FLAG BOTT MANIFOLDS OF GENERAL LIE TYPE AND THEIR EQUIVARIANT COHOMOLOGY RINGS

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Abstract

In this article we introduce flag Bott manifolds of general Lie type as the total spaces of iterated flag bundles. They generalize the notion of flag Bott manifolds and generalized Bott manifolds, and admit nice torus actions. We calculate the torus equivariant cohomology rings of flag Bott manifolds of general Lie type.

1. Introduction

A *Bott tower* $\{M_j \mid 0 \leqslant j \leqslant m\}$ is a sequence of $\mathbb{C}P^1$-fibrations $\mathbb{C}P^1 \hookrightarrow M_j \to M_{j-1}$ such that M_j is the induced projective bundle of the sum of two complex line bundles over M_{j-1}. Each manifold M_j is called *a j-stage Bott manifold* and it is known that M_j is a non-singular projective toric variety. On the other hand, a flag manifold is the orbit space G/P of a complex Lie group G divided by a parabolic subgroup P. A flag manifold is known to be a non-singular projective variety having a nice torus action.

These two families of spaces are closely related by the Bott-Samelson resolution (see [**13, 17**]). Both families have been actively studied as spaces with nice torus actions, and have served as a test ground for various theories and problems. *Schubert calculus* studies the cohomology of flag manifolds, in which topology, algebraic geometry, combinatorics, and representation theory meet together (see [**22**] for a survey). On the other hand, *the cohomological rigidity problem of quasi-toric manifolds* may be regarded as one of the essential problems in toric topology, and some affirmative results are known for Bott manifolds (see [**6, 7, 8, 20**]).

There are two known natural generalizations of Bott manifolds: flag Bott manifolds, and generalized Bott manifolds. The *flag Bott manifolds* extend the relation between Bott manifolds and Bott–Samelson manifolds. We refer the reader to [**23**] for the

definition of flag Bott manifolds, and to [**15**] for this enlarged relation. And the *generalized Bott manifolds* are toric manifolds studied in [**10, 11, 24**].

In this note, we introduce *flag Bott manifolds of general Lie type* which simultaneously generalize flag manifolds, flag Bott manifolds, and generalized Bott manifolds. We give two closely related descriptions of the flag Bott manifold of general Lie type; as the total space of an iterated flag bundle (Definition 3.1) and as an orbit space of a Lie group (Definition 3.5). We see a torus acts on the flag Bott manifold in a natural manner (Definition 4.1). Moreover, we determine its Borel equivariant cohomology ring. Theorem 4.2 unifies the known formulae for the equivariant cohomology of flag manifolds, flag Bott manifolds, and generalized Bott manifolds.

2. Flag bundle and its cohomology

Let K be a compact connected Lie group and $T \subset K$ be a maximal torus of K. Let $Z \subset K$ be the centralizer in K of a circle subgroup of T. Then, Z is known to be connected. Indeed, for any element $g \in Z$ consider the subgroup H which is generated by g and the circle subgroup. Since H is abelian, it is contained in some torus which is contained in Z. Therefore, g is in the identity component of Z. We denote by W (resp. $W(Z)$) the Weyl group of K (resp. Z). The space K/Z of left-cosets is called the *generalized flag manifold*, and there exists the universal flag bundle $K/Z \hookrightarrow BZ \to BK$. For any map $f\colon X \to BK$ from a topological space X, we have the pull-back bundle $K/Z \hookrightarrow F_f(Z) \to X$, which fits in the diagram

$$
\begin{array}{ccc}
K/Z & =\!=\!= & K/Z \\
\downarrow & & \downarrow \\
F_f(Z) & \xrightarrow{\tilde{f}} & BZ \\
\downarrow & & \downarrow \\
X & \xrightarrow{f} & BK
\end{array}
\tag{2.1}
$$

The pull-back bundle is called the *flag bundle over X associated to the classifying map f with fiber K/Z.*

Example 2.1. Let $K = U(n)$ and $f\colon X \to BU(n)$. We denote by $U(1)^n$ the set of all diagonal matrices in K. It is well-known that $U(1)^n$ is a maximal torus in K. We exhibit three examples of the associated flag bundles over X.

Take the circle subgroup $\{\operatorname{diag}(t, \ldots, t, t^2) \mid t \in S^1\} \subset U(1)^n$. Then its centralizer Z_1 is the group of the block diagonal matrices $U(n-1) \times U(1) \subset K$. The associated flag bundle $F_f(Z_1)$ is isomorphic to the projective bundle $\mathbb{P}(E)$ associated to the complex vector bundle E classified by f.

We next take the circle subgroup $\{\operatorname{diag}(t, t^2, \ldots, t^n) \mid t \in S^1\} \subset U(1)^n$. Then its centralizer Z_2 coincides with the group of the diagonal matrices $U(1)^n$. The associated flag bundle $F_f(Z_2)$ is isomorphic to the full flag bundle

$$
\mathcal{F}\ell(E) = \{V_0 \subset V_1 \subset V_2 \subset \cdots \subset V_n = E \mid \operatorname{rank}(V_i) = i\}
$$

associated to E.

More generally, for positive integers $p_1, \ldots, p_k > 0$ with $p_1 + \cdots + p_k = n$, the centralizer Z_3 of the circle subgroup

$$\{\operatorname{diag}(\underbrace{t, \ldots, t}_{p_1}, \underbrace{t^2, \ldots, t^2}_{p_2}, \ldots, \underbrace{t^k, \ldots, t^k}_{p_k}) \mid t \in S^1\} \subset U(1)^n$$

is the group of the block diagonal matrices $U(p_1) \times \cdots \times U(p_k) \subset U(n)$. The associated flag bundle $F_f(Z_3)$ is isomorphic to the partial flag bundle

$$\{V_0 \subset V_1 \subset V_2 \subset \cdots \subset V_k \subset E \mid \operatorname{rank}(V_i) = p_1 + \cdots + p_i\}$$

associated to E.

The cohomology of $F_f(Z)$ can easily be computed for some coefficient rings. A prime p is said to be a torsion prime of K if $H_*(K; \mathbb{Z})$ has p-torsion. Let R be a PID in which torsion primes of K are invertible. For example, if K is $U(n)$ or $Sp(n)$, then we may take R to be $\mathbb{Z}$. When K is simply-connected, the torsion primes are summarized in Table 1 (see [2, §2.5]). Note that if p is not a torsion prime of K, it is not of any circle centralizer Z as well.

Lie type	A	B	C	D	G_2	F_4	E_6	E_7	E_8
torsion primes	$\emptyset$	2	$\emptyset$	2	2	$2, 3$	$2, 3$	$2, 3$	$2, 3, 5$

Table 1: Torsion primes.

Proposition 2.2. *Let R be a PID in which the torsion primes of K are invertible. Then, we have a ring isomorphism*

$$H^*(F_f(Z); R) \cong H^*(X; R) \otimes_{H^*(BK; R)} H^*(BZ; R),$$

where $H^*(X; R)$ has the $H^*(BK; R)$-module structure induced by the map f and $H^*(BZ; R)$ also has the natural $H^*(BK; R)$-module structure induced by the classifying map of the inclusion $Z \to K$.

Proof. Taking the cohomology of the lower square of (2.1) gives rise to a homomorphism

$$\varphi \colon H^*(X; R) \otimes_{H^*(BK; R)} H^*(BZ; R) \longrightarrow H^*(F_f(Z); R),$$

which we will show is an isomorphism. There are elements $\alpha_i \in H^*(BZ; R)$ which restrict to a basis of $H^*(K/Z; R)$ by [2, §4.2]. By the Leray–Hirsch theorem we see $H^*(BZ; R)$ is generated by $\{\alpha_i\}$ over $H^*(BK; R)$ and $H^*(F_f(Z); R)$ is generated by $\{\tilde{f}^*\alpha_i\}$ over $H^*(X; R)$. Since φ is a ring homomorphism which is $H^*(X; R)$-module isomorphism, it is a ring isomorphism. $\qquad\square$

Example 2.3. Let $K = U(3)$ and $W \cong \mathfrak{S}_3 \cong \langle s_1, s_2 \rangle$. Here we may regard s_1 as the permutation matrix of the 1st and the 2nd coordinates and s_2 as that of the 2nd and the 3rd coordinates. Suppose that Z is the centralizer subgroup of the circle subgroup $\{\operatorname{diag}(t, 1, 1) \mid t \in S^1\} \subset U(1)^3$. Note that the Lie algebra of this circle subgroup is

fixed by the permutation s_2, and Z is isomorphic to the block diagonal matrices $U(1) \times U(2)$, thus, we have $K/Z \cong \mathbb{C}P^2$, and

$$H^*(BK; \mathbb{Z}) \cong \mathbb{Z}[x_1, x_2, x_3]^{\mathfrak{S}_3}, \quad H^*(BZ; \mathbb{Z}) \cong \mathbb{Z}[x_1, x_2, x_3]^{W(Z)},$$

where $W(Z) = \langle s_2 \rangle \subset \mathfrak{S}_3$. Let $f \colon X \to BU(3)$ be a map and $E \to X$ be the complex vector bundle of rank 3 classified by f. The induced projective bundle $\mathbb{P}(E)$ coincides with $F_f(Z)$. Hence, by applying Proposition 2.2, we can compute its cohomology as follows:

$$\begin{aligned} H^*(\mathbb{P}(E); \mathbb{Z}) &\cong H^*(X; \mathbb{Z}) \otimes_{H^*(BK; \mathbb{Z})} H^*(BZ; \mathbb{Z}) \\ &\cong H^*(X; \mathbb{Z}) \otimes_{\mathbb{Z}[x_1, x_2, x_3]^{\mathfrak{S}_3}} \mathbb{Z}[x_1, x_2, x_3]^{\langle s_2 \rangle} \\ &\cong (H^*(X; \mathbb{Z}) \otimes \mathbb{Z}[x_1, x_2 + x_3, x_2 x_3])/I, \end{aligned}$$

where $I = \langle x_1 + x_2 + x_3 - c_1(E), x_1 x_2 + x_2 x_3 + x_3 x_1 - c_2(E), x_1 x_2 x_3 - c_3(E) \rangle$. Observe the following relations given by the ideal I:

$$x_2 + x_3 = c_1(E) - x_1,$$
$$x_2 x_3 = c_2(E) - x_1(x_2 + x_3) = c_2(E) - x_1(c_1(E) - x_1) = c_2(E) - x_1 c_1(E) + x_1^2,$$
$$x_1 x_2 x_3 = c_3(E).$$

Denoting x_1 by x and eliminating x_2 and x_3, we get the following formula:

$$H^*(\mathbb{P}(E); \mathbb{Z}) \cong H^*(X; \mathbb{Z})[x]/\langle x^3 - x^2 c_1(E) + x c_2(E) - c_3(E) \rangle.$$

Similarly, for any complex vector bundle $E \to X$ of rank $n + 1$, we have that:

$$H^*(\mathbb{P}(E); \mathbb{Z}) \cong H^*(X; \mathbb{Z})[x] \Big/ \Big\langle \sum_{k=1}^{n+1} (-1)^k x^{n+1-k} c_k(E) \Big\rangle,$$

which recovers the well-known Borel–Hirzebruch formula (see [**3**, Chapter V, §15]).

As a corollary of Proposition 2.2, we obtain a quick proof of [**4**, Proposition 21.17 and Remarks 21.18, 21.19].

Corollary 2.4. *Let $\mathcal{F}\ell(E) \to X$ be the full flag bundle associated to an $(n+1)$-dimensional complex vector bundle. Then, we have that*

$$H^*(\mathcal{F}\ell(E); \mathbb{Z}) \cong H^*(X; \mathbb{Z})[x_1, \ldots, x_{n+1}] \Big/ \Big\langle \prod_{k=1}^{n+1} (1 + x_k) - c(E) \Big\rangle.$$

Proof. The flag bundle $\mathcal{F}\ell(E) \to X$ with fiber $\mathcal{F}\ell(n+1) := \mathcal{F}\ell(\mathbb{C}^{n+1})$ fits into the following diagram:

$$\begin{array}{ccc} \mathcal{F}\ell(n+1) & \!\!=\!\!\!=\!\!\!=\!\! & \mathcal{F}\ell(n+1) \\ \downarrow & & \downarrow \\ \mathcal{F}\ell(E) & \longrightarrow & BT \\ \downarrow & & \downarrow \\ X & \xrightarrow{\ f\ } & BU(n+1) \end{array}$$

where f is the classifying map of the vector bundle $E \to X$ and T is a maximal torus

in $U(n+1)$. Applying Proposition 2.2, we have that

$$H^*(\mathcal{F}\ell(E); \mathbb{Z}) \cong H^*(X; \mathbb{Z}) \otimes_{H^*(BU(n+1);\mathbb{Z})} H^*(BT; \mathbb{Z}).$$

We identify $H^*(BT; \mathbb{Z}) \cong \mathbb{Z}[x_1, x_2, \ldots, x_{n+1}]$ and $H^*(BU(n+1); \mathbb{Z}) \cong \mathbb{Z}[c_1, c_2, \ldots, c_{n+1}]$, where c_i is the ith elementary symmetric polynomial in $x_1, \ldots, x_{n+1}$ for any i. The assertion follows from the fact $f^*(c_i) = c_i(E)$. $\qquad\square$

3. Flag Bott manifold of general Lie type

In this section, we introduce the main object of our study, *flag Bott towers of general Lie type*. We give two closely related definitions of a flag Bott tower of general Lie type, and prove that they are equivalent when relevant groups are simply-connected. For $1 \leqslant j \leqslant m$, let K_j be a compact connected Lie group, $T_j \subset K_j$ be a maximal torus, and $Z_j \subset K_j$ be the centralizer of a circle subgroup of T_j.

Definition 3.1. An *m-stage flag Bott tower* $F_\bullet = \{F_j \mid 0 \leqslant j \leqslant m\}$ *of general Lie type* (or *an m-stage flag Bott tower*) *associated to* $(K_\bullet, Z_\bullet) = \{(K_j, Z_j) \mid 1 \leqslant j \leqslant m\}$ is defined recursively as follows:

1. F_0 is a point.
2. F_j is the flag bundle over F_{j-1} with fiber K_j/Z_j associated to a map

$$f_j \colon F_{j-1} \longrightarrow BK_j,$$

where f_j factors through BT_j.

The requirement for f_j to factor through BT_j means that we consider those bundles which are the sum of line bundles.

Two flag Bott towers $F_\bullet$ and $F_\bullet'$ are *isomorphic* if there is a collection of diffeomorphisms $F_j \to F_j'$ which commute with the projections $p_j \colon F_j \to F_{j-1}$ and $p_j' \colon F_j' \to F_{j-1}'$ for all $1 \leqslant j \leqslant m$.

Note that the fiber of each stage is a flag manifold, which admits a cell decomposition involving only even dimensional cells. Moreover, the total space E of a fiber bundle $F \hookrightarrow E \to B$ has the structure of a CW-complex whose cells are the product of those in the base B and the fiber F (see, for instance, [**25**, p. 105]). More precisely, for the fiber bundle $F \hookrightarrow E \to B$, the total space E can be decomposed into the pull backs of all cells of B. The pull-back of a cell $c \subset B$ is homeomorphic to the product $F \times c$ (because c is contractible), and $F \times c$ is also decomposed into the product of cells of F and c. Because the product of two disks is homeomorphic to the disk, this gives a cell decomposition of the total space E. Therefore, a flag Bott manifold F_j admits a cell decomposition involving only even dimensional cells as well, and in particular, it is simply-connected.

Example 3.2. Let $(K_j, Z_j) = (U(n_j + 1), T^{n_j+1})$, where T^{n_j+1} is a maximal torus of $U(n_j + 1)$. We have $K_j/Z_j \cong \mathcal{F}\ell(n_j + 1)$ for each j, and we get an *m-stage flag Bott tower* which is introduced in [**23**, Definition 2.1]. An m-stage flag Bott tower $F_\bullet = \{F_j \mid 0 \leqslant j \leqslant m\}$ is defined to be an iterated bundle

$$F_m \longrightarrow F_{m-1} \longrightarrow \cdots \longrightarrow F_1 \longrightarrow F_0 = \{\text{a point}\},$$

of manifolds $F_j = \mathcal{F}\ell\left(\bigoplus_{k=1}^{n_j+1} \xi_k^{(j)}\right)$ where $\xi_k^{(j)}$ is a complex line bundle over F_{j-1} for

each $1 \leqslant k \leqslant n_j + 1$ and $1 \leqslant j \leqslant m$. Since the flag Bott tower in Definition 3.1 is the generalization of flag Bott tower in [23], we call the latter a *full flag Bott tower of type A* in this paper whenever we need to specify them.

Example 3.3. Let $(K_j, Z_j) = (U(n_j + 1), U(n_j) \times U(1))$. We have $K_j/Z_j \cong \mathbb{C}P^{n_j}$ for each j, and we get an *m-stage generalized Bott tower* which is defined in [9, 10] to be an iterated bundle

$$B_m \longrightarrow B_{m-1} \longrightarrow \cdots \longrightarrow B_1 \longrightarrow B_0 = \{\text{a point}\},$$

of manifolds $B_j = \mathbb{P}\left(\bigoplus_{k=1}^{n_j+1} \xi_k^{(j)}\right)$ where $\xi_k^{(j)}$ is a complex line bundle over B_{j-1} for each $1 \leqslant k \leqslant n_j + 1$ and $1 \leqslant j \leqslant m$.

Remark 3.4. Iterated flag bundles have been studied well in the literature. For instance, in [19], the author studies the cohomology rings of iterated flag bundles associated to vector bundles which do not necessarily split into line bundles. He called them *Bott towers of flag manifolds*. A Bott tower of flag manifolds admits a torus action if the base space admits one and the vector bundle is equivariant. However, the essential point of our construction is to restrict to vector bundles which are sums of line bundles. In this case, we obtain two views of our flag Bott manifold of general Lie type (in this section §3) and can introduce the bigger torus action (§4), which are non-trivial.

We now give the second definition of flag Bott tower of general Lie type in the form of an orbit space similarly to the full flag Bott tower of type A case (see [23, §2.2]) as follows.

Definition 3.5. Let $(K_\bullet, Z_\bullet) = \{(K_j, Z_j) \mid 1 \leqslant j \leqslant m\}$. Given a family of homomorphisms $\{\varphi_j^{(\ell)} \colon Z_j \to T_\ell \mid 1 \leqslant j < \ell \leqslant m\}$, the space F_m^φ is defined as the orbit space

$$F_m^\varphi := (K_1 \times \cdots \times K_m)/(Z_1 \times \cdots \times Z_m),$$

where $(z_1, \ldots, z_m) \in Z_1 \times \cdots \times Z_m$ acts on $(g_1, \ldots, g_m) \in K_1 \times \cdots \times K_m$ from the right by

$$(g_1, \ldots, g_m) \cdot (z_1, \ldots, z_m) :=$$
$$\left(g_1 z_1, \varphi_1^{(2)}(z_1)^{-1} g_2 z_2, \prod_{j=1}^{2} \varphi_j^{(3)}(z_j)^{-1} g_3 z_3, \ldots, \prod_{j=1}^{m-1} \varphi_j^{(m)}(z_j)^{-1} g_m z_m \right). \quad (3.1)$$

This action is easily seen to be free, and hence, F_m^φ is a smooth manifold. Moreover, the space F_m^φ has the structure of K_m/Z_m-fiber bundle over F_{m-1}^φ whose classifying map f_m is given by the composition

$$F_{m-1}^\varphi \longrightarrow B\left(\prod_{j=1}^{m-1} Z_j\right) \xrightarrow{B(\prod_{j=1}^{m-1} \varphi_j^{(m)})} BT_m \xrightarrow{B\iota} BK_m.$$

Here the first map is the classifying map of the principal $\prod_{j=1}^{m-1} Z_j$-bundle

$$\prod_{j=1}^{m-1} Z_j \hookrightarrow \prod_{j=1}^{m-1} K_j \longrightarrow F_{m-1}^\varphi,$$

and $\iota \colon T_m \to K_m$ is the inclusion. Therefore, $F_\bullet^\varphi := \{F_j^\varphi \mid 0 \leqslant j \leqslant m\}$ is an m-stage flag Bott tower of general Lie type associated to $(K_\bullet, Z_\bullet)$.

Conversely, when K_j are simply-connected for all $1 \leqslant j \leqslant m$, we claim that every flag Bott tower of general Lie type associated to $(K_\bullet, Z_\bullet)$ can be described as the orbit space as in Definition 3.5, which implies that two definitions of flag Bott tower are equivalent.

Proposition 3.6. *Let* $F_\bullet = \{F_j \mid 0 \leqslant j \leqslant m\}$ *be an* m-*stage flag Bott tower of general Lie type associated to* $(K_\bullet, Z_\bullet)$, *where* K_j *are simply-connected for all* $1 \leqslant j \leqslant m$. *Then there exists a family of homomorphisms* $\{\varphi_j^{(\ell)} \colon Z_j \to T_\ell \mid 1 \leqslant j < \ell \leqslant m\}$ *such that* $F_\bullet$ *and* $F_\bullet^\varphi$ *are isomorphic as flag Bott towers.*

Proof. We show by induction. Assume that a flag bundle F over F_{m-1}^φ is associated to the classifying map

$$f_m \colon F_{m-1}^\varphi \longrightarrow BK_m,$$

with fiber K_m/Z_m, and also assume that f_m factors through BT_m, i.e. there exist $\overline{f_m} \colon F_{m-1}^\varphi \to BT_m$ such that the following diagram commutes.

$$
\begin{array}{ccc}
F_{m-1}^\varphi & \xrightarrow{\ f_m\ } & BK_m \\
& \searrow{\scriptstyle \overline{f_m}} \quad \nearrow{\scriptstyle B\iota} & \\
& BT_m &
\end{array}
$$

From the construction of F_{m-1}^φ, we have a principal $\prod_{j=1}^{m-1} Z_j$ bundle. Denote its classifying map by u so that we have the pull-back of the universal $\prod_{j=1}^{m-1} Z_j$ bundle as follows:

$$
\begin{array}{ccc}
\prod_{j=1}^{m-1} Z_j & = & \prod_{j=1}^{m-1} Z_j \\
\downarrow & & \downarrow \\
\prod_{j=1}^{m-1} K_j & \longrightarrow & E \\
\downarrow & & \downarrow \\
F_{m-1}^\varphi & \xrightarrow{\ u\ } & B(\prod_{j=1}^{m-1} Z_j)
\end{array}
$$

where $E = E(\prod_{j=1}^{m-1} K_j)$ can also serve as the universal space for any subgroup of $\prod_{j=1}^m K_j$.

Since $E \to B(\prod_{j=1}^{m-1} Z_j)$ is a fibration, the bottom pull-back square is a homotopy pull-back at the same time. Thus, the bottom square can be understood as restricting u on a contractible space E so that the homotopy fiber of u is $\prod_{j=1}^{m-1} K_j$. That is, we have the following homotopy fibration

$$\prod_{j=1}^{m-1} K_j \longrightarrow F_{m-1}^\varphi \xrightarrow{\ u\ } B\left(\prod_{j=1}^{m-1} Z_j\right). \tag{3.2}$$

Note that each K_j is compact and simply-connected, so that homotopy groups $\pi_1(K_j)$ and $\pi_2(K_j)$ are trivial for $1 \leqslant j \leqslant m$ (see [**5**, Proposition 7.5 in Chapter V]). Hence $H^1(K_j) = H^2(K_j) = 0$ by the Hurewicz isomorphism theorem. Since Z_j is connected, BZ_j is simply-connected for all $1 \leqslant j \leqslant m$. Recall that F_{m-1}^φ admits the

structure of a CW complex with even dimensional cells only. Combining these facts with the Serre spectral sequence with respect to the fibration (3.2) we see that

$$u^*\colon H^2\left(B\left(\prod_{j=1}^{m-1} Z_j\right);\mathbb{Z}\right) \longrightarrow H^2(F_{m-1}^\varphi;\mathbb{Z})$$

is an isomorphism.

Using the identification $[X, BS^1] \cong H^2(X;\mathbb{Z})$ for a topological space X which has the homotopy type of a CW complex (see [**18**, Proposition 3.10]), we see that $\overline{f_m}$ factors up to homotopy as follows:

$$
\begin{array}{ccc}
F_{m-1}^\varphi & \xrightarrow{\;\overline{f_m}\;} & BT_m \\
& \searrow^{u} \qquad \nearrow^{\psi} & \\
& B(\prod_{j=1}^{m-1} Z_j) &
\end{array}
\tag{3.3}
$$

Furthermore, we obtain $\varphi \in \mathrm{Hom}\left(\prod_{j=1}^{m-1} Z_j, T_m\right)$ such that $B\varphi = \psi$ through the following bijection:

$$
\begin{aligned}
\mathrm{Hom}(Z_j, S^1) &\cong \mathrm{Hom}(\pi_1(Z_j), \mathbb{Z}) \quad \text{(see, for example, [\textbf{14}, Proposition 9.4])} \\
&\cong \mathrm{Hom}(\pi_2(BZ_j), \mathbb{Z}) \\
&\cong \mathrm{Hom}(H_2(BZ_j);\mathbb{Z}) \quad \text{(by the Hurewicz isomorphism theorem)} \\
&\cong H^2(BZ_j;\mathbb{Z}) \quad \text{(by the universal coefficient theorem)}.
\end{aligned}
$$

Consider the isomorphism

$$
\mathrm{Hom}\left(\prod_{j=1}^{m-1} Z_j, T_m\right) \xrightarrow{\;\cong\;} \left(\prod_{j=1}^{m-1} \mathrm{Hom}(Z_j, T_m)\right),
\tag{3.4}
$$

and denote the image of φ under the map (3.4) by

$$\varphi_j^{(m)}\colon Z_j \longrightarrow T_m, \quad \text{for } 1 \leqslant j \leqslant m-1.$$

By Definition 3.5, this sequence $(\varphi_j^{(m)})_{1 \leqslant j \leqslant m-1}$ defines a bundle F_m^φ over F_{m-1}^φ, which we will show is isomorphic to F. In fact, F has the classifying map

$$f_m\colon F_{m-1}^\varphi \xrightarrow{\;\overline{f_m}\;} BT_m \xrightarrow{\;B\iota\;} BK_m,$$

and F_m^φ has the classifying map

$$F_{m-1}^\varphi \xrightarrow{\;u\;} B\left(\prod_{j=1}^{m-1} Z_j\right) \xrightarrow{\;B\varphi\;} BT_m \xrightarrow{\;B\iota\;} BK_m.$$

Since $\overline{f_m}$ is homotopic to $B\varphi \circ u$ by (3.3), they define the same bundle (see [**12**, Theorem 2.1]). □

Remark 3.7. Assume that all Z_j's are maximal tori. If we fix the isomorphisms $Z_j \simeq (S^1)^{n_j+1}$, then $\varphi_j^{(\ell)}\colon Z_j \to T_\ell$ can be represented by $(n_\ell + 1) \times (n_j + 1)$ matrices $A_j^{(\ell)} \in M_{n_\ell+1, n_j+1}(\mathbb{Z})$ with integer entries through the obvious identification. For

example, for $(K_j, Z_j) = (U(n_j + 1), T_j)$ we have

$$\varphi_j^{(\ell)}(\mathrm{diag}(e^{\sqrt{-1}t_1}, \ldots, e^{\sqrt{-1}t_{n_j+1}}))$$
$$= \mathrm{diag}(e^{\sqrt{-1}\sum_{h=1}^{n_j+1} A_j^{(\ell)}(1,h)t_h}, \ldots, e^{\sqrt{-1}\sum_{h=1}^{n_j+1} A_j^{(\ell)}(n_\ell+1,h)t_h}),$$

as was introduced in [23], where $A_j^{(\ell)}(k,h)$ is the (k,h)-entry of $A_j^{(\ell)}$ for $1 \leqslant k \leqslant n_\ell + 1$ and $1 \leqslant h \leqslant n_j + 1$.

Remark 3.8. The simply-connectedness assumption on K_j in Proposition 3.6 can be weakened. The assumption is used only to assert u^* is an isomorphism. When u^* is surjective, we can choose a map ψ which makes (3.3) commutative, and hence, homomorphisms $\varphi_j^{(\ell)} \colon Z_j \to T_\ell$ for $1 \leqslant j < \ell \leqslant m$ so that $F_\bullet^\varphi$ is isomorphic to $F_\bullet$ as flag Bott towers. In fact, in [23, Proposition 2.11] the case of full flag Bott manifolds of type A when $(K_j, Z_j) = (U(n_j + 1), T_j)$ is considered and a particular choice for ψ is made.

We now have two descriptions which are equivalent when all K_j are simply-connected of a flag Bott tower of general Lie type. The latter description encodes not only the iterated bundle structure but also an action of a torus as we will see in the next section.

4. Equivariant cohomology rings of flag Bott manifolds of general Lie type

For a topological space X with an action of a topological group G, its equivariant cohomology $H_G^*(X)$ is defined to be the singular cohomology $H^*(X_G)$ of the Borel construction $X_G = EG \times_G X$ of X. Here EG is a contractible space on which G acts freely, and $EG \times_G X = EG \times X / \sim$ where $(h, x) \sim (gh, gx)$ for any $(h, x) \in EG \times X$ and $g \in G$. In this section, we define an action of a torus $\mathbb{T}$ on the flag Bott manifold F_m^φ of general Lie type and compute the equivariant cohomology $H_{\mathbb{T}}^*(F_m^\varphi)$.

For $1 \leqslant j \leqslant m$, let K_j be a compact connected Lie group, $T_j \subset K_j$ be a maximal torus, and $Z_j \subset K_j$ be the centralizer of a circle subgroup of T_j. Consider an m-stage flag Bott tower $\{F_j^\varphi \mid 0 \leqslant j \leqslant m\}$ determined by a family of homomorphisms $\{\varphi_j^{(\ell)} \colon Z_j \to T_\ell \mid 1 \leqslant j < \ell \leqslant m\}$. We define a torus action on F_m^φ as follows:

Definition 4.1. Let $\mathbb{T} = \prod_{j=1}^m T_j$. We have a well-defined action of $\mathbb{T}$ on F_m^φ given by

$$(t_1, \ldots, t_m) \cdot [g_1, \ldots, g_m] = [t_1 g_1, \ldots, t_m g_m],$$

where $(t_1, \ldots, t_m) \in \mathbb{T}$ and $[g_1, \ldots, g_m] \in F_m^\varphi$. The well-definedness can be seen from the fact that the images of $\varphi_j^{(\ell)}$ lie in the commutative groups T_ℓ.

Let us compute the equivariant cohomology ring of F_m^φ with respect to the action of $\mathbb{T}$. Let $\prod_{j=1}^{m-1} T_j$ act on $\prod_{j=1}^{m-1} K_j$ by the left multiplication. The action of $\prod_{j=1}^{m-1} T_j$ on $\prod_{j=1}^{m-1} K_j$ commutes with that of $\prod_{j=1}^{m-1} Z_j$ given by $\{\varphi_j^{(\ell)}\}$ as in (3.1), so we have

the principal $\prod_{j=1}^{m-1} Z_j$-bundle

$$\prod_{j=1}^{m-1} Z_j \longrightarrow E \times_{\prod_{j=1}^{m-1} T_j} \prod_{j=1}^{m-1} K_j \longrightarrow E \times_{\prod_{j=1}^{m-1} T_j} F_{m-1}^\varphi.$$

We denote its classifying map by

$$\overline{u} \colon E \times_{\prod_{j=1}^{m-1} T_j} F_{m-1}^\varphi \longrightarrow B\left(\prod_{j=1}^{m-1} Z_j\right).$$

Since F_m^φ can be identified with the associated flag bundle of $f_m \colon F_{m-1}^\varphi \to BK_m$ with fiber K_m/Z_m, its Borel construction with respect to the $\mathbb{T}$ action fits into the following pull-back diagram

$$
\begin{array}{ccc}
K_m/Z_m & = = = & K_m/Z_m \\
\downarrow & & \downarrow \\
E\mathbb{T} \times_{\mathbb{T}} F_m^\varphi & \longrightarrow & BZ_m \\
\downarrow & & \downarrow \\
E\mathbb{T} \times_{\mathbb{T}} F_{m-1}^\varphi & \xrightarrow{\widetilde{f}_m} & BK_m
\end{array}
\tag{4.1}
$$

where the factor T_m in $\mathbb{T}$ acts trivially on F_{m-1}^φ. Here, $\widetilde{f}_m \colon E\mathbb{T} \times_{\mathbb{T}} F_{m-1}^\varphi \to BK_m$ is the following composition:

$$E\mathbb{T} \times_{\mathbb{T}} F_{m-1}^\varphi \cong BT_m \times E\left(\prod_{j=1}^{m-1} T_j\right) \times_{\prod_{j=1}^{m-1} T_j} F_{m-1}^\varphi \xrightarrow{1 \times \overline{u}}$$

$$B\left(T_m \times \prod_{j=1}^{m-1} Z_j\right) \xrightarrow{B(\mathrm{mul}) \circ B(1 \times \varphi)} BT_m \xrightarrow{B\iota} BK_m,$$

$$\tag{4.2}$$

where mul is the multiplication of T_m and $\varphi := \prod_{j=1}^{m-1} \varphi_j^{(m)}$.

By applying Proposition 2.2 to (4.1) we have the following result:

Theorem 4.2. *Let K_j be a compact connected Lie group, $T_j \subset K_j$ a maximal torus with dimension n_j, and $Z_j \subset K_j$ the centralizer of a circle subgroup of T_j for $1 \leqslant j \leqslant m$. Let $\{F_j^\varphi \mid 0 \leqslant j \leqslant m\}$ be an m-stage flag Bott tower determined by a family of homomorphisms $\{\varphi_j^{(\ell)} \colon Z_j \to T_\ell \mid 1 \leqslant j < \ell \leqslant m\}$. Let R be a PID in which torsion primes of all K_j are invertible. Then the equivariant cohomology ring of F_m^φ with respect to the action of $\mathbb{T}$ defined by Definition 4.1 is*

$$H_{\mathbb{T}}^*(F_m^\varphi; R) \simeq H_{\mathbb{T}}^*(F_{m-1}^\varphi; R) \otimes_{H^*(BK_m;R)} H^*(BZ_m; R),$$

where the $H^*(BK_m; R)$-module structure on $H_{\mathbb{T}}^*(F_{m-1}^\varphi; R)$ is induced by $\widetilde{f}_m$.

We use the following well-known identification to get an explicit formula for $H_{\mathbb{T}}^*(F_m^\varphi; R)$:

$$H^*(BT_m; R) \simeq R[u_{m,k} \mid 1 \leqslant k \leqslant n_m] \stackrel{\text{let}}{=} R[\mathbf{u}_m],$$

$$H^*(BK_m; R) \simeq R[y_{m,k} \mid 1 \leqslant k \leqslant n_m]^{W(K_m)} \overset{\text{let}}{=} R[\mathbf{y}_m]^{W(K_m)},$$

$$H^*(BZ_m; R) \simeq R[y_{m,k} \mid 1 \leqslant k \leqslant n_m]^{W(Z_m)} \overset{\text{let}}{=} R[\mathbf{y}_m]^{W(Z_m)},$$

where $u_{m,k}$ and $y_{m,k}$ are all degree 2 elements for all m, k. Here, we use the fact that the Weyl group $W(Z_m)$ of Z_m may be regarded as the subgroup of the Weyl group $W(K_m)$ of K_m; therefore, the invariant polynomial ring $H^*(BK_m; R) \simeq H^*(BT_m; R)^{W(K_m)}$ is a subring of the invariant polynomial ring $H^*(BZ_m; R) \simeq H^*(BT_m; R)^{W(Z_m)}$ (see [**1**]). We also note that we use the symbol $\mathbf{u}_m$ as the generators defined from the acting torus T_m and the symbol $\mathbf{y}_m$ as the generators defined from the maximal torus $T_m \subset Z_m \subset K_m$.

Since T_m acts trivially on F^{φ}_{m-1}, we have that

$$
\begin{aligned}
H^*_{\mathbb{T}}(F^{\varphi}_m; R) &\simeq H^*_{\mathbb{T}}(F^{\varphi}_{m-1}; R) \otimes_{H^*(BK_m; R)} H^*(BZ_m; R) \\
&\simeq H^*_{\mathbb{T}}(F^{\varphi}_{m-1}; R) \otimes_{R[\mathbf{y}_m]^{W(K_m)}} R[\mathbf{y}_m]^{W(Z_m)} \\
&\simeq \left(H^*_{\mathbb{T}'}(F^{\varphi}_{m-1}; R) \otimes H^*(BT_m) \right) \otimes_{R[\mathbf{y}_m]^{W(K_m)}} R[\mathbf{y}_m]^{W(Z_m)} \\
&\simeq \left(H^*_{\mathbb{T}'}(F^{\varphi}_{m-1}; R) \otimes R[\mathbf{u}_m] \right) \otimes_{R[\mathbf{y}_m]^{W(K_m)}} R[\mathbf{y}_m]^{W(Z_m)},
\end{aligned}
$$

where $\mathbb{T}' = \prod_{j=1}^{m-1} T_j$. It is easy to check that the image of $y_{m,k} \in H^*(BT_m)$ ($k = 1, \dots, n_m$) by the induced homomorphism of the composition maps (4.2) can be written as

$$u_{m,k} + \sum_{j=1}^{m-1} (\varphi_j^{(m)})^*(y_{m,k}),$$

where $(\varphi_j^{(m)})^* \colon H^*(BT_m) \to H^*_{\mathbb{T}'}(F^{\varphi}_{m-1})$ is the induced homomorphism of $B\varphi_j^{(m)} \circ \overline{u}$. Using the inductive application of the above argument, we have the following explicit formula:

Corollary 4.3. *Let $\mathbf{u}_j$ and $\mathbf{y}_j$ stand for $(u_{j,1}, \dots, u_{j,n_j})$ and $(y_{j,1}, \dots, y_{j,n_j})$, respectively for $j = 1, \dots, m$. Then, we have*

$$H^*_{\mathbb{T}}(F^{\varphi}_m; R) \simeq R[\mathbf{u}_1, \dots, \mathbf{u}_m] \otimes_R (R[\mathbf{y}_1, \dots, \mathbf{y}_m])^{W(Z)} / \langle I_1, \dots, I_m \rangle,$$

where $W(Z) = \prod_{j=1}^{m} W(Z_j)$, and I_ℓ, for $1 \leqslant \ell \leqslant m$, is the ideal generated by the polynomials

$$h(\mathbf{y}_\ell) - h\left(\mathbf{u}_\ell + \sum_{j=1}^{\ell-1} \Phi_j^{(\ell)}(\mathbf{y}_\ell) \right),$$

for $h \in R[\mathbf{y}_\ell]^{W(K_\ell)}$ and

$$\Phi_j^{(\ell)}(\mathbf{y}_\ell) := \left((\varphi_j^{(\ell)})^*(y_{\ell,1}), \dots, (\varphi_j^{(\ell)})^*(y_{\ell,n_\ell}) \right).$$

Corollary 4.4. *Suppose that $F_\bullet = \{ F_j \mid 0 \leqslant j \leqslant m \}$ is an m-stage flag Bott manifold of type A defined by a set of integer matrices $\{ A_j^{(\ell)} \in M_{(n_\ell+1) \times (n_j+1)}(\mathbb{Z}) \mid 1 \leqslant j < \ell \leqslant m \}$ as in Remark 3.7. Then, we have that*

$$H^*_{\mathbb{T}}(F_m; \mathbb{Z}) \cong \mathbb{Z}[\mathbf{u}_j, \mathbf{y}_j \mid 1 \leqslant j \leqslant m] / \langle I'_1, \dots, I'_m \rangle,$$

where I'_ℓ is the ideal generated by the polynomials

$$c_k(\mathbf{y}_\ell) - c_k\left(\mathbf{u}_\ell + \sum_{j=1}^{\ell-1} \Phi_j^{(\ell)}(\mathbf{y}_\ell)\right).$$

Here c_k are the kth elementary symmetric polynomials for $1 \leqslant k \leqslant n_\ell + 1$. Since the homomorphism $\varphi_j^{(\ell)}\colon T_j \to T_\ell$ is determined by the matrix $A_j^{(\ell)}\colon H_2(BT_j) = \mathbb{Z}^{n_j+1} \to H_2(BT_\ell) = \mathbb{Z}^{n_\ell+1}$, by letting $A_j^{(\ell)} = (A_j^{(\ell)}(k,h)) \in M_{(n_\ell+1)\times(n_j+1)}(\mathbb{Z})$ for $1 \leqslant k \leqslant n_\ell + 1$ and $1 \leqslant h \leqslant n_j + 1$, the kth entry of $\sum_{j=1}^{\ell-1} \Phi_j^{(\ell)}(\mathbf{y}_\ell)$, $1 \leqslant k \leqslant n_\ell + 1$, is

$$\sum_{j=1}^{\ell-1}\left(\sum_{h=1}^{n_j+1} A_j^{(\ell)}(k,h) \cdot y_{j,h}\right).$$

The next two remarks show that (4.1) may define different torus actions on isomorphic $F_\bullet^\varphi$'s depending on the defining data $(K_\bullet, Z_\bullet)$ and $\{\varphi_j^{(\ell)}\}$.

Remark 4.5. Let $(K_1, Z_1) = (SU(2), T_{SU(2)})$. The corresponding flag Bott manifold of general Lie type is $SU(2)/T_{SU(2)} \simeq \mathbb{C}P^1$. Then $S^1 \simeq T_{SU(2)}$ acts on $SU(2)/T_{SU(2)}$ by the left multiplication (see Definition 4.1). This action is not effective but $\mathrm{diag}(-1,-1) \in T_{SU(2)}$ acts trivially. By Corollary 4.3, the equivariant cohomology of $SU(2)/T_{SU(2)}$ with this action is

$$H^*_{T_{SU(2)}}(SU(2)/T_{SU(2)}; \mathbb{Z}) \simeq \mathbb{Z}[u_{1,1}, y_{1,1}]/I,$$

where I is the ideal generated by

$$u_{1,1}^2 - y_{1,1}^2$$

because $W(SU(2)) \simeq \mathbb{Z}_2$ acts on $H^*(BS^1) \simeq \mathbb{Z}[x]$ by $x \mapsto -x$. Hence, we have that

$$H^*_{T_{SU(2)}}(SU(2)/T_{SU(2)}; \mathbb{Z}) \simeq \mathbb{Z}[u_{1,1}, y_{1,1}]/\langle u_{1,1}^2 - y_{1,1}^2\rangle. \tag{4.3}$$

On the other hand, for $(K_1, Z_1) = (SO(3), T_{SO(3)})$ the corresponding flag Bott manifold of general Lie type is again $SO(3)/T_{SO(3)} \simeq \mathbb{C}P^1$. This time, $S^1 \simeq T_{SO(3)}$ acts effectively on $SO(3)/T_{SO(3)}$ by the left multiplication. Theorem 4.2 is not applicable for $R = \mathbb{Z}$ coefficients since two is the torsion prime of $SO(3)$. However, the standard argument shows

$$H^*_{T_{SO(3)}}(SO(3)/T_{SO(3)}; \mathbb{Z}) \simeq \mathbb{Z}[u, v]/\langle uv\rangle. \tag{4.4}$$

One can see that rings (4.3) and (4.4) are not isomorphic, and this is because they represent the equivariant cohomology rings with different S^1-actions.

Remark 4.6. Consider $(K_\bullet, Z_\bullet) = \{(SU(2), T_1), (SU(2), T_2)\}$. Since $SU(2)/T \simeq \mathbb{C}P^1$ and $\pi_2(BSU(2))$ is trivial, $F_2^\varphi \simeq \mathbb{C}P^1 \times \mathbb{C}P^1$ for any $\varphi_2^{(1)}\colon T_1 \to T_2$. On the other hand, we can see the torus actions defined by (4.1) are distinct for different $\varphi_2^{(1)}$. In this sense, Definition 3.5 encodes more structures (torus equivariant structures) than Definition 3.1.

Remark 4.7. For an m-stage flag Bott manifold F_m associated to $\{(U(n_j + 1), U(1)^{n_j+1}) \mid j = 1, \ldots, m\}$ (see Example 3.2), the torus $\mathbb{T}$ does not act effectively

on F_m. If we write $t_j = \text{diag}(t_{j,1}, \ldots, t_{j,n_j+1}) \in U(1)^{n_j+1} = T_j \subset U(n_j+1)$, the sub-torus

$$\mathbf{T} := \{(t_1, \ldots, t_m) \in \mathbb{T} \mid t_{1,n_1+1} = \cdots = t_{m,n_m+1} = 1\} \cong (S^1)^{n_1+\cdots+n_m}$$

acts effectively on F_m (see [**23**, §3.1]). Then the equivariant cohomology ring $H_{\mathbf{T}}^*(F_m; \mathbb{Z})$ with respect to the effective torus action of $\mathbf{T}$ is given by

$$H_{\mathbf{T}}^*(F_m; \mathbb{Z}) \cong \mathbb{Z}[\mathbf{u}_j, \mathbf{y}_j \mid 1 \leqslant j \leqslant m]/\langle u_{1,n_1+1}, \ldots, u_{m,n_m+1}, I_1', \ldots, I_m' \rangle,$$

where I_j' are defined in Corollary 4.4. Moreover, by ignoring the generators $\mathbf{u}_j$ we get the singular cohomology ring of F_m:

$$H^*(F_m; \mathbb{Z}) \cong \mathbb{Z}[\mathbf{y}_j \mid 1 \leqslant j \leqslant m]/\langle J_1, \ldots, J_m \rangle,$$

where J_ℓ is the ideal generated by the polynomials

$$c_k(\mathbf{y}_\ell) - c_k \left(\sum_{j=1}^{\ell-1} \Phi_j^{(\ell)}(\mathbf{y}_\ell) \right).$$

Example 4.8. Consider $F_3 := Sp(3) \times_{T^3} Sp(3) \times_{T^3} Sp(2)/T^2$ with $\varphi_j^{(\ell)} \colon T^{n_j} \to T^{n_\ell}$ $((n_1, n_2, n_3) = (3, 3, 2))$ for $1 \leqslant j < \ell \leqslant 3$ determined by the matrices $A_j^{(\ell)}$:

$$A_1^{(2)} = \begin{bmatrix} A_1^{(2)}(1,1) & A_1^{(2)}(1,2) & A_1^{(2)}(1,3) \\ A_1^{(2)}(2,1) & A_1^{(2)}(2,2) & A_1^{(2)}(2,3) \\ A_1^{(2)}(3,1) & A_1^{(2)}(3,2) & A_1^{(2)}(3,3) \end{bmatrix} \in M_{3\times 3}(\mathbb{Z}),$$

$$A_1^{(3)} = \begin{bmatrix} A_1^{(3)}(1,1) & A_1^{(3)}(1,2) & A_1^{(3)}(1,3) \\ A_1^{(3)}(2,1) & A_1^{(3)}(2,2) & A_1^{(3)}(2,3) \end{bmatrix} \in M_{2\times 3}(\mathbb{Z}),$$

$$A_2^{(3)} = \begin{bmatrix} A_2^{(3)}(1,1) & A_2^{(3)}(1,2) & A_2^{(3)}(1,3) \\ A_2^{(3)}(2,1) & A_2^{(3)}(2,2) & A_2^{(3)}(2,3) \end{bmatrix} \in M_{2\times 3}(\mathbb{Z}).$$

Here, we think of $Sp(n)$ as matrices with quaternion entries, and its maximal torus is chosen to be the diagonal matrices whose entries are complex numbers with unit lengths. Then by Corollary 4.3 (or by [**16**, §6.2]), we have that

$$H_{\mathbb{T}}^*(F_3; \mathbb{Z}) \cong \mathbb{Z}[\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3]/\langle I_1, I_2, I_3 \rangle,$$

where $\mathbf{y}_j = (y_{j,1}, \ldots, y_{j,n_j})$, $\mathbf{u}_j = (u_{j,1}, \ldots, u_{j,n_j})$, $y_{j,k}$ and $u_{j,k}$ are degree 2 elements, and

$$I_1 = (1 + y_{1,1}^2)(1 + y_{1,2}^2)(1 + y_{1,3}^2) - (1 + u_{1,1}^2)(1 + u_{1,2}^2)(1 + u_{1,3}^2),$$

$$I_2 = (1 + y_{2,1}^2)(1 + y_{2,2}^2)(1 + y_{2,3}^2) - \prod_{k=1}^{3} \left\{ 1 + \left(u_{2,k} + \left(\sum_{h=1}^{3} A_1^{(2)}(k,h) \cdot y_{1,h} \right) \right)^2 \right\},$$

$$I_3 = (1 + y_{3,1}^2)(1 + y_{3,2}^2) - \prod_{k=1}^{2} \left\{ 1 + \left(u_{3,k} + \sum_{j=1}^{2} \left(\sum_{h=1}^{3} A_j^{(3)}(k,h) \cdot y_{j,h} \right) \right)^2 \right\}.$$

Example 4.9. Consider $F_3 := SU(4) \times_{T^3} Sp(3) \times_{T^3} G_2/T^2$ with $\varphi_j^{(\ell)} \colon T^{n_j} \to T^{n_\ell}$ $((n_1, n_2, n_3) = (3, 3, 2))$ for $1 \leqslant j < \ell \leqslant 3$. Since two is the torsion prime of G_2, we can

apply our theorem with coefficients in $R = \mathbb{F}_3$, i.e., the finite field of order 3. Let the homomorphisms $\{\varphi_j^{(\ell)} \mid 1 \leqslant j < \ell \leqslant 3\}$ be determined by matrices $A_j^{(\ell)} \in M_{n_\ell \times n_j}(\mathbb{Z})$, where we identify $T^2 \subset G_2$ with the set of matrices $\mathrm{diag}(t_1, t_2, t_1^{-1}t_2^{-1}) \in SU(3) \subset G_2 \subset SO(7)$ for $SU(3) = G_2 \cap SO(6)$ following [**21**]. Then, by Corollary 4.3 together with [**21**, p. 300], there are $\mathbf{y}_j = (y_{j,1}, \ldots, y_{j,n_j})$, $\mathbf{u}_j = (u_{j,1}, \ldots, u_{j,n_j})$ for $j = 1, 2, 3$ such that

$$H_{\mathbb{T}}^*(F_3; \mathbb{F}_3) \cong \mathbb{F}_3[\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3]/\langle L_1, L_2, L_3 \rangle,$$

where

$$L_1 = (1 + y_{1,1})(1 + y_{1,2})(1 + y_{1,3}) - (1 + u_{1,1})(1 + u_{1,2})(1 + u_{1,3}),$$

$$L_2 = (1 + y_{2,1}^2)(1 + y_{2,2}^2)(1 + y_{2,3}^2) - \prod_{k=1}^{3} \left\{ 1 + \left(u_{2,k} + \left(\sum_{h=1}^{3} A_1^{(2)}(k, h) \cdot y_{1,h} \right) \right)^2 \right\},$$

$$L_3 = (1 + (y_{3,1} - y_{3,2})^2)(1 + y_{3,1}^2 y_{3,2}^2 (y_{3,1} + y_{3,2})^2)$$
$$- \left(1 + h_4 \left(\mathbf{u}_3 + \Phi_1^{(3)}(\mathbf{y}_3) + \Phi_2^{(3)}(\mathbf{y}_3) \right) \right) \left(1 + h_{12} \left(\mathbf{u}_3 + \Phi_1^{(3)}(\mathbf{y}_3) + \Phi_2^{(3)}(\mathbf{y}_3) \right) \right).$$

In the last relation, we use the notation

$$h_4(x_1, x_2) = (x_1 - x_2)^2,$$
$$h_{12}(x_1, x_2) = x_1^2 x_2^2 (x_1 + x_2)^2,$$

for

$$x_1 = u_{3,1} + \left(\sum_{h=1}^{3} A_1^{(3)}(1, h) \cdot y_{1,h} \right) + \left(\sum_{h=1}^{3} A_2^{(3)}(1, h) \cdot y_{2,h} \right),$$

$$x_2 = u_{3,2} + \left(\sum_{h=1}^{3} A_1^{(3)}(2, h) \cdot y_{1,h} \right) + \left(\sum_{h=1}^{3} A_2^{(3)}(2, h) \cdot y_{2,h} \right).$$

Here, $A_j^{(\ell)}(k, h)$ is the mod 3 reduction of the (k, h)-entry of the matrix $A_j^{(\ell)}$.

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