^n = P^{p^k} \cdots P^{p^n}$ for $n \geqslant k \geqslant 0$. Define a Z-monomial to be a product

$$Z^I = Z_{k_1}^{n_1} \cdots Z_{k_m}^{n_m},$$

where I is a sequence of pairs $((n_1, k_1), \dots, (n_m, k_m))$ satisfying two conditions:

(1) $(n_m, k_m) \leqslant_L \cdots \leqslant_L (n_2, k_2) \leqslant_L (n_1, k_1)$,

(2) if in the sequence I there is a subsequence of equal pairs:

$$(n_t, k_t) >_L (n_{t+1}, k_{t+1}) = \cdots = (n_{p+s}, k_{p+s}) >_L (n_{t+s+1}, k_{t+s+1})$$

then $s < p$ for every such a subsequence.

In other words, Z-monomials are monomials of the form

$$Z^I = (Z_{k_1}^{n_1})^{q_1} \cdots (Z_{k_m}^{n_m})^{q_m},$$

where $(n_m, k_m) <_L \cdots <_L (n_2, k_2) <_L (n_1, k_1)$ and $0 < q_j < p$ for all j.

Sometimes it is useful to consider the identity element $1 \in \overline{\mathcal{A}}_p$ as Z^I with empty I.

Theorem 2.8 ([**6**, Theorem 2.3])**.** *The set of all Z-monomials forms an additive basis of $\overline{\mathcal{A}}_p$.*

One of the useful properties of the Z-basis is illustrated by the following statement.

Lemma 2.9. *The algebra $\overline{\mathcal{A}}_p$ is a free right module over $\mathcal{S}_r$.*

Proof. From Theorem 2.8 it follows that the Z-monomials $Z_{k_1}^{n_1} \cdots Z_{k_j}^{n_j}$, where $n_j > r$, and the element $1 = Z^{\emptyset}$ form a free basis of $\overline{\mathcal{A}}_p$ considered as a right $\mathcal{S}_r$-module. □

Therefore instead of the sequence (7) we can consider the sequence

$$\overline{\mathcal{S}}_r \xrightarrow{\alpha_i} \mathcal{S}_r/\mathcal{I}_{r-2} \xrightarrow{\beta} \mathcal{S}_r/\mathcal{I}_{r-1},$$

where $\overline{\mathcal{S}}_r$ is $\mathcal{S}_r$ together with the identity element.

3. Some calculations modulo the ideals $\mathcal{I}_{r-2}$ and $\mathcal{I}_{r-1}$

In this section we make some necessary calculations in the algebra $\mathcal{S}_r$ modulo the ideals $\mathcal{I}_{r-1}$ or $\mathcal{I}_{r-2}$ which will be used later.

Consider a product X of the elements $P^{p^0}, P^{p^1}, \dots, P^{p^r}$ such that P^{p^j} is contained in the product d_j times and such that the rightmost term is P^{p^r}. Let $L = \sum d_j$ be the length of the product. Define a collection of finite sequences $a_{(l)} = (a_{(l),0}, \dots, a_{(l),r})$ where $a_{(l),j}$ is the number of P^{p^j} in the l rightmost terms of the product X. For example, $(a_{(1),0}, \dots, a_{(1),r}) = (0, \dots, 0, 1)$ and $(a_{(L),0}, \dots, a_{(L),r}) = (d_0, \dots, d_r)$.

Lemma 3.1. *Assume $d_0 \leqslant \dots \leqslant d_r \leqslant p-1$. Let $N = \sum d_j p^j$. Then*

$$X = \lambda P^N \bmod \mathcal{I}_{r-1}, \tag{8}$$

for some nonzero $\lambda \in \mathbb{Z}/p$ if and only if all the sequences $a_{(l)}$ are increasing (nonstrictly). Moreover, if $d_r = 1$ and $k \leqslant r-1$ is maximal such that $d_k \neq 0$ then the equality holds modulo the ideal $\mathcal{I}_{k-1}$.

Proof. Denote by Q_l the product of the l rightmost symbols in the product X. We proceed by induction to prove that

$$Q_l = c_l P^{a_{(l),0}p^0 + \dots + a_{(l),r}p^r} \bmod \mathcal{I}_{r-1}, \tag{9}$$

where c_l is equal, up to sign, to a product of some binomial coefficients which we discuss later. For $l = 1$ the equality (9) is trivial.

Let the $(l+1)$-th symbol be P^{p^j}. Then by the Adem relations

$$Q_{l+1} = P^{p^j} Q_l = c_l P^{p^j} P^{a_{(l),r}p^r + \dots + a_{(l),0}p^0} = -b_l c_l P^{p^j + a_{(l),r}p^r + \dots + a_{(l),0}p^0} \bmod \mathcal{I}_{r-1},$$

where $b_l = \binom{(a_{(l),r}p^r + \dots + a_{(l),0}p^0)(p-1)-1}{p^j}$. Therefore (9) holds for all l. Also we have $c_1 = 1$ and $c_{l+1} = -b_l c_l$. Hence we are left to prove that all $b_l \neq 0$ if and only if all the sequences $a_{(l)}$ are nonstrictly increasing.

Fix some l. For simplicity, we write a_i instead of $a_{(l),i}$. If $a_{(l)}$ is increasing we have $a_i \leqslant a_{i+1}$ for all $i = 0, \dots, r-1$. Note also that $a_j + 1 = a_{(l+1),j} \leqslant p-1$, hence $a_j < p-1$ and $a_j < a_{j+1}$. Then we have a p-adic expansion

$$A = (a_r p^r + \dots + a_0 p^0)(p-1) - 1 = (a_r - 1)p^{r+1} + \sum_{i-1}^{r} (p - 1 + a_{i-1} - a_i)p^i + (p - 1 - a_0). \tag{10}$$

Hence $b_l = \binom{A}{p^j}$ is equal either to $\binom{p-1+\alpha_{j-1}-\alpha_j}{1}$ or to $\binom{p-1+\alpha_0}{1}$; in both cases $b_l \neq 0$.

On the contrary choose maximal l such that the sequence $a_{(l)}$ does not increase. As $a_{(l+1)}$ is increasing we have $a_r \geqslant \dots \geqslant a_{j+1} > a_j$, $a_j = a_{j-1} - 1$, $a_{j-1} \geqslant \dots \geqslant a_0$ for some j. From this it follows that (10) is no longer a p-adic expansion of A because the coefficient of p^j is equal to $p - 1 + a_{j-1} - a_j = p$. Modifying two summands in (10) we obtain a p-adic expansion of A in which the coefficient of p^j is equal to zero, hence $b_l = \binom{A}{1} = 0$.

For the last statement of the lemma it is enough to note that in this case all 'error' terms from the Adem relations belong to $\mathcal{I}_{k-1}$. □

Corollary 3.2. *For $r \geqslant 2$ we have $P^{p^{r-1}} P^{p^{r-2}} P^{p^r} \in \mathcal{I}_{r-2}$.*

Proof. The sequence $a_{(2)}$ is equal to $(0,\dots,0,1,0,1)$. Therefore, $P^{p^{r-1}}P^{p^{r-2}}P^{p^r} = \lambda P^{p^{r-2}+p^{r-1}+p^r} \bmod \mathcal{I}_{r-1}$ with $\lambda = 0$ by Lemma 3.1. □

Lemma 3.3.

1. *For $k \leqslant r$ we have*

$$P^{p^k}\cdots P^{p^r} = P^{p^k+\cdots+p^r} \bmod \mathcal{I}_{r-2}. \tag{11}$$

2. *Assume N has the p-adic expansion $\sum_{j=0}^{r} a_j p^j$, $a_r \neq 0$. Then*

$$(P^{p^0})^{a_0}\cdots(P^{p^r})^{a_r} = \lambda P^N \bmod \mathcal{I}_{r-1},\ \textit{where}\ \lambda = \prod_j (a_j!) \neq 0.$$

Moreover, if $a_r = 1$ and $k \leqslant r-1$ is maximal such that $a_k \neq 0$ then the equality holds modulo the ideal $\mathcal{I}_{k-1}$.

Proof. For the product from (1) the sequences $a_{(l)}$ have the form $(0,\dots,0,1,\dots,1)$ for all l. Hence $\lambda \neq 0$ in equation (8) by Lemma 3.1. Moreover, the binomial coefficients from the proof of Lemma 3.1 are easy to calculate:

$$b_l = \binom{(p^r+\cdots+p^{l+1})(p-1)-1}{p^l} = \binom{\cdots+(p-1)p^l+\cdots}{p^l} = -1,$$

therefore $\lambda = 1$. Also note that in the equality (11) all the 'error' terms from the Adem relations belong to $\mathcal{I}_{r-2}$.

For the statement (2) we prove by descending induction on k that for N with p-adic expansion $N = \sum_{j=k}^{r} a_j p^j$ we have

$$(P^{p^k})^{a_k}\cdots(P^{p^r})^{a_r} = \lambda P^N \bmod \mathcal{I}_{r-1},\ \text{where}\ \lambda = \prod_{j=k}^{r}(a_j!).$$

The base of induction $(P^{p^r})^{a_r} = (a_r)!P^{a_r p^r} \bmod \mathcal{I}_{r-1}$ is the same as Lemma 2.2(2).

For the inductive step assume $N = \sum_{j=k}^{r} a_j p^j$, where $0 \leqslant a_j \leqslant p-1$ for $j > k$ and $0 \leqslant a_k < p-1$. Let $N = Mp^{k+1} + a_k p^k$, $M > 0$. Then $P^{p^k}P^N = -\binom{N(p-1)-1}{p^k}P^{N+p^k} \bmod \mathcal{I}_{r-1}$. If $a_k = 0$ then $N(p-1)-1 = Mp^{k+1}(p-1)-1 = \cdots + (p-1)p^k + \cdots$ and if $0 < a_k < p-1$ then $N(p-1)-1 = M(p-1)p^{k+1} + (a_k-1)p^{k+1} + (p-a_k)p^k - 1$. In both cases $\binom{N(p-1)-1}{p^k} = -(a_k+1)$ and, finally, $P^{p^k}P^N = (a_k+1)P^{N+p^k} \bmod \mathcal{I}_{r-1}$. □

Lemma 3.4. *Let $r \geqslant 2$. Then for any Z-monomial $x = Z_{t_1}^{r-1}\cdots Z_{t_p}^{r-1}$ we have*

$$x = \lambda Z_{t_1}^{r-2}\cdots Z_{t_p}^{r-2}P^{p^r} \bmod \mathcal{I}_{r-2},$$

where $\lambda \neq 0$, $\lambda \in \mathbb{Z}/p$.

Proof. From Lemma 3.1 we have

$$Z_{t_2}^{r-1}\cdots Z_{t_p}^{r-1} = \lambda_1 P^N \bmod \mathcal{I}_{r-2} \tag{12}$$

and

$$Z_{t_2}^{r-2} \cdots Z_{t_p}^{r-2} P^{p^r} = \lambda_2 P^{N-p^{r-1}} \bmod \mathcal{I}_{r-2}, \tag{13}$$

where $\lambda_1, \lambda_2 \neq 0$, $\lambda_1, \lambda_2 \in \mathbb{Z}/p$, and N is the grading of the element $Z_{t_2}^{r-1} \cdots Z_{t_p}^{r-1}$.

From (13) we have

$$P^{p^{r-1}} Z_{t_2}^{r-2} \cdots Z_{t_p}^{r-2} P^{p^r} = \lambda_2 P^{p^{r-1}} P^{N-p^{r-1}} \bmod \mathcal{I}_{r-2} = -\lambda_2 b P^N \bmod \mathcal{I}_{r-2}, \tag{14}$$

where b is the binomial coefficient

$$b = \binom{(N-p^{r-1})(p-1)-1}{p^{r-1}}.$$

Assume there are d_k terms P^{p^k} in the product x. From the assumption that x is a Z-monomial it follows that $p = d_{r-1} = \cdots = d_{t_1} > d_{t_1-1} \geqslant \cdots \geqslant d_0 \geqslant 0$. Let d'_j be equal to d_j for $j \geqslant t_1$ and let it be equal to $d - j - 1$ for $j < t_1$. Then we have the p-adic expansion of $N - p^{r-1}$

$$(p-1)p^{r-1} + d'_{r-2}p^{r-2} + \cdots + d'_0 p^0,$$

which we write as $N - p^{r-1} = (p-1)p^{r-1} + d'_{r-2}p^{r-2} + \varepsilon$. One has an estimate $\varepsilon \leqslant d'_{r-3}(p^{r-3} + \cdots + p^0) = d'_{r-3}\frac{p^{r-2}-1}{p-1}$. From this we deduce the necessary information about the p-adic expansion of $(N-p^{r-1})(p-1)-1$, namely about the coefficient of p^{r-1}. We have

$$\begin{aligned}(N-p^{r-1})(p-1)-1 &= (p-2)p^r + p^{r-1} + d'_{r-2}p^{r-2}(p-1) + \varepsilon(p-1) - 1\\ &= (p-2)p^r + d'_{r-2}p^{r-1} + p^{r-2}(p-d'_{r-2}) + \varepsilon(p-1) - 1.\end{aligned}$$

This is all we need to know about the p-adic expansion of $(N-p^{r-1})(p-1)-1$ since $d'_{r-2} \geqslant 1$ and

$$\begin{aligned}p^{r-2}(p-d'_{r-2}) + \varepsilon(p-1) - 1 &\leqslant p^{r-2}(p-d'_{r-2}) + d'_{r-3}(p^{r-2}-1) - 1\\ &= p^{r-1} - p^{r-2}(d'_{r-2} - d'_{r-3}) - d'_{r-3} - 1 \leqslant p^{r-1} - 1.\end{aligned}$$

Therefore

$$b = \binom{(N-p^{r-1})(p-1)-1}{p^{r-1}} = \binom{\cdots + d'_{r-2}p^{r-1} + \cdots}{p^{r-1}} \neq 0.$$

Taking into account equalities (12) and (14) it remains to prove that

$$Z_{t_1}^{r-2} P^N = \lambda_3 P^{N+p^{r-2}+\cdots+p^{t_1}} \bmod \mathcal{I}_{r-2},$$

for some $\lambda_3 \neq 0$, $\lambda_3 \in \mathbb{Z}/p$.

Arguing by induction we see that

$$P^{p^k} P^{N+p^{r-2}+\cdots+p^{k+1}} = -b_k P^{N+p^{r-2}+\cdots+p^{k+1}+p^k} \bmod \mathcal{I}_{r-2},$$

where

$$b_k = \binom{(N+p^{r-2}+\cdots+p^{k+1}+p^k)(p-1)-1}{p^k}.$$

Let us check that $b_k \neq 0$ in $\mathbb{Z}/p$. Let

$$d''_j = \begin{cases} p = d_j & \text{for } r-2 \geqslant j \geqslant k+1, \\ p-1 = d_j - 1 & \text{for } j = k, \\ d_j - 1 & \text{for } k > j \geqslant t_1, \\ d_j & \text{for } j < t_1. \end{cases}$$

Then we have the p-adic expansion $N + p^{r-2} + \cdots + p^{k+1} + p^k = p^{r-1} + \cdots + p^{k+2} + (p-1)p^k + d''_{k-1}p^{k-1} + \cdots + d''_0 p^0$, which we write as

$$Ap^{k+1} + (p-1)p^k + d''_{k-1}p^{k-1} + \varepsilon,$$

where $A > 0$ is an integer and $\varepsilon \leqslant d''_{k-2}(p^{k-2} + \cdots + p^0) = d''_{k-2}(p^{k-1})/(p-1)$. Note that $d''_{k-1} > 0$, otherwise $x = (Z^{r-1}_k)^p$ which contradicts the assumptions of the lemma. It should be stressed that this is the key point of the proof of inequality $b_k \neq 0$. In the next Lemma 3.5 we shall see the converse statement. We need to find the coefficient of p^k in the p-adic expansion of $(N + p^{r-2} + \cdots + p^{k+1} + p^k)(p-1) - 1$. Write this expression as

$$\begin{aligned} (Ap^{k+1} &+ (p-1)p^k + d''_{k-1}p^{k-1} + \varepsilon)(p-1) - 1 \\ &= A(p-1)p^{k+1} + (p-2)p^{k+1} + d''_{k-1}p^{k-1}(p-1) + \varepsilon(p-1) \qquad (15) \\ &= Bp^{k+1} + d''_{k-1}p^k + p^{k-1}(p - d''_{k-1}) + \varepsilon(p-1), \end{aligned}$$

where $B > 0$ is an integer. One has the estimate $p^{k-1}(p - d''_{k-1}) + \varepsilon(p-1) - 1 \leqslant p^{k-1}(p - d''_{k-1}) + d''_{k-2}(p^{k-1}) - 1 = p^k - p^{k-1}(d''_{k-1} - d''_{k-2}) - 1 \leqslant p^k - 1$, from which it follows that the coefficient of p^k in (15) is equal to d''_{k-1}. Hence $b_k = \binom{\cdots + d''_{k-1}p^k + \cdots}{p^k} = d''_{k-1} \neq 0$. □

Lemma 3.5.

1. *Consider a product X of the elements $P^{p^k}, P^{p^{k+1}}, \ldots, P^{p^r}$ such that P^{p^k} is contained in the product at least p times and such that the rightmost term is P^{p^r}. Then*

$$X \in \mathcal{I}_{r-2}.$$

2. *If P^{p^k} is contained in the product X only once (as the rightmost term) then*

$$X = 0 \bmod \mathcal{I}_{r-2}.$$

Proof. Without loss of generality we can assume that P^{p^k} is contained in the product X exactly p times and that the leftmost term is P^{p^k}. Denote by M the grading of X. Then arguing as in Lemma 3.1 or Lemma 3.4 one can easily show that

$$X = \lambda P^{p^k} P^{M-p^k} \bmod \mathcal{I}_{r-1}, \qquad (16)$$

for some $\lambda \in \mathbb{Z}/p$. Note we have no evidence to say whether λ vanishes or not. But

$$P^{p^k} P^{M-p^k} = -bP^M \bmod \mathcal{I}_{r-1}, \qquad (17)$$

where $b = \binom{(M-p^k)(p-1)-1}{p^k}$. From the assumptions on the terms of M we see that $M - p^k = Ap^{k+1} + (p-1)p^k$. Then $(M - p^k)(p-1) - 1 = (Ap^{k+1} + (p-1)p^k)(p-1) - 1$

$= A(p-1)p^{k+1} + p^{k+1}(p-2) + p^k - 1$, hence in its p-adic expansion the coefficient of p^k is zero. Therefore $b = \binom{\cdots+0p^k+\cdots}{p^k} = 0$, and $X = 0 \bmod \mathcal{I}_{r-1}$. This proves (1). For (2) note that the same arguments apply but due to the restrictions on the gradings of terms in X the 'error' terms from the Adem relations in equalities (16) and (17) belong to $\mathcal{I}_{r-2}$. □

Lemma 3.6. *For $r \geqslant 2$ we have*

$$P^{p^r+p^{r-1}-p^{r-2}} P^{p^{r-2}} P^{p^{r-1}} = P^{p^r+p^{r-1}-p^{r-2}} P^{p^{r-2}+p^{r-1}} = 0 \bmod \mathcal{I}_{r-2}.$$

Proof. Note that

$$P^{p^r+p^{r-1}-p^{r-2}} P^{p^{r-2}} P^{p^{r-1}} = P^{p^r+p^{r-1}-p^{r-2}} P^{p^{r-2}+p^{r-1}} \bmod \mathcal{I}_{r-2}$$

by Lemma 3.3.

Apply the Adem relations to the product

$$P^{p^r+(p-1)p^{r-2}} P^{p^{r-1}+p^{r-2}} = \sum_{j=0}^{h} (-1)^{1+j} \binom{(p^{r-1}+p^{r-2}-j)(p-1)-1}{p^r+(p-1)p^{r-2}-pj} P^{p^r+p^{r-1}-j} P^j,$$

where $h = [p^{r-1} + (p-1)p^{r-3}]$. The binomial coefficients for $j \leqslant p^{r-1}$ vanish since for such j the inequality $(p^{r-1} + p^{r-2} - j)(p-1) - 1 < p^r + (p-1)p^{r-2} - pj$ holds. For $r = 2$ there are no other summands hence the product vanishes. For $r > 3$ there are summands with $p^{r-1} + 1 \leqslant j \leqslant p^{r-1} + (p-1)p^{r-3}$. In this case $a_{r-1}(j) = 1$ and $a_{r-2}(j) = 0$. Assume $k \leqslant r-3$ is the greatest integer such that $a_k(j) \neq 0$. Then by Lemma 3.3 we have $P^j = \lambda z P^{p^k} P^{p^{r-1}} \bmod \mathcal{I}_{r-2}$ for some $\lambda \neq 0$ and $z \in \mathcal{S}_k$. Now $P^{p^k} P^{p^{r-1}} \in \mathcal{I}_{r-2}$ by Lemma 2.4, hence $P^j \in \mathcal{I}_{r-2}$ for any integer $j \in [p^{r-1} + 1, p^{r-1} + (p-1)p^{r-3}]$. □

4. Case $r = 0$, exactness

For $r = 0$ we consider $\alpha(x) = x(P^1)^a$ and $\beta(x) = x(P^1)^b$, where $a = 1$ and $b = p-1$. We need to prove that the sequence

$$\overline{\mathcal{A}}_p \xrightarrow{\alpha} \overline{\mathcal{A}}_p \xrightarrow{\beta} \overline{\mathcal{A}}_p \tag{18}$$

is exact.

Proof of Theorem 1.1. From $(P^1)^p = 0$ it follows that $\beta \circ \alpha = 0$.

Consider the set Z' consisting of all Z-monomials of the form $\cdots (Z_0^0)^q$ where $q \geqslant a$ and denote by Z'' the set of all other Z-monomials. Clearly the span of Z' coincides with $\operatorname{im}\alpha$. On the other hand β is an injection of Z'' into the set of all Z-monomials. Hence $\ker\beta$ coincides with $\operatorname{im}\alpha$.

Another proof uses the same argument applied to the admissible basis of $\overline{\mathcal{A}}_p$. □

It is worth noting that the sequence (18) is an analogue of the sequence

$$\mathcal{A}_2 \xrightarrow{\psi} \mathcal{A}_2 \xrightarrow{\psi} \mathcal{A}_2,$$

where $\psi(x) = x \cdot Sq^1$. Here $\phi \circ \phi = 0$ since $Sq^1 Sq^1 = 0$. This sequence is known to be exact, and this can be proved by the same argument as in the proof of Theorem 1.1.

In the algebra $\mathcal{A}_2$ the element Sq^1 coincides with the Bockstein element. On the other hand, in the full mod p Steenrod algebra $\mathcal{A}_p$ the Bockstein element does not belong to the subalgebra $\overline{\mathcal{A}}_p$. The algebra $\mathcal{A}_p$ contains a collection of elements Q_j, $\deg Q_j = 2p^j - 1$, $j \geqslant 0$, which form an exterior algebra: $Q_j^2 = 0$, $Q_jQ_i + Q_iQ_j = 0$ for any $i \neq j$. They can be defined by induction: Q_0 coincides with the Bockstein element, and for $j > 0$ we have $Q_{j+1} = [P^{p^j}, Q_j]$, for details see [**2**].

Define $\psi_j\colon \mathcal{A}_p \to \mathcal{A}_p$ by the formula $\psi_j(x) = xQ_j$.

Proposition 4.1. *The sequence*

$$\mathcal{A}_p \xrightarrow{\psi_j} \mathcal{A}_p \xrightarrow{\psi_j} \mathcal{A}_p$$

is exact.

Proof. The relation $\psi_j \circ \psi_j = 0$ follows from $Q_j^2 = 0$.

Now consider any collection of the elements $\{\theta_k\}$ forming an additive basis of $\overline{\mathcal{A}}_p$. Then the collection of elements $\theta_k Q_{i_1} \cdots Q_{i_m}$, where $m \geqslant 0$, $i_1 < \cdots < i_m$, forms an additive basis of $\mathcal{A}_p$. For example, the modification of the Milnor elements $P(r_1, r_2, \ldots)Q_0^{\varepsilon_0}Q_1^{\varepsilon_1}\cdots$ can be used.

To prove that if $\psi_j(x) = 0$ then $x = \psi_j(y)$ for some y it is enough to consider an arbitrary basis element x. Now the statement easily follows from the commutation relations $Q_j^2 = 0$, $Q_jQ_i + Q_iQ_j = 0$ for any $i \neq j$. $\square$

5. Case $r = 1$, exactness

For $r = 1$ we consider $\alpha(x) = xP^{ap}$ and $\beta(x) = xP^{bp}$, where $a = 1$ and $b = p - 1$. We want to prove the exactness of the sequence

$$\overline{\mathcal{S}}_1^{(k)} \xrightarrow{\alpha} (\mathcal{S}_1)^{(p+k)} \xrightarrow{\beta} (\mathcal{S}_1/\mathcal{I}_0)^{(p^2+k)}, \tag{19}$$

for any $k < p$

Proof of Theorem 1.2(1). For $k < -p$ the modules $\overline{\mathcal{S}}_1^{(k)}, (\mathcal{S}_1)^{(p+k)}$ are trivial and the statement is obvious. Consider the following cases:

(a.1) $-p \leqslant k < 0$: $\operatorname{im}\alpha = 0$ since its domain is trivial. Note $(\mathcal{S}_1)^{(p+k)}$ is one-dimensional and is generated by the Z-monomial $(Z_0^0)^{p+k}$. By Lemma 2.2(2) and Lemma 3.3 we have $\beta((Z_0^0)^{p+k}) = (Z_0^0)^{p+k}P^{(p-1)p} = \lambda P^{p^2+k} \bmod \mathcal{I}_0$ for some $\lambda \in \mathbb{Z}/p$, $\lambda \neq 0$. On the other hand $(\mathcal{S}_1/\mathcal{I}_0)^{(p^2+k)}$ contains the basis element $(Z_1^1)^{-1-k}(Z_0^1)^{p+k}$ which, by Lemma 3.1, is equal to $\mu P^{p^2+k} \bmod \mathcal{I}_0$ for some $\mu \in \mathbb{Z}/p$, $\mu \neq 0$. Hence $\ker\beta = 0$.

(a.2) $k = 0$: we have $\dim(\mathcal{S}_1)^{(p+k)} = 1$ with basis element $Z_1^1 = P^p$. It belongs to $\operatorname{im}\alpha$ since $\alpha(1) = P^p$. Hence in this case $\ker\beta = \operatorname{im}\alpha$.

(a.3) $0 < k < p$: we have $\dim(\mathcal{S}_1)^{(p+k)} = 2$ with basis elements $Z_1^1(Z_0^0)^k$ and $Z_1^0(Z_0^0)^{k-1}$, and $\dim(\mathcal{S}_1)^{(k)} = 1$ with the basis element $(Z_0^0)^k$.

First of all note that $\alpha((Z_0^0)^k) = (Z_0^0)^kP^p$ up to scalar multiple $k!$ coincides with P^kP^p which is equal to $(-1)^k\binom{p-1}{k}P^{p+k}$ and hence is not equal to zero.

For $\ker\beta$ we need to calculate β on the elements $Z_1^1(Z_0^0)^k$ and $Z_1^0(Z_0^0)^{k-1}$. We have

$$\beta(Z_1^1(Z_0^0)^k) = P^p(P^1)^k P^{(p-1)p} = k!P^pP^kP^{(p-1)p} = (-1)^k k!\binom{p-1}{k}P^pP^{(p-1)p+k}$$
$$= (-1)^{k+1}k!\binom{p-1}{k}kP^{p^2+k} \bmod \mathcal{I}_0, \qquad (20)$$

and

$$\beta(Z_0^1(Z_0^0)^{k-1}) = P^1P^p(P^1)^{k-1}P^{(p-1)p} = (k-1)!\binom{p-1}{k-1}(k-1)P^1P^{p^2+k-1}$$
$$= (k-1)!\binom{p-1}{k-1}(k-1)kP^{p^2+k} \bmod \mathcal{I}_0.$$

On the other hand, for the Z-monomial $(Z_1^1)^{p-k}(Z_0^1)^k$ in $\mathcal{S}_1/\mathcal{I}_0$ we have that $(Z_1^1)^{p-k}(Z_0^1)^k = \lambda P^{p^2+k}$ for some nonzero $\lambda \in \mathbb{Z}/p$. Therefore $\beta(Z_1^1(Z_0^0)^k) \neq 0$ in $\mathcal{S}_1/\mathcal{I}_0$ and $\beta(Z_0^1(Z_0^0)^{k-1}) = 0$ iff $k = 1$. From these calculations it follows that $\dim\ker\beta = 1$. From the previous arguments we see that $\ker\beta$ contains nonzero element $\alpha((Z_0^0)^k) = (Z_0^0)^kP^p$. Hence $\ker\beta = \operatorname{im}\alpha$. □

6. Case $r = 1$, non-exactness

For $r = 1$ we consider $\alpha(x) = xP^p$ and $\beta(x) = xP^{(p-1)p}$, and we want to prove that in the sequence (19) for $k = p$ the quotient $(\ker\beta/\operatorname{im}\alpha)^{(p+k)}$ has dimension 1. Moreover, the element $Z_0^1(Z_0^0)^{p-1}$ belongs to $\ker\beta$ and is not contained in the image of α.

Proof of Theorem 1.2(2). One has $\dim(\mathcal{S}_1)^{(p+k)} = 2$ with the basis elements $(Z_1^1)^2$ and $Z_0^1(Z_0^0)^{p-1}$. Both of them belong to $\ker\beta$. Indeed, $\beta((Z_1^1)^2) = (P^p)^{p+1} = P^p(P^p)^p \in \mathcal{I}_0$ by Lemma 2.2, and as in formula (20) one deduces

$$\beta(Z_0^1(Z_0^0)^{p-1}) = P^1\big(P^p(P^1)^{p-1}P^{(p-1)p}\big) = P^1(p-1)!\binom{p-1}{p-1}(p-1)P^{p^2+(p-1)}$$
$$= P^1P^{p^2+(p-1)} = \binom{(p^2+(p-1))(p-1)-1}{1}P^1P^{p^2+(p-1)} = 0 \bmod \mathcal{I}_0.$$

On the other hand, $\dim(\mathcal{S}_1)^{(p)} = 1$ with basis element Z_1^1. Clearly $\alpha(Z_1^1) = P^pP^p = (Z_1^1)^2 \neq 0$. Hence $\dim(\ker\beta/\operatorname{im}\alpha)^{(2p)} = 1$. □

Corollary 6.1. *Choose integers a and b such that $a+b = p$ and $2 \leqslant a \leqslant p-1$. Redefine α and β as $\alpha(x) = xP^{ap}$ (or $\alpha(x) = x(P^p)^a$) and $\beta(x) = xP^{bp}$. Then* $\dim(\ker\beta/\operatorname{im}\alpha)^{(2p)} = 1$ *for the sequence (19).*

Proof. Indeed, applying the same arguments as in the 'non-exactness' part of the proof of Theorem 1.2 one can see that $\dim(\ker\beta)^{(2p)} = 1$ for $a > 2$ while $\operatorname{im}\alpha = 0$.

For $a = 2$ the image of α in the grading $2p$ is generated by $\alpha(1) = P^{2p} \neq 0$ (or by $\alpha(1) = (P^p)^2 = -P^{2p} + P^{2p-1}P^1 \neq 0$). On the other hand β vanishes on $\mathcal{S}_1^{(2p)}$ since $\beta((Z_1^1)^2) = (P^p)^2(P^p)^{p-2} = 0 \bmod \mathcal{I}_0$ and $\beta(Z_0^1(Z_0^0)^{p-1}) = P^1P^p(P^1)^{p-1} = 0 \bmod \mathcal{I}_0$. □

7. Case $r \geqslant 2$, exactness

Consider $\beta(x) = x \cdot P^{(p-1)p^r}$ and $\alpha(x) = x \cdot P^{p^r}$. We want to prove that for $k < 2p^{r-1} + p^{r-2}$ the sequence

$$\overline{\mathcal{S}_r}^{(k)} \xrightarrow{\alpha} (\mathcal{S}_r/\mathcal{I}_{r-2})^{(p^r+k)} \xrightarrow{\beta} (\mathcal{S}_r/\mathcal{I}_{r-1})^{(p^{r+1}+k)} \tag{21}$$

is exact.

Proof of Theorem 1.3(1). Consider the following cases:

1. for $k < -p^r$ we have $\mathcal{S}_r^{(k)} = 0$ and $(\mathcal{S}_r/\mathcal{I}_{r-2})^{(p^r+k)} = 0$, hence the statement is trivial;
2. for $-p^r \leqslant k < 0$ we have $\mathcal{S}_r^{(k)} = 0$ and the statement is $\ker \beta = 0$;
3. for $0 \leqslant k < 2p^{r-1} + p^{r-2}$ all three modules are nontrivial in general, hence the statement is $\ker \beta = \operatorname{im} \alpha$.

We start with (2). The basis of $\mathcal{S}_r/\mathcal{I}_{r-2}$ consists of Z-monomials of the form

$$Z^r_{l_1} \cdots Z^r_{l_s} Z^{r-1}_{k_1} \cdots Z^{r-1}_{k_t}.$$

In the grading $N = p^r - k$, where $-p^r < k < 0$, we have $s = 0$ and $t \leqslant p - 1$. Denote by $a_j(Z)$ the number of P^{p^j}, $0 \leqslant j \leqslant r-1$, in the product $Z = Z^{r-1}_{k_1} \cdots Z^{r-1}_{k_t}$, $t \leqslant p - 1$, of grading N. Then $N = \sum a_j(Z)p^j$ is the p-adic expansion of N. Note $0 \leqslant a_0(Z) \leqslant \cdots \leqslant a_{r-1}(Z) = t \leqslant p-1$, hence Z is uniquely defined by the numbers $a_0, \ldots, a_{r-1}$.

On the other hand, consider the p-adic expansion $N = \sum a_j(N)\, p^j$ for $0 \leqslant N < p^r$. If $0 \leqslant a_0(N) \leqslant \cdots \leqslant a_{r-1}(N) \leqslant p-1$ then there is a unique Z-monomial $Z^{r-1}_{k_1} \cdots Z^{r-1}_{k_t}$ in grading N and $\dim(\mathcal{S}_r/\mathcal{I}_{r-2})^{(N)} = 1$. Otherwise $\dim(\mathcal{S}_r/\mathcal{I}_{r-2})^{(N)} = 0$. Hence it remains to prove that $\beta(Z^{r-1}_{k_1} \cdots Z^{r-1}_{k_t}) \neq 0$ in $\mathcal{S}_r/\mathcal{I}_{r-1}$.

From Lemma 2.2(2) and Lemma 3.1 it follows that

$$\beta(Z^{r-1}_{k_1} \cdots Z^{r-1}_{k_t}) = Z^{r-1}_{k_1} \cdots Z^{r-1}_{k_t} P^{(p-1)p^r} = \lambda P^{N+(p-1)p^r} \bmod \mathcal{I}_{r-1},$$

where $\lambda \neq 0$, $\lambda \in \mathbb{Z}/p$. On the other hand, the basis of $\mathcal{S}_r/\mathcal{I}_{r-1}$ consists of Z-monomials of the form $Z^r_{l_1} \cdots Z^r_{l_s}$. Consider the Z-monomial $(Z^r_r)^{p-1-t} Z^r_{k_1} \cdots Z^r_{k_t} \in \mathcal{S}_r/\mathcal{I}_{r-1}$. From the same Lemma 3.1 it follows that

$$(Z^r_r)^{p-1-t} Z^r_{k_1} \cdots Z^r_{k_t} = \mu P^{N+(p-1)p^r} \bmod \mathcal{I}_{r-1},$$

where $\mu \in \mathbb{Z}/p$ and $\mu \neq 0$. Hence $\beta(Z^{r-1}_{k_1} \cdots Z^{r-1}_{k_t}) \neq 0$.

Now consider case (3). First of all we address the question of how many Z-monomials of the form $Z^r_{l_1} \cdots Z^r_{l_s} Z^{r-1}_{k_1} \cdots Z^{r-1}_{k_t}$ have grading equal to N if $p^k \leqslant N < p^k + 2p^{k-1} + p^{k-2}$.

From the restriction on N it follows that $s \leqslant 1$. If a Z-monomial $x = Z^r_{l_1} \cdots \cdots Z^r_{l_s} Z^{r-1}_{k_1} \cdots Z^{r-1}_{k_t}$ contains P^{p^r} (that is $s = 1$) then it can contain at most two $P^{p^{r-1}}$. Hence the only Z-monomials of this type are

$$Z^r_r,\ Z^r_{r-1} Z^{r-1}_{r-1},\ Z^r_r (Z^{r-1}_{r-1})^2,\ Z^r_r Z^{r-1}_k,$$

where $k \leqslant r-1$. If $s = 0$ (the monomial x contains no P^{p^r}) then from $p^k \leqslant N < p^k + 2p^{k-1} + p^{k-2}$ it follows that $t = p$ or $t = p+1$. It is easy to check that all these monomials have different grading except two of them, namely $Z^r_{r-1} Z^{r-1}_{r-1}$, $Z^r_r (Z^{r-1}_{r-1})^2$ which have grading $p^r + 2p^{r-1}$. In other words we have the following subcases:

(3.1) $p^k \leqslant N < p^k + 2p^{k-1} + p^{k-2}$ and $N \neq p^r + 2p^{r-1}$, then $\dim(\mathcal{S}_r/\mathcal{I}_{r-2})^{(N)} \leqslant 1$;

(3.2) $N = p^r + 2p^{r-1}$, then $\dim(\mathcal{S}_r/\mathcal{I}_{r-2})^{(p^r+2p^{r-1})} = 2$.

(3.3) $Z^{r-1}_{k_1} \cdots Z^{r-1}_{k_t}$, $t = p$ or $t = p+1$.

For (3.1) we prove that the restriction of β on $(\mathcal{S}_r/\mathcal{I}_{r-2})^{(N)}$ has no kernel. Consider $Z^r_r Z^{r-1}_k$ and let $N = p^r + p^{r-1} + \cdots + p^k$. By Lemma 2.2(2) and Lemma 3.1 we have

$$Z^{r-1}_k P^{(p-1)p^r} = \lambda P^{(p-1)p^r + p^{r-1} + \cdots + p^k} \bmod \mathcal{I}_{r-2}.$$

One can show as an exercise in Lucas' Lemma that $\lambda = 1$.

Then

$$\begin{aligned}\beta(Z^r_r Z^{r-1}_k) &= P^{p^r} P^{(p-1)p^r + p^{r-1} + \cdots + p^k} \bmod \mathcal{I}_{r-1}\\ &= -\binom{(p^r(p-1) + p^{r-1} + \cdots + p^k)(p-1) - 1}{p^r} P^{N+(p-1)p^r} \bmod \mathcal{I}_{r-1}\\ &= -P^{N+(p-1)p^r} \bmod \mathcal{I}_{r-1}.\end{aligned}$$

On the other hand in $(\mathcal{S}_r/\mathcal{I}_{r-1})^{(N+(p-1)p^r)}$ there is the (nonzero) Z-monomial $(Z^r_r)^{p-1} Z^r_k$. Arguing as above one can show that in $(\mathcal{S}_r/\mathcal{I}_{r-1})^{(N+(p-1)p^r)}$

$$(Z^r_r)^{p-1} Z^r_k = (p-1)! P^{N+(p-1)p^r} \bmod \mathcal{I}_{r-1}.$$

By Wilson's Theorem $(p-1)! + 1 = 0$ in $\mathbb{Z}/p$. Hence $\beta(Z^r_r Z^{r-1}_k) \neq 0$.

Case (3.2) is more involved. First of all we need to calculate $\beta(Z^r_{r-1} Z^{r-1}_{r-1})$ and $\beta(Z^r_r (Z^{r-1}_{r-1})^2)$. We go along the proof of Lemma 3.1 carefully calculating binomial coefficients. We, consequently, apply the Adem relations and keep summands not from the ideals of factorization:

$$\begin{aligned}\beta(Z^r_{r-1} Z^{r-1}_{r-1}) &= P^{p^{r-1}} P^{p^r} P^{p^{r-1}} P^{(p-1)p^r}\\ &= -P^{p^{r-1}} P^{p^r} \binom{p^r(p-1)^2 - 1}{p^{r-1}} P^{p^{r-1}+(p-1)p^r} \bmod \mathcal{I}_{r-2}\\ &= P^{p^{r-1}} P^{p^r} P^{p^{r-1}+(p-1)p^r} \bmod \mathcal{I}_{r-2} = -P^{p^{r-1}} P^{p^{r-1}+p^{r+1}} \bmod \mathcal{I}_{r-1}\\ &= -2P^{2p^{r-1}+p^{r+1}} \bmod \mathcal{I}_{r-1},\end{aligned}$$

and

$$\begin{aligned}\beta(Z^r_r (Z^{r-1}_{r-1})^2) &= P^{p^r} P^{p^{r-1}} P^{p^{r-1}} P^{(p-1)p^r} = P^{p^r} P^{p^{r-1}} P^{p^{r-1}+(p-1)p^r} \bmod \mathcal{I}_{r-2}\\ &= 2P^{p^r} P^{2p^{r-1}+(p-1)p^r} \bmod \mathcal{I}_{r-2} = -4P^{2p^{r-1}+p^{r+1}} \bmod \mathcal{I}_{r-1}.\end{aligned}$$

In $(\mathcal{S}_r/\mathcal{I}_{r-1})^{(p^{r+1}+2p^{r-1})}$ by Lemma 3.4 we have the equality

$$(Z^r_r)^{p-2} (Z^r_{r-1})^2 = \lambda P^{2p^{r-1}+p^{r+1}} \bmod \mathcal{I}_{r-1},$$

for some $\lambda \neq 0$, $\lambda \in \mathbb{Z}/p$. Indeed by Lemma 3.1

$$(Z^r_r)^{p-3} (Z^r_{r-1})^2 = \lambda P^{(p-1)p^r+2p^{r-1}} \bmod \mathcal{I}_{r-1},$$

for some $\lambda' \neq 0, \lambda' \in \mathbb{Z}/p$. Then

$$(Z_r^r)^{p-2}(Z_{r-1}^r)^2 = \lambda' P^{p^r} P^{(p-1)p^r+2p^{r-1}}$$
$$= -\lambda' \binom{((p-1)p^r + 2p^{r-1})(p-1) - 1}{p^r} P^{p^{r+1}+2p^{r-1}} = -2\lambda' P^{p^{r+1}+2p^{r-1}} \bmod \mathcal{I}_{r-1}.$$

Therefore the kernel of β on $(\mathcal{S}_r/\mathcal{I}_{r-2})^{(p^r+2p^{r-1})}$ has dimension one and is generated by the element $x = Z_r^r(Z_{r-1}^{r-1})^2 - 2Z_{r-1}^r Z_{r-1}^{r-1}$. Now we are going to show that $x \in \operatorname{im} \alpha$. To be more precise we show that $\alpha((Z_{r-1}^{r-1})^2) = -x$.

The main difference with the previous arguments is that now we make calculations in $\mathcal{S}_r$ and apply the Adem relations modulo the ideal $\mathcal{I}_{r-2}$. For the product $P^{p^r} P^{p^{r-1}} P^{p^{r-1}}$ we have

$$P^{p^r} P^{p^{r-1}} P^{p^{r-1}} = 2P^{p^r} P^{2p^{r-1}} = -2\binom{2p^{r-1}(p-1) - 1}{p^r} P^{p^r+2p^{r-1}} + 2P^{p^r+p^{r-1}} P^{p^{r-1}}$$
$$= -2P^{p^r+2p^{r-1}} + 2P^{p^r+p^{r-1}} P^{p^{r-1}} \bmod \mathcal{I}_{r-2}.$$

One can check that $P^{p^{r-1}} P^{p^{r-1}} P^{p^r} = 2P^{p^r+2p^{r-1}} \bmod \mathcal{I}_{r-2}$ (see also Lemma 3.1). Hence it is enough to establish the equality

$$P^{p^r+p^{r-1}} P^{p^{r-1}} = P^{p^{r-1}} P^{p^r} P^{p^{r-1}} \bmod \mathcal{I}_{r-2}.$$

This is the most tricky part of the proof of case (3.2). The Adem relations give

$$P^{p^{r-1}} P^{p^r} P^{p^{r-1}} = (P^{p^r+p^{r-1}} + P^{p^r+p^{r-1}-p^{r-2}} P^{p^{r-2}} + y) P^{p^{r-1}},$$

where $y \in \mathcal{I}_{r-3}$. By Lemma 2.4 we have $yP^{p^{r-1}} \in \mathcal{I}_{r-2}$, and $P^{p^r+p^{r-1}-p^{r-2}} P^{p^{r-2}} P^{p^{r-1}} \in \mathcal{I}_{r-2}$ by Lemma 3.6, which finishes the proof of (3.2).

Case (3.3): An element $Z_{k_1}^{r-1} \cdots Z_{k_t}^{r-1}$, where $t = p$ or $t = p+1$, by Lemma 3.4 belongs to $\operatorname{im} \alpha$. Note that there are no other Z-monomials of the same grading in $\mathcal{S}_r/\mathcal{I}_{r-2}$, hence $\ker \beta = \operatorname{im} \alpha$ in this grading. □

8. Case $r \geqslant 2$, non-exactness

In this section we prove that for the sequence (21) for $k = 2p^{r-1} + p^{r-2}$ the quotient $\ker \beta / \operatorname{im} \alpha$ has dimension one and, moreover, the element $2Z_{r-1}^r Z_{r-2}^{r-1} - Z_{r-2}^r Z_{r-1}^{r-1}$ belongs to $\ker \beta$, but is not contained in $\operatorname{im} \alpha$.

Proof of Theorem 1.3(2). In $(\mathcal{S}_r/\mathcal{S}_{r-2})^{(p^r+2p^{r-1}+p^{r-2})}$ there are three Z-monomials containing P^{p^r}, namely $Z_{r-1}^r Z_{r-2}^{r-1}$, $Z_r^r Z_{r-1}^{r-1} Z_{r-2}^{r-1}$, $Z_{r-2}^r Z_{r-1}^{r-1}$. First of all we calculate the image of these elements under the map $\beta\colon \mathcal{S}_r/\mathcal{I}_{r-2} \to \mathcal{S}_r/\mathcal{I}_{r-1}$ which maps x to $xP^{(p-1)p^r}$. But for future purposes we consider the more general map $\beta_1 : \mathcal{S}_r/\mathcal{I}_{r-2} \to \mathcal{S}_r/\mathcal{I}_{r-1}$ which was defined above by the formula $\beta_1(x) = xP^{bp^r}$ for some fixed integer $1 \leqslant b \leqslant p-1$.

Lemma 8.1. *In $\mathcal{S}_r/\mathcal{I}_{r-1}$ we have $(Z_r^r)^{p-2} Z_{r-1}^r Z_{r-2}^r = P^{p^{r+1}+2p^{r-1}+p^{r-2}}$. Also*

1. $\beta_1(Z_{r-1}^r Z_{r-2}^{r-1}) = bP^{p^{r+1}+2p^{r-1}+p^{r-2}} \bmod \mathcal{I}_{r-1}$,
2. $\beta_1(Z_r^r Z_{r-1}^{r-1} Z_{r-2}^{r-1}) = (b-1)P^{p^{r+1}+2p^{r-1}+p^{r-2}} \bmod \mathcal{I}_{r-1}$,

3. $\beta_1(Z^r_{r-2}Z^{r-1}_{r-1}) = 2bP^{p^{r+1}+2p^{r-1}+p^{r-2}} \bmod \mathcal{I}_{r-1}$.

Proof of Lemma 8.1. Following arguments from the proof of Lemma 3.1 and Lemma 2.2(2) one can check that for the Z-monomial $(Z^r_r)^{p-2}Z^r_{r-1}Z^r_{r-2}$ there is the equality

$$(Z^r_r)^{p-2}Z^r_{r-1}Z^r_{r-2} = (p-2)!P^{p^{r+1}+2p^{r-1}+p^{r-2}} \bmod \mathcal{I}_{r-1}.$$

By Wilson's theorem $(p-1)!+1 = 0 \bmod p$, hence $(p-2)! = 1 \bmod p$.

For (1) one checks that, modulo the ideal $\mathcal{I}_{r-1}$,

$$\begin{aligned}\beta_1(Z^r_{r-1}Z^{r-1}_{r-2}) &= P^{p^{r-1}}P^{p^r}P^{p^{r-2}}P^{p^{r-1}}P^{bp^r} = P^{p^{r-1}}P^{p^r}P^{p^{r-2}}P^{bp^r+p^{r-1}} \\ &= P^{p^{r-1}}P^{p^r}P^{bp^r+p^{r-1}+p^{r-2}} = bP^{p^{r-1}}P^{(b+1)p^r+p^{r-1}+p^{r-2}} \\ &= bP^{(b+1)p^r+2p^{r-1}+p^{r-2}} \bmod \mathcal{I}_{r-1}.\end{aligned}$$

For (2) one checks that, modulo the ideal $\mathcal{I}_{r-1}$,

$$\begin{aligned}\beta_1(Z^r_rZ^{r-1}_{r-1}Z^{r-1}_{r-2}) &= P^{p^r}P^{p^{r-1}}P^{p^{r-2}}P^{p^{r-1}}P^{bp^r} = P^{p^r}P^{p^{r-1}}P^{bp^r+p^{r-1}+p^{r-2}} \\ &= P^{p^r}P^{bp^r+2p^{r-1}+p^{r-2}} = (b-1)P^{(b+1)p^r+2p^{r-1}+p^{r-2}} \bmod \mathcal{I}_{r-1}.\end{aligned}$$

For (3) one checks that, modulo the ideal $\mathcal{I}_{r-1}$,

$$\begin{aligned}\beta_1(Z^r_{r-2}Z^{r-1}_{r-1}) &= P^{p^{r-2}}P^{p^{r-1}}P^{p^r}P^{p^{r-1}}P^{bp^r} = P^{p^{r-2}}P^{p^{r-1}}P^{p^r}P^{bp^r+p^{r-1}} \\ &= bP^{p^{r-2}}P^{p^{r-1}}P^{(b+1)p^r+p^{r-1}} = 2bP^{p^{r-2}}P^{(b+1)p^r+2p^{r-1}} \\ &= 2bP^{(b+1)p^r+2p^{r-1}+p^{r-2}} \bmod \mathcal{I}_{r-1}.\end{aligned}$$

□

From Lemma 8.1 it follows that on the subspace spanned by the Z-monomials $Z^r_{r-1}Z^{r-1}_{r-2}$, $Z^r_rZ^{r-1}_{r-1}Z^{r-1}_{r-2}$, $Z^r_{r-2}Z^{r-1}_{r-1}$ the kernel of β_1 has dimension 2.

Now we calculate the image of α in $(\mathcal{S}_r/\mathcal{I}_{r-2})^{(p^r+2p^{r-1}+p^{r-2})}$. We show that

$$\dim(\operatorname{im}\alpha)^{(p^r+2p^{r-1}+p^{r-2})} = 1$$

by establishing the equality

$$\alpha(Z^{r-1}_{r-1}Z^{r-1}_{r-2}) = -Z^r_rZ^{r-1}_{r-1}Z^{r-1}_{r-2} + Z^r_{r-2}Z^{r-1}_{r-1} \bmod \mathcal{I}_{r-2} \tag{22}$$

and that α vanishes on all other Z-monomials of grading $2p^{r-1}+p^{r-2}$.

First of all note that, by using Lemma 3.3(1) in the second equality,

$$\alpha(Z^{r-1}_{r-1}Z^{r-1}_{r-2}) = P^{p^{r-1}}P^{p^{r-2}}P^{p^{r-1}}P^{p^r} = P^{p^{r-1}}P^{p^{r-2}+p^{r-1}+p^r} = P^{p^r+2p^{r-1}+p^{r-2}} \bmod \mathcal{I}_{r-2}.$$

To prove equality (22) we transform $Z^r_rZ^{r-1}_{r-1}Z^{r-1}_{r-2}$ modulo the ideal $\mathcal{I}_{r-2}$:

$$\begin{aligned}Z^r_rZ^{r-1}_{r-1}Z^{r-1}_{r-2} &= P^{p^r}P^{p^{r-1}}P^{p^{r-2}}P^{p^{r-1}} = P^{p^r}P^{p^{r-1}}P^{p^{r-2}+p^{r-1}} \\ &= P^{p^r}P^{p^{r-2}+2p^{r-1}} = -P^{p^r+2p^{r-1}+p^{r-2}} + P^{p^r+p^{r-1}+p^{r-2}}P^{p^{r-1}} \bmod \mathcal{I}_{r-2}.\end{aligned}$$

Hence we have the equality

$$Z^r_rZ^{r-1}_{r-1}Z^{r-1}_{r-2} = -\alpha(Z^{r-1}_{r-1}Z^{r-1}_{r-2}) + P^{p^r+p^{r-1}+p^{r-2}}P^{p^{r-1}} \bmod \mathcal{I}_{r-2}.$$

It remains to prove the equality

$$P^{p^r+p^{r-1}+p^{r-2}}P^{p^{r-1}}P^{p^{r-1}} = P^{p^{r-2}}P^{p^{r-1}}P^{p^r}P^{p^{r-1}} \bmod \mathcal{I}_{r-2}.$$

By the Adem relations we have

$$P^{p^{r-1}}P^{p^r}P^{p^{r-1}} = (P^{p^{r-1}+p^r} + P^{p^r+p^{r-1}-p^{r-2}}P^{p^{r-2}} + y)P^{p^{r-1}},$$

where $y \in \mathcal{I}_{r-3}$. Then $yP^{p^{r-1}} \in \mathcal{I}_{r-2}$ by Lemma 2.4 and $P^{p^r+p^{r-1}-p^{r-2}}P^{p^{r-2}}P^{p^{r-1}} \in \mathcal{I}_{r-2}$ by Lemma 3.6. Finally, by the Adem relations we have

$$\begin{aligned} P^{p^{r-2}}P^{p^{r-1}}P^{p^r}P^{p^{r-1}} &= P^{p^{r-2}}P^{p^{r-1}+p^r}P^{p^{r-1}} = (P^{p^{r-2}+p^{r-1}+p^r} + y')P^{p^{r-1}} \\ &= P^{p^{r-2}+p^{r-1}+p^r}P^{p^{r-1}} \bmod \mathcal{I}_{r-2}, \end{aligned}$$

since $y' \in \mathcal{I}_{r-3}$ and again $y'P^{p^{r-1}} \in \mathcal{I}_{r-2}$ by Lemma 2.4. Hence (22) is established. Now we check that α vanishes on the other basis elements of grading $2p^{r-1} + p^{r-2}$. There is only one basis element which does not contain P^{p^j} with $j < r-2$ and does not coincide with $P^{p^{r-1}}P^{p^{r-2}}P^{p^{r-1}}$, namely $X = P^{p^{r-1}}P^{p^{r-1}}P^{p^{r-2}}$. For this element X we have $\alpha(X) = 0 \bmod \mathcal{I}_{r-2}$ by Corollary 3.2. Assume X is another basis element of grading $2p^{r-1} + p^{r-2}$. Then it should contain P^{p^j} with $j < r-2$. Let j be the smallest integer such that X contains P^{p^j}. Then X contains P^{p^j} at least p times. By Lemma 3.5(2) we have $\alpha(X) = 0 \bmod \mathcal{I}_{r-2}$.

To finish the proof of part (1) of Theorem 1.3 note that from Lemma 8.1 applied to $b = p-1$ we have $2Z^r_{r-1}Z^{r-1}_{r-2} - Z^r_{r-2}Z^{r-1}_{r-1} \in \ker\beta$, but as we have just checked this element does not belong to $\operatorname{im}\alpha$. □

Corollary 8.2. *Choose integers a and b such that $a + b = p$ and $2 \leqslant a \leqslant p-1$. Redefine α and β for the sequence (4) as $\alpha(x) = xP^{ap}$ (or $\alpha(x) = x(P^{p^r})^a$) and $\beta(x) = xP^{bp^r}$. Then $(\operatorname{im}\alpha)^{(p^r+2p^{r-1}+p^{r-2})} = 0$ and $\dim(\ker\beta)^{(p^r+2p^{r-1}+p^{r-2})} \geqslant 2$. Moreover, $2Z^r_{r-1}Z^{r-1}_{r-2} - Z^r_{r-2}Z^{r-1}_{r-1} \in \ker\beta$.*

Proof. The estimate for the dimension of $\ker\beta$ in grading $p^r + 2p^{r-1} + p^{r-2}$ follows from Lemma 8.1. For $a > 1$ the domain of α in grading $p^r + 2p^{r-1} + p^{r-2} - ap^r$ is trivial, hence $(\operatorname{im}\alpha)^{(p^r+2p^{r-1}+p^{r-2})} = 0$. □

9. Some remarks

Interesting results concerning $\ker\beta$ (and the kernel of the left multiplication by P^{p^r}) for $b = 1$ were obtained in [**4**]. Also it should be noted that in Milnor's paper [**2**] properties of the solutions of the equation $xP^1 = 0$ were discussed. Theorem 1.1 gives another description of this space of the solutions.

The sequence

$$\mathcal{A}_2 \xrightarrow{\hat{\varphi}_r} \mathcal{A}_2/\mathcal{S}_{r-2}\mathcal{A}_2 \xrightarrow{\hat{\varphi}_r} \mathcal{A}_2/\mathcal{S}_{r-1}\mathcal{A}_2,$$

where $\hat{\varphi}_r(x) = Sq^{2^r} \cdot x$, was proved to be exact in [**7**]. One can show that the analogue of this sequence for $p > 2$ is exact for $r = 0$ and is not exact for $r \geqslant 1$.

Also let us explain why similar results (exactness for $r = 0$ and non-exactness for $r \geqslant 1$) are valid for the whole Steenrod algebra $\mathcal{A}_p$. Recall that the algebra $\mathcal{A}_p$ has an additive basis consisting of elements of the form $Q_0^{\varepsilon_0}Q_1^{\varepsilon_1}\cdots P(r_1, r_2, \ldots)$, see [**2**] for details. For us it is enough to know that the elements Q_j, $j \geqslant 0$, form an exterior algebra and the collection of all the elements $P(r_1, r_2, \ldots)$ is a basis of $\overline{\mathcal{A}}_p$. From this it follows that additively $\overline{\mathcal{A}}_p$ is a direct sum of subspaces $Q_0^{\varepsilon_0}Q_1^{\varepsilon_1}\cdots\overline{\mathcal{A}}_p$ which are obviously invariant under right multiplication by any $P^k, k \geqslant 1$. The quotients $\mathcal{A}_p/\mathcal{A}_p\mathcal{S}_{r-1}$ also decompose in a direct sum of similar type. Hence for any $r \geqslant 0$ the sequence

$$\mathcal{A}_p \xrightarrow{\alpha_i} \mathcal{A}_p/\mathcal{A}_p\mathcal{S}_{r-2} \xrightarrow{\beta_j} \mathcal{A}_p/\mathcal{A}_p\mathcal{S}_{r-1}$$

splits into a direct sum of sequences isomorphic, up to a shift of degree, to the sequence (2).

Finally, we note that neither Toda in his famous series of papers on homotopy groups of spheres, nor Wall in [**1**] gave an application of the sequence (1). So now it is quite tempting to find an application of the sequences (1) and (2), which shows a difference between the $p = 2$ and $p > 2$ cases. Presumably it can be found in the Adams spectral sequence for stable homotopy groups of spheres or possibly in certain spectral sequences converging to the E_2 term of the Adams spectral sequence. We prefer to postpone this discussion to a next paper.

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