

ON A CONJECTURE OF MAHOWALD ON THE COHOMOLOGY OF FINITE SUB-HOPF ALGEBRAS OF THE STEENROD ALGEBRA

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Abstract

Mahowald’s conjecture arose as part of a program attempting to view chromatic phenomena in stable homotopy theory through the lens of the classical Adams spectral sequence. The conjecture predicts the existence of nonzero classes in the cohomology of the finite sub-Hopf algebras $A(n)$ of the mod 2 Steenrod algebra that correspond to generators in the homotopy rings of certain periodic spectra. The purpose of this note is to present a proof of the conjecture.

1. Introduction and statement of results

To provide some context for the conjecture, here’s a brief summary of what has been known about how elements that detect periodic phenomena appear in the cohomology of finite Hopf-subalgebras of the Steenrod algebra. For a Hopf-subalgebra B of the mod p Steenrod algebra, we’ll often use the notation $H^*(B)$ as an abbreviation for the cohomology of B , $\text{Ext}_B(\mathbb{F}_p, \mathbb{F}_p)$.

Let $A(n)$ denote the finite sub-Hopf algebra of the Steenrod algebra generated by

$$Sq^0, Sq^1, \dots, Sq^{2^n},$$

if $p = 2$ and by

$$\beta, P^1, \dots, P^{p^{n-1}},$$

if p odd. Let $E(n) = E(Q_0, Q_1, \dots, Q_n)$ denote the subalgebra of $A(n)$ generated by the Milnor generators. Since the Johnson–Wilson spectrum $BP\langle n \rangle$ has $H^*(BP\langle n \rangle; \mathbb{F}_p) = A // E(n)$, we can calculate $\pi_*(BP\langle n \rangle)$ from the classical Adams spectral sequence:

$$\begin{aligned} E_2^{s,t} &= \text{Ext}_A^{s,t}(H^*(BP\langle n \rangle), \mathbb{F}_p) = \text{Ext}_A^{s,t}(A // E(n), \mathbb{F}_p) \cong \text{Ext}_{E(n)}(\mathbb{F}_p, \mathbb{F}_p) \\ &\cong \mathbb{F}_p[q_0, q_1, \dots, q_n]. \end{aligned}$$

We observe that $E_2 = E_\infty$, since the generators are concentrated in even degrees, and

$$\pi_*(BP\langle n \rangle) \cong \mathbb{Z}_{(p)}[v_1, \dots, v_n],$$

where $|v_i| = 2p^i - 2$ (and the generators are in Adams filtration 1). We hereafter

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denote the generators of $\text{Ext}_{E(n)}(\mathbb{F}_p, \mathbb{F}_p)$ by v_i , for $i \geq 0$. The inclusion $i: E(n) \hookrightarrow A(n)$ induces the restriction homomorphism in cohomology

$$i^*: \text{Ext}_{A(n)}(\mathbb{F}_p, \mathbb{F}_p) \rightarrow \text{Ext}_{E(n)}(\mathbb{F}_p, \mathbb{F}_p).$$

When one attempts to understand chromatic phenomena via the classical Adams spectral sequence, odd primes are easier to handle than the $p = 2$ case, as demonstrated by the following result from [13] (although work on the prime 2 case was done first).

Theorem 1.1. *For p any odd prime, there are classes defined and nonzero in $\text{Ext}_{A(n)}(\mathbb{F}_p, \mathbb{F}_p)$ that form a polynomial subalgebra:*

$$\mathbb{F}_p[v_0, v_1^{p^n}, v_2^{p^{n-1}}, \dots, v_n^p] \subset \text{Ext}_{A(n)}(\mathbb{F}_p, \mathbb{F}_p),$$

where the generators restrict to the obvious classes in $H^*(E(n))$.

The proof uses a careful analysis of the Cartan–Eilenberg spectral sequence for the extension of Hopf algebras

$$\mathbb{F}_p \rightarrow P(n) \rightarrow A(n)_* \rightarrow D(n) \rightarrow \mathbb{F}_p,$$

where $P(n)$ is the truncated polynomial algebra on the ξ_i s and $D(n)$ is the exterior algebra on the τ_i s. The spectral sequence collapses for odd primes, as one can see by filtering the dual Steenrod algebra by the number of τ s. This allows one to move directly from understanding the coaction on the cohomology of $D(n)$ to seeing the appropriate classes in $\text{Ext}_{A(n)}(\mathbb{F}_p, \mathbb{F}_p)$.

For the prime 2, the situation is more difficult, but there were partial results available [8]:

Theorem 1.2. *There are classes defined and nonzero in $\text{Ext}_{A(n)}(\mathbb{F}_2, \mathbb{F}_2)$ that form a polynomial subalgebra*

$$\mathbb{F}_2[v_0, v_1^{N_1}, v_2^{N_2}, \dots, v_n^{N_n}] \subset \text{Ext}_{A(n)}(\mathbb{F}_2, \mathbb{F}_2),$$

where the generators restrict to the obvious classes in $H^*(E(n))$.

The proof uses results of Lin [7] and Wilkerson [16] that show that the restriction homomorphism i^* is onto in infinitely many positive degrees. Note that this argument defines these generators $v_i^{N_i}$ only as cosets for $p = 2$ and $1 \leq i \leq n - 1$. The top class was explicitly identified as $v_n^{2^{n+1}}$ and shown to be a non-zero divisor in the cohomology ring $\text{Ext}_{A(n)}(\mathbb{F}_2, \mathbb{F}_2)$, using a spectral sequence based on a Koszul-type resolution of $A(n) // A(n - 1)$. This spectral sequence first appeared in [5] and was referred to as the “Koszul spectral sequence” in [8]. It has recently been rechristened as the Davis–Mahowald spectral sequence by Bruner, Rognes and their coworkers in [3] and [12].

The exponents of the lower v_i s have been more mysterious. Low dimensional calculations led to the following conjecture, the proof of which is the goal of this note.

Conjecture 1.3. *For any natural numbers n and k , the class $v_n^{2^{n+1+k}}$ is defined and nonzero in $\text{Ext}_{A(j)}(\mathbb{F}_2, \mathbb{F}_2)$ for all $j = n, n + 1, \dots, 2n + k$.*

This conjecture is originally due to Mahowald in the late 1970s, although it first appeared in print in [8]. Note that the conjecture predicts that $v_n^{2^{n+1+k}}$ is nonzero in

the cohomology of $A(j)$ for the largest possible j : If $v_n^{2^{n+1+k}}$ is defined and nonzero in $\text{Ext}_{A(j)}(\mathbb{F}_2, \mathbb{F}_2)$ for $j > 2n + k$, then the class would persist to $\text{Ext}_A(\mathbb{F}_2, \mathbb{F}_2)$ by the Adams Approximation Theorem [1]. This class would then be a permanent cycle in the classical Adams spectral sequence for the homotopy ring of the (2-complete) sphere, persisting to a nonnilpotent element in $\pi_0((S^0)^\wedge)$, contradicting Nishida's Theorem [9].

For those who prefer lists, the main theorem can be restated as: For any natural number n , we have

$$\mathbb{F}_2[v_0, v_1^{2^{2n}}, v_2^{2^{2n-1}}, v_3^{2^{2n-2}}, \dots, v_n^{2^{n+1}}, v_{n+1}^{2^{n+2}}, v_{n+2}^{2^{n+3}}, \dots, v_{2n}^{2^{2n+1}}] \subset \text{Ext}_{A(2n)}(\mathbb{F}_2, \mathbb{F}_2)$$

and

$$\mathbb{F}_2[v_0, v_1^{2^{2n+1}}, v_2^{2^{2n}}, v_3^{2^{2n-1}}, \dots, v_n^{2^{n+2}}, v_{n+1}^{2^{n+3}}, v_{n+2}^{2^{n+4}}, \dots, v_{2n+1}^{2^{2n+2}}] \subset \text{Ext}_{A(2n+1)}(\mathbb{F}_2, \mathbb{F}_2).$$

The proof of the conjecture presented here is inductive, using as its main tools the Davis–Mahowald spectral sequence and the classical May spectral sequence (augmented by Nakamura's squaring operations [10]).

A very different approach to the conjecture can be found in chapter 5 of Singer's monograph [14]. He considers the Cartan–Eilenberg spectral sequence for the 2-primary extension of Hopf algebras

$$E(n) \rightarrow A(n) \rightarrow \mathcal{D}A(n-1),$$

where $\mathcal{D}A(n-1)$ has algebra and coalgebra structures identical to $A(n-1)$ but with the gradings doubled. This Cartan–Eilenberg spectral sequence for the cohomology of $A(n)$ fails to collapse for the prime 2, so one needs to look closely at the (right) action of $\mathcal{D}A(n-1)$ on $\text{Ext}_{E(n)}(\mathbb{F}_2, \mathbb{F}_2) \cong \mathbb{F}_2[v_0, v_1, v_2, \dots]$:

$$v_k D(\overline{Sq}^{2^i}) = \begin{cases} v_{k-1} & \text{if } k = i - 1; \\ 0 & \text{otherwise.} \end{cases}$$

One then needs to compute the image of the restriction homomorphism

$$\text{Ext}_{A(n)}(\mathbb{F}_2, \mathbb{F}_2) \rightarrow (\text{Ext}_{E(n)}(\mathbb{F}_2, \mathbb{F}_2))^{\mathcal{D}\overline{A}(n-1)}.$$

Singer observes that the lowest power of v_k invariant under the coaction of $\mathcal{D}\overline{A}(n-1)$ is $v_k^{2^{n+1-k}}$, exactly the lowest power predicted to live in the cohomology of $A(n)$ by Conjecture 1.3. He suggests that a careful analysis of this Cartan–Eilenberg spectral sequence will likely shed light on the conjecture and other questions about the cohomology of finite Hopf-subalgebras of the mod 2 Steenrod algebra.

2. The Davis–Mahowald spectral sequence

The two main tools used in the proof of the conjecture are the now-classical May spectral sequence and the spectral sequence developed by Davis and Mahowald in their study of $\text{Ext}_{A(2)}(P_k, \mathbb{F}_2)$ [5]. Because the May spectral sequence is so ubiquitous, there's no need to include details about its construction here. We note, however, that we'll use the more modern notation for classes in the May spectral sequence, found in such sources as [11]. We provide some basic information on the construction of the Davis–Mahowald spectral sequence, based on their variant of the Koszul resolution.

The following material is based on presentations in [5] and [8], augmented by Rognes' later work in [12].

The idea, originally due to Davis and Mahowald, is to use a sort of “sideways” version of the traditional Koszul resolution to allow one to compute $\text{Ext}_{A(n)}(M, \mathbb{F}_2)$ if one understands $\text{Ext}_{A(n-1)}(M \otimes N, \mathbb{F}_2)$ for certain $A(n-1)$ -modules N . We begin by observing that (at the prime 2) the dual Steenrod algebra

$$A_* \cong \mathbb{F}_2[\xi_1, \xi_2, \dots]$$

is a module over A , with the action given by the total squaring operation

$$Sq\xi_i = \xi_i + \xi_{i-1}^2.$$

Observe that

$$(A(n) \otimes_{A(n-1)} \mathbb{F}_2)^* \cong E(\zeta_1^{2^n}, \zeta_2^{2^{n-1}}, \dots, \zeta_{n+1}),$$

as right $A(n)$ -modules, where the action on the exterior algebra is given by

$$Sq^{2^k} \zeta_{n-j+1}^{2^k} = \zeta_{n-j}^{2^{k+1}}$$

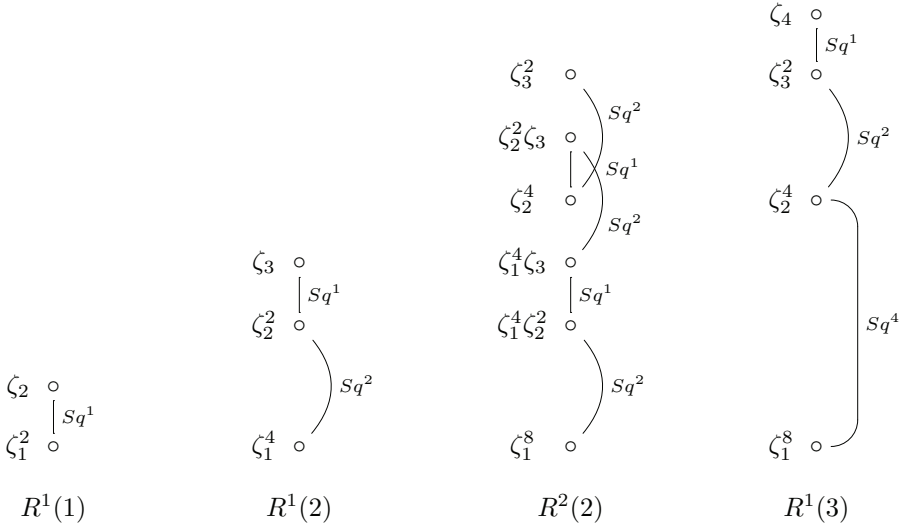
and

$$Sq^{2^n} \zeta_1^{2^n} = 1,$$

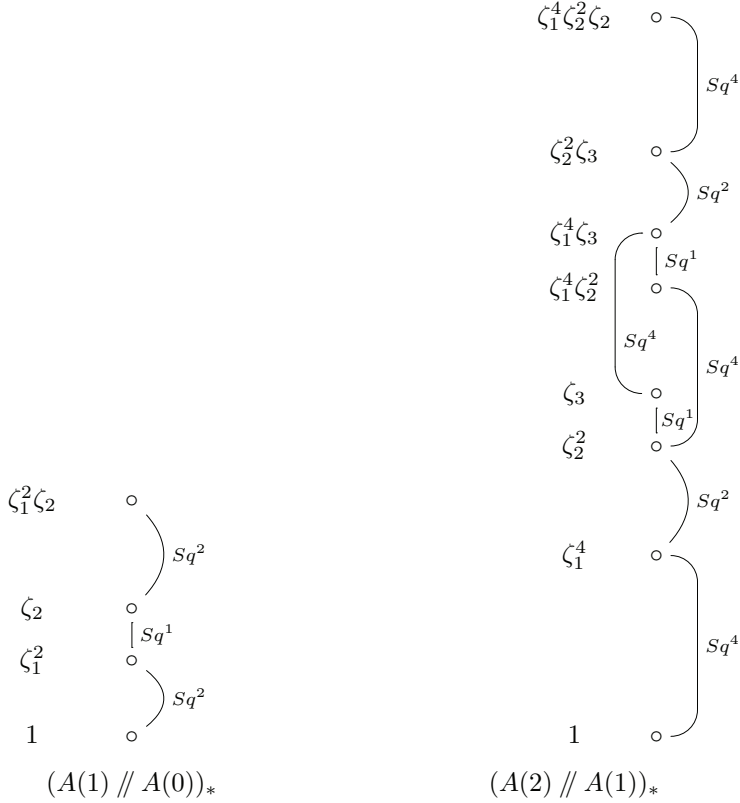
extended by the Cartan formula, where $\zeta_i = \chi(\xi_i)$. We note that $E(\zeta_1^{2^n}, \zeta_2^{2^{n-1}}, \dots, \zeta_{n+1})$ is an $A(n)$ -module, but not an $A(n+1)$ -module, because of the Adem relations. For each n , let

$$R = R(n) = \mathbb{F}_2[\zeta_1^{2^n}, \zeta_2^{2^{n-1}}, \dots, \zeta_{n+1}]$$

with the same A -action on the generators, extended by the Cartan formula. For each $\sigma \geq 0$, let $R^\sigma = R^\sigma(n)$ denote the sub vector space of R spanned by homogeneous polynomials of degree σ . Here are pictures of the first few $R^i(n)$:



For future reference, we also include diagrams of the exterior algebras:



Then for every n we have the following sequence:

$$0 \rightarrow \mathbb{F}_2 \rightarrow (A(n) \otimes_{A(n-1)} \mathbb{F}_2)^* \otimes R^0(n) \rightarrow (A(n) \otimes_{A(n-1)} \mathbb{F}_2)^* \otimes R^1(n) \rightarrow \dots$$

with the maps given by

$$\delta[(x_1 x_2 \dots x_r) \otimes p] = \sum_{j=1}^r (x_1 \dots \widehat{x}_j \dots x_r) \otimes x_j p.$$

This sequence is exact by the standard Koszul resolution argument (see [6, p. 243] or [5]) and is dual to

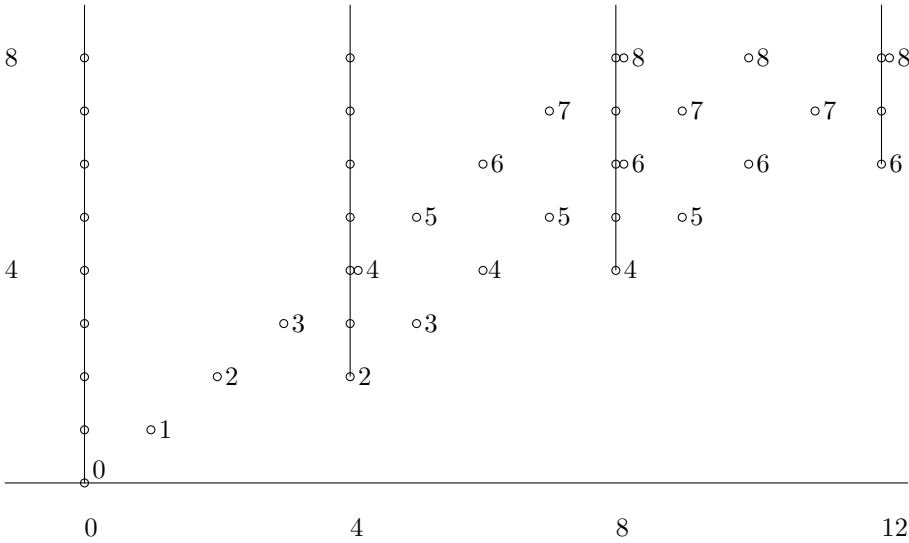
$$0 \leftarrow \mathbb{F}_2 \leftarrow A(n) \otimes_{A(n-1)} (R^0(n))^* \leftarrow A(n) \otimes_{A(n-1)} (R^1(n))^* \leftarrow \dots \quad (1)$$

Given an $A(n)$ -module M , apply the functor $\text{Ext}_{A(n)}(-, \mathbb{F}_2)$ to the resolution above, tensored with M . We obtain a spectral sequence

$$E_1^{\sigma, s, t} = \text{Ext}_{A(n-1)}^{s-\sigma, t}((R^\sigma(n))^* \otimes M, \mathbb{F}_2) \implies \text{Ext}_{A(n)}^{s, t}(M, \mathbb{F}_2).$$

Note that the homomorphisms in the resolution are given by $A(n)$ -module maps which are *not* extended $A(n-1)$ -module maps. Thus d_1 -differentials in the Davis–Mahowald spectral sequence are induced by the Sq^{2^n} -action in $R^\sigma(n)^* \otimes M$.

As an (easy and familiar) example, we'll use this Davis–Mahowald spectral sequence to compute $\text{Ext}_{A(1)}(\mathbb{F}_2, \mathbb{F}_2)$, the E_2 term of the classical Adams spectral sequence for bo_* . Here we use as our starting point the fact that the dual of $A(1) // A(0)$ is isomorphic to $E(\zeta_1^2, \zeta_2)$ as $A(1)$ -coalgebras. Note that the gradings are quite important here: ζ_1^2 is in dimension 2 and ζ_2 is in dimension 3. We set up the Koszul resolution of (1) in the $n = 1$ case, then apply the functor $\text{Ext}_{A(1)}^{*,*}(-, \mathbb{F}_2)$, obtaining the Davis–Mahowald spectral sequence. Here the E_1 term consists of $\text{Ext}_{A(0)}$ s, with the d_1 -differentials induced by the Sq^2 -action on the R^σ s (not the Sq^1 action). A picture of the E_1 term follows, in the standard $(t - s, s)$ chart form, with the σ -filtration of the classes labeled appropriately.



Now we need to sort out the differentials in the Davis–Mahowald spectral sequence. Since $d_1: E_1^{\sigma, s, t} \rightarrow E_1^{\sigma+1, s+2, t}$ (i.e., one up and one to the left on the chart), we see a potential d_1 from the second filtration 2 class to the first filtration 3 class. To check whether or not this is nonzero, we need to look at the homomorphism

$$A(1) // A(0) \otimes R^2(1)^* \xleftarrow{\delta_2^*} A(1) // A(0) \otimes R^3(1)^*,$$

dual to

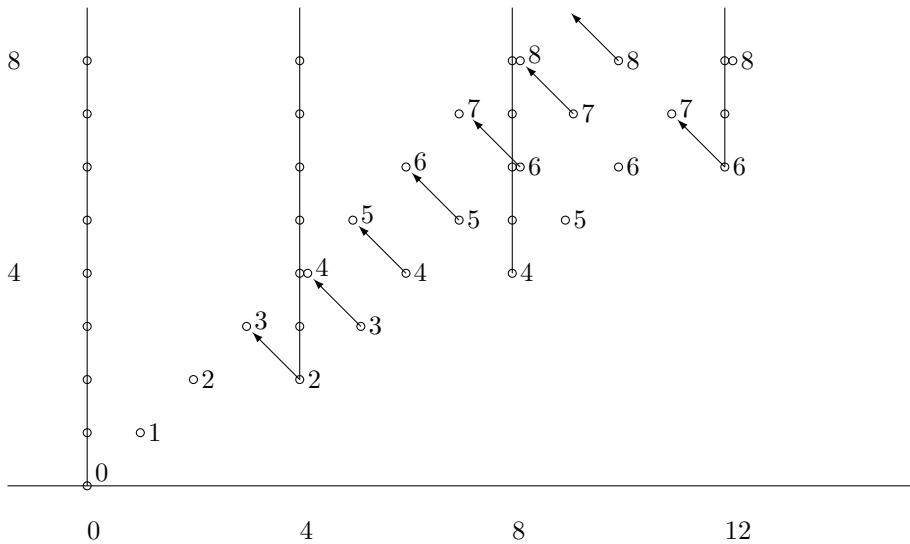
$$E(1) \otimes R^2 \xrightarrow{\delta_2} E(1) \otimes R^3.$$

For any class $p \in R^2$, we see that

$$\begin{aligned} \delta_2(1 \otimes p) &= 0, \\ \delta_2(\zeta_1^2 \otimes p) &= 1 \otimes \zeta_1^2 p, \\ \delta_2(\zeta_2 \otimes p) &= 1 \otimes \zeta_2 p, \\ \delta_2(\zeta_1^2 \zeta_2 \otimes p) &= \zeta_1^2 \otimes \zeta_2 p + \zeta_2 \otimes \zeta_1^2 p. \end{aligned}$$

To see which polynomials p yield a nonzero d_1 , we look at the action of Sq^2 on p . Since $\zeta_1^4 Sq^2 = 0$ and $\zeta_1^2 \zeta_2 Sq^2 = 0$, the classes spawned by these in Ext will have zero d_1 s. But $\zeta_2^2 Sq^2 = \zeta_1^4$, so that the class this gives in Ext (the second filtration 2 class) must hit the first filtration 3 class.

This sort of reasoning establishes all of the d_1 s in the following picture:



This leaves the expected picture for $\text{Ext}_{A(1)}(\mathbb{F}_2, \mathbb{F}_2)$.

The Davis–Mahowald spectral sequence gives an easy proof that $v_n^{2^{n+1}}$ is defined and nonzero in $\text{Ext}_{A(n)}^{2^{n+1}, 2^{n+1}(2^{n+1}-1)}(\mathbb{F}_2, \mathbb{F}_2)$, which we outline here. We note that the top class in $R^{2^{n+1}}(n)$ is $\zeta_{n+1}^{2^{n+1}}$, which “splits off” because $A(n)$ acts trivially on it, yielding the desired class in $H^*A(n)$. The resulting short exact sequence of $A(n)$ -modules yields a splitting of the Koszul-type resolution, so the corresponding class in Ext is a nonzero-divisor in the cohomology ring $H^*A(n)$. See [8] for details.

In [12], Rognes follows up on the work of Davis and Mahowald to fill in all the details for the computation of $\text{Ext}_{A(2)}(\mathbb{F}_2, \mathbb{F}_2)$, the E_2 -term of the classical Adams spectral sequence for $\pi_*(tmf)$. One of his very useful observations (on page 44) is that the Davis–Mahowald spectral sequence for the cocommutative Hopf algebras $A(n)$ is multiplicative, which we’ll exploit in the proof of the conjecture.

3. Strategy of the proof

Now we outline the proof of the conjecture. First, as a notational shortcut, the phrase $y \in \text{Ext}_{A(k)}(\mathbb{F}_2, \mathbb{F}_2)$ should be read as “the class y is defined and nonzero in $\text{Ext}_{A(k)}(\mathbb{F}_2, \mathbb{F}_2)$.”

We begin with a simple observation:

Lemma 3.1. *If $v_j^k \in \text{Ext}_{A(m)}(\mathbb{F}_2, \mathbb{F}_2)$, then $v_j^k \in \text{Ext}_{A(i)}(\mathbb{F}_2, \mathbb{F}_2)$ for $i = j, j + 1, \dots, m$.*

Proof. Simply notice that the restriction maps in cohomology commute:

$$\begin{array}{ccc}
 \text{Ext}_{A(m)}(\mathbb{F}_2, \mathbb{F}_2) & \xrightarrow{\text{restr}} & \text{Ext}_{E(m)}(\mathbb{F}_2, \mathbb{F}_2) \\
 \downarrow \text{restr} & & \downarrow \text{restr} \\
 \text{Ext}_{A(i)}(\mathbb{F}_2, \mathbb{F}_2) & \xrightarrow{\text{restr}} & \text{Ext}_{E(i)}(\mathbb{F}_2, \mathbb{F}_2).
 \end{array}
 \quad \square$$

We will prove the conjecture by induction, in a manner that might be paraphrased as follows: Assume that the conjecture “works for v_{n-1} ,” then prove that it must also “work for v_n .”

To begin, we assume that $v_{n-1}^{2^{n+1}}$ is defined and nonzero in $\text{Ext}_{A(2n-1)}(\mathbb{F}_2, \mathbb{F}_2)$. As a (quite relevant) aside, we could actually assume as a “base case” that $v_{n-1}^{2^n}$ is nonzero in $\text{Ext}_{A(2n-2)}(\mathbb{F}_2, \mathbb{F}_2)$, which is isomorphic to the $E_1^{0,*,*}$ -term of the Davis–Mahowald spectral sequence for $\text{Ext}_{A(2n-1)}(\mathbb{F}_2, \mathbb{F}_2)$. Because $\text{Ext}_{A(2n-1)}(\mathbb{F}_2, \mathbb{F}_2)$ is isomorphic to $\text{Ext}_A(\mathbb{F}_2, \mathbb{F}_2)$ in this range, this class must be killed in the Davis–Mahowald spectral sequence, and the only possible differential is

$$d_1^{DM}(v_{n-1}^{2^n}) = v_{n-2}^{2^n} h_{2n-1}.$$

By the work of Rognes [12, p. 44], the cocommutativity of each $A(r)$ implies that the Davis–Mahowald spectral sequence is multiplicative, so $d_1(v_{n-1}^{2^{n+1}}) = 0$, since $v_{n-1}^{2^n}$ is a cycle. Note that the bottom cell of $R^2(2n-1)$ is in dimension 2^{2n+1} , too high to be the target of a d_2^{DM} . Since the higher $R^\sigma(2n-1)$ s are even more highly connected, we conclude that $v_{n-1}^{2^n}$ is nonzero in $\text{Ext}_{A(2n-2)}(\mathbb{F}_2, \mathbb{F}_2)$ actually implies that $v_{n-1}^{2^{n+1}}$ is nonzero in $\text{Ext}_{A(2n-1)}(\mathbb{F}_2, \mathbb{F}_2)$. In any case, we will assume that $v_{n-1}^{2^{n+1}}$ lives where we want it to.

Next, we use the class $v_{n-1}^{2^{n+1}} \in \text{Ext}_{A(2n-1)}(\mathbb{F}_2, \mathbb{F}_2)$ to construct a nonzero product $v_{n-1}^{2^{n+1}} h_{2n+1} \in \text{Ext}_{A(2n)}(R^1(2n+1), \mathbb{F}_2)$, the E_1 -term of the Davis–Mahowald spectral sequence converging to $\text{Ext}_{A(2n+1)}(\mathbb{F}_2, \mathbb{F}_2)$. In this range $\text{Ext}_{A(2n+1)}(\mathbb{F}_2, \mathbb{F}_2)$ is isomorphic to $\text{Ext}_A(\mathbb{F}_2, \mathbb{F}_2)$, so we can use a classical May spectral sequence argument to show that $v_{n-1}^{2^{n+1}} h_{2n+1}$ cannot live there.

We will conclude that there must be a Davis–Mahowald spectral sequence class $x \in E_1^{0,*,*} = \text{Ext}_{A(2n)}(\mathbb{F}_2, \mathbb{F}_2)$ such that $d_1^{DM}(x) = v_{n-1}^{2^{n+1}} h_{2n+1}$. We then use the multiplicativity of the spectral sequence to rule out all other possibilities and conclude that this class x must, in fact, be the desired $v_n^{2^{n+1}} \in \text{Ext}_{A(2n)}(\mathbb{F}_2, \mathbb{F}_2)$.

To continue the proof from this base case, we will use the inductive hypothesis that $v_{n-1}^{2^{n+1+k}}$ is nonzero in $\text{Ext}_{A(2n-1+k)}(\mathbb{F}_2, \mathbb{F}_2)$ to produce a nonzero class corresponding to $v_{n-1}^{2^{n+1+k}} h_{2n+1+k}$ in the E_1 -term of the Davis–Mahowald spectral sequence for $\text{Ext}_{A(2n+1+k)}(\mathbb{F}_2, \mathbb{F}_2)$ that must be killed by a differential – otherwise it would persist to $\text{Ext}_A(\mathbb{F}_2, \mathbb{F}_2)$. We then show that the only way that this class can be killed in the Davis–Mahowald spectral sequence is if $v_n^{2^{n+1+k}}$ is nonzero in $\text{Ext}_{A(2n+k)}(\mathbb{F}_2, \mathbb{F}_2)$.

4. An example

Given the complexity of the notation, it’s best to work through an “early” example before dealing with the details of the proof of the general case. Here we detail the step from “the conjecture is true for appropriate powers of v_1 ” to “it must be true for the appropriate powers of v_2 .”

Note that no power of v_1 can be nonzero in $\text{Ext}_A(\mathbb{F}_2, \mathbb{F}_2)$: Such a class would be above the “Adams edge,” so there would be no possible targets for differentials on it in the classical Adams spectral sequence, yielding a nonzero nonnilpotent homotopy class in violation of Nishida’s Theorem on the nilpotence of the stable homotopy ring of the sphere, per [1] and [9].

First, we observe that the conjecture holds for all appropriate powers of v_1 , by using well known calculations in the May spectral sequence. Note that v_1^2 is represented in the May spectral sequence by the class $b_{2,0}$, where we use the more modern notation from [11].

This is as good a place as any to address concerns about whether we can identify these classes in a precise way in both the May and Davis–Mahowald spectral sequences. For arbitrarily chosen classes in $\text{Ext}_{A(k)}(\mathbb{F}_2, \mathbb{F}_2)$, there might be ambiguity in how one would choose representatives in these two spectral sequences. For the particular classes we work with in this example (and in the proof of the general case), we can resolve this issue easily. First, we note that $h_k \in \text{Ext}_{A(k)}(\mathbb{F}_2, \mathbb{F}_2)$ shows up in the Davis–Mahowald spectral sequence in the $\sigma = 1$ filtration, given by the class dual to $\zeta_1^{2^k} \in R^1(k)$, which has an obvious counterpart in the May filtration of the cobar complex. For the “smaller” h_j s in $\text{Ext}_{A(k)}(\mathbb{F}_2, \mathbb{F}_2)$, they show up in the Davis–Mahowald spectral sequence in filtration 0 (alias $\text{Ext}_{A(k-1)}(\mathbb{F}_2, \mathbb{F}_2)$), so a simple induction “back to” the $H^*(A(j))$ case does the trick. The compatibility of the spectral sequence representatives for the v_j^k s for $j < n$ is even easier to see: First, note that the “last case” $v_n^{2^{n+1}} \in H^*A(n)$ is easily resolved by the fact that there is only one May spectral sequence generator in that bidegree, namely $b_{n-1,0}^{2^n}$. For the lower cases, the May filtration on $A(n)$ is compatible with the inclusion $E(n) \hookrightarrow A(n)$, and the classes $v_j^k \in H^*A(n)$ are actually defined in terms of the resulting restriction homomorphism.

Computations of Tangora, following May, in [15] show that

$$d_2^{May}(b_{2,0}) = b_{1,1}h_1 + b_{1,0}h_2 = h_1^3 + h_0^2h_2.$$

This differential is propagated by Nakamura’s squaring operations [10] (using the “dual” versions Sq_i as in [2]). In particular, we will use the simplest case of these operations: $d_{2^r}^{May}(Sq_0(x)) = Sq_1(d_r^{May}(x))$, so the potential indeterminacy in the squaring operations is not an issue.

We conclude that

$$\begin{aligned} d_4^{May}(b_{2,0}^2) &= d_4^{May}(Sq_0b_{2,0}) \\ &= Sq_1(h_1^3 + h_0^2h_2) \\ &= Sq_1h_1^2Sq_0h_1 + Sq_1h_1^2Sq_0h_1 + Sq_1h_0^2Sq_0h_2 + Sq_0h_0^2Sq_1h_2 \\ &\quad \text{(by the Cartan formula)} \\ &= h_1^4h_1 + h_0^4h_3 = h_0^4h_3 \quad (\text{since } Sq_1x^2 = 0). \end{aligned}$$

We continue this process, to obtain

$$d_{2^n}^{May}(b_{2,0}^{2^{n-1}}) = h_0^{2^n}h_{n+1}.$$

Tangora notes that in the range of his calculations, this May differential is necessary to truncate the h_0 -tower on h_{n+1} in $\text{Ext}_A(\mathbb{F}_2, \mathbb{F}_2)$ at the desired height.

In fact, Tangora’s reasoning can be used to see that v_1^{3+k} “lives” as expected in $\text{Ext}_{A(1)}(\mathbb{F}_2, \mathbb{F}_2)$ through $\text{Ext}_{A(3+k)}(\mathbb{F}_2, \mathbb{F}_2)$, meaning that the class is defined and nonzero in the cohomology of those $A(i)$ s. Precisely, if v_1^{3+k} is not a nonzero class in $\text{Ext}_{A(3+k)}(\mathbb{F}_2, \mathbb{F}_2)$, then there is no possible differential in the Davis–Mahowald spectral sequence that can kill the class $h_0^{2^{3+k}}h_{4+k} \in E_1^{0,*,*}$. But if $h_0^{2^{3+k}}h_{4+k}$ persists

to a class in $\text{Ext}_{A(4+k)}(\mathbb{F}_2, \mathbb{F}_2)$, it must also show up as a nonzero class in $\text{Ext}_A(\mathbb{F}_2, \mathbb{F}_2)$, by the Adams Approximation Theorem, contradicting the family of May differentials starting with Tangora's computation.

An appropriate version of this reasoning is at the heart of the proof of the higher cases of the conjecture.

We will now show that the conjecture holding for all the appropriate powers of v_1 implies that $v_2^{2^{3+k}}$ must be nonzero in $\text{Ext}_{A(4+k)}(\mathbb{F}_2, \mathbb{F}_2)$ for all $k \geq 0$. The first nontrivial case is the following:

$$v_1^8 \in \text{Ext}_{A(3)}(\mathbb{F}_2, \mathbb{F}_2) \text{ implies } v_2^8 \in \text{Ext}_{A(4)}(\mathbb{F}_2, \mathbb{F}_2).$$

We first show that $v_1^8 \in \text{Ext}_{A(3)}(\mathbb{F}_2, \mathbb{F}_2)$ implies $v_1^8 h_5$ is nonzero in the E_1 -term for $\text{Ext}_{A(5)}(\mathbb{F}_2, \mathbb{F}_2)$. To see why, observe that the Davis–Mahowald spectral sequence for $\text{Ext}_{A(5)}(\mathbb{F}_2, \mathbb{F}_2)$ has

$$E_1^{\sigma, s, t} = \text{Ext}_{A(4)}^{s-\sigma, t}((R^\sigma(5))^*, \mathbb{F}_2),$$

for all $\sigma \geq 0$. We note that $E_1^{0, *, *}$ is just $\text{Ext}_{A(4)}^{*, *}(\mathbb{F}_2, \mathbb{F}_2)$, and the class $h_5 \in \text{Ext}_{A(5)}(\mathbb{F}_2, \mathbb{F}_2)$ is given by the bottom cell of $R^1(5)$, via the map $\Sigma^{32}\mathbb{F}_2 \rightarrow R^1(5)$. We also note that $R^1(5)$ is isomorphic to $\Sigma^{32}A(4) // A(3)$ as $A(4)_*$ -comodules, through dimension 62, which can be seen easily by comparing the cell diagrams. More precisely, recall that for $E_5 = E(\zeta_1^{32}, \zeta_2^{16}, \dots, \zeta_6)$, we have the following long exact sequence of $A(5)$ -comodules, analogous to the Koszul resolution:

$$0 \rightarrow \mathbb{F}_2 \rightarrow E_5 \otimes R^0(5) \rightarrow E_5 \otimes R^1(5) \rightarrow E_5 \otimes R^2(5) \dots$$

where $R^2(5)$ begins in dimension 64. Thus we have an $A(5)$ -comodule isomorphism through degree 63 from $\mathbb{F}_2\{\xi_1^{32}, \dots, \xi_6\} \subset E_5 = E_5 \otimes R^0(5)$ to $\mathbb{F}_2\{x_{32}, x_{48}, x_{56}, x_{60}, x_{62}, x_{63}\} \subset E_5 \otimes R^1(5)$. So we know that $\text{Ext}_{A(4)}(R^1(5), \mathbb{F}_2) \cong \text{Ext}_{A(3)}(\mathbb{F}_2, \mathbb{F}_2)$ in this range, and we must have a nonzero class $v_1^8 h_5$ corresponding to “ $\Sigma^{32}v_1^8$ ” $\in \text{Ext}_{A(4)}(R^1(5), \mathbb{F}_2)$.

We can easily observe that our class $v_1^8 h_5$ cannot persist through the Davis–Mahowald spectral sequence be a nonzero class in $\text{Ext}_{A(5)}(\mathbb{F}_2, \mathbb{F}_2)$ (and hence in $\text{Ext}_A(\mathbb{F}_2, \mathbb{F}_2)$) by looking at the “next” family of May spectral sequence differentials, starting with $b_{3,0}$ corresponding to v_2^2 :

$$d_2^{May}(b_{3,0}) = b_{2,1}h_1 + b_{2,0}h_3. \quad (2)$$

But $d_2^{May}b_{2,1} = h_2^3 + h_1^2h_3$, so it follows that $b_{2,1}h_1 = h_1^3h_3$. Applying the Nakamura Sq_0 twice to equation 2, we obtain

$$d_8^{May}(b_{3,0}^4) = b_{2,0}^4 h_5,$$

which shows that $v_1^8 h_5$ cannot be a nonzero class in $\text{Ext}_A(\mathbb{F}_2, \mathbb{F}_2)$, as we hoped.

A quick check of the dimensions of the modules $R^\sigma(5)$ and the trajectories of the differentials in the Davis–Mahowald spectral sequence shows that the only possible way to “kill” the class $v_1^8 h_5 \in \text{Ext}_{A(4)}(R^1(5), \mathbb{F}_2)$ is to have a class $x \in E_1^{0,8,56} = \text{Ext}_{A(4)}(\mathbb{F}_2, \mathbb{F}_2)$ which could bear a d_1 . We hope that the only candidate for such a class would be v_2^8 , IF we knew that it persisted to $\text{Ext}_{A(4)}(\mathbb{F}_2, \mathbb{F}_2)$.

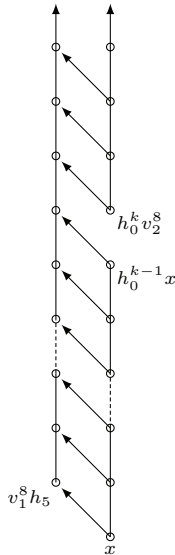
So we know that there must exist a class $x \in \text{Ext}_{A(4)}^{8,56}(\mathbb{F}_2, \mathbb{F}_2) = E_1^{0,8,56}$ such that $d_1^{DM}(x) = v_1^8 h_5$. We need to show that this class x is exactly the desired $v_2^8 \in$

$\text{Ext}_{A(4)}(\mathbb{F}_2, \mathbb{F}_2)$. First, we recall from the construction that the Davis–Mahowald spectral sequence is multiplicative, since $A(k)$ is a cocommutative Hopf algebra. (See [12] for details.) Next, note that $v_1^8 h_5$ bears an h_0 -tower in $\text{Ext}_{A(4)}(R^1(5), \mathbb{F}_2)$, so whatever classes and differentials are involved in killing it in the Davis–Mahowald spectral sequence must also account for the entire h_0 -tower. Recall that for a subalgebra B of A , the h_0 -towers in $\text{Ext}_B(\mathbb{F}_2, \mathbb{F}_2)$ are in one-to-one correspondence with the Q_0 -homology $H(A // B; Q_0)$, as detailed in [4]¹. In particular, since

$$(A // A(4))_* \cong \mathbb{F}_2[\xi_1^{32}, \xi_2^{16}, \xi_3^8, \xi_4^4, \xi_5^2, \xi_6, \xi_7, \dots],$$

we see that $H((A // A(4))_*; Q_0)$ is spanned as an \mathbb{F}_2 -vector space by $\chi \text{Sq}(32i, 16j, 8k, 4l)$, in dimensions $32i + 48j + 56k + 60l$, for $i, j, k, l \geq 0$. So there's a unique tower in $\text{Ext}_{A(4)}(\mathbb{F}_2, \mathbb{F}_2)$ in $t - s = 48$. Note that the restriction homomorphism $\text{restr}: \text{Ext}_{A(4)}(\mathbb{F}_2, \mathbb{F}_2) \rightarrow \text{Ext}_{E(4)}(\mathbb{F}_2, \mathbb{F}_2)$ sends the elements in this tower to $h_0^m v_2^8 \in \text{Ext}_{E(4)}(\mathbb{F}_2, \mathbb{F}_2)$ for some m (and all its h_0 -multiples). This does not show that v_2^8 itself is present in $\text{Ext}_{A(4)}(\mathbb{F}_2, \mathbb{F}_2)$, but $h_0^k v_2^8$ and all its h_0 -multiples must be nonzero for some (possibly large) k .

We might worry that the class we've detected, $x \in \text{Ext}_{A(4)}(\mathbb{F}_2, \mathbb{F}_2)$, could be h_0 -torsion, but that the Davis–Mahowald spectral sequence d_1 -differentials might still wipe out the whole tower on $v_1^8 h_5$ in the manner given by the following diagram:



However, the multiplicativity in the Davis–Mahowald spectral sequence prohibits

¹Because this reference is not available online, we present an outline of Davis's proof. For any A -module M , recall that the Q_0 -homology of M is given by $H(M; Q_0) = \ker(\text{Sq}^1) / \text{im}(\text{Sq}^1)$. Construct an epimorphism of A -modules $M \xrightarrow{\phi} N = (\bigoplus A) \oplus (\bigoplus A // A(0))$, sending the A -module generators of M to the generators of the first summand of N and the generators of $H(M; Q_0)$ to the generators of the second summand. Davis observes that $L = \ker(\phi)$ has zero Q_0 -homology, so that $\text{Ext}_A^{s,t}(L, \mathbb{F}_2)$ is zero above the line $3s - t + 6$ by the vanishing theorem of Adams [1]. Thus $\text{Ext}_A^{s,t}(M, \mathbb{F}_2) \cong \text{Ext}_A^{s,t}(N, \mathbb{F}_2)$ above that line, as we hoped.

this: If x is h_0 -torsion and $d_1(x) = v_1^8 h_5$, then

$$0 = d_1(h_0^L x) = h_0^L v_1^8 h_5 \neq 0,$$

for some large L . We conclude that the class x we detected in $\text{Ext}_{A(4)}^{8,56}(\mathbb{F}_2, \mathbb{F}_2)$ must be exactly v_2^8 , as we hoped.

This process can be continued to show that $v_1^{16} \in \text{Ext}_{A(4)}(\mathbb{F}_2, \mathbb{F}_2)$ implies $v_2^{16} \in \text{Ext}_{A(5)}(\mathbb{F}_2, \mathbb{F}_2)$. The argument is now easy to see: First, we observe that $R^1(6) \cong \Sigma^{64} A(5) // A(4)$ as $A(5)_*$ -comodules in the relevant range of dimensions, so $v_1^{16} h_6$ must be a nonzero class in $\text{Ext}_{A(5)}(R^1(6), \mathbb{F}_2)$, which contributes via the Davis–Mahowald spectral sequence to $\text{Ext}_{A(6)}(\mathbb{F}_2, \mathbb{F}_2)$. If this class survives the Davis–Mahowald spectral sequence, it must also live in $\text{Ext}_A(\mathbb{F}_2, \mathbb{F}_2)$, by the Adams Approximation Theorem. We use the May spectral sequence differential $d_{16}^{May}(b_{3,0}^8) = b_{2,0}^8 h_6$ to see that $v_1^{16} h_6$ cannot persist to $\text{Ext}_A(\mathbb{F}_2, \mathbb{F}_2)$. The only way to kill $v_1^{16} h_6$ in the Davis–Mahowald spectral sequence for $\text{Ext}_{A(6)}(\mathbb{F}_2, \mathbb{F}_2)$ is for some class x to be nonzero in $E_1^{0,16,112} \cong \text{Ext}_{A(5)}(\mathbb{F}_2, \mathbb{F}_2)$, thus providing the source for the d_1^{DMSS} differential to kill $v_1^{16} h_6$. We see that x must indeed be exactly v_2^{16} using the multiplicativity of the spectral sequence, by examining the Q_0 -homology of $A // A(5)$ and seeing that there is only one h_0 -tower in $\text{Ext}_{A(5)}^{t-s=96}(\mathbb{F}_2, \mathbb{F}_2)$, corresponding to $h_0^k v_2^{16}$.

The other powers of v_2 are shown to live in the appropriate Exts similarly.

5. Proof of the general case

We prove that there is a nonzero class

$$v_n^{2^{n+1+k}} \in \text{Ext}_{A(2n+k)}(\mathbb{F}_2, \mathbb{F}_2),$$

by using induction on n . We assume inductively that there is a nonzero element

$$v_{n-1}^{2^{n+1+k}} \in \text{Ext}_{A(2n-1+k)}(\mathbb{F}_2, \mathbb{F}_2).$$

We look closely at the Davis–Mahowald spectral sequence for $\text{Ext}_{A(2n+1+k)}(\mathbb{F}_2, \mathbb{F}_2)$, focusing particularly on $E_1^{1,*,*} \cong \text{Ext}_{A(2n+k)}(R^1(2n+1+k), \mathbb{F}_2)$. The class $h_{2n+1+k} \in \text{Ext}_{A(2n+1+k)}(\mathbb{F}_2, \mathbb{F}_2)$ is detected in the $E_1^{1,*,*}$ -term by the bottom cell of $R^1(2n+1+k)$. More precisely, h_{2n+1+k} is given by applying $\text{Ext}_{A(2n+k)}(-, \mathbb{F}_2)$ to the homomorphism $\Sigma^{2^{2n+1+k}} \mathbb{F}_2 \rightarrow R^1(2n+1+k)$. We know that $R^1(2n+1+k) \cong \Sigma^{2^{2n+1+k}} A(2n+k) // A(2n-1+k)$ as $A(2n+k)_*$ -comodules through dimension $2^{2n+2+k} - 2$, by examining the cell diagrams or by the following argument: Observe, as in the example, that for $E_r = E(\zeta_1^{2^r}, \zeta_2^{2^{r-1}}, \dots, \zeta_{r+1})$, we have the following long exact sequence of $A(r)$ -comodules, analogous to the Koszul resolution:

$$0 \rightarrow \mathbb{F}_2 \rightarrow E_r \otimes R^0(r) \rightarrow E_r \otimes R^1(r) \rightarrow E_r \otimes R^2(r) \cdots,$$

where $R^2(r)$ begins in dimension 2^{r+1} . Thus we have an $A(r)$ -comodule isomorphism through degree $2^{r+1} - 1$ from $\mathbb{F}_2\{\xi_1^{2^r}, \dots, \xi_{2^{r+1}-1}\} \subset E_r = E_r \otimes R^0(r)$ to $\mathbb{F}_2\{x_{2^r}, \dots, x_{2^{r+1}-1}\} \subset E_r \otimes R^1(r)$.

We conclude, then, that $v_{n-1}^{2^{n+1+k}} h_{2n+1+k}$ is nonzero in the E_1 -term of the Davis–Mahowald spectral sequence for $H^* A(2n+1+k)$. If this class survived the spectral sequence to the cohomology of $A(2n+1+k)$, then the Adams Approximation Theorem tells us that it would be a nonzero class in $\text{Ext}_A(\mathbb{F}_2, \mathbb{F}_2)$. But in

the May spectral sequence for the cohomology of A , we have the family of differentials starting with $d_2^{May}(b_{n+1,0}) = b_{n,1}h_1 + b_{n,0}h_{n+1}$. We observe that $d_2^{May}(b_{n,1}) = b_{n-1,2}h_2 + b_{n-1,1}h_{n+1}$ and $d_2^{May}(b_{n,1}h_1) = 0 + b_{n-2,1}h_1h_{n+1}$, so the “extra” class in the d_2 on $b_{n+1,0}$ is dead by the E_3 -term. We propagate this differential using Nakamura’s squaring operations to obtain

$$d_{2^{n+k}}^{May}(b_{n+1,0}^{2^n}) = v_{n-1}^{2^{n+1+k}} h_{2n+1+k}.$$

So the class $v_{n-1}^{2^{n+1+k}} h_{2n+1+k}$ cannot survive to $\text{Ext}_A(\mathbb{F}_2, \mathbb{F}_2)$ (and hence, to $\text{Ext}_{A(2n+k+1)}(\mathbb{F}_2, \mathbb{F}_2)$), and there must exist a class $x \in \text{Ext}_{A(2n+k)}(\mathbb{F}_2, \mathbb{F}_2) \cong E_1^{0,*,*}$ such that $d_1^{DM}(x) = v_{n-1}^{2^{n+1+k}} h_{2n+1+k}$.

We show that the class x must be exactly $v_n^{2^{n+1+k}}$ by examining the Q_0 -homology of $(A // A(2n+k))_* \cong \mathbb{F}_2[\xi_1^{2^{n+1+k}}, \xi_2^{2^{n+k}}, \dots]$ and observing the presence of a unique class in dimension $2^{n+k} \times 3$, which must map under the restriction homomorphism to the class $v_n^{2^{n+1+k}} \in \text{Ext}_{E(2n+k)}(\mathbb{F}_2, \mathbb{F}_2)$. Thus $\text{Ext}_{A(2n+k)}(\mathbb{F}_2, \mathbb{F}_2)$ contains an h_0 -tower starting at some h_0 multiple of $v_n^{2^{n+1+k}}$. As in the $n = 2$ example above, the h_0 -linearity of the Davis–Mahowald differentials demonstrates that the class x must be exactly $v_n^{2^{n+1+k}}$, as we wished. \square

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References

- [1] Adams, J.F., A periodicity theorem in homological algebra, *Math. Proc. Cambridge Philos. Soc.*, **62** (1966) pp. 365–377.
- [2] Bhattacharya, P., Egger, P. and Mahowald, M., On the periodic v_2 -self map of A_1 , preprint, 2016.
- [3] Bruner, R.B. and Rognes, J., The Adams spectral sequence for topological modular forms, preprint (May 2019, 542 pages).
- [4] Davis, D.M., The cohomology of the spectrum bJ , *Bol. Soc. Mat. Mex.* **20** (1975) pp. 6–11.
- [5] Davis, D.M., and Mahowald, M.E., Ext over the subalgebra A_2 of the Steenrod algebra for stunted projective spaces, *Current Trends in Algebraic Topology*, CMS Conf. Proc **2** part 1, Amer. Math. Soc. (1982) pp. 297–342.
- [6] Hilton, P.J. and Stammbach, U., *A Course in Homological Algebra*, Springer-Verlag, 1971.
- [7] Lin, W.-H., Cohomology of sub-Hopf algebras of the Steenrod algebra, *J. Pure Appl. Alg.* **10** (1977) pp. 101–114.
- [8] Mahowald, M.E., and Shick, P.L., Periodic phenomena in the classical Adams spectral sequence, *Trans. Amer. Math. Soc.* **300** (1987) pp. 191–206.

- [9] Nishida, G., The nilpotency of elements of the stable homotopy groups of spheres, *J. Math. Soc. Japan* **25** (1973) pp. 707–732.
- [10] Nakamura, O., On squaring operations in the May spectral sequence, *Mem. Fac. Sci., Kyushu Univ., Ser. A* **26** no. 2 (1972) pp. 293–308.
- [11] Ravenel, D.C., *Complex Cobordism and Stable Homotopy*, second edition, AMS/Chelsea, 2004.
- [12] Rognes, J., The Davis–Mahowald spectral sequence, presentation at Nils Baas conference 2016, <https://folk.uio.no/rognes/papers/dmss.pdf>.
- [13] Shick, P., Odd primary periodic phenomena in the classical Adams spectral sequence, *Trans. Amer. Math. Soc.*, **309** (1988) pp. 77–86.
- [14] Singer, W., *Steenrod Operations in Spectral Sequences*, Amer. Math Soc. Surveys and Monographs 129 (2006).
- [15] Tangora, M.C., On the cohomology of the Steenrod algebra, *Math. Z.* **116** (1970) pp. 18–64.
- [16] Wilkerson, C., The cohomology algebras of finite dimensional Hopf algebras, *Trans. Amer. Math. Soc.*, **264** (1981) pp. 137–150.

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