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ALGEBRAIC COBORDISM IN MIXED CHARACTERISTIC

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Abstract

We compute the geometric part of algebraic cobordism over Dedekind domains of mixed characteristic after inverting the positive residue characteristics and prove cases of a Conjecture of Voevodsky relating this geometric part to the Lazard ring for regular local bases. The method is by analyzing the slice tower of algebraic cobordism, relying on the Hopkins-Morel isomorphism from the quotient of the algebraic cobordism spectrum by the generators of the Lazard ring to the motivic Eilenberg-MacLane spectrum, again after inverting the positive residue characteristics.

1. Introduction

Algebraic cobordism is a theory for smooth schemes over a base scheme S defined by a motivic ring spectrum MGL_S in the stable motivic homotopy category $\mathsf{SH}(S)$. It is the motivic counterpart of complex cobordism MU. A famous Theorem of Quillen states that the natural map from the Lazard ring L_* classifying formal group laws to the coefficients of MU is an isomorphism, moreover $L_* \cong \mathbb{Z}[x_1, x_2, x_3, \ldots]$ with $\deg(x_i) = i$ (here we divide the usual topological grading by 2).

For an oriented motivic ring spectrum E the geometric part $E_{(2,1)*}$ of the coefficients also carries a formal group law constructed in the exact same way as in topology by evaluating the theory on $\mathbb{P}^\infty$ and using that $\mathbb{P}^\infty$ is naturally endowed with a multiplication.

Thus there is a classifying map $L_* \to E_{(2,1)*}$. It is known that for $E = \mathsf{MGL}_k$ for a field k of characteristic 0 this map is an isomorphism using the Hopkins-Morel isomorphism, see [**4**, Proposition 8.2]. Also in [**5**] it is shown that over such fields the Levine-Morel algebraic cobordism $\Omega^*(-)$ is isomorphic to $\mathsf{MGL}_k^{(2,1)*}(-)$ on smooth schemes over k (this generalizes the previous statement since $L_* \cong \Omega^{-*}(\mathrm{Spec}(k))$). If the base field k has positive characteristic the map $L_* \to \mathsf{MGL}_{k,(2,1)*}$ becomes at least an isomorphism after inverting the characteristic, see again [**4**, Proposition 8.2].

The main ingredient in the proof is that the Hopkins-Morel isomorphism yields a computation of the slices of MGL_S with respect to Voevodsky's slice filtration, that MGL_S is complete with respect to this filtration and that the slices have a simple form, namely they are shifted twists of the motivic Eilenberg-MacLane spectrum.

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The facts about the slices of MGL_S hold more generally true over spectra S of Dedekind domains of mixed characteristic (after inverting the positive residue characteristics), using the motivic Eilenberg-MacLane spectrum introduced in [**9**]. The main new input of this note is that in this case MGL_S is also complete with respect to the slice filtration (Corollary 5.9), a consequence of the fact that MGL_S is connective with respect to the homotopy sheaves, see Proposition 5.8.

This yields a computation of the geometric part of the homotopy groups of MGL_S (Theorem 6.5), again after inverting the residue characteristics. In our formulation we always assume a Hopkins-Morel isomorphism for the given coefficients, hoping that the Hopkins-Morel isomorphism will be settled completely in the future.

We prove weakened versions of cases of a Conjecture of Voevodsky ([**10**, Conjecture 1]), see Theorem 6.7, comparing the Lazard ring to $(\mathsf{MGL}_S)_{(2,1)*}$ for S the spectrum of a regular local ring.

We also give applications to some homotopy groups or sheaves of MGL_S outside the geometric diagonal, see section 7, and discuss generalizations of our results to motivic Landweber spectra.

We note that the observation that the Hopkins-Morel isomorphism yields the computation of the zero-slice of the sphere spectrum (after inverting suitable primes), see Theorem 3.1, was independently made by Oliver Röndigs.

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2. Preliminaries

By a base scheme we always mean a separated Noetherian scheme of finite Krull dimension. For a base scheme S we let $\mathsf{SH}(S)$ be the stable motivic homotopy category (see [**10**]).

We let $\mathsf{M}\mathbb{Z}_S \in \mathsf{SH}(S)$ be the motivic Eilenberg-MacLane spectrum over S constructed in [**9**] (see Definition 4.27 in loc. cit.). Also we let $\mathcal{M}(r) \in \mathsf{D}(\mathrm{Sh}(\mathrm{Sm}_{S,Zar}, \mathbb{Z}))$ (for notation see [**9**, Section 2]) be the motivic complexes of weight $r \in \mathbb{Z}$, so as a $\mathbb{G}_{m,S}$-spectrum $\mathsf{M}\mathbb{Z}_S$ has $\mathcal{M}(r)[r]$ in level r. If S is the spectrum of a Dedekind domain of mixed characteristic we note that $\mathcal{M}(0) = S^0\underline{\mathbb{Z}}$, thus for $X \in \mathrm{Sm}_S$ we have $H^{0,0}(X,\mathbb{Z}) = \mathbb{Z}^{\pi_0(X)}$. Also $\mathcal{M}(1) \cong \mathcal{O}^*[-1]$, so $H^{1,1}(X,\mathbb{Z}) \cong \mathcal{O}^*(X)$ and $H^{2,1}(X,\mathbb{Z}) \cong \mathrm{Pic}(X)$. We have $\mathcal{M}(r) \cong 0$ for $r < 0$.

For general S we denote by $\mathsf{MGL}_S \in \mathsf{SH}(S)$ the algebraic cobordism spectrum (see [**10**]). There is a natural map $L_* \to (\mathsf{MGL}_S)_{(2,1)*}$, where L_* denotes the Lazard ring. Fixing generators $x_i \in L_i$ there is a map

$$\Phi_S \colon \mathsf{MGL}_S/(x_1, x_2, \ldots)\mathsf{MGL}_S \longrightarrow \mathsf{M}\mathbb{Z}_S,$$

see [**9**, §10.1], which is an isomorphism after inverting all positive residue characteristics of S, see [**9**, Theorem 10.3].

For any ring or abelian group R we let $M_{R,S} \in \mathsf{SH}(S)$ be the Moore spectrum on R and $\mathsf{M}R_S$ the version of $\mathsf{M}\mathbb{Z}_S$ with R-coefficients. If no confusion can arise we also leave out the S in the notation.

3. Slices

For $i \in \mathbb{Z}$ denote by f_i resp. l_i the i-th colocalization resp. localization functor for Voevodsky's motivic slice filtration on $\mathsf{SH}(S)$ (see [**11**]). For any $E \in \mathsf{SH}(S)$ and $k \geqslant n$ we set $E\langle n, k\rangle := l_{k+1}(f_n(E))$. Thus we have exact triangles

$$f_{k+1}(E) \longrightarrow f_n(E) \longrightarrow E\langle n,k\rangle \longrightarrow f_{k+1}(E)[1]$$

and $s_n(E) = E\langle n, n\rangle$.

We note that all these functors commute with homotopy colimits since the effective subcategories of $\mathsf{SH}(S)$ are generated by compact objects as localizing triangulated subcategories (see also [**7**, Corollary 4.6]).

It follows then, e.g., that for $E \in \mathsf{SH}(S)$ and an abelian group A we have $f_i(E) \wedge M_A \cong f_i(E \wedge M_A)$.

Theorem 3.1. *Let X be an essentially smooth scheme over a Dedekind domain of mixed characteristic and R a localization of $\mathbb{Z}$ such that $\Phi_X \wedge M_{R,X}$ is an isomorphism (e.g., if every positive residue characteristic of X is invertible in R). Then*

$$s_0 M_{R,X} \cong s_0(\mathsf{MGL}_X \wedge M_{R,X}) \cong \mathsf{M}R_X.$$

More generally

$$s_n(\mathsf{MGL}_X \wedge M_{R,X}) \cong \Sigma^{2n,n}\mathsf{M}R_X \otimes L_n.$$

Proof. The first isomorphism of the first line follows from the fact that the unit map of MGL_X induces an isomorphism on zero slices, see [**7**, Corollary 3.3]. From the assumption that $\Phi_X \wedge M_R$ is an isomorphism it follows that the map $\mathsf{MGL}_X \wedge M_R \to \mathsf{M}R_X$ induces an isomorphism on zero-slices and that $\mathsf{M}R_X$ is effective. Moreover, $l_1\mathsf{M}\mathbb{Z}_X \cong \mathsf{M}\mathbb{Z}_X$, since negative weight motivic cohomology vanishes in our situation (this is the statement that $\mathcal{M}(r) \cong 0$ for $r < 0$ from section 2, which follows, e.g., from the comparison to Bloch-Levine motivic cohomology, see [**9**, Corollary 7.19], or also from the construction of $\mathsf{M}\mathbb{Z}$ in [**9**, Chapter 4]). Thus the second isomorphism of the first line follows. The second line is a version of [**7**, Theorem 4.7] with R-coefficients. □

Remark 3.2. It is then also possible to determine the slices of motivic Landweber spectra with R-coefficients, see [**8**], for example of $\mathsf{KGL}_X \wedge M_R$.

4. Subcategories of the stable motivic homotopy category

Fix a base scheme S. We let $\mathsf{SH}(S)_{\geqslant n}$ be the $\geqslant n$ part (in the homological sense) of $\mathsf{SH}(S)$ with respect to the homotopy t-structure, see, e.g., [**4**, §2.1]. Thus $\mathsf{SH}(S)_{\geqslant n}$ is generated by homotopy colimits and extensions by the objects $\Sigma^{p,q}\Sigma^\infty_+ X$ for $X \in \mathrm{Sm}_S$ and $p - q \geqslant n$. Here $\Sigma^{p,q} = \Sigma^{p-q}_{S^1}\Sigma^q_{\mathbb{G}_m}$.

For each $E \in \mathsf{SH}(S)$ we let $\underline{\pi}^{\mathrm{pre}}_{p,q}(E)$ be the presheaf

$$X \longmapsto \mathrm{Hom}_{\mathsf{SH}(S)}(\Sigma^{p,q}\Sigma^\infty_+ X, E)$$

on Sm_S. Let $\underline{\pi}_{p,q}(E)$ be the sheafification of $\underline{\pi}^{\mathrm{pre}}_{p,q}(E)$ with respect to the Nisnevich topology. We also set $\pi_{p,q}(E) := \underline{\pi}^{\mathrm{pre}}_{p,q}(E)(S) = E_{p,q}$.

We let $\mathsf{SH}(S)_{h \geqslant n}$ be the full subcategory of $\mathsf{SH}(S)$ of objects E such that $\underline{\pi}_{p,q}(E) = 0$ for $p - q < n$.

Lemma 4.1. *The categories $\mathsf{SH}(S)_{h\geqslant n}$ are closed under homotopy colimits and extensions in $\mathsf{SH}(S)$.*

Proof. The functors $\underline{\pi}_{p,q}$ respect sums. Moreover, the long exact sequences of homotopy sheaves associated to an exact triangle in $\mathsf{SH}(S)$ show that $\mathsf{SH}(S)_{h\geqslant n}$ is closed under cofibers and extensions. This shows the claim. □

In the following, we use the functor $f_* \colon \mathsf{SH}(X) \to \mathsf{SH}(Y)$ for a morphism $f \colon X \to Y$ between base schemes (for a construction see [**1**, Chapter 4.5]).

Proposition 4.2. *Let $i \colon Z \hookrightarrow S$ be a closed inclusion of base schemes. Then*

$$i_*(\mathsf{SH}(Z)_{h\geqslant n}) \subset \mathsf{SH}(S)_{h\geqslant n}.$$

Proof. Let $E \in \mathsf{SH}(Z)_{h\geqslant n}$. Let Y be the spectrum of the henselization of a local ring of a scheme from Sm_S. Then $Y_Z := Y \times_S Z$ is also the spectrum of a henselian local ring, and $\underline{\pi}^{\mathrm{pre}}_{p,q}(i_*E)(Y) \cong \underline{\pi}^{\mathrm{pre}}_{p,q}(E)(Y_Z) = 0$ for $p - q < n$ (the first isomorphism holds since i_* commutes with homotopy colimits). □

We let $\mathsf{SH}^{S^1}_s(S)$ be the homotopy category of presheaves of S^1-spectra on Sm_S localized with respect to the Nisnevich topology, and $\mathsf{SH}^{S^1}(S)$ the further $\mathbb{A}^1$-localization of that category.

We let $\mathsf{SH}^{S^1}_s(S)_{\geqslant n}$ be the $\geqslant n$ part (in the homological sense) of $\mathsf{SH}^{S^1}_s(S)$ with respect to the standard t-structure, and for $E \in \mathsf{SH}^{S^1}_s(S)$ we let $E_{\geqslant n}$ and $E_{\leqslant n}$ be the corresponding truncations. We let $E_{=n} := (E_{\geqslant n})_{\leqslant n}$.

As above for $E \in \mathsf{SH}^{S^1}_s(S)$ we have the presheaves $\underline{\pi}^{\mathrm{pre}}_p(E)$ and the sheaves $\underline{\pi}_p(E)$. For $E \in \mathsf{SH}^{S^1}_s(S)$ we have $E \in \mathsf{SH}^{S^1}_s(S)_{\geqslant n}$ if and only if $\underline{\pi}_k(E) = 0$ for $k < n$.

Note that $\mathsf{SH}^{S^1}_s(S)_{\geqslant n}$ is generated by homotopy colimits and extensions by the objects $\Sigma^n\Sigma^\infty_+ X$, $X \in \mathrm{Sm}_S$, thus the canonical functor $\sigma \colon \mathsf{SH}^{S^1}_s(S) \to \mathsf{SH}(S)$ sends $\mathsf{SH}^{S^1}_s(S)_{\geqslant n}$ to $\mathsf{SH}(S)_{\geqslant n}$.

Lemma 4.3. *We have $\mathsf{SH}(S)_{h\geqslant n} \subset \mathsf{SH}(S)_{\geqslant n}$. If S is the spectrum of a field then the inclusion is an equality.*

Proof. Let $E \in \mathsf{SH}(S)_{h\geqslant n}$. For any $i \in \mathbb{N}$ let E_i be the image of $\Sigma^{i,i}E$ in $\mathsf{SH}^{S^1}_s(S)$. By assumption we have $E_i \in \mathsf{SH}^{S^1}_s(S)_{\geqslant n}$. Thus $\Sigma^{-i,-i}\sigma(E_i) \in \mathsf{SH}(S)_{\geqslant n}$. The proof of the first statement concludes by noting that $E \cong \mathrm{hocolim}_{i\to\infty}\Sigma^{-i,-i}\sigma(E_i)$ (the proof of this statement is analogous to the proof of [**2**, Lemma 6.1]).

The second statement is [**4**, Theorem 2.3]. □

Lemma 4.4. *Let $E \in \mathsf{SH}^{S^1}_s(S)$. Then $E \to \mathrm{holim}_{n\to\infty} E_{\leqslant n}$ is an isomorphism.*

Proof. We show that for all $n \in \mathbb{Z}$ we have $\underline{\pi}_n(E) \cong \underline{\pi}_n(\mathrm{holim}_{k\to\infty} E_{\leqslant k})$. Fix $n \in \mathbb{Z}$ and let $X \in \mathrm{Sm}_S$ be of dimension d. We are ready if we show

$$\underline{\pi}_n(E)|_{X_{Nis}} \cong \underline{\pi}_n(\mathrm{holim}_{k\to\infty} E_{\leqslant k})|_{X_{Nis}} \quad (*).$$

For $m > n + d$ we have $\underline{\pi}_n^{\mathrm{pre}}(E_{=m}[j])(Y) = 0$ for $Y \in X_{Nis}$ and $j \geqslant 0$, so homing out of $\Sigma_+^\infty Y$ into the exact triangle

$$E_{=m} \longrightarrow E_{\leqslant m} \longrightarrow E_{\leqslant (m-1)} \longrightarrow E_{=m}[1]$$

shows that $\underline{\pi}_n^{\mathrm{pre}}(E_{\leqslant m})(Y) \cong \underline{\pi}_n^{\mathrm{pre}}(E_{\leqslant (m-1)})(Y)$. Using the Milnor short exact sequence this shows that

$$\underline{\pi}_n^{\mathrm{pre}}(\mathrm{holim}_{k\to\infty} E_{\leqslant k})|_{X_{Nis}} \cong \underline{\pi}_n^{\mathrm{pre}}(E_{\leqslant m})|_{X_{Nis}}$$

for $m \geqslant n + d$. Sheafifiying proves $(*)$ (note that the map $\underline{\pi}_n(E) \to \underline{\pi}_n(E_{\leqslant m})$ is an isomorphism for $n \leqslant m$). ☐

Corollary 4.5. *Let*

$$\cdots \longrightarrow E_{i+1} \longrightarrow E_i \longrightarrow E_{i-1} \longrightarrow \cdots \longrightarrow E_1 \longrightarrow E_0$$

be an inverse system of objects in $\mathsf{SH}(S)$*. Suppose for each* $n \in \mathbb{N}$ *there is an* $N \in \mathbb{N}$ *such that* $E_i \in \mathsf{SH}(S)_{h\geqslant n}$ *for* $i \geqslant N$*. Then* $\mathrm{holim}_{i\to\infty} E_i \cong 0$.

Proof. Fix $q \in \mathbb{Z}$ and let F_i be the image of $\Sigma^{q,q} E_i$ in $\mathsf{SH}_s^{S^1}(S)$. We are ready if we show $\mathrm{holim}_{i\to\infty} F_i \cong 0$. By assumption for every $n \in \mathbb{N}$ there is a $N \in \mathbb{N}$ such that $F_i \in \mathsf{SH}_s^{S^1}(S)_{\geqslant n}$ for each $i \geqslant N$. By Lemma 4.4 we have $F_i \cong \mathrm{holim}_{k\to\infty}(F_i)_{\leqslant k}$. Thus

$$\mathrm{holim}_{i\to\infty} F_i \cong \mathrm{holim}_i \mathrm{holim}_k (F_i)_{\leqslant k} \cong \mathrm{holim}_k \mathrm{holim}_i (F_i)_{\leqslant k} \cong \mathrm{holim}_k 0 \cong 0. \quad ☐$$

We also have

Corollary 4.6. *Let* $E \in \mathsf{SH}(S)_{h\geqslant n}$ *and* $X \in \mathrm{Sm}_S$ *of dimension* d*. Then*

$$\underline{\pi}_{p,q}^{\mathrm{pre}}(E)(X) = 0$$

for $p - q < n - d$.

Proof. Let F be the image of $\Sigma^{-q,-q} E$ in $\mathsf{SH}_s^{S^1}(S)$. Then F lies in $\mathsf{SH}(S)_s^{S^1}(S)_{\geqslant n}$ and $\underline{\pi}_{p,q}^{\mathrm{pre}}(E)(X) \cong \underline{\pi}_{p-q}^{\mathrm{pre}}(F)(X)$. For $m \geqslant n$ the object $F_{\leqslant m}$ is a finite iterated extension of the objects $F_{=r}$ for $n \leqslant r \leqslant m$. For each such r we have $\underline{\pi}_{p-q}^{\mathrm{pre}}(F_{=r})(X) = 0$ for $p - q < n - d$, since the Nisnevich cohomological dimension of X is bounded by d, so we also have $\underline{\pi}_{p-q}^{\mathrm{pre}}(F_{\leqslant m})(X) = 0$ for each $m \geqslant n$. Now the claim follows from Lemma 4.4 and the Milnor short exact sequence. ☐

Proposition 4.7. *Let*

$$\cdots \longrightarrow E_{i+1} \longrightarrow E_i \longrightarrow E_{i-1} \longrightarrow \cdots \longrightarrow E_1 \longrightarrow E_0$$

be an inverse system of objects in $\mathsf{SH}(S)_{h\geqslant n}$*. Suppose for each* $p, q \in \mathbb{Z}$ *and* $d \in \mathbb{N}$ *there is an* $N \in \mathbb{N}$ *such that for* $X \in \mathrm{Sm}_S$ *of dimension* d *the map*

$$\underline{\pi}_{p,q}^{\mathrm{pre}}(E_{i+1})(X) \to \underline{\pi}_{p,q}^{\mathrm{pre}}(E_i)(X)$$

is an isomorphism for all $i \geqslant N$*. Then* $\mathrm{holim}_{i\to\infty} E_i \in \mathsf{SH}(S)_{h\geqslant n}$*. (Here the latter homotopy limit is computed in* $\mathsf{SH}(S)$*.)*

Proof. Let $p, q \in \mathbb{Z}$, $d \in \mathbb{N}$ and $X \in \mathrm{Sm}_S$ of dimension d. Choose $N \in \mathbb{N}$ such that for any $Y \in \mathrm{Sm}_S$ of dimension $\leqslant d$ the map

$$\underline{\pi}_{p,q}^{\mathrm{pre}}(E_{i+1})(Y) \longrightarrow \underline{\pi}_{p,q}^{\mathrm{pre}}(E_i)(Y)$$

is an isomorphism for all $i \geqslant N$. We claim that

$$\underline{\pi}_{p,q}(\mathrm{holim}_k E_k)|_{X_{Nis}} \cong \underline{\pi}_{p,q}(E_i)|_{X_{Nis}}$$

for all $i \geqslant N$. For every $Y \in X_{Nis}$ we have the Milnor short exact sequence

$$0 \to \lim_i{}^1 \underline{\pi}_{p+1,q}^{\mathrm{pre}}(E_i)(Y) \longrightarrow \underline{\pi}_{p,q}^{\mathrm{pre}}(\mathrm{holim}_i E_i)(Y) \longrightarrow \lim_i \underline{\pi}_{p,q}^{\mathrm{pre}}(E_i)(Y) \to 0.$$

The $\lim^1$-term vanishes because the inverse system of abelian groups stabilizes by assumption. Sheafifying we see that $\underline{\pi}_{p,q}(\mathrm{holim}_k E_k)|_{X_{Nis}} \cong \underline{\pi}_{p,q}(E_i)|_{X_{Nis}}$ for $i \geqslant N$, in particular $\underline{\pi}_{p,q}(\mathrm{holim}_k E_k)|_{X_{Nis}} = 0$ in case $p - q < n$. Since this is true for all $X \in \mathrm{Sm}_S$ we conclude $\underline{\pi}_{p,q}(\mathrm{holim}_k E_k) = 0$ for $p - q < n$. □

5. Connectivity of algebraic cobordism

Lemma 5.1. *Let X be a smooth scheme over a Dedekind domain of mixed characteristic or over a field. Then for any abelian group A we have $\mathsf{M}A_X \in \mathsf{SH}(X)_{h\geqslant 0}$.*

Proof. This follows from the fact that the motivic complexes $\mathcal{M}(r)$ have vanishing i-th cohomology sheaf for $i > r$, see [**3**, Corollary 4.4]. (Note that for $Y \in \mathrm{Sm}_X$ the restriction $\mathcal{M}(r)|_{Y_{Zar}}$ is canonically equivalent to the complex of sheaves $\mathbb{Z}(r)_{Zar}$ on Y_{Zar} considered in [**3**] by the comparison result [**9**, Theorem 7.18].) □

Proposition 5.2. *Let S be the spectrum of a discrete valuation ring of mixed characteristic, $j\colon \eta \to S$ the inclusion of the generic point. Then for any abelian group A we have $j_*\mathsf{M}A_\eta \in \mathsf{SH}(S)_{h\geqslant 0}$.*

Proof. Let $i\colon s \to S$ be the inclusion of the closed point. We have an exact triangle

$$i_! i^! \mathsf{M}A_S \longrightarrow \mathsf{M}A_S \longrightarrow j_*\mathsf{M}A_\eta \longrightarrow i_! i^! \mathsf{M}A_S[1]$$

and an isomorphism $i^!\mathsf{M}A_S \cong \mathsf{M}A_s(-1)[-2] \in \mathsf{SH}(s)_{h\geqslant -1}$, see [**9**, Theorem 7.4]. We conclude with Proposition 4.2 and Lemma 5.1. □

Lemma 5.3. *Let the situation be as in Proposition 5.2. Then*

$$j_*\mathsf{MGL}_\eta \langle 0, n\rangle \wedge M_A \in \mathsf{SH}(S)_{h\geqslant 0}$$

for all $n \geqslant 0$.

Proof. We can assume $A = \mathbb{Z}$. Since η is of characteristic 0 we have

$$s_n \mathsf{MGL}_\eta \cong \Sigma^{2n,n} \mathsf{M}\mathbb{Z} \otimes L_n.$$

Moreover, we have exact triangles

$$s_n\mathsf{MGL}_\eta \longrightarrow \mathsf{MGL}_\eta \langle 0, n\rangle \longrightarrow \mathsf{MGL}_\eta \langle 0, n-1\rangle \longrightarrow s_n\mathsf{MGL}_\eta[1].$$

Applying j_* to these triangles and using Proposition 5.2 one concludes by induction on n. □

Lemma 5.4. *Let the situation be as in Proposition 5.2. Let $p, q \in \mathbb{Z}$ and $X \in \mathrm{Sm}_S$ of dimension d. Then*

$$\underline{\pi}^{\mathrm{pre}}_{p,q}(j_*\mathsf{MGL}_\eta \langle 0, n+1\rangle)(X) \longrightarrow \underline{\pi}^{\mathrm{pre}}_{p,q}(j_*\mathsf{MGL}_\eta \langle 0, n\rangle)(X)$$

is an isomorphism for $n \geqslant p - q + d$.

Proof. Consider the exact triangle

$$j_*s_{n+1}\mathsf{MGL}_\eta \longrightarrow j_*\mathsf{MGL}_\eta \langle 0, n+1\rangle \longrightarrow j_*\mathsf{MGL}_\eta \langle 0, n\rangle \longrightarrow s_{n+1}\mathsf{MGL}_\eta[1].$$

We have

$$\underline{\pi}^{\mathrm{pre}}_{p,q}(j_*s_{n+1}\mathsf{MGL}_\eta)(X) = H^{2(n+1)-p, n+1-q}_{\mathrm{mot}}(X_\eta, L_{n+1}).$$

The latter group vanishes for $2(n+1) - p > n + 1 - q + d$, showing the claim. □

Lemma 5.5. *Let the situation be as in Proposition 5.2. Then $j_*\mathsf{MGL}_\eta \in \mathsf{SH}(S)_{\mathsf{h}\geqslant 0}$.*

Proof. Consider the inverse system

$$\cdots \longrightarrow j_*\mathsf{MGL}_\eta \langle 0, n+1\rangle \longrightarrow j_*\mathsf{MGL}_\eta \langle 0, n\rangle \longrightarrow \cdots \longrightarrow j_*s_0\mathsf{MGL}_\eta$$

in $\mathsf{SH}(S)$. Since j_* preserves homotopy limits the homotopy limit over this system is $j_*\mathsf{MGL}_\eta$, using [**4**, Corollary 2.4 and Lemma 8.10 or Theorem 8.12]. By Lemma 5.3 every object of this system is in $\mathsf{SH}(S)_{h\geqslant 0}$. Moreover, by Lemma 5.4 the assumptions of Proposition 4.7 are satisfied. Thus this Proposition implies the claim. □

Proposition 5.6. *Let the situation be as in Proposition 5.2 and let $i\colon s \to S$ be the inclusion of the closed point. Then $i^!\mathsf{MGL}_S \in \mathsf{SH}(s)_{\geqslant -1}$.*

Proof. Note first that i^* sends $\mathsf{SH}(S)_{\geqslant 0}$ to $\mathsf{SH}(s)_{\geqslant 0}$. We have $\mathsf{MGL} \in \mathsf{SH}(S)_{\geqslant 0}$ and by Lemma 5.5 also $j_*\mathsf{MGL}_\eta \in \mathsf{SH}(S)_{h\geqslant 0} \subset \mathsf{SH}(S)_{\geqslant 0}$. Applying i^* to the exact triangle

$$i_!i^!\mathsf{MGL}_S \longrightarrow \mathsf{MGL}_S \longrightarrow j_*\mathsf{MGL}_\eta \longrightarrow i_!i^!\mathsf{MGL}_S[1]$$

shows the claim. □

Lemma 5.7. *Let S be the spectrum of a discrete valuation ring of mixed characteristic. Then $\mathsf{MGL}_S \in \mathsf{SH}(S)_{h\geqslant -1}$.*

Proof. Let the notation be as above. The claim follows from the exact triangle

$$i_!i^!\mathsf{MGL}_S \longrightarrow \mathsf{MGL}_S \longrightarrow j_*\mathsf{MGL}_\eta \longrightarrow i_!i^!\mathsf{MGL}_S[1],$$

Lemma 5.5, Proposition 5.6 and Proposition 4.2. □

Proposition 5.8. *Let S be the spectrum of a Dedekind domain of mixed characteristic. Then $\mathsf{MGL}_S \in \mathsf{SH}(S)_{h\geqslant -1}$.*

Proof. The henselization of a local ring of a scheme in Sm_S lies over a local ring of S, thus the claim follows from Lemma 5.7. □

Compare the following result to [**11**, Conjecture 15].

Corollary 5.9. *Let S be the spectrum of a Dedekind domain of mixed characteristic. Let R be a localization of $\mathbb{Z}$ such that $\Phi_S \wedge M_R$ is an isomorphism. Then for any R-module A we have*

$$f_n\mathsf{MGL}_S \wedge M_A \cong \mathrm{holim}_{k\to\infty}\mathsf{MGL}_S \langle n, k\rangle \wedge M_A.$$

Proof. Under the assumption we have $f_k\mathsf{MGL}_S \wedge M_A \in \mathsf{SH}(S)_{h\geqslant k-1}$, since this is a homotopy colimit of objects of the form $\Sigma^{2i,i}\mathsf{MGL}_S \wedge M_A$ with $i \geqslant k$, see the proof of [**7**, Theorem 4.7], using Proposition 5.8. Thus by Corollary 4.5 we have

$$\mathrm{holim}_{k\to\infty} f_k\mathsf{MGL}_S \wedge M_A \cong 0$$

implying the claim. □

Remark 5.10. A similar result holds for motivic Landweber spectra using the same argument as in the proof of [**4**, Lemma 8.11]. For example, $\mathsf{KGL}_S \wedge M_A$ is complete with respect to the slice filtration.

6. The geometric part of algebraic cobordism

Lemma 6.1. *Let S be the spectrum of a Dedekind domain of mixed characteristic. Let R be a localization of $\mathbb{Z}$ such that $\Phi_S \wedge M_R$ is an isomorphism. Let $p, q \in \mathbb{Z}$ and $X \in \mathrm{Sm}_S$. Then for any R-module A the inverse system of abelian groups $(\underline{\pi}^{\mathrm{pre}}_{p,q}(\mathsf{MGL}_S\langle 0,k\rangle \wedge M_A)(X))_k$ eventually becomes constant for $k \to \infty$.*

Proof. This follows from the exact triangle

$$s_k\mathsf{MGL}_S \wedge M_A \longrightarrow \mathsf{MGL}_S\langle 0,k\rangle \wedge M_A \longrightarrow \mathsf{MGL}_S\langle 0,k-1\rangle \wedge M_A \longrightarrow s_k\mathsf{MGL}_S \wedge M_A[1]$$

and $s_k\mathsf{MGL}_S \wedge M_A \cong \Sigma^{2k,k}\mathsf{M}A \otimes L_k$ since $\underline{\pi}^{\mathrm{pre}}_{p,q}(\Sigma^{2k+j,k}\mathsf{M}A)(X) = 0$, $j \geqslant 0$, for k big enough. □

Corollary 6.2. *Let S be the spectrum of a Dedekind domain of mixed characteristic. Let R be a localization of $\mathbb{Z}$ such that $\Phi_S \wedge M_R$ is an isomorphism. Let $p, q \in \mathbb{Z}$ and $X \in \mathrm{Sm}_S$. Then for any R-module A the canonical map*

$$\underline{\pi}^{\mathrm{pre}}_{p,q}(\mathsf{MGL}_S \wedge M_A)(X) \longrightarrow \lim_k \underline{\pi}^{\mathrm{pre}}_{p,q}(\mathsf{MGL}_S\langle 0,k\rangle \wedge M_A)(X)$$

is an isomorphism.

Proof. This follows from Corollary 5.9, the Milnor short exact sequence and Lemma 6.1. □

Lemma 6.3. *Let S be the spectrum of a Dedekind domain of mixed characteristic. Let R be a localization of $\mathbb{Z}$ such that $\Phi_S \wedge M_R$ is an isomorphism. Let $n \in \mathbb{Z}$. Then for $k \geqslant n+1$ and any R-module A the natural map*

$$\pi_{2n,n}\mathsf{MGL}_S\langle n,k+1\rangle \wedge M_A \longrightarrow \pi_{2n,n}\mathsf{MGL}_S\langle n,k\rangle \wedge M_A$$

is an isomorphism.

Proof. This follows from the exact sequence

$$\pi_{2n,n}s_{k+1}\mathsf{MGL}_S \wedge M_A \longrightarrow \pi_{2n,n}\mathsf{MGL}_S\langle n,k+1\rangle \wedge M_A \longrightarrow \pi_{2n,n}\mathsf{MGL}_S\langle n,k\rangle \wedge M_A \longrightarrow \pi_{2n-1,n}s_k\mathsf{MGL}_S \wedge M_A$$

and the fact that the two outer terms are 0 for $k \geqslant n+1$. □

Corollary 6.4. *Let S be the spectrum of a Dedekind domain of mixed characteristic. Let R be a localization of $\mathbb{Z}$ such that $\Phi_S \wedge M_R$ is an isomorphism. Then for any R-module A the canonical map*

$$\pi_{2n,n}\mathsf{MGL}_S \wedge M_A \longrightarrow \pi_{2n,n}\mathsf{MGL}_S \langle n, n+1\rangle \wedge M_A$$

is an isomorphism.

Proof. This follows from Corollary 6.2 and Lemma 6.3. □

Theorem 6.5. *Let S be the spectrum of a Dedekind domain of mixed characteristic. Let R be a localization of $\mathbb{Z}$ such that $\Phi_S \wedge M_R$ is an isomorphism. Then for every $n \in \mathbb{Z}$ and R-module A there is a canonical isomorphism*

$$\pi_{2n,n}\mathsf{MGL}_S \wedge M_A \cong L_n \otimes A \oplus L_{n+1} \otimes \mathrm{Pic}(S) \otimes A.$$

Proof. We have the exact sequence

$$\pi_{2n+1,n}s_n\mathsf{MGL}_S \wedge M_A \longrightarrow \pi_{2n,n}s_{n+1}\mathsf{MGL}_S \wedge M_A \longrightarrow \pi_{2n,n}\mathsf{MGL}_S \langle n, n+1\rangle \wedge M_A$$
$$\longrightarrow \pi_{2n,n}s_n\mathsf{MGL}_S \wedge M_A \longrightarrow \pi_{2n-1,n}s_{n+1}\mathsf{MGL}_S \wedge M_A.$$

The two outer terms are 0. Also $\pi_{0,0}\Sigma^{2,1}\mathsf{M}A_S \cong \mathrm{Pic}(S) \otimes A$. Moreover, there is a canonical map $L_n \otimes A \to \pi_{2n,n}\mathsf{MGL}_S \wedge M_A$ splitting the resulting short exact sequence, whence the claim follows form Corollary 6.4. □

Corollary 6.6. *Let S be the spectrum of a Dedekind domain of mixed characteristic and R the localization of $\mathbb{Z}$ obtained by inverting all positive residue characteristics of S. Then*

$$(\pi_{2n,n}\mathsf{MGL}_S) \otimes R \cong (L_n \oplus L_{n+1} \otimes \mathrm{Pic}(S)) \otimes R.$$

We have the following weakened versions (because we have to invert the residue characteristic) of cases of a Conjecture of Voevodsky (see [**10**, Conjecture 1]):

Theorem 6.7. *Let $S = \mathrm{Spec}(R)$, where R is a (regular) Noetherian local ring which is regular over some discrete valuation ring of mixed characteristic. Then the natural map*

$$L_* \longrightarrow (\mathsf{MGL}_S)_{(2,1)*}$$

becomes an isomorphism after inverting the residue characteristic of the closed point of S.

Proof. By Popescu's Theorem R is a filtered colimit of smooth algebras over a discrete valuation ring V of mixed characteristic. Thus we are reduced to the case where R is the local ring of a scheme $X \in \mathrm{Sm}_{\mathrm{Spec}(V)}$ by a colimit argument. Let l be the residue characteristic of the closed point of $\mathrm{Spec}(V)$. By the same type of argument as above and the vanishing of (p,q)-motivic cohomology of such local rings for $p > q$ we have

$$(\mathsf{MGL}_S)_{2n,n}[1/l] \cong (s_n\mathsf{MGL}_S)_{2n,n}[1/l] \cong L_n[1/l],$$

using that for a fixed dimension only a fixed finite number of slices of $\mathsf{MGL}_S[1/l]$ contribute to the value of $\pi^{\mathrm{pre}}_{2n,n}(\mathsf{MGL}_S[1/l])$ on schemes of that dimension. □

More generally, we have

Proposition 6.8. *Let S be as in the previous Theorem and $E \in \mathsf{SH}(S)$ a motivic Landweber spectrum modelled on E^{top}_{2*} (see [**6**]). Then the natural map*

$$E^{\mathrm{top}}_{2*} \longrightarrow E_{(2,1)*}$$

is an isomorphism after inverting the residue characteristic of the closed point of S.

Proof. This follows from the definition of motivic Landweber spectrum, since it represents the assignment $\mathsf{SH}(S) \ni X \mapsto \mathsf{MGL}_{S,**}(X) \otimes_{\mathsf{MU}_*} E^{\mathrm{top}}_{2*}$ (see [**6**, Theorem 8.7]), using Theorem 6.7. □

7. Some other parts of algebraic cobordism

We have the following vanishing result:

Proposition 7.1. *Let S be the spectrum of a Dedekind domain of mixed characteristic. Let R be a localization of $\mathbb{Z}$ such that $\Phi_S \wedge M_R$ is an isomorphism. Then for any $p, q \in \mathbb{Z}$ and R-module A we have $\underline{\pi}_{p,q}\mathsf{MGL}_S \wedge M_A \cong 0$ for $p < 2q$ or $p < q$. In particular, we have $\mathsf{MGL}_S \wedge M_R \in \mathsf{SH}(S)_{h \geqslant 0}$.*

Proof. Let $p, q \in \mathbb{Z}$. Let $d \in \mathbb{N}$. Then there is a $N \geqslant q$ such that for any scheme of dimension $\leqslant d$ and $k \geqslant N$ the map

$$\underline{\pi}^{\mathrm{pre}}_{p,q}(\mathsf{MGL}_S \wedge M_A)(X) \longrightarrow \underline{\pi}^{\mathrm{pre}}_{p,q}(\mathsf{MGL}_S\,\langle 0, k\rangle \wedge M_A)(X) \qquad (1)$$

is an isomorphism. The assertion then follows by an induction argument on i showing that $\underline{\pi}_{p,q}(\mathsf{MGL}_S\,\langle q, q+i\rangle \wedge M_A) = 0$ if p, q satisfy the condition of the statement. □

Lemma 7.2. *Let S be the spectrum of a Dedekind domain of mixed characteristic. Let R be a localization of $\mathbb{Z}$ such that $\Phi_S \wedge M_R$ is an isomorphism. Then for any $p, q \in \mathbb{Z}$ and R-module A we have*

$$\underline{\pi}_{p,q}(\mathsf{MGL}_S \wedge M_A) \cong \lim_k \underline{\pi}_{p,q}(\mathsf{MGL}_S\,\langle 0, k\rangle \wedge M_A) \cong \underline{\pi}_{p,q}(\mathsf{MGL}_S\,\langle \max(0,q), n\rangle \wedge M_A)$$

for $n \geqslant p - q$ or $n \geqslant p - 2q$.

Proof. The first isomorphism follows from (1), the second isomorphism follows then by noting that

$$\underline{\pi}_{p,q}(\mathsf{MGL}_S\,\langle 0, k\rangle \wedge M_A) \cong \underline{\pi}_{p,q}(\mathsf{MGL}_S\,\langle 0, n\rangle \wedge M_A) \cong \underline{\pi}_{p,q}(\mathsf{MGL}_S\,\langle \max(0,q), n\rangle \wedge M_A)$$

for $k \geqslant n$ and $n \geqslant p - q$ or $n \geqslant p - 2q$. □

Corollary 7.3. *Let S be the spectrum of a Dedekind domain of mixed characteristic. Let R be a localization of $\mathbb{Z}$ such that $\Phi_S \wedge M_R$ is an isomorphism. Then for any R-module A and $n \in \mathbb{Z}$ we have $\underline{\pi}_{n,n}(\mathsf{MGL}_S \wedge M_A) \cong \underline{K}^M_{-n} \otimes A$, where $\underline{K}^M_{-n}$ is the $(-n)$-th Milnor-K-theory sheaf defined via the degree $(-n,-n)$-motivic cohomology.*

Proof. It follows from Lemma 7.2 that

$$\underline{\pi}_{n,n}(\mathsf{MGL}_S \wedge M_A) \cong \underline{\pi}_{n,n}(s_0(\mathsf{MGL}_S \wedge M_A)) \cong \underline{\pi}_{n,n}(\mathsf{M}A_S),$$

whence the claim. □

Corollary 7.4. *Let S be the spectrum of a Dedekind domain of mixed characteristic. Let R be a localization of $\mathbb{Z}$ such that $\Phi_S \wedge M_R$ is an isomorphism. Then for any R-module A and $n \in \mathbb{Z}$ we have $\underline{\pi}_{2n,n}(\mathsf{MGL}_S \wedge M_A) \cong \underline{L}_n \otimes A$, where the latter sheaf is the constant sheaf on $L_n \otimes A$.*

Proof. It follows from Lemma 7.2 that

$$\underline{\pi}_{2n,n}(\mathsf{MGL}_S \wedge M_A) \cong \underline{\pi}_{2n,n}(s_n(\mathsf{MGL}_S \wedge M_A)) \cong \underline{\pi}_{0,0}(\mathsf{M}A_S \otimes L_n). \qquad \square$$

Corollary 7.5. *Let S be the spectrum of a Dedekind domain of mixed characteristic. Let R be a localization of $\mathbb{Z}$ such that $\Phi_S \wedge M_R$ is an isomorphism. Then for any R-module A and $n \in \mathbb{Z}$ we have $\underline{\pi}_{2n+1,n}(\mathsf{MGL}_S \wedge M_A) \cong \mathcal{O}^* \otimes L_{n+1} \otimes A$.*

Proof. By Lemma 7.2 we have

$$\underline{\pi}_{2n+1,n}(\mathsf{MGL}_S \wedge M_A) \cong \underline{\pi}_{2n+1,n}(\mathsf{MGL}_S \langle n, n+1 \rangle \wedge M_A).$$

The long exact sequence of sheaves associated to the exact triangle

$$s_{n+1}\mathsf{MGL}_S \wedge M_A \longrightarrow \mathsf{MGL}_S \langle n, n+1 \rangle \wedge M_A \longrightarrow s_n\mathsf{MGL}_S \wedge M_A \longrightarrow s_{n+1}\mathsf{MGL}_S \wedge M_A[1]$$

together with

$$\underline{\pi}_{2n+1,n}(s_n\mathsf{MGL}_S \wedge M_A[-1]) = \underline{\pi}_{2n+1,n}(s_n\mathsf{MGL}_S \wedge M_A) = 0$$

and

$$\underline{\pi}_{0,0}(\Sigma^{1,1}\mathsf{M}A_S) \cong \mathcal{O}^* \otimes A$$

gives the result. $\square$

Corollary 7.6. *Let S be the spectrum of a Dedekind domain of mixed characteristic. Let R be a localization of $\mathbb{Z}$ such that $\Phi_S \wedge M_R$ is an isomorphism. Then for any R-module A and $n \in \mathbb{Z}$ there is an exact sequence*

$$\underline{K}^M_{1-n} \otimes A \longrightarrow \underline{\pi}_{n+1,n}(\mathsf{MGL}_S \wedge M_A) \longrightarrow \mathcal{H}^{-n-1,-n}_{\mathrm{mot}}(-, A) \longrightarrow 0,$$

where the latter group denotes the motivic cohomology sheaf in degrees $(-n-1, -n)$ and A-coefficients.

Proof. By Lemma 7.2 we have

$$\underline{\pi}_{n+1,n}(\mathsf{MGL}_S \wedge M_A) \cong \underline{\pi}_{n+1,n}(\mathsf{MGL}_S \langle 0, 1 \rangle \wedge M_A).$$

The long exact sequence of sheaves associated to the exact triangle

$$s_1\mathsf{MGL}_S \wedge M_A \longrightarrow \mathsf{MGL}_S \langle 0, 1 \rangle \wedge M_A \longrightarrow s_0\mathsf{MGL}_S \wedge M_A \longrightarrow s_1\mathsf{MGL}_S \wedge M_A[1]$$

together with

$$\underline{\pi}_{n+1,n} s_1(\mathsf{MGL}_S \wedge M_A[1]) = 0$$

shows the claim. $\square$

We also have

Proposition 7.7. *Let S be the spectrum of a Dedekind domain of mixed characteristic. Let R be a localization of $\mathbb{Z}$ such that $\Phi_S \wedge M_R$ is an isomorphism. Then for any R-module A and $n \in \mathbb{Z}$ there is an exact sequences*

$$H^{3,2}(S) \otimes A \otimes L_{n+2} \longrightarrow \pi_{2n+1,n}(\mathsf{MGL}_S \wedge M_A) \longrightarrow H^{1,1}(S,A) \otimes L_{n+1} \longrightarrow 0.$$

If A is torsion free the first map is also injective.

Proof. We have $\pi_{2n+1,n}(\mathsf{MGL}_S \wedge M_A) \cong \pi_{2n+1,n}(\mathsf{MGL}_S \langle n, n+2\rangle \wedge M_A)$ since higher slices do not contribute. The long exact sequence of homotopy groups associated to the exact triangle

$$s_{n+2}(\mathsf{MGL}_S \wedge M_A) \longrightarrow \mathsf{MGL}_S \langle n, n+2\rangle \wedge M_A \longrightarrow \mathsf{MGL}_S \langle n, n+1\rangle \longrightarrow s_{n+2}(\mathsf{MGL}_S \wedge M_A)[1]$$

shows the claim. □

Proposition 7.8. *Let S be the spectrum of a Dedekind domain of mixed characteristic. Let R be a localization of $\mathbb{Z}$ such that $\Phi_S \wedge M_R$ is an isomorphism. Let X be an essentially smooth scheme over S. Then for any R-module A, $n \in \mathbb{Z}$ and $i \geqslant 2$ we have $\mathsf{MGL}_S^{2n+i,n}(X, A) = 0$.*

Proof. By (1) only finitely many slices of $\mathsf{MGL}_S \wedge M_A$ contribute to this group, so the claim follows from the fact that for $X \in \mathrm{Sm}_S$ we have $H^{p,q}(X) = 0$ for $p \geqslant 2q+2$, since motivic cohomology is computed as hypercohomology over S of the Bloch-Levine cycle complexes and these complexes vanish in degrees greater than $2q$. □

We leave proofs of the assertions

$$\pi_{2n-1,n}\mathsf{MGL}_S \wedge M_A = 0,$$
$$\pi_{n-1,n}\mathsf{MGL}_S \wedge M_A \cong H^{-n+1,-n}(S) \otimes A$$

for all $n \in \mathbb{Z}$, $\pi_{n,n}\mathsf{MGL}_S \wedge M_A = 0$ for $n > 0$, and the existence of an exact sequence

$$H^{-n+2,-n+1}(S) \otimes A \longrightarrow \pi_{n,n}\mathsf{MGL}_S \wedge M_A \longrightarrow H^{-n,-n}(S,A) \longrightarrow 0$$

for $n < 0$ to the interested reader.

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