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# A DG LIE MODEL FOR RELATIVE HOMOTOPY AUTOMORPHISMS

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*Abstract*

We construct a dg Lie algebra model for the universal cover of the classifying space of the grouplike monoid of homotopy automorphisms of a space that fix a given subspace. We derive the model from a known model for based homotopy automorphisms together with general result on rational models for geometric bar constructions.

## 1. Introduction

The classifying space of the monoid of homotopy automorphisms of a space $X$ classifies fibrations with fiber homotopy equivalent to $X$. Given a subspace $A \subset X$, the classifying space of the monoid $\operatorname{aut}_A(X)$ of homotopy automorphisms that restrict to the identity on $A$ classifies all fibrations $E \to B$ with fiber homotopy equivalent to $X$ under the trivial fibration $A \times B \to B$, such that, over each $b \in B$, the canonical map from $A$ to $\operatorname{im}(A \to E_b)$ is a weak equivalence. The special case in which $X$ is a manifold with a non-empty boundary and where $A = \partial X$ is the boundary, has been of interest in the study of homological stability for homotopy automorphisms of manifolds (see [**BM13, BM20, Gre19**]).

The main result of this paper is a proof of the following theorem:

**Theorem 1.1** ([**BM20**, Theorem 3.4])**.** *Let $A \subset X$ be a cofibration of simply connected spaces with homotopy types of finite CW-complexes, and let $i\colon \mathbb{L}_A \to \mathbb{L}_X$ be a cofibration that models the inclusion $A \subset X$ and where $\mathbb{L}_A$ and $\mathbb{L}_X$ are cofibrant Lie models for $A$ and $X$ respectively. A Lie model for the universal covering of $B\operatorname{aut}_A(X)$ is given by the positive truncation of the dg Lie algebra of derivations on $\mathbb{L}_X$ that vanish on $\mathbb{L}_A$, denoted by* $\operatorname{Der}(\mathbb{L}_X \| \mathbb{L}_A)\langle 1\rangle$.

This theorem is stated in [**BM20**] together with the suggestion that a proof can be given by generalizing [**Tan83**, Chapitre VII], but no detailed proof exists in the literature. One purpose of this paper is to fill this gap. However, instead of following the suggested route (which seems to yield a rather tedious proof), we give a proof that is perhaps more interesting. Namely, we show that the model for relative homotopy automorphisms can be derived from the known model for based homotopy

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automorphisms together with general result on rational models for geometric bar constructions.

### 1.1. Standing assumptions and notation

- Throughout the paper, $X$ is a pointed simply connected space and $A$ is a simply connected subspace that contains the basepoint of $X$. The inclusion $A \subset X$ is assumed be a cofibration. We let $\mathbb{L}_A$ and $\mathbb{L}_X$ denote cofibrant dg Lie models over $\mathbb{Q}$ for $A$ and $X$ respectively. A dg Lie algebra is cofibrant if and only if its underlying graded Lie algebra is a free graded Lie algebra $\mathbb{L}(V)$ on a graded vector space $V$. We let $i\colon \mathbb{L}_A \to \mathbb{L}_X$ denote a cofibration that models the inclusion $A \subset X$. Recall that a map of free dg Lie algebras is a cofibration if and only if it is a free map (see the remark after Proposition 5.5 in [**Qui69**]).
- All dg Lie algebras and dg coalgebras are homologically graded, which means that the differential lowers the degree. All dg associative algebras are cohomologically graded. Note that if $L$ is a dg Lie algebra and $\Omega$ is a commutative dg associative algebra then $L \otimes \Omega$ is a dg Lie algebra over $\Omega$ with homological grading given by
$$(L \otimes \Omega)_n = \bigoplus_{p-q=n} L_p \otimes \Omega^q.$$
An analogous statement holds for dg coalgebras.
- The suspension $sV$ of a homologically graded dg vector space $V$ is a dg vector space with grading given by $(sV)_n = V_{n-1}$ and with differential given by $d(sa) = -sd(a)$.
- If $\mathfrak{h}$ is a dg Lie algebra, we define its $n$-connected cover $\mathfrak{h}\langle n\rangle \subseteq \mathfrak{h}$ to be the dg Lie subalgebra given by
$$\mathfrak{h}\langle n\rangle_i = \begin{cases} \mathfrak{h}_i & \text{if } i > n, \\ \ker(\mathfrak{h}_n \xrightarrow{d} \mathfrak{h}_{n-1}) & \text{if } i = n, \\ 0 & \text{if } i < n. \end{cases}$$
We say that $\mathfrak{h}$ is connected if $\mathfrak{h} = \mathfrak{h}\langle 0\rangle$ and we say that $\mathfrak{h}$ is simply connected if $\mathfrak{h} = \mathfrak{h}\langle 1\rangle$.
- Given a dg Lie algebra $(L, d)$, let $(\mathrm{Der}(L), D)$ denote the dg Lie algebra of derivations on $L$. We remind the reader that a derivation on $L$ is a linear map $\theta\colon L \to L$ that satisfy the equality $\theta[x,y] = [\theta(x), y] + (-1)^{|\theta||x|}[x, \theta(y)]$. The Lie bracket and the differential on $\mathrm{Der}(L)$ are given by
$$[\theta, \varphi] = \theta \circ \varphi - (-1)^{|\theta||\varphi|}\varphi \circ \theta, \qquad D(\theta) = d \circ \theta - (-1)^{|\theta|}\theta \circ d.$$
- The connected component of the identity map in $\mathrm{aut}_*(X)$ and $\mathrm{aut}_A(X)$ are denoted by $\mathrm{aut}_{*,\circ}(X)$ and $\mathrm{aut}_{A,\circ}(X)$ respectively. The connected component of the inclusion map $\iota\colon A \hookrightarrow X$ in $\mathrm{map}_*(A, X)$ is denoted by $\mathrm{map}_*^{\iota}(A, X)$.

### 1.2. Strategy for the proof

We observe that the universal cover of $B\,\mathrm{aut}_A(X)$ is homotopy equivalent to $B\,\mathrm{aut}_{A,\circ}(X)$ (if $G$ is a topological group and $G^\circ$ is the connected component of the identity, then $BG^\circ \to BG \to B\pi_0(G) \cong K(\pi_0(G), 1)$ is equivalent to a fibration, giving that $BG^\circ \to BG$ induces isomorphisms $\pi_k(BG^\circ) \xrightarrow{\cong} \pi_k(BG)$ for $k \geqslant 2$, which implies that $BG^\circ \simeq \widetilde{BG}$).

In Section 2, we show that $B\operatorname{aut}_{A,\circ}(X) \simeq B(*, \operatorname{aut}_{*,\circ}(X), \operatorname{map}_*^\iota(A,X))$, where the right-hand side is the geometric bar construction of $\operatorname{aut}_{*,\circ}(X)$ and the left $\operatorname{aut}_{*,\circ}(X)$-space $\operatorname{map}_*^\iota(A,X)$. The rational homotopy type of $B\operatorname{aut}_{*,\circ}(X)$ and $\operatorname{map}_*^\iota(A,X)$ are known and the identification of $B\operatorname{aut}_{A,\circ}(X)$ with the geometric bar construction above gives us a way of expressing the Lie model for $B\operatorname{aut}_{A,\circ}(X)$ in terms of the Lie models for $B\operatorname{aut}_{*,\circ}(X)$ and $\operatorname{map}_*^\iota(A,X)$.

Briefly, if a grouplike monoid $G$ acts on $X$ from the left, then $B(*,G,X)$ is modelled by a twisted semidirect product $\mathfrak{g} \ltimes_\xi L$ where $\mathfrak{g}$ is a Lie model for $BG$ and $L$ is a Lie model for $X$. This is treated in Section 3.

In Section 4 we apply the theory of Section 3 to

$$B\operatorname{aut}_{A,\circ}(X) \simeq B(*, \operatorname{aut}_{*,\circ}(X), \operatorname{map}_*^\iota(A,X)),$$

and get that a Lie model for $B\operatorname{aut}_{A,\circ}(X)$ is given by a twisted semidirect product $\operatorname{Der}(\mathbb{L}_X)\langle 1\rangle \ltimes_{\tau_*} \operatorname{Hom}^\tau(\bar{\mathcal{C}}(\mathbb{L}_A), \mathbb{L}_X)\langle 0\rangle$ of the Lie model for $B\operatorname{aut}_{*,\circ}(X)$ and the Lie model for $\operatorname{map}_*^\iota(A,X)$. We also prove that $\operatorname{Der}(\mathbb{L}_X)\langle 1\rangle \ltimes_{\tau_*} \operatorname{Hom}^\tau(\bar{\mathcal{C}}(\mathbb{L}_A), \mathbb{L}_X)\langle 0\rangle \simeq \operatorname{Der}(\mathbb{L}_X \| \mathbb{L}_A)\langle 1\rangle$, which completes the proof of the main theorem, Theorem 1.1.

## 2. The geometric bar construction

The geometric bar construction, introduced by May (see [**May72, May75**]), is a construction that generalizes the classifying space functor. May forms a category $\mathscr{K}$, whose objects are triples $(X,G,Y)$, where $G$ is a topological monoid and $X$ and $Y$ are right and left $G$-spaces, respectively. A morphism between two objects, $(X,G,Y)$ and $(X',G',Y')$, in $\mathscr{K}$ is a triple $(i,f,j)$ where $f\colon G \to G'$ is a map of monoids, and $i\colon X \to X'$ and $j\colon Y \to Y'$ are equivariant with respect to $f$. We say that $(i,f,j)$ is a weak equivalence if $i$, $f$ and $j$ are weak equivalences, and two objects in $\mathscr{K}$ are called weakly equivalent if there is a zig-zag of weak equivalences connecting these two objects. The geometric bar construction $B(X,G,Y)$ on a triple $(X,G,Y)$ in $\mathscr{K}$ is a topological space and defines a functor from $\mathscr{K}$ to the category of topological spaces.

We recall some classical facts about classifying spaces of grouplike monoids (recall that a topological monoid $G$ is called grouplike if $\pi_0(G)$ is a group). The classifying space of a grouplike monoid $G$ is a space $BG$ that classifies all principal $G$-bundles in the following sense: The set of isomorphism classes of principal $G$-bundles over a space $X$, is in one-to-one correspondence with the set $[X,BG]$ of homotopy classes of maps from $X$ to $BG$.

A classifying space $BG$ of a grouplike monoid $G$ may also be recognized as a homotopy orbit space $EG /\!\!/ G$ where $EG$ is any contractible space on which $G$ acts freely on from the left. Note that classifying spaces are only unique up to homotopy equivalences.

We also recall from [**May75**] that the 'homotopy-correct' definition of the left coset space $G/H$ associated to an inclusion of monoids $H \subset G$ is given by

$$G/H = B(G,H,*).$$

We list some of the properties related to the geometric bar construction that are relevant for this paper.

**Proposition 2.1** ([**May75**]). *Let $G$ be a topological monoid.*

*(a)* $BG = B(*, G, *)$ *is a classifying space of* $G$.

*(b)* $B(X, *, *)$ *is homeomorphic to* $X$.

*(c) If* $(X, G, Y)$ *and* $(X', G', Y')$ *are weakly equivalent in* $\mathscr{K}$ *then* $B(X, G, Y)$ *and* $B(X', G', Y')$ *are weakly equivalent as spaces.*

*(d) If* $G$ *is a grouplike monoid, then* $EG = B(*, G, G)$ *is a contractible space on which* $G$ *acts freely from the right.*

*(e) If* $G$ *is a grouplike monoid, and* $Y$ *is a left* $G$*-space, then* $B(*, G, Y) \simeq EG \times_G Y$.

*(f) If* $H \subset G$ *is an inclusion of grouplike monoids then* $BH \simeq B(*, G, G/H)$.

Applying Proposition 2.1 *(f)* to $\operatorname{aut}_{A,\circ}(X) \subset \operatorname{aut}_{*,\circ}(X)$ we get that

$$B \operatorname{aut}_{A,\circ}(X) \simeq B(*, \operatorname{aut}_{*,\circ}(X), \operatorname{aut}_{*,\circ}(X)/\operatorname{aut}_{A,\circ}(X))$$

(note that since $A \subset X$ is a cofibration, any homotopy automorphism of $X$ that fixes $A$ has a homotopy inverse that also fixes $A$ (see [**May99**, Section 6.5]), which makes $\operatorname{aut}_A(X)$ into a grouplike monoid).

**Lemma 2.2.** *There is a weak equivalence of left* $\operatorname{aut}_{*,\circ}(X)$*-spaces*

$$\operatorname{map}_*^\iota(A, X) \simeq \operatorname{aut}_{*,\circ}(X)/\operatorname{aut}_{A,\circ}(X).$$

*Proof.* We will throughout this proof use that $X$ and $B(X, *, *)$ are interchangeable. We have that

- The map $\operatorname{aut}_{*,\circ}(X) \to \operatorname{aut}_{*,\circ}(X)/\operatorname{aut}_{A,\circ}(X)$ given by
$$B(\mathrm{id}, *, *) \colon B(\operatorname{aut}_{*,\circ}(X), *, *) \to B(\operatorname{aut}_{*,\circ}(X), \operatorname{aut}_{A,\circ}(X), *)$$
is a quasifibration with fiber $\operatorname{aut}_{A,\circ}(X)$ (see [**May75**, Proposition 7.9]).
- The restriction map $\operatorname{res}_A \colon \operatorname{aut}_{*,\circ}(X) \to \operatorname{map}_*^\iota(A, X)$ is a fibration, since the functor $\operatorname{map}_*(-, X)$ turns cofibrations into fibrations.
- The restriction map $\operatorname{res}_A \colon \operatorname{aut}_{*,\circ}(X) \to \operatorname{map}_*^\iota(A, X)$ is invariant under the right action of $\operatorname{aut}_{A,\circ}(X)$ on $\operatorname{aut}_{*,\circ}(X)$ and therefore the triple
$$(\operatorname{res}_A, *, *) \colon (\operatorname{aut}_{*,\circ}(X), \operatorname{aut}_{A,\circ}(X), *) \to (\operatorname{map}_*^\iota(A, X), *, *)$$
defines a map in $\mathscr{K}$. Thus
$$B(\operatorname{res}_A, *, *) \colon B(\operatorname{aut}_{*,\circ}(X), \operatorname{aut}_{A,\circ}(X), *) \to B(\operatorname{map}_*^\iota(A, X), *, *)$$
is a well-defined map.

It follows that the restriction map $\operatorname{aut}_{*,\circ}(X) \to \operatorname{map}_*^\iota(A, X)$ factors through $\operatorname{aut}_{*,\circ}(X)/\operatorname{aut}_{A,\circ}(X)$. Hence, there is a commutative diagram with rows being quasi-fibrations:

$$\begin{array}{ccccc}
\operatorname{aut}_{A,\circ}(X) & \xrightarrow{incl} & \operatorname{aut}_{*,\circ}(X) & \longrightarrow & \operatorname{aut}_{*,\circ}(X)/\operatorname{aut}_{A,\circ}(X) \\
\| & & \| & & \downarrow \\
\operatorname{aut}_{A,\circ}(X) & \xrightarrow{incl} & \operatorname{aut}_{*,\circ}(X) & \longrightarrow & \operatorname{map}_*^\iota(A, X)
\end{array}$$

By the functoriality of the long exact sequence of homotopy groups associated to a quasifibration and by the five lemma, it follows that there is a weak equivalence of spaces $\operatorname{map}_*^\iota(A, X) \simeq \operatorname{aut}_{*,\circ}(X)/\operatorname{aut}_{A,\circ}(X)$.

Moreover, the restriction map $\mathrm{aut}_{*,\circ}(X)/\mathrm{aut}_{A,\circ}(X) \to \mathrm{map}_*^\iota(A,X)$ respects the left $\mathrm{aut}_{*,\circ}(X)$-action. This completes the proof. □

**Corollary 2.3.** *Let $A \subset X$ be cofibration. There is a weak equivalence of spaces*

$$B\,\mathrm{aut}_{A,\circ}(X) \simeq B(*, \mathrm{aut}_{*,\circ}(X), \mathrm{map}_*^\iota(A,X)).$$

*Proof.* This is an immediate consequence of Proposition 2.1 *(c), (f)* and Lemma 2.2. □

**Proposition 2.4.** *The rationalization of $B\,\mathrm{aut}_{A,\circ}(X)$ is given by*

$$B\,\mathrm{aut}_{A_{\mathbb{Q}},\circ}(X_{\mathbb{Q}}) \simeq B(*, \mathrm{aut}_{*,\circ}(X_{\mathbb{Q}}), \mathrm{map}_*^{\iota_{\mathbb{Q}}}(A_{\mathbb{Q}}, X_{\mathbb{Q}}))$$

*Proof.* This may be obtained by 'dualizing' the proof [**Ber17**, Lemma 3.1]. □

*Remark 2.5.* By Proposition 2.4 it is enough to prove the assertion in Theorem 1.1 for rational spaces $X$ and $A$ in order to get a full proof of the theorem.

## 3. Rational homotopy of grouplike monoid actions

### 3.1. Preliminaries: degree-wise nilpotency and completness of dg Lie algebras

Nilpotent spaces are modelled by the so called degree-wise nilpotent dg Lie algebras.

**Definition 3.1.** The lower central series of a dg Lie algebra $L$ is the descending filtration

$$L = \Gamma^0 L \supseteq \Gamma^1 L \supseteq \Gamma^2 L \supseteq \cdots,$$

where $\Gamma^0 L = L$ and $\Gamma^{k+1} L = [\Gamma^k L, L]$. We say that $L$ is degree-wise nilpotent if for every $n \in \mathbb{Z}$ there exists some $k$ such that $(\Gamma^k L)_n = 0$.

**Definition 3.2.** Let $\Omega_\bullet$ denote the simplicial commutative dg algebra in which $\Omega_n$ is the Sullivan-de Rham algebra of polynomial differential forms on the $n$-simplex, see [**FHT01**, Section 10 (c)]. The geometric realization of a degree-wise nilpotent dg Lie algebra $L$, is defined to be the simplicial set $\mathrm{MC}(L \otimes \Omega_\bullet)$ of Maurer–Cartan elements of the simplicial dg Lie algebra $L \otimes \Omega_\bullet$, denoted by $\mathrm{MC}_\bullet(L)$. We say that that a degree-wise nilpotent dg Lie algebra $L$ is a Lie model for a nilpotent space $X$ if there exists a rational homotopy equivalence between the geometric realization $\mathrm{MC}_\bullet(L)$ and $X$.

In [**Ber15**], the geometric realization functor is extended to the so called complete dg Lie algebras.

**Definition 3.3.** A dg Lie algebra $\mathfrak{h}$ equipped with a filtration

$$\mathfrak{h} = F^1\mathfrak{h} \supseteq F^2\mathfrak{h} \supseteq \cdots$$

is called complete if

(i) each quotient $\mathfrak{h}/F^i\mathfrak{h}$ is a nilpotent dg Lie algebra, and
(ii) the canonical map $\mathfrak{h} \to \varprojlim \mathfrak{h}/F^i\mathfrak{h}$ is an isomorphism.

**Definition 3.4.** Given a complete dg Lie algebra $\mathfrak{h}$ we define its geometric realization to be the inverse limit

$$\widehat{\mathrm{MC}}_\bullet(\mathfrak{h}) := \varprojlim \mathrm{MC}_\bullet(\mathfrak{h}/F^r\mathfrak{h}).$$

We say that $\mathfrak{h}$ is a Lie model for $X$ if the realization of $\mathfrak{h}$ is rationally equivalent to $X$.

*Remark 3.5.* A degree-wise nilpotent dg Lie algebra $L$ together with its lower central series, makes $L$ into a complete dg Lie algebra, and we have that $\widehat{\mathrm{MC}}_\bullet(L) = \mathrm{MC}_\bullet(L)$. From this we may view the functor $\widehat{\mathrm{MC}}_\bullet$ as an extension of the functor $\mathrm{MC}_\bullet$.

*Example 3.6.* Let $C$ be a commutative dg coalgebra concentrated in non-negative degrees, with coproduct $\Delta\colon C \to C \otimes C$, and let $L$ be a connected degree-wise nilpotent dg Lie algebra of finite type with Lie bracket $\ell\colon L \otimes L \to L$. The convolution dg Lie algebra $\mathrm{Hom}(C, L)$ is a dg Lie algebra with differential and Lie bracket given by

$$\partial(f) = d_L \circ f - (-1)^{|f|} f \circ d_C,$$
$$[f, g] = \ell \circ f \otimes g \circ \Delta.$$

The convolution dg Lie algebra together $\mathrm{Hom}(C, L)$ with the filtration

$$\mathrm{Hom}(C, L) \supseteq \mathrm{Hom}(C, L\langle 1\rangle) \supseteq \mathrm{Hom}(C, L\langle 2\rangle) \supseteq \cdots$$

is a complete dg Lie algebra.

### 3.2. Outer actions and exponentials

We start by recalling some of the background for the notion of outer actions, as discussed in [**Ber17**]. By the theory of Schlessinger–Stasheff [**SS12**] and Tanré [**Tan83**], we have that if $L$ is a cofibrant Lie model for $X$, then a Lie model for the universal cover of $B\,\mathrm{aut}(X)$, or equivalently, a Lie model for $B\,\mathrm{aut}_\circ(X)$ where $\mathrm{aut}_\circ(X)$ is the connected component of the identity map (see the beginning of Section 1.2 for a motivation for this equivalence), is given by the semidirect product $\mathrm{Der}(L)\langle 1\rangle \ltimes^{\mathrm{ad}} sL$ where $\mathrm{Der}(L)\langle 1\rangle$ is the 1-connected cover of the dg Lie algebra of derivations on $L$, and $sL$ the abelian dg Lie algebra with the underlying dg vector space structure given by the suspension of $L$. The differential on the semidirect product is twisted by the adjoint map $\mathrm{ad}\colon sL \to \mathrm{Der}(L)\langle 1\rangle$, $sl \mapsto \mathrm{ad}_l = [l, -]$. That is, $\mathrm{Der}(L)\langle 1\rangle \ltimes^{\mathrm{ad}} sL$ is a dg Lie algebra with bracket and differential given by

$$[(\theta, sx), (\varphi, sy)] = ([\theta, \varphi], (-1)^{|\theta|} s\theta(y) - (-1)^{|\varphi||x|} s\varphi(x))$$

and

$$\partial(\theta, sx) = (D(\theta) + \mathrm{ad}_x, -sdx).$$

The set of homotopy classes of maps from a simply connected dg Lie algebra $\mathfrak{g}$ to $\mathrm{Der}(L)\langle 1\rangle \ltimes^{\mathrm{ad}} sL$ is thus in bijection with equivalence classes of $\mathrm{MC}_\bullet(L)$-fibrations over $\mathrm{MC}_\bullet(\mathfrak{g})$ in the category of simply connected rational spaces. Given a map $\psi\colon \mathfrak{g} \to \mathrm{Der}(L)\langle 1\rangle \ltimes^{\mathrm{ad}} sL$, the composition of $\psi$ with the projection on $\mathrm{Der}(L)\langle 1\rangle$ gives a map $\mathfrak{g} \to \mathrm{Der}(L)\langle 1\rangle$ which induces a map of graded vector spaces $\alpha\colon \mathfrak{g} \otimes L \to L$ of degree 0 (this is not necessarily a chain map), and the composition of $\psi$ with the projection on $sL$ gives a map $\mathfrak{g} \to sL$ which is equivalent to having a map $\xi\colon \mathfrak{g} \to L$ of degree $-1$. These two maps encode a so called outer action of $\mathfrak{g}$ on $L$.

**Definition 3.7** ([**Ber17**])**.** An outer action of $\mathfrak{g}$ on $L$ consists of a pair of maps $(\alpha, \xi)$, where $\alpha\colon \mathfrak{g} \otimes L \to L$ is a map of degree 0 and $\alpha(x \otimes a)$ is denoted by $x.a$, and where $\xi\colon \mathfrak{g} \to L$ is a map of degree $-1$, such that $\alpha$ and $\xi$ satisfy the following conditions

(I) $[x, y].a = x.(y.a) - (-1)^{|x||y|}y.(x.a)$,
(II) $x.[a, b] = [x.a, b] + (-1)^{|x||a|}[a, x.b]$,
(III) $\xi$ is a chain map, i.e. $d\xi = -\xi d$,
(IV) $\xi[x, y] = -(-1)^{|y||\xi(x)|}y.\xi(x) + (-1)^{|x|}x.\xi(y)$,
(V) $d(x.a) = d(x).a + (-1)^{|x|}x.d(a) + [\xi(x), a]$.

**Proposition 3.8.** *Specifying an outer action of $\mathfrak{g}$ on $L$ is tantamount to specifying a morphism of dg Lie algebras $\mathfrak{g} \to \mathrm{Der}(L)\langle 1\rangle \ltimes^{\mathrm{ad}} sL$.*

**Definition 3.9.** Given an outer action of $\mathfrak{g}$ on $L$, the twisted semidirect product $\mathfrak{g} \ltimes_\xi L$ of $\mathfrak{g}$ and $L$ is a dg Lie algebra with the underlying graded vector space given by $(\mathfrak{g} \ltimes_\xi L)_n = \mathfrak{g}_n \times L_n$. The Lie bracket and the differential on $\mathfrak{g} \ltimes_\xi L$ are given by

$$[(x, a), (y, b)] = ([x, y], [a, b] + x.b - (-1)^{|y||a|}y.a)$$

and

$$\partial^\xi(x, a) = (dx, da + \xi(x)).$$

Next, we associate to an outer action $(\alpha, \xi)$ of $\mathfrak{g}$ on $L$ an action of a group $\exp_\bullet(\mathfrak{g})$ on the realization $\mathrm{MC}_\bullet(L)$.

**Definition 3.10** ([**Ber17**])**.** The exponential $\exp(\mathfrak{h})$ of a nilpotent Lie algebra $\mathfrak{h}$ concentrated in degree zero is the nilpotent group with the underlying set given by $\mathfrak{h}$ and with multiplication given by the Campbell–Baker–Hausdorff formula. The exponential of a connected degree-wise nilpotent dg Lie algebra $\mathfrak{g}$, $\exp_\bullet(\mathfrak{g})$, is defined to be the exponential $\exp(Z_0(\mathfrak{g} \otimes \Omega_\bullet))$ of zero cycles in $\mathfrak{g} \otimes \Omega_\bullet$.

**Proposition 3.11.** *Let $\mathfrak{g}$ be a simply connected dg Lie algebra and let $(\alpha, \xi)$ define an outer action of $\mathfrak{g}$ on a dg Lie algebra $L$. The action of $\exp_\bullet(\mathfrak{g})$ on $\mathrm{MC}_\bullet(L)$ corresponding to the outer action $(\alpha, \xi)$ is given by*

$$\exp(x).a = a + \sum_{n \geqslant 0} \frac{\theta_x^n(\theta_x(a) - \xi(x))}{(n+1)!},$$

*where $\theta_x(a) = x.a$.*

*Proof.* We start by recalling some of the theory of the so called gauge actions. Suppose that $\mathfrak{h} = \bigoplus \mathfrak{h}_i$ is a dg Lie algebra with differential $d_\mathfrak{h}$ and suppose that there exists some nilpotent Lie subalgebra $\mathfrak{h}_0' \subseteq \mathfrak{h}_0$, such that $\mathfrak{h}$ becomes a nilpotent $\mathfrak{h}_0'$-module (under the adjoint action). Then, there exists a group action of $\exp(\mathfrak{h}_0')$ on $\mathrm{MC}(\mathfrak{h})$ called the gauge action, and is given by

$$\exp(X).A = A + \sum_{n \geqslant 0} \frac{[X, -]^n}{(n+1)!}([X, A] - d_\mathfrak{h} X),$$

where $X \in \mathfrak{h}_0'$ and $A \in \mathrm{MC}(\mathfrak{h})$ (see [**Man04**, Section 5.4] for details on the gauge action).

Given a connected and bounded commutative dg algebra $\Omega = \oplus_{i=0}^n \Omega^i$, we have that $(\mathfrak{g} \ltimes_\xi L) \otimes \Omega \cong (\mathfrak{g} \otimes \Omega) \ltimes_{\xi \otimes \mathrm{id}} (L \otimes \Omega)$. Since $\mathfrak{g}$ is simply connected, it follows that the adjoint action of $(\mathfrak{g} \otimes \Omega)_0 = (\mathfrak{g}_1 \otimes \Omega^1) \oplus \cdots \oplus (\mathfrak{g}_n \otimes \Omega^n)$ on $(\mathfrak{g} \otimes \Omega) \ltimes_{\xi\otimes\mathrm{id}} (L \otimes \Omega)$ is nilpotent. Hence the action of the subalgebra of zero cycles $Z_0(\mathfrak{g} \otimes \Omega)$ has also a nilpotent adjoint action on $(\mathfrak{g} \otimes \Omega) \ltimes_{\xi\otimes\mathrm{id}} (L \otimes \Omega)$. Note that if $x \in Z_0(\mathfrak{g} \otimes \Omega)$ then $\partial^{\xi\otimes\mathrm{id}}(x) = (\xi \otimes \mathrm{id})(x)$.

Moreover, straightforward calculations give that $a \in L \otimes \Omega$ is a Maurer–Cartan element in $L \otimes \Omega$ if and only if it is a Maurer–Cartan element in $(\mathfrak{g} \otimes \Omega) \ltimes_{\xi\otimes\mathrm{id}} L(\otimes\Omega)$. We have that if $x \in Z_0(\mathfrak{g} \otimes \Omega)$ and $a \in L \otimes \Omega$, then both $[x, a]$ and $\partial^{\xi\otimes\mathrm{id}}(x) = (\xi \otimes \mathrm{id})(x)$ are elements of $L \otimes \Omega \subset (\mathfrak{g} \otimes \Omega) \ltimes_{\xi\otimes\mathrm{id}} (L \otimes \Omega)$, and therefore the gauge action above defines an action of $\exp(Z_0(\mathfrak{g} \otimes \Omega))$ on $\mathrm{MC}(L \otimes \Omega)$. In particular, we have that there exists an action of $\exp_\bullet(\mathfrak{g})$ on $\mathrm{MC}_\bullet(L)$ given by the formula in the proposition. □

**Corollary 3.12.** *If $\xi$ in the previous proposition is trivial, then the action of $\exp_\bullet(\mathfrak{g})$ on $\mathrm{MC}_\bullet(L)$ is basepoint preserving, where $0 \in \mathrm{MC}_\bullet(L)$ is the basepoint.*

*Proof.* This follows immediately from the explicit formula for the action, given in Proposition 3.11. □

We present some properties of $\exp_\bullet(\mathfrak{g})$.

**Proposition 3.13** ([**Ber17**, Corollary 3.10 and Theorem 3.15]). *Let $\mathfrak{g}$ be a simply connected dg Lie algebra of finite type and let $L$ be a dg Lie algebra. Suppose that $(\alpha, \xi)$ defines an outer action of $\mathfrak{g}$ on $L$.*

*(a) $\mathrm{MC}_\bullet(\mathfrak{g})$ is a delooping of $\exp_\bullet(\mathfrak{g})$.*

*(b) The twisted semidirect product $\mathfrak{g} \ltimes_\xi L$ is a Lie model for the Borel construction $B(*, \exp_\bullet(\mathfrak{g}), \mathrm{MC}_\bullet(L))$.*

### 3.3. Mapping spaces

In [**Ber17**] it is shown that if $L$ is a connected degree-wise nilpotent dg Lie algebra of finite type, and $\Pi$ is connected dg Lie algebra then there is a weak equivalence

$$\widehat{\mathrm{MC}}_\bullet(\mathrm{Hom}(\mathcal{C}(\Pi), L)) \to \mathrm{map}(\mathrm{MC}_\bullet(\Pi), \mathrm{MC}_\bullet(L)),$$

where $\mathcal{C}(\Pi)$ is the Chevalley–Eilenberg coalgebra construction on $\Pi$ and $\mathrm{Hom}(\mathcal{C}(\Pi), L)$ is the convolution dg Lie algebra. In particular, $\mathrm{Hom}(\mathcal{C}(\Pi), L)$ is a Lie model for $\mathrm{map}(\mathrm{MC}_\bullet(\Pi), \mathrm{MC}_\bullet(L))$ in the sense of Definition 3.4. We want to show that this weak equivalence is equivariant with respect to the action of $\exp_\bullet(\mathfrak{g})$.

**Lemma 3.14.** *An outer action of $\mathfrak{g}$ on $L$ induces an outer action of $\mathfrak{g}$ on the convolution dg Lie algebra $\mathrm{Hom}(C, L)$ for any counital cocommutative dg coalgebra $C$.*

*Proof.* We define maps $\widetilde{\alpha}\colon \mathfrak{g} \otimes \mathrm{Hom}(C, L) \to \mathrm{Hom}(C, L)$ and $\widetilde{\xi}\colon \mathfrak{g} \to \mathrm{Hom}(C, L)$. We denote $\widetilde{\alpha}(x \otimes f)$ by $x.f$ and is given by

$$\widetilde{\alpha}(x \otimes f)(c) = (x.f)(c) = x.f(c).$$

Let $\varepsilon\colon C \to \mathbb{Q}$ be the counit. We define $\widetilde{\xi}$ as the composition

$$\mathfrak{g} \xrightarrow{\xi} L \xrightarrow{\cong} \mathrm{Hom}(\mathbb{Q}, L) \xrightarrow{\varepsilon^*} \mathrm{Hom}(C, L),$$

so $(\widetilde{\xi}(x))(c) = \varepsilon(c) \cdot \xi(x)$.

It is straightforward to show that $\widetilde{\alpha}$ and $\widetilde{\xi}$ satisfies properties (I)–(V) in Definition 3.7. □

**Proposition 3.15.** *Let $\mathfrak{g}$, $L$, and $\Pi$ be connected nilpotent dg Lie algebras, where $L$ is of finite type. Let $(\alpha, \xi)$ be an outer action of $\mathfrak{g}$ on $L$. The evaluation map of [**Ber17**, Theorem 3.16]*

$$E\colon \mathrm{MC}(\mathrm{Hom}_{\Omega_\bullet}(\mathcal{C}_{\Omega_\bullet}(\Pi\otimes\Omega_\bullet), L\otimes\Omega_\bullet)) \times \mathcal{G}(\mathcal{C}_{\Omega_\bullet}(\Pi\otimes\Omega_\bullet)) \to \mathrm{MC}(L\otimes\Omega_\bullet)$$

*is $\exp_\bullet(\mathfrak{g})$-equivariant.*

*Proof.* That the image of $E$ really lands in $\mathrm{MC}(L\otimes\Omega_\bullet)$ is proved in [**Ber17**]. We prove that $E$ is $\exp_\bullet(\mathfrak{g})$-equivariant. Using Proposition 3.11, we get

$$\begin{aligned}
E(\exp(x).(f,c)) &= E(\exp(x).f, c) = (\exp(x).f)(c) \\
&= \left(f + \sum_{n\geqslant 0} \frac{\theta_x^n(\theta_x(f) - \widetilde{\xi}(x))}{(n+1)!}\right)(c) \\
&= f(c) + \sum_{n\geqslant 0} \frac{\theta_x^n(\theta_x(f(c)) - \widetilde{\xi}(x)(c))}{(n+1)!} \\
&= f(c) + \sum_{n\geqslant 0} \frac{\theta_x^n(\theta_x(f(c)) - \varepsilon(c)\xi(x))}{(n+1)!} \\
&= [c \text{ is a grouplike element} \Rightarrow \varepsilon(c) = 1] \\
&= f(c) + \sum_{n\geqslant 0} \frac{\theta_x^n(\theta_x(f(c)) - \xi(x))}{(n+1)!} \\
&= \exp(x).(f(c)) = \exp(x).E(f,c).
\end{aligned}$$

□

**Corollary 3.16.** *There exists an $\exp_\bullet(\mathfrak{g})$-equivariant weak equivalence*

$$\widehat{\mathrm{MC}}(\mathrm{Hom}(\mathcal{C}(\Pi), L)) \simeq \mathrm{map}(\mathrm{MC}_\bullet(\Pi), \mathrm{MC}_\bullet(L)),$$

*that is natural in $\Pi$ and $L$.*

*Proof.* By Proposition 3.15 we have that the adjoint map of $E$

$$\mathrm{MC}(\mathrm{Hom}_\Omega(\mathcal{C}_\Omega(\Pi\otimes\Omega_\bullet), L\otimes\Omega_\bullet)) \to \mathrm{map}(\mathrm{MC}_\bullet(\Pi), \mathrm{MC}_\bullet(L))$$

is $\exp_\bullet(\mathfrak{g})$ equivariant. By [**Ber17**, Theorem 3.16] this is also a weak equivalence that is natural in $\Pi$ and $L$. Following [**Ber17**, Remark 3.17], there exists a natural isomorphism

$$\widehat{\mathrm{MC}}_\bullet(\mathrm{Hom}(\mathcal{C}(\Pi), L)) \cong \mathrm{MC}(\mathrm{Hom}_\Omega(\mathcal{C}_\Omega(\Pi\otimes\Omega_\bullet), L\otimes\Omega_\bullet)).$$

It is straightforward to show that this isomorphism respects the $\exp_\bullet(\mathfrak{g})$-action. □

**Corollary 3.17.** *There is a weak equivalence of spaces*

$$\widehat{\mathrm{MC}}_\bullet(\mathrm{Hom}(\bar{\mathcal{C}}(\Pi), L)) \to \mathrm{map}_*(\mathrm{MC}_\bullet(\Pi), \mathrm{MC}_\bullet(L)).$$

*Moreover, for every outer action $(\alpha, \xi)$ of $\mathfrak{g}$ on $L$ where $\xi$ is trivial, the map above is $\exp_\bullet(\mathfrak{g})$-equivariant.*

*Proof.* By [**Ber15**, Proposition 5.4], the functor $\widehat{\mathrm{MC}}_\bullet$ takes surjections of complete dg Lie algebras to (Kan) fibrations. In particular, the surjection $\mathrm{Hom}(\mathcal{C}(\Pi), L) \to \mathrm{Hom}(\mathbb{Q}, L) \cong L$ induces a fibration $\widehat{\mathrm{MC}}_\bullet(\mathrm{Hom}(\mathcal{C}(\Pi), L)) \to \mathrm{MC}_\bullet(L)$, which has fiber $\widehat{\mathrm{MC}}_\bullet(\mathrm{Hom}(\bar{\mathcal{C}}(\Pi), L))$.

Moreover, the map $\mathrm{map}(\mathrm{MC}_\bullet(\Pi), \mathrm{MC}_\bullet(L)) \to \mathrm{map}(*, \mathrm{MC}_\bullet(L)) \cong \mathrm{MC}_\bullet(L)$ is a fibration, which has fiber $\mathrm{map}_*(\mathrm{MC}_\bullet(\Pi), \mathrm{MC}_\bullet(L))$, and thus we get a commuting diagram

$$\begin{array}{ccccc} \widehat{\mathrm{MC}}_\bullet(\mathrm{Hom}(\bar{\mathcal{C}}(\Pi), L)) & \longrightarrow & \widehat{\mathrm{MC}}_\bullet(\mathrm{Hom}(\mathcal{C}(\Pi), L)) & \longrightarrow & \mathrm{MC}_\bullet(L) \\ \downarrow & & \downarrow & & \| \\ \mathrm{map}_*(\mathrm{MC}_\bullet(\Pi), \mathrm{MC}_\bullet(L)) & \longrightarrow & \mathrm{map}(\mathrm{MC}_\bullet(\Pi), \mathrm{MC}_\bullet(L)) & \longrightarrow & \mathrm{MC}_\bullet(L) \end{array}$$

with rows being fibrations. The long exact sequence of homotopy groups yields now the weak equivalence $\widehat{\mathrm{MC}}_\bullet(\mathrm{Hom}(\bar{\mathcal{C}}(\Pi), L)) \to \mathrm{map}_*(\mathrm{MC}_\bullet(\Pi), \mathrm{MC}_\bullet(L))$. This completes the proof for the first part of the statement.

For the second part, we just recall that the triviality of $\xi$ gives that the induced $\exp_\bullet(\mathfrak{g})$-action on $\mathrm{MC}_\bullet(L)$ is basepoint preserving, see Corollary 3.12, and will therefore induce an action on the based mapping space $\mathrm{map}_*(\mathrm{MC}_\bullet(\Pi), \mathrm{MC}_\bullet(L))$. □

**Proposition 3.18.** *Let* $\mathfrak{g} = \mathrm{Der}(L)\langle 1\rangle$. *There exists an outer action* $(\alpha, \xi)$ *of* $\mathfrak{g}$ *on* $L$, *where* $\alpha(\theta, x) = \theta(x)$ *and where* $\xi = 0$. *The action of* $\exp_\bullet(\mathfrak{g})$ *on* $\mathrm{MC}_\bullet(L)$ *yields a map* $\exp_\bullet(\mathfrak{g}) \to \mathrm{aut}_{*,\circ}(\mathrm{MC}_\bullet(L))$, *which is a weak equivalence. In particular, the triples* $(*, \exp_\bullet(\mathfrak{g}), \widehat{\mathrm{MC}}_\bullet(\mathrm{Hom}(\bar{\mathcal{C}}(\Pi), L)))$ *and* $(*, \mathrm{aut}_{*,\circ}(\mathrm{MC}_\bullet(L)), \mathrm{map}_*(\mathrm{MC}_\bullet(\Pi), \mathrm{MC}_\bullet(L)))$ *are weakly equivalent in the category* $\mathscr{K}$ *(discussed in Section 2). A Lie model for*

$$B(*, \mathrm{aut}_{*,\circ}(\mathrm{MC}_\bullet(L)), \mathrm{map}_*(\mathrm{MC}_\bullet(\Pi), \mathrm{MC}_\bullet(L)))$$

*is given by* $\mathfrak{g} \ltimes \mathrm{Hom}(\bar{\mathcal{C}}(\Pi), L)$.

*Proof.* It follows by the theory of Schlessinger–Stasheff [**SS12**] and Tanré [**Tan83**] that if $L$ is a cofibrant dg Lie algebra, then a Lie model for $B\,\mathrm{aut}_{*,\circ}(\mathrm{MC}_\bullet(L))$ is given by $\mathrm{Der}(L)\langle 1\rangle$. By Proposition 3.13 *(a)* it follows that $\exp_\bullet(\mathfrak{g})$ is weakly equivalent to $\mathrm{aut}_{*,\circ}(\mathrm{MC}_\bullet(L))$. This fact, together with Corollary 3.17, gives the equivalence of triples mentioned in the proposition.

The statement regarding the Lie model is a consequence of Proposition 3.13 *(b)*. □

## 4. Modelling homotopy automorphisms with derivations

The ultimate goal of this paper is to study the rational homotopy of

$$B\,\mathrm{aut}_{A,\circ}(X) \simeq B(*, \mathrm{aut}_{*,\circ}(X), \mathrm{map}_*^\iota(A, X)),$$

which is a connected component in $B(*, \mathrm{aut}_{*,\circ}(X), \mathrm{map}_*(A, X))$. The disconnected space $B(*, \mathrm{aut}_{*,\circ}(X), \mathrm{map}_*(A, X))$ is modelled by the complete dg Lie algebra

$$\mathrm{Der}(\mathbb{L}_X)\langle 1\rangle \ltimes \mathrm{Hom}(\bar{\mathcal{C}}(\mathbb{L}_A), \mathbb{L}_X)$$

(see Proposition 3.18). If $\mathfrak{h}$ is a complete dg Lie algebra model for a disconnected space $W$, one may extract a dg Lie algebra model for a connected component $W^\tau \subset W$ by the following proposition:

**Proposition 4.1** ([**Ber15**, Theorem 5.5])**.** *Let $\mathfrak{h}$ be a complete dg Lie algebra and let $\tau$ be a Maurer–Cartan element in $\mathfrak{h}$. The connected component of $\widehat{\mathrm{MC}}_\bullet(\mathfrak{h})$ that contains $\tau$ is weakly equivalent to $\mathrm{MC}_\bullet(\mathfrak{h}^\tau\langle 0\rangle)$ where $\mathfrak{h}^\tau$ is the dg Lie algebra whose underlying graded Lie algebra structure coincides with the one of $\mathfrak{h}$ but with a twisted differential $\partial^\tau = \partial + \mathrm{ad}_\tau$.*

We apply this proposition in order to get a Lie model for $B\,\mathrm{aut}_{A,\circ}(X)$:

**Proposition 4.2.** *Let $\tau \in \mathrm{Hom}(\bar{\mathcal{C}}(\mathbb{L}_A), \mathbb{L}_X)$ be the Maurer–Cartan element given by the composition*

$$\tau\colon \bar{\mathcal{C}}(\mathbb{L}_A) \xrightarrow{\pi_A} \mathbb{L}_A \xrightarrow{i} \mathbb{L}_X,$$

*where $\pi_A\colon \bar{\mathcal{C}}(\mathbb{L}_A) \to \mathbb{L}_A$ is the universal twisting morphism and $i\colon \mathbb{L}_A \to \mathbb{L}_X$ is a cofibration that model the inclusion $\iota\colon A \hookrightarrow X$. A Lie model for*

$$B(*, \mathrm{aut}_{*,\circ}(X), \mathrm{map}_*^\iota(A, X))$$

*is given by*

$$\big(\mathrm{Der}(\mathbb{L}_X)\langle 1\rangle \ltimes \mathrm{Hom}(\bar{\mathcal{C}}(\mathbb{L}_A), \mathbb{L}_X)\langle 0\rangle\big)^{(0,\tau)} = \mathrm{Der}(\mathbb{L}_X)\langle 1\rangle \ltimes_{\tau_*} \mathrm{Hom}^\tau(\bar{\mathcal{C}}(\mathbb{L}_A), \mathbb{L}_X)\langle 0\rangle, \tag{1}$$

*where $\tau_*(\theta) = -(-1)^{|\theta|}\theta \circ \tau$.*

*Proof.* The connected component of the Maurer–Cartan element

$$(0, \tau) \in \mathrm{Der}(\mathbb{L}_X)\langle 1\rangle \ltimes \mathrm{Hom}(\bar{\mathcal{C}}(\mathbb{L}_A), \mathbb{L}_X)$$

in the realization corresponds to the connected component $B(*, \mathrm{aut}_{*,\circ}(X), \mathrm{map}_*^\iota(A, X))$ in $B(*, \mathrm{aut}_{*,\circ}(X), \mathrm{map}_*(A, X))$. By Proposition 4.1, a Lie model for

$$B(*, \mathrm{aut}_{*,\circ}(X), \mathrm{map}_*^\iota(A, X))$$

is given by the left hand-side of (1). The equality (1) may now be checked by hand. □

**Definition 4.3.** Let $f\colon \mathfrak{h} \to \Pi$ be a morphism of dg Lie algebras, define $\mathrm{Der}_f(\mathfrak{h}, \Pi)$ to be the dg vector space of so called $f$-derivations from $\mathfrak{h}$ to $\Pi$. An $f$-derivation is a linear map $\theta\colon \mathfrak{h} \to \Pi$ that satisfies

$$\theta[x, y] = [\theta(x), f(y)] + (-1)^{|\theta||x|}[f(x), \theta(y)].$$

**Proposition 4.4.** *Let $\tau\colon \bar{\mathcal{C}}\mathbb{L}_A \to \mathbb{L}_X$ be as in Proposition 4.2. The map*

$$s\pi_A^*\colon \mathrm{Der}_i(\mathbb{L}_A, \mathbb{L}_X)\langle 1\rangle \to s(\mathrm{Hom}^\tau(\bar{\mathcal{C}}\mathbb{L}_A, \mathbb{L}_X)\langle 0\rangle)$$

*given by $s\pi_A^*(\theta) = (-1)^{|\theta|+1}s(\theta \circ \pi_A)$ is a quasi-isomorphism.*

*Proof.* It is enough to show that $\pi_A^*\colon \mathrm{Der}_i(\mathbb{L}_A, \mathbb{L}_X)\langle 1\rangle \to \mathrm{Hom}^\tau(\bar{\mathcal{C}}\mathbb{L}_A, \mathbb{L}_X)\langle 0\rangle$, defined by $\pi_A^*(\theta) = (-1)^{|\theta|+1}\theta \circ \pi_A$, induces isomorphisms in shifted homology, i.e.

$$H(\pi_A^*)\colon H_{p+1}(\mathrm{Der}_i(\mathbb{L}_A, \mathbb{L}_X)\langle 1\rangle) \xrightarrow{\cong} H_p(\mathrm{Hom}^\tau(\bar{\mathcal{C}}\mathbb{L}_A, \mathbb{L}_X)\langle 0\rangle).$$

We will use that universal twisting morphism $\pi_A$ satisfies the Maurer–Cartan equation

$$\pi_A \circ d = -d \circ \pi_A - \frac{1}{2}[\pi_A, \pi_A]. \tag{2}$$

We have that

$$
\begin{aligned}
D(\pi_A^*(\theta)) &= (-1)^{|\theta|+1} d\circ\theta\circ\pi_A - \theta\circ\pi_A\circ d + (-1)^{|\theta|+1}[\tau,\theta\circ\pi_A] \\
&\overset{(2)}{=} (-1)^{|\theta|+1} d\circ\theta\circ\pi_A + (-1)^{|\theta|+1}[i\circ\pi_A,\theta\circ\pi_A] - \theta\circ\big(-d\circ\pi_A - \frac{1}{2}[\pi_A,\pi_A]\big) \\
&= (-1)^{|\theta|+1} d\circ\theta\circ\pi_A + (-1)^{|\theta|+1}[i\circ\pi_A,\theta\circ\pi_A] + \theta\circ d\circ\pi_A \\
&\quad + \left(\frac{1}{2}[\theta\circ\pi_A, i\circ\pi_A] + (-1)^{|\theta|}\frac{1}{2}[i\circ\pi_A,\theta\circ\pi_A]\right) \\
&= (-1)^{|\theta|+1} d\circ\theta\circ\pi_A + (-1)^{|\theta|+1}[i\circ\pi_A,\theta\circ\pi_A] + \theta\circ d\circ\pi_A \\
&\quad + (-1)^{|\theta|}[i\circ\pi_A,\theta\circ\pi_A] \\
&= (-1)^{|\theta|+1} d\circ\theta\circ\pi_A + \theta\circ d\circ\pi_A \\
&= -\pi_A^*(D(\theta)),
\end{aligned}
$$

where the third equality uses that $\theta$ is an $i$-derivation. This proves that $\pi_A^*$ is a chain map.

Now we prove that $\pi_A^*$ is a quasi-isomorphism (up to a degree shift). If $L$ is a connected dg Lie algebra, let $Q(L) = L/[L,L]$ denote the chain complex of the indecomposable elements in $L$.

**Lemma 4.5** ([**FHT01**, Proposition 22.8])**.** *The composition*

$$\bar{\mathcal{C}}\mathbb{L}_A \xrightarrow{\pi_A} \mathbb{L}_A \to Q(\mathbb{L}_A)$$

*induces isomorphisms in homology*

$$H_{p+1}(\bar{\mathcal{C}}\mathbb{L}_A) \xrightarrow{\cong} H_p(Q(\mathbb{L}_A)).$$

Now we consider the complete filtration $\mathrm{Der}_i(\mathbb{L}_A,\mathbb{L}_X)\langle 1\rangle = F^1 \supseteq F^2 \supseteq \cdots$, where $F^p$ is the subcomplex of $i$-derivations that vanish on elements of degree $< p$, and the complete filtration $\mathrm{Hom}^\tau(\bar{\mathcal{C}}\mathbb{L}_A,\mathbb{L}_X)\langle 0\rangle = \hat{F}^1 \supseteq \hat{F}^2 \supseteq \cdots$, where $\hat{F}^p$ is the subcomplex of linear maps that vanish on elements of degree $< p+1$.

With respect to these filtrations, $\pi_A^*$ becomes a map of filtered complexes and induces a map of spectral sequences. We have that the first filtration gives rise to a first quadrant spectral sequence with $E_2$-term

$$E_2^{p,-q} = \mathrm{Hom}(H_p(Q(\mathbb{L}_A)), H_q(\mathbb{L}_X)),$$

and the second filtration gives rise to a first quadrant spectral sequence with $\hat{E}_2$-term

$$\hat{E}_2^{p+1,-q} = \mathrm{Hom}(H_{p+1}(\bar{\mathcal{C}}\mathbb{L}_A), H_q(\mathbb{L}_X)).$$

By Lemma 4.5, the induced map $E_2(\pi_A^*)\colon E_2^{p,-q} \to \hat{E}_2^{p+1,-q}$ is an isomorphism. The comparison theorem (see for instance [**Wei94**]) gives now that $\pi_A^*$ is indeed a quasi-isomorphism up to a degree shift. □

We are left to show that there exists a weak equivalence of dg Lie algebras

$$\mathrm{Der}(\mathbb{L}_X \| \mathbb{L}_A)\langle 1\rangle \simeq \mathrm{Der}(\mathbb{L}_X)\langle 1\rangle \ltimes_{\tau_*} \mathrm{Hom}^\tau(\bar{\mathcal{C}}\mathbb{L}_A,\mathbb{L}_X)\langle 0\rangle$$

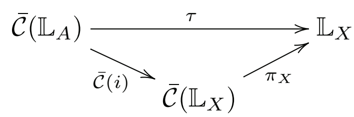
in order to complete the proof of the main theorem, Theorem 1.1.

**Proposition 4.6.** *Let $i\colon \mathbb{L}_A \to \mathbb{L}_X$ be a cofibration (i.e. a free map). Then we may view $\mathbb{L}_A$ as a subalgebra of $\mathbb{L}_X$. Let $\operatorname{Der}(\mathbb{L}_X\|\mathbb{L}_A)$ be the dg Lie algebra of derivations on $\mathbb{L}_X$ that vanish on $\mathbb{L}_A$. The map*

$$\zeta\colon \operatorname{Der}(\mathbb{L}_X\|\mathbb{L}_A)\langle 1\rangle \to \operatorname{Der}(\mathbb{L}_X)\langle 1\rangle \ltimes_{\tau_*} \operatorname{Hom}^\tau(\bar{\mathcal{C}}\mathbb{L}_A, \mathbb{L}_X)\langle 0\rangle$$

*given by inclusion into the first term is a quasi-isomorphism of dg Lie algebras.*

*Proof.* It is straightforward to show that $\zeta$ is a map of Lie algebras. We want to show that $\zeta$ is a chain map. This is equivalent to having that $\tau_*|_{\operatorname{Der}(\mathbb{L}_X\|\mathbb{L}_A)\langle 1\rangle} = 0$. We have that $\tau\colon \bar{\mathcal{C}}(\mathbb{L}_A) \to \mathbb{L}_X$ factors through $\bar{\mathcal{C}}(\mathbb{L}_X)$




where $\pi_X\colon \bar{\mathcal{C}}(\mathbb{L}_X) \to \mathbb{L}_X$ is the universal twisting morphism. Given a derivation $\theta \in \operatorname{Der}(\mathbb{L}_X)$, it induces a coderivation $\Theta \in \operatorname{Coder}(\bar{\mathcal{C}}(\mathbb{L}_X))$ given by

$$\Theta(sx_1 \wedge \cdots \wedge sx_k) = \sum \pm sx_1 \wedge \cdots \wedge s\theta(x_i) \wedge \cdots \wedge sx_k$$

so that $\pi_X \circ \Theta = (-1)^{|\theta|}\theta \circ \pi_X$. Now assume that $\theta \in \operatorname{Der}(\mathbb{L}_X\|\mathbb{L}_A)\langle 1\rangle$, i.e. $\theta \circ i = 0$, then we have that $\Theta \circ \bar{\mathcal{C}}(i) = 0$ and, in particular,

$$\tau_*(\theta) = \theta \circ \tau = \theta \circ \pi_X \circ \bar{\mathcal{C}}(i) = (-1)^{|\theta|}\pi_X \circ \Theta \circ \bar{\mathcal{C}}(i) = 0$$

This gives that $\zeta\colon \operatorname{Der}(\mathbb{L}_X\|\mathbb{L}_A)\langle 1\rangle \to \operatorname{Der}(\mathbb{L}_X)\langle 1\rangle \ltimes_{\tau_*} \operatorname{Hom}^\tau(\bar{\mathcal{C}}\mathbb{L}_A, \mathbb{L}_X)\langle 0\rangle$ is a chain map, and therefore also a map of dg Lie algebras.

Now we show that $\zeta$ is a quasi-isomorphism. In the model category of chain complexes we have that the homotopy cofiber of the projection map

$$\rho\colon \operatorname{Der}(\mathbb{L}_X)\langle 1\rangle \ltimes_{\tau_*} \operatorname{Hom}^\tau(\bar{\mathcal{C}}(\mathbb{L}_A, \mathbb{L}_X)\langle 0\rangle) \to \operatorname{Der}(\mathbb{L}_X)\langle 1\rangle$$

is the mapping cone $\operatorname{Der}(\mathbb{L}_X)\langle 1\rangle \oplus s(\operatorname{Der}(\mathbb{L}_X)\langle 1\rangle \ltimes_{\tau_*} \operatorname{Hom}^\tau(\bar{\mathcal{C}}(\mathbb{L}_A), \mathbb{L}_A)\langle 0\rangle)$, denoted by *cone*$(\rho)$, equipped with the differential given by $d(\theta, s\psi, s\eta) = (d\theta - \psi, -s\partial^\tau(\psi, \eta))$. In particular, we have that

$$\operatorname{Der}(\mathbb{L}_X)\langle 1\rangle \ltimes_{\tau_*} \operatorname{Hom}^\tau(\bar{\mathcal{C}}(\mathbb{L}_A, \mathbb{L}_X)\langle 0\rangle) \xrightarrow{\rho} \operatorname{Der}(\mathbb{L}_X)\langle 1\rangle \to \mathit{cone}(\rho)$$

is equivalent to a homotopy cofibration.

Moreover, we have that there is a short exact sequence of chain complexes

$$0 \to s(\operatorname{Hom}^\tau(\bar{\mathcal{C}}(\mathbb{L}_A), \mathbb{L}_A)\langle 0\rangle) \xrightarrow{incl} \mathit{cone}(\rho) \to \operatorname{Der}(\mathbb{L}_X)\langle 1\rangle \oplus s\operatorname{Der}(\mathbb{L}_X)\langle 1\rangle \to 0$$

We have that $\operatorname{Der}(\mathbb{L}_X)\langle 1\rangle \oplus s(\operatorname{Der}(\mathbb{L}_X)\langle 1\rangle) \simeq 0$ (since $\operatorname{Der}(\mathbb{L}_X)\langle 1\rangle \oplus s(\operatorname{Der}(\mathbb{L}_X)\langle 1\rangle)$ is the mapping cone on the identity map), so it follows that

$$s(\operatorname{Hom}^\tau(\bar{\mathcal{C}}(\mathbb{L}_A), \mathbb{L}_A)\langle 0\rangle) \xrightarrow{incl} \mathit{cone}(\rho)$$

is a homotopy equivalence. It follows now that the composition of homotopy equivalences

$$\alpha\colon \operatorname{Der}_i(\mathbb{L}_A, \mathbb{L}_X)\langle 1\rangle \xrightarrow{s\pi_A^*} s(\operatorname{Hom}^\tau(\bar{\mathcal{C}}(\mathbb{L}_A), \mathbb{L}_X)\langle 0\rangle) \xrightarrow{incl} \mathit{cone}(\rho)$$

is a homotopy equivalence. Since $i\colon \mathbb{L}_A \to \mathbb{L}_X$ is a free map, the restriction map $\operatorname{Der}(\mathbb{L}_X)\langle 1\rangle \to \operatorname{Der}_i(\mathbb{L}_A, \mathbb{L}_X)\langle 1\rangle$ is onto with kernel $\operatorname{Der}(\mathbb{L}_X\|\mathbb{L}_A)\langle 1\rangle$. In particular, we have a short exact sequence

$$0 \to \mathrm{Der}(\mathbb{L}_X \| \mathbb{L}_A)\langle 1\rangle \to \mathrm{Der}(\mathbb{L}_X)\langle 1\rangle \to \mathrm{Der}_i(\mathbb{L}_A, \mathbb{L}_X)\langle 1\rangle \to 0.$$

Now consider the following (non-commuting) diagram with rows being homotopy cofibrations

$$\begin{array}{ccccc} \mathrm{Der}(\mathbb{L}_X \| \mathbb{L}_A)\langle 1\rangle & \longrightarrow & \mathrm{Der}(\mathbb{L}_X)\langle 1\rangle & \longrightarrow & \mathrm{Der}_i(\mathbb{L}_A, \mathbb{L}_X)\langle 1\rangle \\ \zeta\downarrow & & \| & & \wr\downarrow -\alpha \\ \mathrm{Der}(\mathbb{L}_X)\langle 1\rangle \ltimes_{\tau_*} \mathrm{Hom}^\tau(\bar{\mathcal{C}}\mathbb{L}_A, \mathbb{L}_X)\langle 0\rangle & \longrightarrow & \mathrm{Der}(\mathbb{L}_X)\langle 1\rangle & \longrightarrow & cone(\rho) \end{array} \quad (3)$$

**Claim 4.7.** *The diagram above commutes up to homotopy.*

*Proof.* The left square commutes strictly, so we are left to show that the right square commutes up to homotopy. In other words, we want to show that

$$\begin{aligned} \Phi, \Psi \colon \mathrm{Der}\,\mathbb{L}_X\langle 1\rangle \to cone(\rho) = {} & \mathrm{Der}(\mathbb{L}_X)\langle 1\rangle \oplus s(\mathrm{Der}(\mathbb{L}_X)\langle 1\rangle \\ & \ltimes_{\tau_*} \mathrm{Hom}^\tau(\bar{\mathcal{C}}(\mathbb{L}_A), \mathbb{L}_A)\langle 0\rangle) \end{aligned}$$

given by $\Phi(\theta) = (\theta, 0, 0)$ and $\Psi(\theta) = (0, 0, -s\pi_A^*(\theta \circ i))$ are homotopic.

Let $H \colon \mathrm{Der}(\mathbb{L}_X)\langle 1\rangle \to \mathrm{Der}(\mathbb{L}_X)\langle 1\rangle \oplus s(\mathrm{Der}(\mathbb{L}_X)\langle 1\rangle \ltimes_{\tau_*} \mathrm{Hom}^\tau(\bar{\mathcal{C}}(\mathbb{L}_A), \mathbb{L}_A)\langle 0\rangle)$ be given by

$$H(\theta) = (0, s\theta, 0).$$

We have that

$$\begin{aligned} (dH + Hd)(\theta) &= d(0, s\theta, 0) + H(d(\theta)) \\ &= (-\theta, -sd(\theta), -s\tau_*(\theta)) + (0, sd(\theta), 0) = (-\theta, 0, -s\tau_*(\theta)) \end{aligned}$$

which is $(\Psi - \Phi)(\theta)$. ☐

Now as we have that (3) commutes up to homotopy, it induces a (strict) map of the long exact sequences associated to the cofibrations. Since $\alpha \colon \mathrm{Der}_i(\mathbb{L}_A, \mathbb{L}_X)\langle 1\rangle \to cone(\rho)$ and $\mathrm{id} \colon \mathrm{Der}(\mathbb{L}_X)\langle 1\rangle \to \mathrm{Der}(\mathbb{L}_X)\langle 1\rangle$ induces isomorphisms in homology, it follows by the five lemma that $\zeta$ also induces isomorphisms in homology. ☐

## 5. Examples

*Example 5.1.* A Lie model for $\mathbb{C}P^k$, $k \geqslant 1$ is given by the cofibrant dg Lie algebra $\mathbb{L}(x_1, \ldots, x_k)$ on the free graded vector space $\mathrm{span}_{\mathbb{Q}}(x_1, \ldots, x_k)$ where $|x_i| = 2i - 1$ and where the differential is given by $d(x_i) = \frac{1}{2}\sum_{p+q=i}[x_p, x_q]$ (see [**FHT01**, §24.(f)]). The inclusion of $\mathbb{C}P^k \to \mathbb{C}P^n$, $1 \leqslant k < n$ is modelled by the free map induced by the inclusion

$$\mathrm{span}_{\mathbb{Q}}(x_1, \ldots, x_k) \hookrightarrow \mathrm{span}_{\mathbb{Q}}(x_1, \ldots, x_n).$$

In particular, we have that the underlying dg vector space of $\mathrm{Der}(\mathbb{L}_{\mathbb{C}P^n} \| \mathbb{L}_{\mathbb{C}P^k})$ is isomorphic to

$$\mathrm{Hom}(\mathrm{span}_{\mathbb{Q}}(x_{k+1}, \ldots, x_n), \mathbb{L}(x_1, \ldots, x_n)),$$

where the differential on a map $f$ of homogeneous degree $|f|$ is given by

$$d(f)(x_q) = d \circ f(x_q) - (-1)^{|f|}\frac{1}{2}\sum_{i+j=q}[f(x_i), x_j] - \frac{1}{2}\sum_{i+j=q}[x_i, f(x_j)],$$

where we set $f(x_i) = 0$ if $i \leqslant k$. We observe that if $n = k+1$ then only first term in the differential above survives. In this particular case we have an isomorphism of chain complexes

$$\mathrm{Der}(\mathbb{L}_{\mathbb{C}P^{k+1}} \| \mathbb{L}_{\mathbb{C}P^k}) = s^{-2k-1}\mathbb{L}_{\mathbb{C}P^{k+1}}.$$

Hence, we see that

$$\pi^{\mathbb{Q}}_{*+2k+1}(B\,\mathrm{aut}_{\mathbb{C}P^k}(\mathbb{C}P^{k+1})) = \pi^{\mathbb{Q}}_{*}(\mathbb{C}P^{k+1}).$$

*Example 5.2.* Every simply connected topological space $X$ admits a minimal Lie model of the form $(\mathbb{L}(s^{-1}\widetilde{H}_*(X;\mathbb{Q})), d)$, where the generating vector space is the desuspension of the reduced rational homology of $X$ (see [**FHT01**, Chapter 24]). In particular, a minimal Lie model for the sphere $S^{n-1}$ is given by $\mathbb{L}(u)$, where $|u| = n-2$.

If $X$ is an $n$-dimensional simply connected compact manifold with boundary $\partial X \cong S^{n-1}$, then for every basis $B = \{x_1, \ldots, x_m\}$ of $s^{-1}\widetilde{H}_*(X;\mathbb{Q})$ there exists a 'dual basis' $B^{\#} = \{x_1^{\#}, \ldots, x_m^{\#}\}$ such that $|x_i^{\#}| + |x_i| = n-2$ and such that the inclusion $S^{n-1} \to X$ is modelled by the dg Lie algebra morphism

$$\mathbb{L}(u) \to \mathbb{L}(s^{-1}\widetilde{H}_*(X;\mathbb{Q})), \qquad u \mapsto \omega = \frac{1}{2}\sum_{i=1}^{m}[x_i^{\#}, x_i]$$

(see [**Sta83**] and [**BM20**, §3.5] for details).

Note that the dg Lie algebra map above is not a cofibration. In order to be able to apply Theorem 1.1, one needs to replace the map by a cofibration. This can be done by adding generators $u$ and $v$ to the Lie model of $X$ and define $d(u) = 0$ and $d(v) = u - \omega$. Then the inclusion $\mathbb{L}(u) \to \mathbb{L}(s^{-1}\widetilde{H}_*(X), u, v)$ is a cofibration that models the inclusion. Theorem 1.1 shows that $\mathrm{Der}(\mathbb{L}(s^{-1}\widetilde{H}_*(X), u, v)\|u)$ is a Lie model for the universal cover of $B\,\mathrm{aut}_\partial(X)$. In [**BM20**], the authors go further and show that $\mathrm{Der}(\mathbb{L}(s^{-1}\widetilde{H}_*(X), u, v)\|u)$ is quasi-isomorphic to $\mathrm{Der}(\mathbb{L}(s^{-1}\widetilde{H}_*(X))\|\omega)$. However, in general, if $j\colon \mathbb{L}_A \to \mathbb{L}_X$ models a cofibration $A \subset X$ where $\mathbb{L}_A$ and $\mathbb{L}_X$ are cofibrant dg Lie algebras, but where $j$ is not a cofibration, then it is not necessarily true that the Lie subalgebra of $\mathrm{Der}(\mathbb{L}_X)$ of derivations that vanish on the image of $j$ is a Lie model for the universal cover of $B\,\mathrm{aut}_A(X)$. We will see this in the next example.

*Example 5.3.* In Theorem 1.1 it is required that the Lie algebra map $i\colon \mathbb{L}_A \to \mathbb{L}_X$ that models the inclusion $A \subset X$ is a cofibration. In this example we show that this condition is necessary.

Consider the inclusion $S^3 \subset D^4$. A cofibration between cofibrant dg Lie algebras that models the inclusion is given by

$$(\mathbb{L}(u), |u| = 2) \to (\mathbb{L}(u,v),\ dv = u,\ |u| = 2,\ |v| = 3).$$

Hence we know that a Lie model for $B\,\mathrm{aut}_{S^3,\circ}(D^4)$ is given by $\mathrm{Der}(\mathbb{L}(u,v)\|\mathbb{L}(u))$ which one easily shows is homotopically trivial. Let us now model the inclusion

$S^3 \subset D^4$ by a Lie map which is not a cofibration. We let a cofibrant model for $S^3$ be given by the abelian dg Lie algebra

$$\mathbb{L}_{S^3} = (\mathbb{L}(u),\ |u| = 2, du = 0).$$

Since $D^4$ is contractible, any homotopically trivial dg Lie algebra is a Lie model for $D^4$, and any map from $\mathbb{L}_{S^3}$ to that Lie model of $D^4$ is a model for the inclusion $S^3 \subset D^4$. We let

$$\mathbb{L}_{D^4} = (\mathbb{L}(a,b),\ |a| = 1,\ |b| = 2,\ db = a)$$

be Lie model for $D^4$ and we let the inclusion $S^3 \subset D^4$ be modelled by the map $i\colon \mathbb{L}(u) \to \mathbb{L}(a,b)$, $i(u) = [a,a]$. Now we show that $\mathrm{Der}(\mathbb{L}(a,b)\|[a,a])$ is not weakly equivalent to the trivial dg Lie algebra. Let $\mathrm{ad}_a \in \mathrm{Der}(\mathbb{L}(a,b)\|[a,a])$ be given by $\mathrm{ad}_a(x) = [a,x]$ ($\mathrm{ad}_a$ vanishes on $[a,a]$ since $[a,[a,a]] = 0$ by the graded Jacobi identity). Straightforward calculations give that $\mathrm{ad}_a$ is a cycle in $\mathrm{Der}(\mathbb{L}(a,b)\|[a,a])$. Now we show that $\mathrm{ad}_a$ is not a boundary in $\mathrm{Der}(\mathbb{L}(a,b)\|[a,a])$. We have that any $g \in \mathrm{Der}(\mathbb{L}(a,b))$ of degree 2 is determined by its images on the generators. We have that $g(a) = \alpha[b,a]$ for some $\alpha \in \mathbb{Q}$ since the degree three part of $\mathbb{L}(a,b)$ is spanned by $[b,a]$. We also have that $g(b) = \beta[b,[a,a]]$ for some $\beta \in \mathbb{Q}$ since the degree four part of $\mathbb{L}(a,b)$ is spanned by $[b,[a,a]]$. Solving the equation $Dg = \mathrm{ad}_a$ gives that $\alpha = 1$ and that $\beta$ can be chosen arbitrary. However, we get that

$$g[a,a] = 2[[b,a],a] \neq 0$$

showing that $g \notin \mathrm{Der}(\mathbb{L}(a,b)\|[a,a])$, so $\mathrm{ad}_a$ is not a boundary in $\mathrm{Der}(\mathbb{L}(a,b)\|[a,a])$. We conclude that $\mathrm{Der}(\mathbb{L}(a,b)\|[a,a])$ is not homotopically trivial, and therefore not a Lie model for $B\,\mathrm{aut}_{S^3,\circ}(D^4)$.

## References

**[Ber15]** A. Berglund, *Rational homotopy theory of mapping spaces via Lie theory for $L_\infty$-algebras*, Homology Homotopy Appl. **17** (2015), no. 2, 343–369.

**[Ber17]** ———, *Rational models for automorphisms of fiber bundles*, to appear in Doc. Math. Preprint available at arXiv:1703.03747 [math.AT].

**[BM13]** A. Berglund and I. Madsen, *Homological stability of diffeomorphism groups*, Pure Appl. Math. Q. **9** (2013), no. 1, 1–48.

**[BM20]** ———, *Rational homotopy theory of automorphisms of manifolds*, to appear in Acta Math. (2020). Preprint available at arXiv:1401.4096v3 [math.AT].

**[FHT01]** Y. Félix, S. Halperin, and J.-C. Thomas, *Rational Homotopy Theory*, Graduate Texts in Mathematics, vol. 205, Springer, New York, 2001.

**[Gre19]** M. Grey, *On rational homological stability for block automorphisms of connected sums of products of spheres*, Algebr. Geom. Topol. **19** (2019), no. 7, 3359–3407.

**[Man04]** M. Manetti, *Lectures on deformations of complex manifolds*, Rend. Mat. **24** (2004), no. 7, 1–183.

**[May72]** J.P. May, *The Geometry of Iterated Loop Spaces*, Lecture Notes in Mathematics, vol. 271, Springer-Verlag, Berlin, 1972.

[**May75**] ______, *Classifying spaces and fibrations*, Mem. Amer. Math. Soc. **1** (1975), no. 155.

[**May99**] ______, *A concise course in algebraic topology*, University of Chicago Press, 1999.

[**Qui69**] D. Quillen, *Rational homotopy theory*, Ann. Math. **90** (1969), no. 2, 205–295.

[**SS12**] M. Schlessinger and J. Stasheff, *Deformation theory and rational homotopy type*, arXiv:1211.1647 [math.QA].

[**Sta83**] J. Stasheff, *Rational Poincaré duality spaces*, Illinois J. Math. **27** (1983), no. 1, 104–109.

[**Tan83**] D. Tanré, *Homotopie rationnelle: modèles de Chen, Quillen, Sullivan*, Lecture Notes in Mathematics, vol. 1025, Springer-Verlag, Berlin, 1983.

[**Wei94**] C. Weibel, *An introduction to homological algebra*, Cambridge Studies in Advanced Mathematics, vol. 38, Cambridge University Press, Cambridge, 1994.

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