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GRADIENT VERSUS PROPER GRADIENT HOMOTOPIES

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Abstract

We compare the sets of homotopy classes of gradient and proper gradient vector fields in the plane. Namely, we show that gradient and proper gradient homotopy classifications are essentially different. We provide a complete description of the sets of homotopy classes of gradient maps from $\mathbb{R}^n$ to $\mathbb{R}^n$ and proper gradient maps from $\mathbb{R}^2$ to $\mathbb{R}^2$ with the Brouwer degree greater or equal to zero.

1. Introduction

The search for new homotopy invariants in the class of gradient maps has a long history. In 1985, to obtain new bifurcation results, Dancer [**4**] introduced a new topological invariant for S^1-equivariant gradient maps. In turn, Parusiński [**8**] showed that if two gradient vector fields on the unit disc D^n nonvanishing on S^{n-1} are homotopic, i.e., have the same Brouwer degree, then they are also gradient homotopic. Similarly, the authors of this paper proved in [**1, 2**] that there is no better invariant than the Brouwer degree for gradient and proper gradient otopies in $\mathbb{R}^n$. Recall that otopy was introduced by Becker and Gottlieb [**3**] as a very useful generalization of the concept of homotopy.

However, quite surprisingly, Starostka [**9**] showed that for $n \geqslant 2$ there exist proper gradient vector fields in $\mathbb{R}^n$ which are homotopic but not proper gradient homotopic. Roughly speaking, he proved that the identity and the minus identity on the plane are not proper gradient homotopic and then generalized this result to $\mathbb{R}^n$. Since his reasoning is nice and elegant, we will present it briefly here. First, recall that the linear source and sink in the plane are isolated invariant sets with different homological Conley indices. Namely, since $(D_r(0), \partial D_r(0))$ and $(D_r(0), \emptyset)$, where $D_r(0)$ denotes the r-disc at the origin, are index pairs for the source and sink respectively, the respective homological Conley indices are (see Figure 1)

$$\mathrm{CH}_*(D_r(0), \eta_{\mathrm{id}}) = H_*(S^2, \mathrm{pt}) \quad \text{and} \quad \mathrm{CH}_*(D_r(0), \eta_{-\mathrm{id}}) = H_*(S^0, \mathrm{pt}),$$

where η_f denotes the flow generated by the vector field f. Suppose now that there is a proper gradient homotopy connecting id to $-\mathrm{id}$. Such a homotopy determines a continuation between the gradient flows η_{id} and $\eta_{-\mathrm{id}}$ for which a sufficiently large

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disc $D_r(0)$ is a common isolating neighbourhood for all parameter values of the continuation (this is true for proper gradient vector fields and homotopies). Thus, by the continuation property of the Conley index, $\mathrm{CH}_*(D_r(0), \eta_{\mathrm{id}}) = \mathrm{CH}_*(D_r(0), \eta_{-\mathrm{id}})$, a contradiction. To summarize, if we restrict ourselves to proper gradient vector fields and homotopies, then the Conley index is a better invariant than the Brouwer degree.

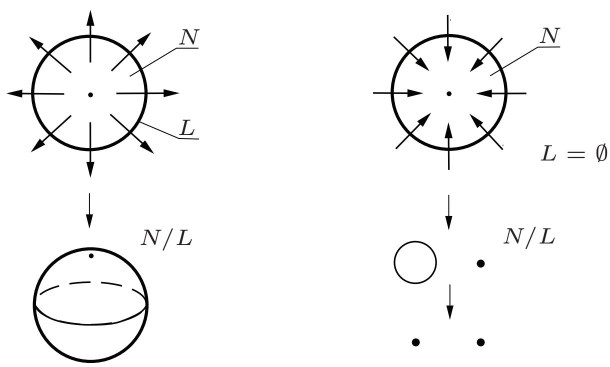

Figure 1: Conley indices of a source and sink

In this paper we strengthen and complement Starostka's result. We present the comparison of two homotopy classifications of gradient vector fields in the plane: gradient and proper gradient. Namely, we show that the set of homotopy classes of gradient vector fields in $\mathbb{R}^n$ having the same Brouwer degree is a singleton (a Parusiński-type theorem). On the other hand, the set of homotopy classes of proper gradient vector fields in $\mathbb{R}^2$ with the same Brouwer degree is empty if the degree is greater than 1, has exactly two elements if the degree is equal to 1 and has one element if the degree is equal to 0. What is still lacking is a description of this set for the degree less than 0. It also would be desirable to provide the proper gradient homotopy classification for the general case of $\mathbb{R}^n$.

The organization of the paper is as follows. Section 2 contains some preliminaries. Our main four theorems are stated in Section 3. These results are proved subsequently in Sections 4–7. Finally, Appendix A presents a series of technical results needed in previous sections.

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2. Preliminaries

In what follows, a map denotes always a continuous function and deg denotes the classical Brouwer degree.

2.1. Gradient and proper gradient maps

Recall that a map f is called *gradient* if there is a C^1 function $\varphi\colon \mathbb{R}^n \to \mathbb{R}$ such that $f = \nabla\varphi$ and is called *proper* if preimages of compact sets for f are compact. Let $I := [0,1]$.

We write $f \in \mathcal{V}^\nabla(\mathbb{R}^n)$ ($f \in \mathcal{P}^\nabla(\mathbb{R}^n)$) if

1. f is gradient,
2. $f^{-1}(0)$ is compact (f is proper).

Moreover, let $\mathcal{V}_k^\nabla(\mathbb{R}^n) := \{f \in \mathcal{V}^\nabla(\mathbb{R}^n) \mid \deg f = k\}$ and $\mathcal{P}_k^\nabla(\mathbb{R}^n) := \{f \in \mathcal{P}^\nabla(\mathbb{R}^n) \mid \deg f = k\}$.

2.2. Gradient and proper gradient homotopies

Apart from maps we consider two classes of homotopies: gradient and proper gradient. Namely, a map $h\colon I \times \mathbb{R}^n \to \mathbb{R}^n$ is called a *(proper) gradient homotopy* if

1. $h_t(\cdot) := h(t,\cdot)$ is gradient for each $t \in I$,
2. $h^{-1}(0)$ is compact (h is proper).

If h is a (proper) gradient homotopy, we say that h_0 and h_1 are *(proper) gradient homotopic*. The relation of being (proper) gradient homotopic is an equivalence relation in $\mathcal{V}_k^\nabla(\mathbb{R}^n)$ ($\mathcal{P}_k^\nabla(\mathbb{R}^n)$). The sets of homotopy classes of the respective relation will be denoted by $\mathcal{V}_k^\nabla[\mathbb{R}^n]$ and $\mathcal{P}_k^\nabla[\mathbb{R}^n]$.

2.3. Hessian maps

Let us consider a C^2 function $\varphi\colon \mathbb{R}^2 \to \mathbb{R}$. Assume that $p \in \mathbb{R}^2$ is a nondegenerate critical point of φ. Let $\mathrm{Hess}_p\varphi$ denote the Hessian of φ at p. In that situation, the Hessian is nondegenerate bilinear symmetric form and, in consequence, its matrix is invertible symmetric. Let us make two simple observations.

Lemma 2.1. *Any two elements of the space of invertible symmetric matrices are in the same path-connected component if and only if they have the same signature.*

Corollary 2.2. *Let $\varphi\colon \mathbb{R}^2 \to \mathbb{R}$ and p be a nondegenerate critical point of φ. Then the Hessian map* $\mathrm{Hess}_p\varphi\colon \mathbb{R}^2 \to \mathbb{R}^2$ *is proper gradient homotopic to* $\mathrm{id}_{\mathbb{R}^2}$ *if* $\operatorname{sign}\mathrm{Hess}_p\varphi = 2$ *or to* $-\mathrm{id}_{\mathbb{R}^2}$ *if* $\operatorname{sign}\mathrm{Hess}_p\varphi = -2$.

2.4. Local flows

A map $\eta\colon A \to \mathbb{R}^2$ is called a *local flow* on $\mathbb{R}^2$ if:

- $A \subset \mathbb{R} \times \mathbb{R}^2$ is open and contains $\{0\} \times \mathbb{R}^2$,
- for each $x \in \mathbb{R}^2$ there are $\alpha_x, \omega_x \in \mathbb{R} \cup \{\pm\infty\}$ such that $(\alpha_x, \omega_x) = \{t \in \mathbb{R} \mid (t,x) \in A\}$,
- $\eta(0,x) = x$ and $\eta(s, \eta(t,x)) = \eta(s+t, x)$ for all $x \in \mathbb{R}^2$ and $s, t \in (\alpha_x, \omega_x)$ such that $s + t \in (\alpha_x, \omega_x)$.

Remark 2.3. Assume that $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ is a C^1 vector field. It is well-known that if $t \to \eta(t, x_0)$ is a solution of the initial value problem

$$\dot{x} = f(x), \quad x(0) = x_0$$

and $(\alpha_{x_0}, \omega_{x_0})$ is the maximal interval of existence of the solution of the initial value problem, then the map η is a local flow on $\mathbb{R}^2$.

2.5. Generic maps

A map $f = \nabla\varphi \in \mathcal{P}^\nabla(\mathbb{R}^2)$ is called *generic* if φ is C^2 Morse function. This implies that φ has a finite number of nondegenerate critical points.

Proposition 2.4. *Any element of $\mathcal{P}^\nabla(\mathbb{R}^2)$ is proper gradient homotopic to a generic map.*

Proof. Let $\nabla\varphi \in \mathcal{P}^\nabla(\mathbb{R}^2)$. Since, by [**6**, Ch. 6 Th. 1.2], Morse functions are dense in C^2 functions in the strong topology and, by [**6**, Ch. 2 Th. 2.2], C^2 functions are dense in C^1 ones, we can choose ψ such that $\nabla\psi$ is generic and $|\nabla\varphi(x) - \nabla\psi(x)| < 1$ for all $x \in \mathbb{R}^2$. In consequence, the straight-line homotopy between $\nabla\varphi$ and $\nabla\psi$ is proper gradient. $\square$

Let $p \in \mathbb{R}^2$ be a critical point of a generic map f and η be the flow of f. The *stable* and *unstable manifolds* of p are defined to be

$$\begin{aligned} W^s(p) &:= \{x \in \mathbb{R}^2 \mid \omega_x = \infty \text{ and } \lim_{t\to\infty} \eta^t(x) = p\}, \\ W^u(p) &:= \{x \in \mathbb{R}^2 \mid \alpha_x = \infty \text{ and } \lim_{t\to-\infty} \eta^t(x) = p\}. \end{aligned}$$

2.6. Notation

Let us denote by $B_r(p)$ ($D_r(p)$) the open (closed) r-ball in $\mathbb{R}^n$ around p.

3. Main results

Let us formulate the main results of our paper.

Theorem 3.1. *$\mathcal{V}_k^\nabla[\mathbb{R}^n]$ is a singleton for each $k \in \mathbb{Z}$ and $n \geqslant 2$.*

Theorem 3.2. *$\mathcal{P}_k^\nabla[\mathbb{R}^2]$ is empty for $k > 1$.*

Theorem 3.3. *$\mathcal{P}_0^\nabla[\mathbb{R}^2]$ is a singleton.*

Theorem 3.4. *$\mathcal{P}_1^\nabla[\mathbb{R}^2]$ has at most two elements.*

Combining Theorem 3.4 with the theorem of Starostka (see [**9**, Main Theorem]) gives immediately the following result.

Corollary 3.5. *$\mathcal{P}_1^\nabla[\mathbb{R}^2]$ has exactly two elements: the class of the identity and the minus identity.*

We close this section with the following conjecture and open problem.

Conjecture. *$\mathcal{P}_k^\nabla[\mathbb{R}^2]$ is a singleton for $k < 0$.*

Open problem. Give the description of the set $\mathcal{P}_k^\nabla[\mathbb{R}^n]$ for any $k \in \mathbb{Z}$ and $n > 2$.

4. Proof of Theorem 3.1

A slight modification of the reasoning presented in the proof of Lemma 4 in [**8**] shows that the sets $\mathcal{V}_k^{\nabla}[\mathbb{R}^n]$ are nonempty. Now we prove that $\mathcal{V}_k^{\nabla}[\mathbb{R}^n]$ consist of only one element. Let $\nabla\varphi, \nabla\psi \in \mathcal{V}_k^{\nabla}(\mathbb{R}^n)$. There is $r > 0$ such that $(\nabla\varphi)^{-1}(0) \cup (\nabla\psi)^{-1}(0) \Subset B_r(0)$. By the Parusiński theorem ([**8**, Th. 1]), there is a C^1 function $\zeta\colon I \times D_r(0) \to \mathbb{R}$ such that

- $\nabla_x\zeta(t,x) \neq 0$ for all $t \in I$ and $x \in \partial D_r(0)$,
- $\nabla\zeta_0 = \nabla\big(\varphi\restriction_{D_r(0)}\big)$,
- $\nabla\zeta_1 = \nabla\big(\psi\restriction_{D_r(0)}\big)$.

Assume that θ is a diffeotopy from Lemma A.1. Let us define three homotopies $h^i\colon I \times \mathbb{R}^n \to \mathbb{R}^n$ $(i = 1,2,3)$ by the formulas

$$\begin{aligned} h_t^1(t,x) &= \nabla_x\varphi(\theta(t,x)),\\ h_t^2(t,x) &= \nabla_x\psi(\theta(t,x)),\\ h_t^3(t,x) &= \nabla_x(\zeta(t,\theta_1(x))). \end{aligned}$$

By Lemma A.2, h^1 and h^2 are gradient homotopies and by Lemma A.3, h^3 is a gradient homotopy. Thus we obtain the following sequence of the gradient homotopy relations

$$\nabla\varphi = h_0^1 \sim h_1^1 = h_0^3 \sim h_1^3 = h_1^2 \sim h_0^2 = \nabla\psi,$$

which completes the proof. □

5. Proof of Theorem 3.4

The following two propositions are crucial for the proof of Theorem 3.4. Assume that $f = \nabla\varphi$ is generic. Let $\sim$ denote the relation of proper gradient homotopy.

Proposition 5.1. *Let $f^{-1}(0) = \{p\}$. If p is a source then $f \sim \mathrm{id}_{\mathbb{R}^2}$. If p is a sink then $f \sim -\mathrm{id}_{\mathbb{R}^2}$.*

Proof. Assume that p is a source. By Corollary A.6, f is proper gradient homotopic to the Hessian map $\mathrm{Hess}_p\varphi\colon \mathbb{R}^2 \to \mathbb{R}^2$ and by Corollary 2.2, $\mathrm{Hess}_p\varphi$ is proper gradient homotopic to $\mathrm{id}_{\mathbb{R}^2}$. The same reasoning applies to the case of a sink. □

Let A_f^- (A_f^+) denote the set of sources (sinks) of f, $A_f = A_f^- \cup A_f^+$ and B_f the set of saddles.

Proposition 5.2. *If A_f and B_f are nonempty then there is a generic map f' such that $f \sim f'$, $|A_{f'}| < |A_f|$ and $|B_{f'}| < |B_f|$.*

The proof of Proposition 5.2 will be preceded by a series of lemmas. From now on, $\eta(t,z) = \eta^t(z)$ denotes the local flow generated by f. Let us start with the following notation. Assume that $x \in A_f^-$. Define

$$\begin{aligned} L_x^\epsilon &:= \{z \in W^u(x) \mid \varphi(z) = \varphi(x) + \epsilon\}, \\ E &:= \{\epsilon > 0 \mid \ L_x^\epsilon \text{ is homeomorphic to } S^1\}, \\ L &:= \cup_{\epsilon \in E} L_x^\epsilon. \end{aligned}$$

Note that E is an open interval starting at 0. Moreover, all L_x^ϵ for $\epsilon \in E$ can be naturally identified along trajectories, which intersect them transversally. Thus the set $\mathcal{S}_x$ consisting of all trajectories $\eta^t(z)$ emanating from the point x can be equipped with the topology of S^1. Finally, for $y \in A_f^+ \cup B_f \cup \{\infty\}$ denote by V_x^y the set $\{\eta^t(z) \in \mathcal{S}_x \mid z \in W^s(y)\}$.

The following lemma describes properties of sets V_x^y.

Lemma 5.3. *Assume that $y \in A_f^+ \cup \{\infty\}$. Then*

1. *V_x^y is an open subset of $\mathcal{S}_x \approx S^1$,*
2. *if $V_x^y = \mathcal{S}_x$ then $x \in A_f^-$ is the only stationary point of f and $y = \infty$.*

Proof. Ad (1). Assume that $y \in A_f^+$. There is a neighbourhood U of y such that if $\eta^t(z) \in U$ then $\eta^s(z) \in U$ for $s > t$. Let $\eta^t(z_0) \in V_x^y \subset \mathcal{S}_x$ with $z_0 \in L$. There is $T \in \mathbb{R}$ such that $\eta^T(z_0) \in U$. Therefore, we can choose a neighbourhood W of z_0 in L such that $\eta^T(z) \in U$ for all $z \in W$. Hence $\{\eta^t(z) \in \mathcal{S}_x \mid z \in W\} \subset V_x^y$ is open, and in consequence, so is V_x^y.

Now let $y = \infty$. Since f is proper gradient, there is $\rho > 0$ such that for $z \notin D_\rho(0)$ $\lim_{t\to-\infty} \eta^t(z) = x$ implies $\lim_{t\to\infty} \eta^t(z) = \infty$ (see [**9**, Prop. 2.4]). Let $\eta^t(z_0) \in V_x^y \subset \mathcal{S}_x$ with $z_0 \in L$. Then there is $T > 0$ such that $\eta^T(z_0) \notin D_\rho(0)$. Therefore, there exists a neighbourhood W of z_0 in L such that $\eta^T(z) \notin D_\rho(0)$ for all $z \in W$, which proves that V_x^∞ is open.

Ad (2). Without loss of generality we can assume that $\varphi(x) = 0$ and $1 \in E$.

We begin by proving that $y = \infty$. Conversely, suppose that y is a sink. Once again, let U be a small neighbourhood of y such that if $\eta^t(z) \in U$ then $\eta^s(z) \in U$ for $s > t$. By compactness of L_x^1, while increasing α is approaching $\varphi(y)$, we have $\alpha \in E$ and $L_x^\alpha \subset U$. From this observation we obtain $\operatorname{Ind}(L_x^1, x) = 1$ and $\operatorname{Ind}(L_x^1, x) = 0$, where Ind denotes the winding number. This leads to a contradiction and forces that $y = \infty$.

It remains to prove that $A_f \cup B_f = \{x\}$. There is no loss of generality in assuming that $x = 0$. Note that for any $z \in L$ the trajectory $\eta^t(z)$ tends to infinity. Since $\nabla\varphi$ is proper, we have $|\nabla\varphi(\eta^t(z))| \geqslant 1$ for t large enough. Hence $\lim_{t\to\omega_z} \varphi(\eta^t(z)) = \infty$. Therefore, $E = (0, \infty)$. We show that any $z \neq 0$ belongs to L_x^α for some $\alpha > 0$ and, in consequence, is not a critical point of φ. For $z \in L$ it is obvious. Let $z \notin L$ and $\alpha_0 = \max\{\varphi(w) \mid |w| \leqslant |z|\}$. Choose arbitrary $\alpha_1 > \alpha_0$. Observe that $\operatorname{Ind}(L_x^1, z) = 0$ and $\operatorname{Ind}(L_x^{\alpha_1}, z) = \operatorname{Ind}(L_x^{\alpha_1}, 0) = 1$. Suppose, contrary to our claim, that $z \notin L_x^\alpha$ for $1 < \alpha < \alpha_1$. Then $\operatorname{Ind}(L_x^{\alpha_1}, z) = \operatorname{Ind}(L_x^{\alpha_1}, z)$, a contradiction. This completes the proof. □

The next four lemmas are of utmost importance for the proof of Proposition 5.2.

Lemma 5.4. *Let $x \in A_f^-$ and B_f is nonempty. Then there is $y \in B_f$ such that x and y are connected by a trajectory of η.*

Proof. Write $V_x^Y := \cup_{y\in Y} V_x^y$ for $Y \subset A_f^+ \cup B_f \cup \{\infty\}$. Since f is generic, we have

$$V_x^\infty \cup V_x^{A_f^+} \cup V_x^{B_f} = \mathcal{S}_x \approx S^1.$$

By Lemma 5.3, $V_x^\infty \cup V_x^{A_f^+}$ is an open strict subset of $\mathcal{S}_x$. Hence $V_x^{B_f} \neq \emptyset$, which is our claim. □

Assume that $x \in A_f^-$. Write

$$A_f^+(x) := \{y \in A_f^+ \mid V_x^y \neq \emptyset\},$$
$$B_f(x) := \{y \in B_f \mid V_x^y \neq \emptyset\}.$$

Lemma 5.5. *If $A_f^+(x) \neq \emptyset$ and $B_f(x) \neq \emptyset$ then*

$$\min\{\varphi(y) \mid y \in A_f^+(x)\} \geqslant \min\{\varphi(y) \mid y \in B_f(x)\}.$$

Proof. Let $y_1 \in A_f^+(x)$ such that $\varphi(y_1) = \min\{\varphi(y) \mid y \in A_f^+(x)\}$. By the above and Lemma 5.3(2), $\emptyset \neq V_x^{y_1} \neq \mathcal{S}_x$. Let C denote a connected component of $V_x^{y_1}$. Moreover, let $a' = \eta^t(z')$ be one of the ends of the arc C and $a = \eta^t(z)$ be any point of C. By Lemma 5.3, there is $y_0 \in B_f(x)$ such that $a' \in V_x^{y_0}$.

Suppose that $\varphi(y_0) > \varphi(y_1)$. Therefore, there are disjoint neighbourhoods U_0 of y_0 and U_1 of y_1 such that for all $z \in U_0$ and $w \in U_1$ we have $\varphi(z) > \varphi(w)$ and, moreover, no trajectory leaves U_1. Observe that if a is close enough to a' then there is t_0 such that $\eta^{t_0}(z) \in U_0$. Moreover, since $a \in C$, there is t_1 such that $\eta^{t_1}(z) \in U_1$. Note that $t_1 > t_0$, because for $t \geqslant t_1$ we have $\eta^t(z) \in U_1$. Hence

$$\varphi\big(\eta^{t_0}(z)\big) < \varphi\big(\eta^{t_1}(z)\big),$$

a contradiction. This gives our assertion. □

The following result, which can be found in [**5**, Sec. 1], is devoted to the question of cancelling a pair of critical points.

Lemma 5.6. *Assume that $\varphi\colon \mathbb{R}^2 \to \mathbb{R}$ is a C^2 function such that $\nabla\varphi$ is generic and η is the local flow of $\nabla\varphi$. Let p and q be two critical points of φ satisfying the following conditions:*

- *$W^u(p)$ and $W^s(q)$ intersect transversely and the intersection consists of one orbit l of η,*
- *for some $\epsilon > 0$, each orbit of η in $W^u(p)$ distinct from l crosses the level set $\varphi^{-1}\big(\varphi(q)+\epsilon\big)$.*

Let U denote an open neighbourhood of the closure of $W^u(p) \cap \{\varphi \leqslant \varphi(q)+\epsilon\}$ such that the only critical points in $\operatorname{cl} U$ are p and q. Then there is a path of smooth functions $\{\varphi_t\}_{t\in I}$, such that:

- *$\varphi_0 = \varphi$,*
- *for every $t \in I$, φ_t coincides with φ on $\mathbb{R}^2 \setminus U$,*
- *the function $\varphi_t|_U$ has two nondegenerate critical points when $0 \leqslant t < 1/2$, one degenerate critical point when $t = 1/2$ and no critical points when $1/2 < t \leqslant 1$.*

Lemma 5.7. *Assume that $f = \nabla\varphi$ is generic. Let $x \in A_f^-$ and $y \in B_f$. If there are two trajectories of η connecting x to y then there is generic f' otopic to f such that $|A_{f'}| < |A_f|$ and $|B_{f'}| < |B_f|$.*

Proof. Let us denote by Γ_1 and Γ_2 two trajectories of η (smooth curves) connecting x to y. Notice that these trajectories form a straight angle at y (see Figure 2). Let G stand for the domain bounded by Γ_1 and Γ_2. Since x is a source, we can choose points $x_1 \in \Gamma_1$, $x_2 \in \Gamma_2$ and a level subset $\Pi_x \subset G$ of φ connecting them close enough to x. Similarly, since y is a saddle, we can choose $y_1 \in \Gamma_1$, $y_2 \in \Gamma_2$ such that $\varphi(y_1) = \varphi(y_2)$ and a smooth curve $\Pi_y \subset G$ perpendicular to Γ_i at y_i for $i = 1, 2$. Let us denote by G_1 the domain bounded by Π_x, Π_y and trajectories connecting x_i to y_i.

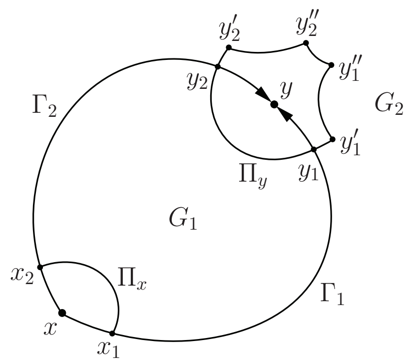

Figure 2: Domains G_1 and G_2

Let us extend Π_y to a little longer smooth curve $\Pi_y' = \operatorname{arc} y_1' y_2'$ such that segments $\operatorname{arc} y_i y_i'$ are contained in level sets of φ. In the neighbourhood of the saddle y choose points y_i'' for $i = 1, 2$ on the trajectory starting from y_i' such that $\varphi(y_1'') = \varphi(y_2'')$. Points y_1'' and y_2'' are connected by a short segment of a level set of φ. Let us denote by G_2 the domain bounded by Π_y', trajectories connecting y_i' to y_i'' and this short segment of a level set connecting y_1'' and y_2''.

Since f is defined and nonvanishing on $\partial(G_1 \cup G_2)$, we can extend $f\restriction_{\partial(G_1\cup G_2)}$ to nonvanishing smooth vector field $\widetilde{f}\colon \partial G_1 \cup \partial G_2 \to \mathbb{R}^2$ such that $\widetilde{f}$ is perpendicular to the curve Π_y. It is easy to check that $\widetilde{f}$ satisfies the assumptions of Lemma A.13 for both G_1 and G_2. As a conclusion we obtain that $\widetilde{f}$ can be extended to nonvanishing gradient vector field $\widehat{f}\colon G_1 \cup G_2 \to \mathbb{R}^2$. Finally, define $f'\colon \mathbb{R}^2 \to \mathbb{R}^2$ by the formula

$$f'(z) = \begin{cases} f(z) & \text{if } z \notin G_1 \cup G_2, \\ \widehat{f}(z) & \text{if } z \in G_1 \cup G_2. \end{cases}$$

The map f' satisfies the assertion of Lemma 5.7. □

Proof of Proposition 5.2. Without loss of generality we can assume that $A_f^- \neq \emptyset$ (in the case $A_f^+ \neq \emptyset$ the proof is analogous). Let $x \in A_f^-$. By Lemma 5.4, $B_f(x) \neq \emptyset$. Moreover, by Lemma 5.5, there is $y \in B_f(x)$ which realizes the minimum of $F = \{\varphi(z) \mid z \in A_f^+(x) \cup B_f(x)\}$. Applying Lemma A.7 we can assume that y is the only minimum in F. There are two possibilities, i.e., there is either one or two trajectories connecting x to y. In the second case it is enough to apply Lemma 5.7 to obtain at once the desired conclusion. Now let us consider the first case. Observe that all assumptions of Lemma 5.6 are satisfied. In consequence, a proper gradient homotopy allows us to cancel both critical points x and y, which is our assertion. $\square$

It occurs that Theorem 3.4 is now a consequence of Propositions 2.4, 5.1 and 5.2.

Proof of Theorem 3.4. Let $f \in \mathcal{P}_1^{\nabla}(\mathbb{R}^2)$. By Proposition 2.4, without loss of generality we can assume that f is generic. Moreover, $\deg f = |A_f| - |B_f| = 1$. By Proposition 5.2, there is f' generic such that $f \sim f'$, $f'^{-1}(0) = \{p\}$ and p is a source or sink. Hence, by Proposition 5.1, $f \sim \mathrm{id}_{\mathbb{R}^2}$ or $f \sim -\mathrm{id}_{\mathbb{R}^2}$. $\square$

6. Proof of Theorem 3.3

The main result of this section is the following lemma.

Lemma 6.1. *If $f, f' \in \mathcal{P}^{\nabla}(\mathbb{R}^2)$ have no zeroes then $f \sim f'$.*

Proof. Let $f = \nabla\varphi$. For $t \in [1/2, 1]$ write $f_t(x) := f((2-2t)x)$. Set $c := \min\{|f(x)| \mid x \in \mathbb{R}^2\}$. Observe that $c > 0$ and for each $t \in [1/2, 1]$

- f_t are gradient maps,
- $\min\{|f_t(x)| \mid x \in \mathbb{R}^2\} \geqslant c$.

Next for $t \in [0, 1/2]$ put $\xi_t(x) := (1 + t\,|x|)x$ and

$$\Xi_t(f)(x) := \nabla\big(\varphi(\xi_t(x))\big) = D\xi_t^T(x)\nabla\varphi(\xi_t(x)) = D\xi_t^T(x) f(\xi_t(x)).$$

Let us check that following inequalities

1. $|\xi_t(x)| \geqslant |x|$,
2. $\big|D\xi_t^T(x)v\big| \geqslant (1 + t\,|x|)\,|v|$,
3. $|\Xi_t(f)(x)| \geqslant |f(\xi_t(x))|$,
4. $\big|\Xi_{\frac{1}{2}}(f_t)(x)\big| \geqslant \big(1 + \frac{1}{2}\,|x|\,\big)c$ for $t \in [1/2, 1]$.

The first one is obvious. The second follows from the fact that for a given x the matrix $D\xi_t(x)$ is diagonal in some basis with the elements $(1 + 2t\,|x|)$ and $(1 + t\,|x|)$ on the diagonal. The third and fourth follow immediately from the second.

Finally, define a homotopy

$$h_t(x) = \begin{cases} \Xi_t(f)(x) & \text{if } t \in [0, 1/2], \\ \Xi_{\frac{1}{2}}(f_t)(x) & \text{if } t \in [1/2, 1]. \end{cases}$$

The homotopy h_t is obviously gradient. Moreover, it is proper. Namely, the first part of the homotopy is proper from (1) and (3) and the properness of f, and the second part is proper from (4).

Observe that the homotopy h_t connects f to $\Xi_{\frac{1}{2}}(f(0))$, where $f(0)$ denotes a constant vector field on $\mathbb{R}^2$. What is left is to show that

$$\Xi_{\frac{1}{2}}(f(0)) \sim \Xi_{\frac{1}{2}}(f'(0)).$$

Note that there is a homotopy g_t between $f(0)$ and $f'(0)$ consisting of nonzero constant vector fields. It is immediate that the homotopy $\Xi_{\frac{1}{2}}(g_t)$ is proper gradient, which completes the proof. $\square$

Remark 6.2. The last lemma is true for $\mathcal{P}^{\nabla}(\mathbb{R}^n)$ $(n \geqslant 2)$ with the same proof.

Proof of Theorem 3.3. Let $f, f' \in \mathcal{P}_0^{\nabla}(\mathbb{R}^2)$. By Proposition 2.4, we can assume that f and f' are generic, and hence, by Proposition 5.2, that they have no zeroes. Lemma 6.1 now shows that $f \sim f'$. □

7. Proof of Theorem 3.2

As previously, Proposition 2.4 guarantees that $f \in \mathcal{P}_k^{\nabla}(\mathbb{R}^2)$ $(k > 1)$ can be chosen generic. By Proposition 5.2, we can assume that f has no saddles. Observe that to complete the proof it is enough to show that if $A_f^- \neq \emptyset$ then $\left|A_f^-\right| = |A_f| = 1$, which contradicts our assumption $k > 1$ (the case $A_f^+ \neq \emptyset$ is analogous).

Let $x \in A_f^-$. Since $\cup_{y \in A_f^+ \cup \{\infty\}} V_x^y = \mathcal{S}_x$ and, by Lemma 5.3(1), the sets V_x^y are open, we have $V_x^y = \mathcal{S}_x$ for some $y \in A_f^+ \cup \{\infty\}$. Hence, by Lemma 5.3(2), $y = \infty$ and x is the only stationary point of f, i.e., $\left|A_f^-\right| = |A_f| = 1$. □

Appendix A.

A.1. Diffeotopies

Lemma A.1. *There is a diffeotopy on the image $\theta\colon I \times \mathbb{R}^n \to \mathbb{R}^n$ such that $\theta_0 = \mathrm{id}_{\mathbb{R}^n}$, $\theta_1(\mathbb{R}^n) = B_r(0)$ and $B_r(0) \subset \theta_t(\mathbb{R}^n)$ for all $t \in I$.*

Proof. Let us consider a function $\mu\colon [0,\infty) \to [0,r)$ of the form $\mu(s) = \frac{2r}{\pi} \arctan s$. Define a straightline homotopy $\mu_t(s) = (1-t)s + t\mu(s)$. The function $\theta\colon I \times \mathbb{R}^n \to \mathbb{R}^n$ given by $\theta(t,x) = \mu_t(|x|)\frac{x}{|x|}$ is a diffeotopy with the desired properties. □

Lemma A.2. *Let $\varphi\colon \mathbb{R}^n \to \mathbb{R}$ be a C^1 function, $\nabla\varphi \in \mathcal{V}^{\nabla}(\mathbb{R}^n)$ and $\theta\colon I \times \mathbb{R}^n \to \mathbb{R}^n$ be a diffeotopy on the image such that $(\nabla\varphi)^{-1}(0) \subset \theta_t(\mathbb{R}^n)$ for all $t \in I$. Then $h\colon I \times \mathbb{R}^n \to \mathbb{R}^n$ given by*

$$h(t,x) = \nabla_x(\varphi(\theta(t,x)))$$

is a gradient homotopy.

Proof. It is enough to check that $h^{-1}(0)$ is compact. Let us define $\widetilde{\theta}\colon I \times \mathbb{R}^n \to I \times \mathbb{R}^n$ by $\widetilde{\theta}(t,x) = (t, \theta(t,x))$. Observe that $\widetilde{\theta}$ is a homeomorphism on the image and

$$h^{-1}(0) = \widetilde{\theta}^{-1}(I \times (\nabla\varphi)^{-1}(0)),$$

which proves the lemma. □

Lemma A.3. *Assume that $\gamma\colon \mathbb{R}^n \to B_r(0)$ is a diffeomorphism and $\zeta\colon I \times D_r(0) \to \mathbb{R}$ is continuous and C^1 with respect to x such that $\nabla_x\zeta(t,x) \neq 0$ for all $t \in I$ and $x \in \partial D_r(0)$. Then the function $h\colon I \times \mathbb{R}^n \to \mathbb{R}^n$ given by*

$$h(t,x) = \nabla_x(\zeta(t,\gamma(x)))$$

is a gradient homotopy.

Proof. The map $\widetilde{\gamma}\colon I \times \mathbb{R}^n \to I \times B_r(0)$ given by $\widetilde{\gamma}(t,x) = (t, \gamma(x))$ is a homeomorphism, the set $(\nabla_x \zeta)^{-1}(0)$ is compact and

$$h^{-1}(0) = \widetilde{\gamma}^{-1}\big((\nabla_x \zeta)^{-1}(0)\big).$$

Hence the last set is compact. □

A.2. Milnor's trick

Let $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ be a map differentiable at 0 and $f(0) = 0$. We define a function $h\colon I \times \mathbb{R}^2 \to \mathbb{R}^2$ by the formula

$$h(t,x) = \begin{cases} \dfrac{f(tx)}{t} & \text{if } t \neq 0, \\ Df(0)x & \text{if } t = 0. \end{cases}$$

Lemma A.4.

1. *The function h is continuous.*
2. *If $Df(0)$ is nonsingular, f is proper and $f^{-1}(0) = \{0\}$ then h is proper.*

Proof. Ad (1). We need to prove that for any $(t_0, x_0) \in I \times \mathbb{R}^2$ and $\epsilon > 0$ there is a neighbourhood U of (t_0, x_0) such that $|h(t,x) - h(t_0,x_0)| < \epsilon$ for any $(t,x) \in U$. Set $\epsilon > 0$. If $t_0 \neq 0$ the claim is obvious. Let $t_0 = 0$ and $x_0 \in \mathbb{R}^2$. We show that there are $\rho > 0$ and $\delta > 0$ such that $|h(t,x) - h(0,x_0)| < \epsilon$ for $t < \rho$ and $|x - x_0| < \delta$. Since for $t = 0$ we have $|h(0,x) - h(0,x_0)| = |Df(0)(x - x_0)|$, we can assume that $t \neq 0$. By the differentiability of f at $x = 0$, we have $\lim_{|x| \to 0} \frac{f(x) - Df(0)x}{|x|} = 0$. Observe that for sufficiently small both t and $|x - x_0|$ we have

$$\begin{aligned} |h(t,x) - h(0,x_0)| &= \left| \frac{f(tx)}{t} - Df(0)x \right| \\ &\leqslant \frac{|f(tx) - Df(0)tx|}{|tx|}\,|x| + |Df(0)x - Df(0)x_0| \leqslant \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon, \end{aligned}$$

which is our claim.

Ad (2). It is enough to show that for every $m > 0$ there is $l > 0$ such that $|h(t,x)| > m$ for all $t \in I$ and $|x| > l$. Observe that

(a) there is $\epsilon > 0$ such that $|Df(0)x| \geqslant \epsilon\,|x|$ for any $x \in \mathbb{R}^2$,

(b) there is $\delta_1 > 0$ such that $|f(x) - Df(0)x| \leqslant \frac{\epsilon}{2}\,|x|$ for any $|x| < \delta_1$,

(c) from (a) and (b) for any $|x| < \delta_1$ we have $|f(x) - Df(0)x| \leqslant \frac{1}{2}\,|Df(0)x|$ and, in consequence, $|f(x)| \geqslant \frac{1}{2}\,|Df(0)x|$,

(d) there is $m_1 \in (0,m)$ such that $|f(x)| > m_1$ for any $|x| \geqslant \delta_1$,

(e) there is $\delta_2 > 0$ such that $|f(x)| > m$ for any $|x| \geqslant \delta_2$.

Set $t_1 := m_1/m$ and $l =: \max\{\delta_2/t_1, 2m/\epsilon\}$. Let $|x| > l$. Now we only need to consider the following three cases.

Case $t_1 \leqslant t \leqslant 1$. Since $|tx| > \delta_2$, we get $|h(t,x)| \geqslant |f(tx)| > m$, by (e).

Case $0 < t < t_1$. If $|tx| \geqslant \delta_1$ then $|h(t,x)| > \frac{m_1}{t_1} = m$ from (d) and the definition of t_1. If $|tx| < \delta_1$ then

$$|h(t,x)| = \left|\frac{f(tx)}{t}\right| \geqslant \frac{1}{2}\left|\frac{Df(0)tx}{t}\right| = \frac{1}{2}\,|Df(0)x| > m$$

from (a), (c) and the definition of l.

Case $t = 0$. We have $|h(0,x)| = |Df(0)x| > 2m > m$ from (a) and the definition of l. □

Remark A.5. Lemma A.4 is true for maps $f\colon \mathbb{R}^n \to \mathbb{R}^k$, but the assumption that $Df(0)$ is nonsingular must be replaced with $\operatorname{rank} Df(0) = n$.

Corollary A.6. *Let $f = \nabla\varphi$ be generic and $f^{-1}(0) = \{p\}$. Then f is proper gradient homotopic to the Hessian map $\mathrm{Hess}_p\varphi\colon \mathbb{R}^2 \to \mathbb{R}^2$.*

A.3. Raising and lowering critical points

Lemma A.7. *Let $\varphi\colon U \subset \mathbb{R}^2 \to \mathbb{R}$ be a C^2 function such that $\nabla\varphi$ is generic and 0 is a saddle point of φ. Then there is a neighbourhood $U' \subset U$ of 0 such that $\operatorname{cl} U' \subset U$ and a Morse function $\psi\colon U \subset \mathbb{R}^2 \to \mathbb{R}$ such that*

- *0 is a saddle and the only critical point of ψ in U',*
- *$\psi\restriction_{U \setminus \operatorname{cl} U'} = \varphi\restriction_{U \setminus \operatorname{cl} U'}$,*
- *$\psi(0) < \varphi(0)$.*

Proof. Choose a sufficiently small disc around 0 and define a bump function μ centered at 0 with compact support contained in that disc. The function $\psi = \varphi - \mu$ has required properties (compare [**7**, Th. 2.34]). □

A.4. Gradient fields on curvilinear quadrangles

In the next lemmas $A = x_1x_2x_3x_4$ denotes a smooth curvilinear quadrangle in the plane with right angles at the corners, $D = \{x_1, x_2, x_3, x_4\}$ (the set of the corners of A) and $B = X_1X_2X_3X_4 = I \times I$ (the unit square). The following results concern maps defined on such quadrangles.

For the needs of the next lemma, assume that $\partial A = \cup_{i=1}^4 L_i$, where L_i are sides of A. The proof of this lemma requires the following extension procedure. For $i = 1, \ldots, 4$ let L_i' denote a C^1 curve being a small extension of L_i such that $L_i' \cap A = L_i$ (see Figure 3). Assume that $f\colon \partial A \to \mathbb{R}$ is continuous and C^1 on each L_i. Denote by $\widehat{f}\colon \cup_{i=1}^4 L_i' \to \mathbb{R}$ any extension of f which is C^1 on each L_i'.

Lemma A.8. *The function $F\colon A \to \mathbb{R}$*

$$F(P) = \begin{cases} f(P) & \text{for } P \in \partial A, \\ \sum_{i=1}^4 \int_{L_i'} \frac{\widehat{f}(Q_i)\,ds}{|Q_i - P|^2} \Big/ \sum_{i=1}^4 \int_{L_i'} \frac{ds}{|Q_i - P|^2} & \text{for } P \in \operatorname{int} A, \end{cases} \tag{A.1}$$

where Q_i are any parametrizations of L_i' and ds is the element of length,

1. *is continuous on A,*
2. *is C^1 on $A \setminus D$,*

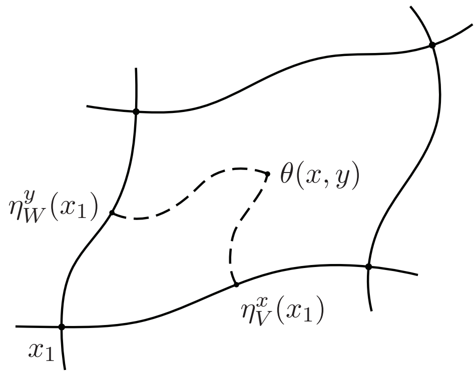

Figure 3: Quadrangle A

3. *satisfies the condition: there is $M \in \mathbb{R}$ such that for any $P \in A \setminus D$ and $w \in S^1$ we have $\left|\frac{\partial F}{\partial w}(P)\right| \leqslant M$.*

Proof. Assertion (1) follows immediately. C^1 smoothness of F on $\operatorname{int} A$ may be concluded directly from the integral form of A.1. To show (2) on $\partial A \setminus D$, let us replace

- a point $P \in L_i \setminus D$ by the origin $(0,0)$,
- a small neighbourhood of P by the rectangle $U = [-1,1] \times [0,1]$,
- the part of L_i contained in U by $[-1,1] \times \{0\}$,
- the function f by $c \cdot x$ with $c = \frac{df}{dx}(0)$.

Since the expressions cx/y and $1/y$ provide good approximations of the integrals $\int_{-1}^{1} \frac{ctdt}{(t-x)^2+y^2}$ and $\int_{-1}^{1} \frac{dt}{(t-x)^2+y^2}$ for $(x,y) \in U$, we have

$$F(x,y) \approx \frac{\frac{cx}{y} + g(x,y)}{\frac{1}{y} + k(x,y)},$$

where $g(x,y)$ and $k(x,y)$ correspond to the parts of the formula (A.1) calculated outside U. It is easy to check that $\lim_{(x,y)\to P} DF(x,y)$ exists. Hence $DF(P)$ exists and F is C^1 on $A \setminus D$.

To prove (3) we proceed similarly. A small neighbourhood of $P \in D$ is replaced by the square $U = [0,1] \times [0,1]$ with $P = (0,0)$ and curves L_i' intersecting at P by the intervals $[-1,1] \times \{0\}$ and $\{0\} \times [-1,1]$. If g and k are the respective integral sums outside U (similarly as above) then one can check that for the function

$$F(x,y) \approx \frac{\frac{cx}{y} + \frac{ey}{x} + g(x,y)}{\frac{1}{x} + \frac{1}{y} + k(x,y)}$$

the derivatives $\frac{\partial F}{\partial x}$ and $\frac{\partial F}{\partial x}$ are bounded for $(x,y) \in \operatorname{int} U$, which proves (3). $\square$

Remark A.9. It is easy to check that the function F from Lemma A.8 is lipschitzian on the whole A.

Lemma A.10. *There is a diffeomorphism $\theta\colon B \to A$ such that $\theta(X_i) = x_i$ for $i = 1,2,3,4$.*

Proof. We start with the description of the construction of θ. Consider a continuous vector field $v\colon \partial A \to \mathbb{R}^2$ such that

- $|v| = 1$,
- v is tangent to $\operatorname{arc} x_1x_2$ and $\operatorname{arc} x_3x_4$,
- v is perpendicular to $\operatorname{arc} x_2x_3$ and $\operatorname{arc} x_1x_4$,
- $v(x_1)$ agrees with the orientation of $\operatorname{arc} x_1x_2$.

Observe that $\mathrm{i}_v(\partial A) = 0$, where $\mathrm{i}_v(C)$ denotes the index of a closed curve C relative to the vector field v. Write $\arg(x) := \arg(v(x))$. Note that $\arg\colon \partial A \to \mathbb{R}$ is determined uniquely up to multiple of 2π and satisfies the assumptions of Lemma A.8. Let $\operatorname{Arg}\colon A \to \mathbb{R}$ denote the extension of the function arg given by Lemma A.8 and $V\colon A \to \mathbb{R}^2$ be given by $V(P) := (\cos \operatorname{Arg}(P), \sin \operatorname{Arg}(P))$. Obviously, V is a lipschitzian extension of v. Moreover, let $W\colon A \to \mathbb{R}^2$ be the vector field V rotated by $\pi/2$. Finally, let η_V and η_W be the flows of the vector fields V and W, respectively. The flows η_V and η_W are well defined, because V and W are lipschitzian on A.

Without loss of generality we can assume that $x_2 = \eta_V^1(x_1)$ and $x_4 = \eta_W^1(x_1)$. Observe that for any $(x,y) \in B$ there are unique $s = s(x,y)$ and $t = t(x,y)$ such that

$$\eta_W^s\big(\eta_V^x(x_1)\big) = \eta_V^t\big(\eta_W^y(x_1)\big).$$

This allows us to define $\theta\colon B \to A$ by the formula

$$\theta(x,y) = \eta_W^{s(x,y)}(a_x) = \eta_V^{t(x,y)}(b_y),$$

where $a_x = \eta_V^x(x_1)$ and $b_y = \eta_W^y(x_1)$ (see Figure 3). A standard reasoning shows that θ is a continuous bijection. Now we prove that θ is also a diffeomorphism.

Recall that the symbol $X(f)(P)$ stands for the directional derivative of a function f in the direction of a vector field X at a point P. The following formulas hold

$$\begin{aligned}
\frac{\partial\theta}{\partial x}(x,y) &= \exp\Bigg(-\int_0^s V(\operatorname{Arg})\big(\eta_W^\tau(a_x)\big)d\tau\Bigg)\cdot V(\theta(x,y)),\\
\frac{\partial\theta}{\partial y}(x,y) &= \exp\Bigg(\int_0^t W(\operatorname{Arg})\big(\eta_V^\tau(b_y)\big)d\tau\Bigg)\cdot W(\theta(x,y)),
\end{aligned}\tag{A.2}$$

where $s = s(x,y)$ and $t = t(x,y)$. A direct computation using the formulas (A.2) shows the continuity of $\frac{\partial\theta}{\partial x}$ and $\frac{\partial\theta}{\partial y}$ on $\operatorname{int} B$, while the continuity on ∂B follows from (A.2) and the boundedness of $V(\operatorname{Arg})$ and $W(\operatorname{Arg})$ on $A \setminus D$, which is guaranteed by Lemma A.8. $\square$

Remark A.11. The assertion of Lemma A.10 can be strengthened in the following way. Let $\gamma'\colon I \to \operatorname{arc} x_1x_4$ and $\gamma''\colon I \to \operatorname{arc} x_2x_3$ be any smooth parametrizations. We can guarantee that the diffeomorphism θ satisfies for all $s \in I$ the following conditions:

- $\theta(0,s) = \gamma'(s)$ and $\theta(1,s) = \gamma''(s)$,
- the curves $\theta(s \times I)$ and $\theta(I \times s)$ are perpendicular to the respective sides of A.

Lemma A.12. *Assume that $w\colon \partial B \to \mathbb{R}^2$ is a continuous nonvanishing vector field such that*

- *$w(0, y) = w(1, y) = (0, 1)$ for all $y \in I$,*
- *$w(x, 0) = (0, w'(x))$ and $w(x, 1) = (0, w''(x))$, where $w', w'' \colon I \to \mathbb{R}$ are C^1 functions.*

Then there is a C^1 function $\psi \colon B \to \mathbb{R}$ such that $\nabla\psi$ is nonvanishing on B and $\nabla\psi\restriction_{\partial B} = w$.

Proof. Let $m := \max\{w'(x), w''(x) \mid x \in I\}$, $a(x) := x(1 - x)$ and $\mu \colon I \to \mathbb{R}$ be a continuous function such that $\mu(0) = 1$, $\mu(1) = 0$, $-1/m \leqslant \mu \leqslant 1$ and $\int_0^1 \mu(t)dt = 0$. Set $C := \{(x, y) \in B \mid y \leqslant a(x)\}$. Let us define a function $u \colon C \to \mathbb{R}$ by

$$u(x, y) = \begin{cases} 0 & \text{if } x \in \{0, 1\}, \\ \int_0^y \mu(t/a(x))dt & \text{otherwise,} \end{cases}$$

and a function $\psi \colon B \to \mathbb{R}$ by

$$\psi(x, y) = y + \begin{cases} (w'(x) - 1)u(x, y) & \text{if } y \leqslant a(x), \\ 0 & \text{if } a(x) < y < 1 - a(x), \\ (1 - w''(x))u(x, 1 - y) & \text{if } 1 - a(x) \leqslant y. \end{cases}$$

One can check that ψ satisfies the assertion of our lemma. In particular, $\nabla\psi$ is nonvanishing, because $\frac{\partial\psi}{\partial y} > 0$ for $(x, y) \in B$. □

Lemma A.13. *Let $v \colon \partial A \to \mathbb{R}^2$ be a continuous nonvanishing vector field such that*

- *$v\restriction_{\operatorname{arc} x_1x_2}$ and $v\restriction_{\operatorname{arc} x_3x_4}$ are perpendicular to ∂A and smooth,*
- *$v\restriction_{\operatorname{arc} x_1x_4}$ and $v\restriction_{\operatorname{arc} x_2x_3}$ are tangent to ∂A,*
- *$\int_{\operatorname{arc} x_1x_4} |v|\, ds = \int_{\operatorname{arc} x_2x_3} |v|\, ds = 1$.*

Then there is a gradient nonvanishing extension of v to A.

Proof. First, we uniquely fix parametrizations $\gamma' \colon I \to \operatorname{arc} x_1x_4$ and $\gamma'' \colon I \to \operatorname{arc} x_2x_3$ so that for each $c \in I$ we have

$$\int\limits_{\operatorname{arc} x_1\gamma'(c)} |v|\, ds = \int\limits_{\operatorname{arc} x_2\gamma''(c)} |v|\, ds = c.$$

Let $\theta \colon B \to A$ be a diffeomorphism satisfying the conditions mentioned in Lemma A.10 and Remark A.11. Without loss of generality we may assume that $v(x_1)$ agrees with the orientation of $\operatorname{arc} x_1x_4$.

Define $w \colon \partial B \to \mathbb{R}^2$ by $w(z) = (D\theta(z))^T \cdot v(\theta(z))$. Note that w satisfies the assumptions of Lemma A.12. Hence there is a C^1 function $\psi \colon B \to \mathbb{R}$ such that $\nabla\psi$ is nonvanishing on B and $\nabla\psi\restriction_{\partial B} = w$.

Finally, define $\varphi \colon A \to \mathbb{R}$ by $\varphi(z) = \psi(\theta^{-1}(z))$. Obviously $\nabla\varphi$ is nonvanishing on A. Moreover, for $z \in \partial A$

$$\nabla\varphi(z) = (D\theta^{-1}(z))^T \cdot \nabla\psi(\theta^{-1}(z)) = (D\theta^{-1}(z))^T \cdot w(\theta^{-1}(z)) = v(z),$$

which completes the proof. □

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