COHOMOLOGY OF THE CLASSIFYING SPACES OF $U(n)$-GAUGE GROUPS OVER THE 2-SPHERE

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Abstract
A gauge group is the topological group of automorphisms of a principal bundle. We compute the integral cohomology ring of the classifying spaces of gauge groups of principal $U(n)$-bundles over the 2-sphere by generalizing the operation for free loop spaces, called the free double suspension.

1.   Introduction

Let G be a topological group, and $P \to X$ be a principal G-bundle over a base X. An automorphism of P is, by definition, a G-equivariant self-map of P covering the identity map of X. The gauge group of P, denoted as $\mathcal{G}(P)$, is defined as the topological group of automorphisms of P.

As in [**1**, **5**], there is a natural equivalence

$$B\mathcal{G}(P) \simeq \mathrm{Map}(X, BG; \alpha), \tag{1}$$

where $\mathrm{Map}(X, Y; f)$ is the path-component of the space of maps $\mathrm{Map}(X, Y)$ containing a map $f\colon X \to Y$ and $\alpha\colon X \to BG$ is a classifying map of P. This connection enables us to employ new techniques and insights, specifically fiberwise homotopy theory and group theory, to study the homotopy theory of mapping spaces and to import rich tools in the homotopy theory of mapping spaces to gauge groups. Moreover, since the classifying space $B\mathcal{G}(P)$ is homotopy equivalent to the moduli space of connections of P in the smooth case, the homotopy theory of gauge groups potentially has application in geometry and physics.

In this paper, we determine the integral cohomology ring of the classifying spaces of gauge groups of principal $U(n)$-bundle over S^2. Although the (co)homology of the classifying spaces is an obviously important object in topology and has possible applications to geometry and physics, there are only a few previous works. Mod-p homology is computed in [**4**, **7**], and partial cohomology calculations are done in [**8**, **9**, **12**]. In [**1**] the rational Poincaré series are determined and the generators of the integral cohomology ring are described.

Thus our result is the first complete determination of the integral cohomology ring of the classifying space of gauge groups in the nontrivial case.

To state the main theorem we set notation. Let $e\colon \mathcal{G}(P) \to G$ be a homomorphism given by the evaluation map at the base point of X. Then one gets the induced map $B\mathcal{G}(P) \to BG$ which is denoted by the same symbol e. Let $P_{n,k}$ be a principal $U(n)$-bundle over S^2 such that $c_1(P_{n,k}) = k \in \mathbb{Z} \cong H^2(S^2; \mathbb{Z})$. Recall that $H^*(BU(n)) \cong \mathbb{Z}[c_1, c_2, \ldots, c_n]$, when c_i is the i-th Chern class of the universal bundle. As we will see later, $e\colon B\mathcal{G}(P) \to BG$ is an injection in cohomology, and so we abbreviate $e^*(c_i)$ by c_i. Let $e_i(a_1, a_2, \ldots) \in \mathbb{Z}[[a_1, a_2, \ldots]]$ be the i-th elementary symmetric function in $a_1, a_2, \ldots$, and s_i be the i-th Newton polynomial defined by

$$s_i(e_1, \ldots, e_i) = \sum_j a_j^i,$$

where we abbreviate $e_i(a_1, a_2, \ldots)$ and $s_i(x_1, \ldots, x_i)$ by e_i and s_i respectively when the indeterminates are obvious.

Now we state the main theorem.

Theorem 1.1. *There is a ring isomorphism*

$$H^*(B\mathcal{G}(P_{n,k}); \mathbb{Z}) \cong \mathbb{Z}[c_1, \ldots, c_n, x_1, x_2, \ldots]/(h_n, h_{n+1}, \ldots),$$

where

$$h_i = kc_i + \sum_{1 \leqslant j \leqslant i} (-1)^j s_j(x_1, x_2, \ldots, x_j) c_{i-j}.$$

Moreover, there is $\zeta \in K(B\mathcal{G}(P_{n,k}))$ such that $c_i(\zeta) = x_i$.

To prove this theorem we will generalize a certain map in the cohomology of free loop spaces which is defined in [**6**] and called the free loop suspension. Since the homotopy equivalence (1) is natural with respect to X, the map $e\colon B\mathcal{G}(P) \to BG$ coincides with the evaluation map $\mathrm{Map}(X, BG; \alpha) \to BG$ at the base point of X which is ambiguously denoted by the same symbol e. Specifically in the case of $P_{n,k}$ there is an evaluation fibration

$$\Omega_k^2 BU(n) \to \mathrm{Map}(S^2, BU(n); k) \xrightarrow{e} BU(n),$$

where $\Omega_k^2 BU(n)$ and $\mathrm{Map}(S^2, BU(n); k)$ are the connected component of the double loop space of $BU(n)$ and $\mathrm{Map}(S^2, BU(n))$ containing a degree k map respectively. Since $\Omega_k^2 BU(n) \simeq \Omega_0 U(n)$, $\Omega_k^2 BU(n)$ and $BU(n)$ have only even cells. Thus the associated Serre spectral sequence of this evaluation fibration collapses at the E_2-term, hence there is an isomorphism as $H^*(BU(n))$-modules

$$H^*(\mathrm{Map}(S^2, BU(n); k)) \cong H^*(\Omega_k^2 BU(n)) \otimes H^*(BU(n)),$$

and so, in particular, e is an injection in cohomology. Thus it remains to determine the ring structure by using the free double suspension.

2. Free double suspension

Let $LX := \mathrm{Map}(S^1, X)$, the free loop space of a space X. In [**6**], a map $\hat{\sigma}\colon H^*(X) \to H^{*-1}(LX)$ is constructed as an extension of the cohomology suspension $\sigma\colon H^*(X) \to$

$H^{*-1}(\Omega X)$ and applied to the evaluation fibration

$$\Omega X \to LX \to X$$

to determine the cohomology of LX. In this section, we generalize the free suspension to a mapping space $\mathrm{Map}(S^2, X; f)$ and show its basic properties that we are going to use. Since in [**6**] the space X is assumed to be simply connected, it is not necessary to consider the components of LX in this case. In our case, we have to consider the path components of $\mathrm{Map}(S^2, X)$.

Let $\hat{e} \colon S^2 \times \mathrm{Map}(S^2, X; f) \to X$ be the evaluation map defined by $\hat{e}(s, g) = g(s)$ for $(s, g) \in S^2 \times \mathrm{Map}(S^2, X)$. We define the *free double suspension*

$$\hat{\sigma}_f^2 \colon H^*(X) \to H^{*-2}(\mathrm{Map}(S^2, X; f)) \quad \text{by} \quad \hat{\sigma}_f^2(x) = \hat{e}^*(x)/v$$

for $x \in H^*(X)$, where $/$ means the slant product and $v \in H_2(S^2)$ is the Hurewicz image of the identity map of S^2.

To state properties of free double suspensions, we set notation. Let $e \colon \mathrm{Map}(S^2, X; f) \to X$ be the evaluation map at the basepoint of S^2. Let $\sigma_f^2 \colon H^*(X) \to H^{*-2}(\Omega_f^2 X)$ be the composite of cohomology suspensions and the inclusion map

$$H^*(X) \xrightarrow{\sigma} H^{*-1}(\Omega X) \xrightarrow{\sigma} H^{*-2}(\Omega^2 X) \to H^{*-2}(\Omega_f^2 X),$$

where $f \in \Omega^2 X$. Let $j \colon \Omega_f^2 X \to \mathrm{Map}(S^2, X; f)$ be the inclusion.

Proposition 2.1. *Let $f \in \Omega^2(X)$. Free double suspensions have the following properties:*

1. *$\hat{\sigma}_f^2$ restricts to σ_f^2 such that*

$$j^* \circ \hat{\sigma}_f^2 = \sigma_f^2.$$

2. *$\hat{\sigma}_f^2$ is a derivation such that for $x, y \in H^*(X)$*

$$\hat{\sigma}_f^2(xy) = \hat{\sigma}_f^2(x)e^*(y) + e^*(x)\hat{\sigma}_f^2(y).$$

3. *Suppose that X is a path-connected H-group with a multiplication $\mu \colon X \times X \to X$. If $\mu^*(y) = \sum_i y_i \times y_i'$ for $y \in H^*(X)$, then*

$$\hat{\sigma}_f^2(y) = \sum_i \alpha^*(\sigma_f^2(y_i))e^*(y_i'),$$

   where $\alpha \colon \mathrm{Map}(S^2, X; f) \to \Omega_f^2 X$ is given by $\alpha(g)(t) = \mu(g(t), e(g)^{-1})$ for $t \in S^2$ and $g \in \mathrm{Map}(S^2, X; f)$.

Proof. (1) Let $\bar{e} \colon S^2 \times \Omega_f^2 X \to X$ be the restriction of $\hat{e}$, that is, $\bar{e} = \hat{e} \circ (1 \times j)$. There is a homotopy commutative diagram

$$
\begin{array}{ccccccc}
S^1 \times S^1 \times \Omega_f^2 X & \xrightarrow{1 \times 1 \times l} & S^1 \times S^1 \times \Omega^2 X & \xrightarrow{1 \times \omega} & S^1 \times \Omega X & \xrightarrow{\omega} & X \\
{\scriptstyle q \times 1} \downarrow & & \downarrow {\scriptstyle q \times 1} & & & & \parallel \\
S^2 \times \Omega_f^2 X & \xrightarrow{1 \times l} & S^2 \times \Omega^2 X & & \xrightarrow{\bar{e}'} & & X,
\end{array}
$$

where q is a quotient map, ω and $\bar{e}'$ are the evaluation map and l is the inclusion

map. Then for $x \in H^*(X)$,

$$\begin{aligned}
\sigma_f^2(x) &= l^* \circ \sigma(\omega^*(x)/w) \\
&= l^* \circ (\omega^* \circ (\omega^*(x)/w))/w \\
&= l^* \circ ((1 \times \omega^*) \circ (\omega^*(x)))/(w \times w) \\
&= \bar{e}^*(x)/v,
\end{aligned}$$

where $w \in H_*(S^1)$ is a generator. Thus

$$j^* \circ \hat{\sigma}_f^2(x) = j^*(\hat{e}^*(x)/v) = ((1 \times j)^* \circ \hat{e}^*(x))/v = \bar{e}^*(x)/v = \sigma_f^2(x).$$

(2) By definition, $\hat{e}^*(x) = 1 \times e^*(x) + u \times \hat{\sigma}_f^2(x)$ for $x \in H^*(X)$, where $u \in H^2(S^2)$ is the Kronecker dual of v. Then for $x, y \in H^*(X)$,

$$\begin{aligned}
\hat{e}^*(xy) &= \hat{e}^*(x)\hat{e}^*(y) = (1 \times e^*(x) + u \times \hat{\sigma}_f^2(x))(1 \times e^*(y) + u \times \hat{\sigma}_f^2(y)) \\
&= 1 \times e^*(xy) + u \times (\hat{\sigma}_f^2(x)e^*(y) + e^*(x)\hat{\sigma}_f^2(y)).
\end{aligned}$$

Thus one gets the desired equality by taking the slant product with v.

(3) The map $\alpha \times e \colon \mathrm{Map}(S^2, X; f) \to \Omega_f^2 X \times X$ is obviously a homotopy equivalence and satisfies a homotopy commutative diagram

$$\begin{CD}
S^2 \times \mathrm{Map}(S^2, X; f) @>{1 \times (\alpha \times e)}>> S^2 \times \Omega_f^2 X \times X \\
@V{\hat{e}}VV @VV{\bar{e} \times 1}V \\
X @<<{\mu}< X \times X.
\end{CD}$$

As well as $\hat{e}^*(y)$, one has $\bar{e}^*(y) = 1 \times j^* \circ e^*(y) + u \times \sigma_f^2(y)$ for $y \in H^*(X)$. Thus if $\mu^*(y) = \sum_i y_i \otimes y_i'$, then

$$\begin{aligned}
\hat{e}^*(y) &= (1 \times (\alpha \times e)^*) \circ (\bar{e}^* \times 1) \circ \mu^*(y) = (1 \times \alpha^* \times e^*) \circ (\bar{e}^* \times 1)\left(\sum_i y_i \times y_i'\right) \\
&= (1 \times (\alpha \times e)^*)\left(\sum_i (1 \times j^* \circ e^*(y_i) + u \times \sigma_f^2(y_i)) \times y_i'\right) \\
&= \sum_i (1 \times \alpha^* \circ j^* \circ e^*(y_i) + u \times \alpha^*(\sigma_f^2(y_i)))e^*(y_i').
\end{aligned}$$

Therefore the desired equality is obtained by taking the slant product with v. $\square$

We observe the relation between cohomology suspensions and component shifts. Let $\bar{f} \colon \Omega_0^2 X \to \Omega_f^2 X$ denote the map given by adding a map $f \colon S^2 \to X$. Then $\bar{f}$ is a homotopy equivalence.

Lemma 2.2. *For a map $f \colon S^2 \to X$ and $x \in H^*(X)$,*

$$\sigma_f^2(x) = (f^*(x) \times 1)/v + (\bar{f}^*)^{-1} \circ \sigma_0^2(x).$$

Proof. Let $\phi_f \colon (S^2 \vee S^2) \times \Omega_0^2 X \to X$ be the map defined by $\phi_f(s, *, g) = f(s)$ and $\phi_f(*, t, g) = g(t)$ for $s, t \in S^2$ and $g \in \Omega_0 X$, and let $\eta \colon S^2 \to S^2 \vee S^2$ be the comultiplication. Since $\bar{e} \circ (1 \times \bar{f}) = \phi_f \circ (\eta \times 1)$, there is a homotopy commutative

diagram

$$
\begin{array}{ccc}
S^2 \times \Omega_f^2 X & \xrightarrow{\ 1 \times \bar{f}^{-1}\ } & S^2 \times \Omega_0^2 X \\
{\scriptstyle \bar{e}} \downarrow & & \downarrow {\scriptstyle \eta \times 1} \\
X & \xleftarrow{\quad \phi_f \quad} & (S^2 \vee S^2) \times \Omega_0^2 X.
\end{array}
$$

Thus

$$
\begin{aligned}
\bar{e}^*(x) &= (1 \times (\bar{f}^*)^{-1}) \circ (\eta \times 1)^* \circ \phi_f^*(x) \\
&= (1 \times (\bar{f}^*)^{-1}) \circ (\eta \times 1)^* ((f^*(x) \vee 1) \times 1 + (1 \vee u) \times \sigma_0^2(x) + 1 \times j^* \circ e^*(x)) \\
&= f^*(x) \times 1 + u \times (\bar{f}^*)^{-1}(\sigma_0^2(x)) + 1 \times (\bar{f}^*)^{-1} \circ j^* \circ e^*(x).
\end{aligned}
$$

The desired equality is obtained by taking the slant product with v. $\qquad\square$

3. Proof of the main theorem

In this section, we prove the main theorem. Let $i_n \colon SU(n) \to SU(\infty)$ be the inclusion.

Proposition 3.1.

1. *There is an isomorphism*

$$
H^*(\Omega SU(n)) \cong \mathbb{Z}[y_1, y_2, \dots]/(s_n, s_{n+1}, \dots), \quad |y_i| = 2i.
$$

2. *The map $\Omega i_n \colon \Omega SU(n) \to \Omega SU(\infty)$ is a surjection in cohomology.*

Proof.

1. This follows from the result of Bott [**3**, Proposition 8.1].

2. By the construction of Bott [**3**], the isomorphism of (1) is natural with respect to the inclusion $i_n \colon \Omega SU(n) \to \Omega SU(n+1)$. Namely, $(i_n)^*(y_i) = y_i$ for each i. $\quad\square$

We set notation. Let $\iota \in \widetilde{K}(S^2)$ be a generator such that $\mathrm{ch}(\iota) = u$, where $\mathrm{ch} \colon K(X) \to H^{**}(X; \mathbb{Q})$ denotes the Chern character and u is as in Section 2. Let $\xi_n \in \widetilde{K}(BU(n))$ be the universal bundle over $BU(n)$ minus the rank n trivial bundle, and $\xi_\infty := \mathrm{colim}\,\xi_n$. We define $\beta \colon BU(\infty) \to \Omega_0^2 BU(\infty)$ as the adjoint of $\iota \wedge \xi_\infty \colon S^2 \wedge BU(\infty) \to BU(\infty)$. Let $j_n \colon \mathrm{Map}(S^2, BU(n); k) \to \mathrm{Map}(S^2, BU(\infty); k)$ be the map induced by the inclusion $i_n \colon BU(n) \to BU(\infty)$.

Lemma 3.2. *In $H^*(\Omega_0^2 BU(\infty))$,*

$$
\sigma_0^2(c_m) = \begin{cases} 0 & (m = 1), \\ (-1)^{m-1}(\beta^*)^{-1}(s_{m-1}(c_1, \dots, c_{m-1})) & (m \geqslant 2). \end{cases}
$$

Proof. There is a homotopy commutative diagram

$$S^2 \wedge BU(\infty) \xrightarrow{\iota \wedge \xi_\infty} BU(\infty),$$

$$1 \times \beta \downarrow \qquad \nearrow \bar{e}$$

$$S^2 \wedge \Omega_0^2 BU(\infty)$$

where $\bar{e}$ is as in Section 2. Then it follows that

$$u \times \mathrm{ch}(\xi_\infty) = \mathrm{ch}(\iota) \times \mathrm{ch}(\xi_\infty) = (\iota \wedge \xi_\infty)^*(\mathrm{ch}(\xi_\infty))$$
$$= (1 \times \beta)^* \circ \bar{e}^*(\mathrm{ch}(\xi_\infty)) = u \times \beta^*(\sigma_0^2(ch(\xi_\infty))),$$

in the rational cohomology. Thus since $\mathrm{ch}(\xi_\infty) = \sum_{i \geqslant 1} \frac{1}{i!} s_i$,

$$u \times s_m = u \times \beta^*(\sigma_0^2((-1)^m c_{m+1}))$$

in integral cohomology, completing the proof. $\qquad \square$

Let $\gamma_k \colon \Omega_0^2 BU(n) \to \Omega_k^2 BU(n)$ be the map given by the concatenation with the degree k map $S^2 \to BU(n)$. Then γ_k is a homotopy equivalence.

Proof of Theorem 1.1. First, we define

$$\bar{x}_m \coloneqq \alpha^* \circ (\gamma_k{}^*)^{-1} \circ (\beta^*)^{-1}(c_m) \quad \text{and} \quad x_m \coloneqq j_n^*(\bar{x}_m),$$

where the map $\alpha \colon \mathrm{Map}(S^2, BU(n); k) \to \Omega_k^2 BU(n)$ is as in Section 2. We show these x_i become the Chern classes of a virtual bundle in the latter half of this proof.

There is a homotopy commutative diagram

$$\Omega_0^2 BU(n) \xrightarrow{\Omega_0^2 i_n} \Omega_0^2 BU(\infty)$$

$$\gamma_k \downarrow \qquad\qquad \downarrow \gamma_k$$

$$\Omega_k^2 BU(n) \xrightarrow{\Omega_k^2 i_n} \Omega_k^2 BU(\infty)$$

in which maps γ_k are homotopy equivalences. Then by Lemma 3.1 (2), $\Omega_k^2 i_n$ is surjective in cohomology. Thus since $H^*(\Omega_k^2 BU(\infty))$ is generated by $(\gamma_k^*)^{-1} \circ (\beta^*)^{-1}(c_i)$ for $i \geqslant 1$, $H^*(\Omega_k^2 BU(n))$ is generated by $(\Omega_k^2 i_n)^* \circ (\gamma_k^*)^{-1} \circ (\beta^*)^{-1}(c_i)$ for $i \geqslant 1$. There is a homotopy commutative diagram

$$\Omega_k^2 BU(n) \xrightarrow{\Omega_k^2 i_n} \Omega_k^2 BU(\infty) =\!=\!=\!=\!=\!= \Omega_k^2 BU(\infty)$$

$$\downarrow \qquad\qquad\qquad \downarrow \qquad\qquad\qquad \|$$

$$\mathrm{Map}(S^2, BU(n); k) \xrightarrow{j_n} \mathrm{Map}(S^2, BU(\infty); k) \xrightarrow{\alpha} \Omega_k^2 BU(\infty).$$

Then x_i restricts to $(\Omega_k^2 i_n)^* \circ (\gamma_k^*)^{-1} \circ (\beta^*)^{-1}(c_i)$. Now we apply the Leray-Hirsch theorem to the evaluation fibration $\Omega_k^2(BU(n)) \xrightarrow{j} \mathrm{Map}(S^2, BU(n); k) \xrightarrow{e} BU(n)$, we obtain that

$$\Phi \colon \mathbb{Z}[c_1, c_2, \ldots, c_n, x_1, x_2, \ldots] \to H^*(\mathrm{Map}(S^2, BU(n); k))$$

is surjective.

We next show $h_i \in \mathrm{Ker}(\Phi)$. Now we know the equation in $H^*(S^2 \times \mathrm{Map}(S^2, BU(n); k))$

$$(f^*(c_i) \times 1)/v = \begin{cases} k & (i = 1), \\ 0 & (\text{otherwise}), \end{cases}$$

where $f \colon S^2 \to BU(n)$ is a degree k-map. From this equation together with Proposition 2.1 (3) and Lemmas 2.2 and 3.2, it follows that in $H^*(\mathrm{Map}(S^2, BU(\infty); k))$, one gets

$$\hat{\sigma}_k^2(c_i) = \sum_{1 \leqslant j \leqslant i} \alpha^*(\sigma_k^2(c_j)) c_{i-j} = k c_{i-1} + \sum_{2 \leqslant j \leqslant i} \alpha^* \circ (\gamma_k^*)^{-1}(\sigma_0^2(c_j)) c_{i-j}$$
$$= k c_{i-1} + \sum_{2 \leqslant j \leqslant i} (-1)^{j-1} s_{j-1}(\bar{x}_1, \dots, \bar{x}_{j-1}) c_{i-j}.$$

Then for $i \geqslant n$

$$\Phi(h_i) = j_n^*(\hat{\sigma}_k^2(c_{i+1})) = \hat{\sigma}_k^2(i_n^*(c_{i+1})) = 0,$$

where $j_n^*(\bar{x}_i) = x_i$. Thus Φ induces a surjection

$$\bar{\Phi} \colon \mathbb{Z}[c_1, c_2, \dots, c_n, x_1, x_2, \dots]/(h_n, h_{n+1}, \dots) \to H^*(\mathrm{Map}(S^2, BU(n); k)).$$

We next show that $\bar{\Phi}$ is an isomorphism. Let $\mathbb{F}$ be an arbitrary field. We calculate the Poincaré series of $A_{\mathbb{F}} \coloneqq \mathbb{F}[c_1, c_2, \dots, c_n, x_1, x_2, \dots]/(h_n, h_{n+1}, \dots)$. Let $P_t(V)$ be the Poincaré series of a graded vector space V. Since $h_i \equiv s_i(x_1, x_2, \dots, x_i) \mod(c_1, c_2, \dots, c_n)$ and $c_1, c_2, \dots, c_n$ is a regular sequence in $A_{\mathbb{F}}$,

$$P_t(\mathbb{F}[c_1, c_2, \dots, c_n, x_1, x_2, \dots]/(h_n, h_{n+1}, \dots)) = \frac{P_t(\mathbb{F}[x_1, x_2, \dots]/(s_n, s_{n+1}, \dots))}{(1 - t^2)(1 - t^4) \cdots (1 - t^{2n})}.$$

Then by Lemma 3.1,

$$P_t(A_{\mathbb{F}}) = \frac{P_t(H^*(\Omega SU(n); \mathbb{F}))}{(1 - t^2)(1 - t^4) \cdots (1 - t^{2n})}.$$

On the other hand as in Section 1 the Serre spectral sequence of the evaluation fibration $\Omega_k^2(BU(n)) \to \mathrm{Map}(S^2, BU(n); k) \xrightarrow{e} BU(n)$ collapses at the E_2-term, and so

$$P_t(H^*(\mathrm{Map}(S^2, BU(n); k); \mathbb{F})) = P_t(H^*(BU(n); \mathbb{F})) \times P_t(H^*(\Omega SU(n); \mathbb{F}))$$
$$= \frac{P_t(H^*(\Omega SU(n); \mathbb{F}))}{(1 - t^2)(1 - t^4) \cdots (1 - t^{2n})}.$$

Then we get the equality

$$P_t(A_{\mathbb{F}}) = P_t(H^*(\mathrm{Map}(S^2, BU(n); k); \mathbb{F})).$$

Since the source and target of $\bar{\Phi}$ is of finite type, $\bar{\Phi}$ is an isomorphism over an arbitrary field. Thus $\bar{\Phi}$ is an isomorphism over $\mathbb{Z}$.

It remains to show that the classes x_i can be represented as the Chern classes of a virtual bundle. Since the Künneth formula holds as

$$K(S^2 \times \mathrm{Map}(S^2, BU(\infty); k)) \cong K(S^2) \otimes K(\mathrm{Map}(S^2, BU(\infty); k)),$$

we can define the K-theoretic free double suspension $\hat{\sigma}_k^2$. We define

$$\hat{\sigma}_f^2 \colon K(BU(\infty)) \to K(\mathrm{Map}(S^2, BU(\infty); f)) \quad \text{by} \quad \hat{e}^*(x) = 1 \otimes e^*(x) + \iota \otimes \hat{\sigma}_f^2(x),$$

for $x \in K(BU(\infty))$. By the same argument as in the first half of the proof of this theorem,

$$\hat{\sigma}_k^2(\xi_\infty) = k + \alpha^* \circ (\gamma_k^*)^{-1} \circ (\beta^*)^{-1}(\xi_\infty).$$

If we put $\zeta_\infty := \hat{\sigma}_k^2(\xi_\infty)$, then $c_i(\zeta_\infty) = \bar{x}_i$ for $i \geqslant 1$. Let $\zeta_n := j_n^*(\zeta_\infty)$ then $c_i(\zeta_n) = j_n^*(c_i(\zeta_\infty)) = x_i$ as desired. Therefore the proof is complete. $\qquad \square$

Acknowledgments

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