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THE TRACE OF THE LOCAL $\mathbb{A}^1$-DEGREE

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Abstract

We prove that the local $\mathbb{A}^1$-degree of a polynomial function at an isolated zero with finite separable residue field is given by the trace of the local $\mathbb{A}^1$-degree over the residue field. This fact was originally suggested by Morel's work on motivic transfers, and by Kass and Wickelgren's work on the Scheja–Storch bilinear form. As a corollary, we generalize a result of Kass and Wickelgren relating the Scheja–Storch form and the local $\mathbb{A}^1$-degree.

1. Introduction

The $\mathbb{A}^1$-degree, first defined by Morel [**Mor04, Mor12**], provides a foundational tool for solving problems in $\mathbb{A}^1$-*enumerative geometry*.[1] In contrast to classical notions of degree, the local $\mathbb{A}^1$-degree is not integer valued: given a polynomial function $f\colon \mathbb{A}^n_k \to \mathbb{A}^n_k$ with isolated zero p, the local $\mathbb{A}^1$-degree of f at p, denoted by $\deg_p^{\mathbb{A}^1}(f)$, is defined to be an element of the Grothendieck–Witt group of the ground field.

Definition 1.1. Let k be a field. The *Grothendieck–Witt group* $\mathrm{GW}(k)$ is defined to be the group completion of the monoid of isomorphism classes of symmetric non-degenerate bilinear forms over k. The group operation is the direct sum of bilinear forms. We may also give $\mathrm{GW}(k)$ a ring structure by taking tensor products of bilinear forms for our multiplication.

The local $\mathbb{A}^1$-degree, which will be defined in Definition 2.9, can be related to other important invariants at rational points. The Scheja–Storch form (Definition 2.15) is another $\mathrm{GW}(k)$-valued invariant defined via a duality on the local complete intersection cut out by the components of a given polynomial map (see Subsection 2.3 for details). Kass and Wickelgren show that the isomorphism class of the Scheja–Storch bilinear form [**SS75**] is equal to the local $\mathbb{A}^1$-degree at rational points [**KW19**]. Kass and Wickelgren also show that at points with finite separable residue field, the Scheja–Storch form is given by taking the trace of the Scheja–Storch form over the residue field [**KW17**, Proposition 32].

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[1] $\mathbb{A}^1$-enumerative geometry is the application of $\mathbb{A}^1$-homotopy theory to the study of enumerative geometry over arbitrary fields. For details, see the expository paper [**WW19**], as well as the exposition found in [**KW17, Lev17, BKW18, SW18, KW19, LV19**].

In practice, one may need to consider the local $\mathbb{A}^1$-degree at non-rational points. This is the case of interest to us. At points whose residue field is a finite extension of the ground field, Morel's work on cohomological transfer maps [**Mor12**] suggests the following formula: the local $\mathbb{A}^1$-degree at a non-rational point should be computed by first taking the local $\mathbb{A}^1$-degree over the residue field, and then by post-composing with a field trace. This suggestion is supported by the aforementioned results of Kass–Wickelgren on the Scheja–Storch form [**KW17**, Proposition 32]. Our main result is to confirm this formula. We state our result precisely in Theorem 1.3, after introducing necessary terminology.

Definition 1.2. Given a separable field extension L/k of finite degree, the *trace*

$$\mathrm{Tr}_{L/k}\colon \mathrm{GW}(L) \to \mathrm{GW}(k)$$

is given by post-composing the field trace (which we also denote $\mathrm{Tr}_{L/k}$). That is, if $\beta\colon V \times V \to L$ is a representative of an isomorphism class of symmetric non-degenerate bilinear forms over L, then its trace is the isomorphism class of the following symmetric bilinear form over k

$$\mathrm{Tr}_{L/k}\,\beta\colon V \times V \to L \xrightarrow{\mathrm{Tr}_{L/k}} k.$$

When p is not a k-rational point, we can lift f to a function $f_{k(p)}\colon \mathbb{A}^n_{k(p)} \to \mathbb{A}^n_{k(p)}$ after fixing a choice of field embedding $k \hookrightarrow k(p)$. Moreover, we may lift p to an isolated $k(p)$-rational zero $\widetilde{p}$ of $f_{k(p)}$, and we thus obtain the local degree $\deg^{\mathbb{A}^1}_{\widetilde{p}}(f_{k(p)}) \in \mathrm{GW}(k(p))$.

We can now state our main result.

Theorem 1.3. *Let k be a field, $f\colon \mathbb{A}^n_k \to \mathbb{A}^n_k$ be an endomorphism of affine space, and let $p \in \mathbb{A}^n_k$ be an isolated zero of f such that $k(p)$ is a separable extension of finite degree over k. Let $\widetilde{p}$ denote the canonical point above p. Then*

$$\deg^{\mathbb{A}^1}_p(f) = \mathrm{Tr}_{k(p)/k}\deg^{\mathbb{A}^1}_{\widetilde{p}}\left(f_{k(p)}\right)$$

in $\mathrm{GW}(k)$.

As a corollary, we strengthen Kass and Wickelgren's result relating the local $\mathbb{A}^1$-degree and the Scheja–Storch form [**KW19**] by weakening the requirement that the point be rational.

Corollary 1.4. *At points whose residue fields are finite separable extensions of the ground field, the local $\mathbb{A}^1$-degree coincides with the isomorphism class of the Scheja–Storch form.*

In this paper we utilize the machinery of stable $\mathbb{A}^1$-homotopy theory, initially developed by Morel and Voevodsky [**MV99**], as well as the six functors formalism in this setting [**Ayo07, CD19**]. We also rely heavily on results of Hoyois [**Hoy15b**] to prove our main result. After working in the stable $\mathbb{A}^1$-homotopy category, we apply Morel's $\mathbb{A}^1$-degree to obtain the desired equality in $\mathrm{GW}(k)$.

Conventions 1.5. Throughout, we adopt the following conventions:

- We will use k to denote a general field. If p is a point of a k-scheme, with residue field $k(p)$ such that $k(p)/k$ a separable extension of finite degree, we call p a *finite separable point*. We may also say that p has a *finite separable residue field* in this context. We remark that all such points are closed points.
- Whenever a closed point p of a k-scheme X is chosen, we denote by $\rho\colon \operatorname{Spec} k(p) \to \operatorname{Spec} k$ the composite of the morphism $\operatorname{Spec} k(p) \xrightarrow{p} X$ defining the point p, and the structure map $X \to \operatorname{Spec} k$. This fixes a field embedding $k \hookrightarrow k(p)$.
- Given a scheme X over k, we denote the base change $X \times_{\operatorname{Spec} k} \operatorname{Spec} k(p)$ by $X_{k(p)}$, and given a morphism of k-schemes $f\colon X \to Y$, we denote its base change by $f_{k(p)}\colon X_{k(p)} \to Y_{k(p)}$.
- The structure map ρ allows us to define the canonical $k(p)$-rational point in $\mathbb{A}^n_{k(p)}$ sitting above p, which we denote by $\tilde{p}$.

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2. Preliminaries

In this section, we introduce the main notions necessary to state and prove Theorem 1.3. We begin in Subsection 2.1 by defining the local $\mathbb{A}^1$-degree. In Subsection 2.2, we highlight key properties of the stable motivic homotopy category in the form that we will need them. Finally, in Subsection 2.3, we discuss the Scheja–Storch form.

We will assume some familiarity with motivic homotopy theory. For more detail about the category of motivic spaces $\mathrm{Spc}_k^{\mathbb{A}^1}$ and the unstable motivic homotopy category $\mathcal{H}(k)$, we refer the reader to the excellent expository articles [**AE16, WW19**]. For the construction of the stable motivic homotopy category $\mathcal{SH}(k)$, we refer the reader to [**Mor04**].

Notation 2.1. We denote by $\mathcal{SH}(k)$ the stable motivic homotopy category over the scheme $\operatorname{Spec} k$. The sphere spectrum in this category will be denoted by $\mathbf{1}_k$. We will also use $[-,-]_{\mathbb{A}^1}$ to denote $\mathbb{A}^1$-weak equivalence classes of maps between two motivic spaces, by which we mean a hom-set in the homotopy category $\mathcal{H}(k)$.

2.1. The local $\mathbb{A}^1$-degree

Given an endomorphism of affine space $f\colon \mathbb{A}^n_k \to \mathbb{A}^n_k$ with an isolated zero at a point p, we describe how to obtain an endomorphism of the sphere spectrum in the stable motivic homotopy category $\mathcal{SH}(k)$, following the exposition of [**KW19**, pp. 438–439]. We remind the reader of Conventions 1.5, which we use in what follows.

Since p is an isolated zero of f, we may find an open neighborhood $U \subseteq \mathbb{A}^n_k$ for which $f^{-1}(0) \cap U = \{p\}$, that is, an open neighborhood containing no other zeros of f. Viewing $U \subseteq \mathbb{A}^n_k \subseteq \mathbb{P}^n_k$ as an open subset of projective space via a standard affine chart, we may take a Nisnevich-local pushout diagram in $\mathrm{Spc}^{\mathbb{A}^1}_k$:

$$\begin{tikzcd} U \smallsetminus \{p\} \arrow[r] \arrow[d] & \mathbb{P}^n_k \smallsetminus \{p\} \arrow[d] \\ U \arrow[r] \arrow[ur, phantom, "\ulcorner"] & \mathbb{P}^n_k. \end{tikzcd}$$

This induces an $\mathbb{A}^1$-weak equivalence on cofibers

$$\frac{U}{U \smallsetminus \{p\}} \xrightarrow{\sim} \frac{\mathbb{P}^n_k}{\mathbb{P}^n_k \smallsetminus \{p\}}.$$

We now appeal to the purity theorem [**MV99**, Theorem 2.23], a fundamental result in $\mathbb{A}^1$-homotopy, which we record for future use. While the purity theorem holds for smooth schemes over a sufficiently nice base scheme, we only need the result for smooth schemes over a field.

Theorem 2.2 (Morel–Voevodsky)**.** *Let $Z \to X$ be a closed embedding of smooth schemes over a field k. Let $N_{X,Z}$ denote the normal bundle of Z in X. Then there is a canonical weak equivalence of motivic spaces:*

$$X/(X \smallsetminus Z) \simeq \mathrm{Th}(N_{X,Z}).$$

Returning to the situation above, we remark that projective space is endowed with a local trivialization of the tangent bundle of $\mathbb{P}^n_k$ around p, arising from the trivialization of the tangent bundle of affine space. We thus obtain canonical $\mathbb{A}^1$-weak equivalences identifying the Thom space of the trivial rank n bundle over $\mathrm{Spec}\, k(p)$ with the object of study:

$$\frac{U}{U \smallsetminus \{p\}} \xrightarrow{\sim} \frac{\mathbb{P}^n_k}{\mathbb{P}^n_k \smallsetminus \{p\}} \simeq \mathrm{Th}(\mathcal{O}^n_{k(p)}) \simeq \left(\mathbb{P}^n_k/\mathbb{P}^{n-1}_k\right) \wedge \mathrm{Spec}\, k(p)_+.$$

We remark that U was chosen to satisfy $f(U \smallsetminus \{p\}) \subseteq \mathbb{A}^n_k \smallsetminus \{0\}$, and we can perform a completely analogous procedure to obtain $\mathbb{A}^1$-weak equivalences

$$\frac{\mathbb{A}^n_k}{\mathbb{A}^n_k \smallsetminus \{0\}} \xrightarrow{\sim} \frac{\mathbb{P}^n_k}{\mathbb{P}^n_k \smallsetminus \{0\}} \simeq \mathrm{Th}(\mathcal{O}^n_k) \simeq \mathbb{P}^n_k/\mathbb{P}^{n-1}_k.$$

Recall that in differential topology, the local degree is defined as the homotopy class of an induced map of spheres about a point. The space $\mathbb{P}^n_k/\mathbb{P}^{n-1}_k$ analogously plays the role of a sphere in $\mathrm{Spc}^{\mathbb{A}^1}_k$ when constructing the local $\mathbb{A}^1$-degree.

Definition 2.3. The *collapse map* is the map $c_p \colon \mathbb{P}^n_k/\mathbb{P}^{n-1}_k \to \mathbb{P}^n_k/(\mathbb{P}^n_k \smallsetminus \{p\})$ induced by the inclusion $\mathbb{P}^{n-1}_k \subseteq \mathbb{P}^n_k \smallsetminus \{p\}$.

Definition 2.4. For any $f \colon \mathbb{A}^n_k \to \mathbb{A}^n_k$ with an isolated zero at p, we denote by f_p the $\mathbb{A}^1$-homotopy class in the unstable motivic homotopy category assigned to the composite

$$\mathbb{P}^n_k/\mathbb{P}^{n-1}_k \xrightarrow{c_p} \frac{\mathbb{P}^n_k}{\mathbb{P}^n_k \smallsetminus \{p\}} \xleftarrow{\sim} \frac{U}{U \smallsetminus \{p\}} \xrightarrow{\overline{f}} \frac{\mathbb{A}^n_k}{\mathbb{A}^n_k \smallsetminus \{0\}} \xrightarrow{\sim} \mathbb{P}^n_k/\mathbb{P}^{n-1}_k.$$

Remark 2.5. Despite notational similarities, we remark that f_p and $f_{k(p)}$ are essentially unrelated. The notation f_p is consistent with that used in [**KW19**, Definition 11], while $f_{k(p)}$ is common notation for the base change of a morphism of schemes.

Remark 2.6. When p is k-rational, one can avoid the collapse map by applying the purity theorem to obtain the composite

$$\mathbb{P}^n_k/\mathbb{P}^{n-1}_k \simeq \frac{U}{U \smallsetminus \{p\}} \xrightarrow{\overline{f}} \frac{\mathbb{A}^n_k}{\mathbb{A}^n_k \smallsetminus \{0\}} \xrightarrow{\sim} \mathbb{P}^n_k/\mathbb{P}^{n-1}_k.$$

This composite yields the same element of $\left[\mathbb{P}^n_k/\mathbb{P}^{n-1}_k, \mathbb{P}^n_k/\mathbb{P}^{n-1}_k\right]_{\mathbb{A}^1}$ as in Definition 2.4. Indeed, by [**KW19**, Lemma 10], the composite of the collapse map with the canonical $\mathbb{A}^1$-weak equivalence $\mathbb{P}^n_k/(\mathbb{P}^n_k \smallsetminus \{0\}) \xrightarrow{\sim} \mathbb{P}^n_k/\mathbb{P}^{n-1}_k$ is the class of the identity map in $\left[\mathbb{P}^n_k/\mathbb{P}^{n-1}_k, \mathbb{P}^n_k/\mathbb{P}^{n-1}_k\right]_{\mathbb{A}^1}$.

Remark 2.7. The $\mathbb{A}^1$-homotopy class of f_p does not depend upon the original choice of open neighborhood $U \ni p$, provided that U contains no other zeros of f besides p. This follows immediately from our ability to provide an $\mathbb{A}^1$-weak equivalence between the cofiber $U/\left(U \smallsetminus \{p\}\right)$ and the Thom space $\mathrm{Th}(\mathcal{O}^n_k)$.

We now describe how to obtain an endomorphism of the sphere spectrum in $\mathcal{SH}(k)$ from the class f_p defined above. By [**MV99**, Proposition 2.17], we have a canonical $\mathbb{A}^1$-weak equivalence

$$\mathbb{P}^n_k/\mathbb{P}^{n-1}_k \simeq \left(\mathbb{P}^1_k\right)^{\wedge n}.$$

We recall also that $\mathbb{P}^1 \simeq S^1 \wedge \mathbb{G}_m$ as elements of the stable homotopy category. In particular by following the indexing convention of [**Mor04**] for motivic spheres, we see that $\Sigma^\infty \mathbb{P}^1_k = \Sigma^{2,1}\mathbf{1}_k$ in $\mathcal{SH}(k)$, where $\mathbf{1}_k$ denotes the sphere spectrum. We therefore have that

$$\Sigma^\infty \mathbb{P}^n_k/\mathbb{P}^{n-1}_k \simeq \Sigma^{2n,n}\mathbf{1}_k$$

in $\mathcal{SH}(k)$. It is immediate that, by desuspending, we obtain a canonical isomorphism

$$\mathrm{End}_{\mathcal{SH}(k)}\left(\Sigma^{2n,n}\mathbf{1}_k\right) \cong \mathrm{End}_{\mathcal{SH}(k)}(\mathbf{1}_k)$$

in the stable homotopy category. Collecting these facts together, we see that f_p determines an element in $\mathrm{End}_{\mathcal{SH}(k)}(\mathbf{1}_k)$. Abusing notation, we will refer to this endomorphism of the sphere spectrum as f_p.

Theorem 2.8 (Morel)**.** *For any field k, there is an isomorphism*

$$\deg^{\mathbb{A}^1} : \mathrm{End}_{\mathcal{SH}(k)}(\mathbf{1}_k) \cong \mathrm{GW}(k). \tag{1}$$

*Morel initially required the assumption that k be perfect [**Mor12**], however this can be removed via work of Hoyois [**Hoy15a**, Appendix A].*

Definition 2.9. With notation as above, the image of f_p in $\mathrm{GW}(k)$ under $\deg^{\mathbb{A}^1}$ is the *local $\mathbb{A}^1$-degree* of f at p, denoted $\deg^{\mathbb{A}^1}_p(f)$.

2.2. Stable motivic homotopy theory

We begin by recalling a few concepts and results from stable motivic homotopy theory that will play a role in the proof of Theorem 1.3.

The category theory of $\mathcal{SH}(k)$ supports a six functor formalism, the general exposition of which we defer to [**Hoy15b**, §2]. Indeed, for the purposes of this paper, we need only consider this formalism in the case of functors induced by maps $\rho\colon \operatorname{Spec} k(p) \to \operatorname{Spec} k$, where $k(p)/k$ is a finite separable field extension. We recall that we have an adjunction

$$\rho^*\colon \mathcal{SH}(k) \rightleftarrows \mathcal{SH}(k(p)) : \rho_*.$$

Since ρ is separated and finite type, we also have an exceptional adjunction

$$\rho_!\colon \mathcal{SH}(k(p)) \rightleftarrows \mathcal{SH}(k) : \rho^!.$$

We denote by η the unit of the adjunction between the direct and inverse image functors, and by ϵ the counit of the exceptional adjunction. That is, we have natural transformations:

$$\begin{aligned} \eta\colon &\ \mathrm{id}_{\mathcal{SH}(k)} \Rightarrow \rho_*\rho^*, \\ \epsilon\colon &\ \rho_!\rho^! \Rightarrow \mathrm{id}_{\mathcal{SH}(k)}. \end{aligned}$$

Remark 2.10. To facilitate exposition, we pause here to provide references for a few basic facts about six functors which we will make use of in this paper. Let ρ be as in Conventions 1.5.

1. Since ρ is smooth, ρ^* admits a left adjoint, denoted $\rho_\sharp$. As ρ is furthermore finite and étale, we have a canonical equivalence $\rho_* \simeq \rho_\sharp$ [**Hoy15b**, p. 21].
2. We have a canonical isomorphism $\rho_*\rho^*\mathbf{1}_k \simeq \rho_*\mathbf{1}_{k(p)}$. This is due to [**MV99**, p. 112, Proposition 2.17(3)]. See also [**KW19**, Equation 11].
3. Under our assumptions on ρ, we have canonical natural isomorphisms $\rho^! \simeq \rho^*$ and $\rho_! \simeq \rho_\sharp \simeq \rho_*$. This may be found in [**Hoy15b**, p. 21]. In particular, we remark that $\rho_\sharp$ can be interpreted as a forgetful functor under the structure map ρ.
4. There is a canonical equivalence $\rho_!\mathbf{1}_{k(x)} \cong \operatorname{Spec} k(x)_+$ in $\mathcal{SH}(k)$. See [**KW19**, p. 441].

We are now in a position to recall a description of the collapse map at the level of the stable motivic homotopy category.

Lemma 2.11 ([**Hoy15b, KW19**])**.** *In the stable homotopy category $\mathcal{SH}(k)$, the collapse map of Definition 2.3*

$$c_p\colon \mathbb{P}^n_k/\mathbb{P}^{n-1}_k \to \mathbb{P}^n_k/(\mathbb{P}^n_k \smallsetminus \{p\}) \cong \left(\mathbb{P}^n_k/\mathbb{P}^{n-1}_k\right) \wedge \mathit{Spec}(k(p))_+$$

is computed by applying $\mathbb{P}^n_k/\mathbb{P}^{n-1}_k \wedge (-)$ to the component of the unit η at the sphere spectrum:

$$\eta_{\mathbf{1}_k}\colon \mathbf{1}_k \to \rho_*\rho^*\mathbf{1}_k \cong \rho_*\mathbf{1}_{k(p)}.$$

Proof. The case $n = 1$ may be found in [**Hoy15b**, Lemma 5.5], and the proof generalizes to higher n as in [**KW19**, Lemma 13]. $\square$

Remark 2.12. We can furthermore describe $f_p \in \operatorname{End}_{\mathcal{SH}(k)}(\mathbf{1}_k)$ in the following way.

Recall that f induces a map

$$\overline{f}\colon U/(U \smallsetminus \{p\}) \to \mathbb{A}^n_k/(\mathbb{A}^n_k \smallsetminus \{0\}).$$

As above, we have $\mathbb{A}^1$-weak equivalences

$$U/(U \smallsetminus \{p\}) \simeq \left(\mathbb{P}^n_k/\mathbb{P}^{n-1}_k\right) \wedge \operatorname{Spec}(k(p))_+$$

and

$$\mathbb{A}^n_k/(\mathbb{A}^n_k \smallsetminus \{0\}) \simeq \mathbb{P}^n_k/\mathbb{P}^{n-1}_k.$$

We may thus identify $\overline{f}$ with the composite $\left(\mathbb{P}^n_k/\mathbb{P}^{n-1}_k\right) \wedge \operatorname{Spec}(k(p))_+ \to \mathbb{P}^n_k/\mathbb{P}^{n-1}_k$ in $\mathcal{SH}(k)$. By Definition 2.4, we have that f_p is the composite of $\overline{f}$ and the collapse map. In $\mathcal{SH}(k)$, we can record this via the following commutative diagram:

$$\begin{tikzcd}
\left(\mathbb{P}^n_k/\mathbb{P}^{n-1}_k\right) \wedge \operatorname{Spec}(k(p))_+ \arrow[r, "\overline{f}"] & \mathbb{P}^n_k/\mathbb{P}^{n-1}_k \\
\mathbb{P}^n_k/\mathbb{P}^{n-1}_k \arrow[u, "\mathbb{P}^n_k/\mathbb{P}^{n-1}_k \wedge (\eta_{\mathbf{1}_k})"] \arrow[ur, bend right, "f_p"'] &
\end{tikzcd}$$

We now recall that the trace $\operatorname{Tr}_{k(p)/k}\colon \operatorname{GW}(k(p)) \to \operatorname{GW}(k)$ can be described purely in terms of maps in the motivic homotopy category, under the isomorphism of Theorem 2.8.

Definition 2.13. The *transfer*

$$\operatorname{Tr}_{k(p)/k}\colon \operatorname{End}_{\mathcal{SH}(k(p))}(\mathbf{1}_{k(p)}) \to \operatorname{End}_{\mathcal{SH}(k)}(\mathbf{1}_k)$$

is defined by sending $\omega \in \operatorname{End}_{\mathcal{SH}(k(p))}(\mathbf{1}_{k(p)})$ to the composite

$$\mathbf{1}_k \xrightarrow{\eta_{\mathbf{1}_k}} \rho_*\mathbf{1}_{k(p)} \simeq \rho_\sharp \mathbf{1}_{k(p)} \xrightarrow{\rho_\sharp\omega} \rho_\sharp \mathbf{1}_{k(p)} \simeq \rho_! \rho^! \mathbf{1}_k \xrightarrow{\epsilon_{\mathbf{1}_k}} \mathbf{1}_k.$$

Lemma 2.14 ([**Hoy15b**, Proposition 5.2, Lemma 5.3])**.** *The transfer agrees with the field trace. That is, the diagram*

$$\begin{tikzcd}
\operatorname{End}_{\mathcal{SH}(k(p))}\left(\mathbf{1}_{k(p)}\right) \arrow[r, "\operatorname{Tr}_{k(p)/k}"] \arrow[d, "\cong"'] & \operatorname{End}_{\mathcal{SH}(k)}\left(\mathbf{1}_k\right) \arrow[d, "\cong"] \\
\operatorname{GW}(k(p)) \arrow[r, "\operatorname{Tr}_{k(p)/k}"'] & \operatorname{GW}(k)
\end{tikzcd}$$

commutes, where the vertical maps are given by Morel's degree isomorphism (Equation 1), the top map is the transfer (Definition 2.13), and the bottom map is the trace map on the Grothendieck–Witt group of k (Definition 1.2).

2.3. The Scheja-Storch bilinear form

We give a brief description of the Scheja–Storch bilinear form (see also [**SS75**], [**KW19**], and [**KW17**, Section 4]). Given a polynomial map

$$f = (f_1, \dots, f_n)\colon \mathbb{A}^n_k \to \mathbb{A}^n_k$$

with isolated zero p, let $\mathfrak{m}$ be the maximal ideal of $k[x_1, \dots, x_n]$ corresponding to the point p. Consider the local algebra $Q_p = \frac{k[x_1,\dots,x_n]_{\mathfrak{m}}}{(f_1,\dots,f_n)}$. As a local complete intersection,

Q_p is isomorphic to its dual $\mathrm{Hom}_k(Q_p, k)$. Scheja and Storch construct an explicit Q_p-linear isomorphism $\Theta\colon \mathrm{Hom}_k(Q_p, k) \to Q_p$ realizing this self-duality, which gives us a distinguished homomorphism $\eta := \Theta^{-1}(1)\colon Q_p \to k$.

Definition 2.15. Given a polynomical function $f\colon \mathbb{A}^n_k \to \mathbb{A}^n_k$, the *Scheja–Storch bilinear form* $\beta_p(f)\colon Q_p \times Q_p \to k$ is given by $\beta_p(f)(x, y) = \eta(xy)$, where η is defined in the preceeding paragraph.

Since Q_p is commutative, the Scheja–Storch bilinear form is symmetric. By [**KW19**, Lemma 28] and [**EL77**, Proposition 3.4], the Scheja–Storch form is non-degenerate. Thus the Scheja–Storch form gives a class in $\mathrm{GW}(k)$, which we denote by $\mathrm{ind}_p(f)$.

Kass–Wickelgren show that if p is k-rational or if f is étale at p, then $\deg^{\mathbb{A}^1}_p(f) = \mathrm{ind}_p(f)$ [**KW19**]. They also show that if p is a finite separable point, then

$$\mathrm{ind}_p(f) = \mathrm{Tr}_{k(p)/k}\, \mathrm{ind}_{\widetilde{p}}(f_{k(p)}),$$

where $\widetilde{p}$ is the canonical $k(p)$-point above p and $f_{k(p)}\colon \mathbb{A}^n_{k(p)} \to \mathbb{A}^n_{k(p)}$ is the base change of f [**KW17**, Proposition 32]. Given these two results, one would expect Theorem 1.3 to be true.

3. Main results

We now proceed to the proof of the main theorem, as stated in Theorem 1.3. Our first step is to apply the machinery of Subsection 2.2 to frame our problem in terms of motivic homotopy theory. Recall from Conventions 1.5 that we have already fixed a choice of field embedding $k \hookrightarrow k(p)$. Thus, for any k-scheme X and any point $p \in X$, we have the canonical $k(p)$-rational point $\widetilde{p} \in X_{k(p)}$ sitting over p, defined via the following pullback diagram:

$$\begin{tikzcd}
\operatorname{Spec} k(p) \arrow[dr, dashed, "\widetilde{p}"] \arrow[drr, bend left, "\mathrm{id}"] \arrow[ddr, bend right, "p"'] & & \\
& X_{k(p)} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \operatorname{Spec} k(p) \arrow[d, "\rho"] \\
& X \arrow[r] & \operatorname{Spec} k.
\end{tikzcd}$$

We write $f_{k(p)}$ and $\pi\colon \mathbb{A}^n_{k(p)} \to \mathbb{A}^n_k$ for the morphisms induced by base change. We then may consider the following diagram of k-schemes:

$$\begin{tikzcd}
\mathbb{A}^n_{k(p)} \arrow[r, "f_{k(p)}"] \arrow[d, "\pi"'] & \mathbb{A}^n_{k(p)} \arrow[d, "\pi"] \\
\mathbb{A}^n_k \arrow[r, "f"'] & \mathbb{A}^n_k
\end{tikzcd} \quad \text{which maps} \quad \begin{tikzcd}
\widetilde{p} \arrow[r, mapsto] \arrow[d, mapsto] & 0 \arrow[d, mapsto] \\
p \arrow[r, mapsto] & 0.
\end{tikzcd}$$

Note that the point $\widetilde{p}$ is a root of $f_{k(p)}$, so $f_{k(p)}$ has an isolated rational zero at $\widetilde{p}$. Let $U \subseteq \mathbb{A}^n_{k(p)}$ be an open neighborhood containing $\widetilde{p}$ and no other zeros of $f_{k(p)}$. As the structure map $\mathbb{A}^n_k \to \operatorname{Spec} k$ is universally open, π is an open morphism of schemes. Thus, $\pi(U)$ is an open neighborhood of p and contains no other zeros of f

by construction. Taking cofibers, we obtain an induced diagram of motivic spaces

$$\begin{tikzcd}
\dfrac{U}{U \smallsetminus \{\widetilde{p}\}} \arrow[r, "\overline{f}_{k(p)}"] \arrow[d, "\overline{\pi}_p"'] & \dfrac{\mathbb{A}^n_{k(p)}}{\mathbb{A}^n_{k(p)} \smallsetminus \{0\}} \arrow[d, "\overline{\pi}_0"] \\
\dfrac{\pi(U)}{\pi(U) \smallsetminus \{p\}} \arrow[r, "\overline{f}"'] & \dfrac{\mathbb{A}^n_k}{\mathbb{A}^n_k \smallsetminus \{0\}}.
\end{tikzcd} \tag{2}$$

As discussed in Section 2.1, we have the following $\mathbb{A}^1$-weak equivalences:

$$\begin{aligned}
\frac{U}{U \smallsetminus \{\widetilde{p}\}} &\xrightarrow{\sim} \frac{\mathbb{P}^n_{k(p)}}{\mathbb{P}^n_{k(p)} \smallsetminus \{\widetilde{p}\}} \simeq \left(\mathbb{P}^n_k/\mathbb{P}^{n-1}_k\right) \wedge \operatorname{Spec}(k(p))_+, \\
\frac{\mathbb{A}^n_{k(p)}}{\mathbb{A}^n_{k(p)} \smallsetminus \{0\}} &\xrightarrow{\sim} \frac{\mathbb{P}^n_{k(p)}}{\mathbb{P}^n_{k(p)} \smallsetminus \{0\}} \simeq \left(\mathbb{P}^n_k/\mathbb{P}^{n-1}_k\right) \wedge \operatorname{Spec}(k(p))_+, \\
\frac{\pi(U)}{\pi(U) \smallsetminus \{p\}} &\xrightarrow{\sim} \frac{\mathbb{P}^n_k}{\mathbb{P}^n_k \smallsetminus \{p\}} \simeq \left(\mathbb{P}^n_k/\mathbb{P}^{n-1}_k\right) \wedge \operatorname{Spec}(k(p))_+, \\
\frac{\mathbb{A}^n_k}{\mathbb{A}^n_k \smallsetminus \{0\}} &\xrightarrow{\sim} \frac{\mathbb{P}^n_k}{\mathbb{P}^n_k \smallsetminus \{0\}} \simeq \mathbb{P}^n_k/\mathbb{P}^{n-1}_k.
\end{aligned}$$

Appending these $\mathbb{A}^1$-weak equivalences to Diagram (2), we obtain the following diagram in the unstable homotopy category $\mathcal{H}(k)$.

$$\begin{tikzcd}
 & \left(\mathbb{P}^n_k/\mathbb{P}^{n-1}_k\right) \wedge \operatorname{Spec}(k(p))_+ \arrow[rr, dashed] \arrow[dd, dashed] & & \left(\mathbb{P}^n_k/\mathbb{P}^{n-1}_k\right) \wedge \operatorname{Spec}(k(p))_+ \arrow[dd, dashed] \\
\frac{U}{U \smallsetminus \{\widetilde{p}\}} \arrow[ur, "\sim"] \arrow[rr, "\overline{f}_{k(p)}"] \arrow[dd, "\overline{\pi}_p"'] & & \frac{\mathbb{A}^n_{k(p)}}{\mathbb{A}^n_{k(p)} \smallsetminus \{0\}} \arrow[ur, "\sim"] \arrow[dd, "\overline{\pi}_0"] & \\
 & \left(\mathbb{P}^n_k/\mathbb{P}^{n-1}_k\right) \wedge \operatorname{Spec}(k(p))_+ \arrow[rr, dashed] & & \mathbb{P}^n_k/\mathbb{P}^{n-1}_k \\
\frac{\pi(U)}{\pi(U) \smallsetminus \{p\}} \arrow[ur, "\sim"] \arrow[rr, "\overline{f}"'] & & \frac{\mathbb{A}^n_k}{\mathbb{A}^n_k \smallsetminus \{0\}} \arrow[ur, "\sim"] &
\end{tikzcd} \tag{3}$$

The dashed arrows above are obtained by inverting $\mathbb{A}^1$-weak equivalences.

We now turn our attention to the dashed face of the cube (3). We obtain the class f_p from Definition 2.4 by pre-composing with the collapse map (see Diagram (4)). Working in the stable homotopy category, the top edge of the dashed face is exactly the image of $\left(f_{k(p)}\right)_{\widetilde{p}}$ under $\rho_* \colon \mathcal{SH}(k(p)) \to \mathcal{SH}(k)$. Taking suspension spectra, we get the following diagram in $\mathcal{SH}(k)$.

$$\begin{tikzcd}
\left(\mathbb{P}^n_k/\mathbb{P}^{n-1}_k\right) \wedge \operatorname{Spec}(k(p))_+ \arrow[r, "\rho_*(f_{k(p)})_{\widetilde{p}}"] \arrow[d, "r"'] & \left(\mathbb{P}^n_k/\mathbb{P}^{n-1}_k\right) \wedge \operatorname{Spec}(k(p))_+ \arrow[d, "g"] \\
\left(\mathbb{P}^n_k/\mathbb{P}^{n-1}_k\right) \wedge \operatorname{Spec}(k(p))_+ \arrow[r] & \mathbb{P}^n_k/\mathbb{P}^{n-1}_k \\
\mathbb{P}^n_k/\mathbb{P}^{n-1}_k \arrow[u, "\left(\mathbb{P}^n_k/\mathbb{P}^{n-1}_k\right) \wedge \eta_{\mathbf{1}_k}"] \arrow[ur, bend right, "f_p"'] &
\end{tikzcd} \tag{4}$$

The rest of our paper will center around the diagram above. In proving Theorem 1.3, we will show that r in Diagram (4) is invertible in $\mathcal{SH}(k)$, which allows us to rewrite f_p by exploiting the commutativity of this diagram.

3.1. The stable classes of r and g

In order to analyze Diagram (4) we state the following two lemmas, which allow us to characterize the $\mathcal{SH}(k)$-classes of r and g, respectively.

Lemma 3.1. *Let $p \in \mathbb{A}^n_k$ be a closed point with finite separable residue field $k(p)/k$. Let $\pi\colon \mathbb{A}^n_{k(p)} \to \mathbb{A}^n_k$ be the projection map induced by the structure map $\rho\colon \mathrm{Spec}\,k(p) \to \mathrm{Spec}\,k$, and let $\widetilde{p} \in \mathbb{A}^n_{k(p)}$ be the canonical $k(p)$-rational point above p. Then for any open neighborhood U about $\widetilde{p}$ such that $U \cap \pi^{-1}(p) = \{\widetilde{p}\}$, the stable class in $\mathcal{SH}(k)$ of the map*

$$\frac{U}{U \smallsetminus \{\widetilde{p}\}} \to \frac{\pi(U)}{\pi(U) \smallsetminus \{p\}} \tag{5}$$

is given by $(\mathbb{P}^n_k/\mathbb{P}^{n-1}_k) \wedge \rho_* \mathrm{id}_{\mathbf{1}_{k(p)}}$.

Proof. The base change $\pi\colon \mathbb{A}^n_{k(p)} \to \mathbb{A}^n_k$ is simply Spec applied to the k-algebra homomorphism $\iota\colon k[x_1, \dots, x_n] \hookrightarrow k(p)[x_1, \dots, x_n]$. As $\iota(x_i) = x_i$, we get an induced map $T\mathbb{A}^n_{k(p)} \to \pi^* T\mathbb{A}^n_k$ which in turn induces an isomorphism $(T\mathbb{A}^n_{k(p)})_{\widetilde{p}} \xrightarrow{\sim} (\pi^* T\mathbb{A}^n_k)_{\widetilde{p}}$. The right hand side is easily seen to be $T_p\mathbb{A}^n_k \otimes_k k(p)$. As $k(p)$-vector spaces, we have isomorphisms $T_{\widetilde{p}}\mathbb{A}^n_{k(p)} \cong T_p\mathbb{A}^n_k \otimes_k k(p) \cong \mathbb{A}^n_{k(p)}$.

Next, we consider the Thom spaces $\mathrm{Th}(T_{\widetilde{p}}\mathbb{A}^n_{k(p)})$ and $\mathrm{Th}(T_p\mathbb{A}^n_k \otimes_k k(p))$. Via the purity isomorphism in Theorem 2.2, we have weak equivalences of k-motivic spaces

$$\frac{U}{U \smallsetminus \{\widetilde{p}\}} \simeq \mathrm{Th}\left(T_{\widetilde{p}}\mathbb{A}^n_{k(p)}\right) \xrightarrow{\sim} \left(\mathbb{P}^n_k/\mathbb{P}^{n-1}_k\right) \wedge \mathrm{Spec}\,(k(p))_+,$$
$$\frac{\pi(U)}{\pi(U) \smallsetminus \{p\}} \simeq \mathrm{Th}\left(T_p\mathbb{A}^n_k\right) \xrightarrow{\sim} \left(\mathbb{P}^n_k/\mathbb{P}^{n-1}_k\right) \wedge \mathrm{Spec}\,(k(p))_+.$$

Since base change is a left adjoint and Thom spaces are obtained by taking colimits, we deduce

$$\mathrm{Th}(T_p\mathbb{A}^n_k \otimes_k k(p)) \simeq \rho^* \,\mathrm{Th}(T_p\mathbb{A}^n_k)$$

as $k(p)$-motivic spaces. Moreover, the purity theorem implies that $\mathrm{Th}(T_p\mathbb{A}^n_k)$ is a $k(p)$-motivic space, so its base change to $k(p)$ is canonically identified with itself in the homotopy category of $k(p)$-motivic spaces. That is, the class in $\mathcal{SH}(k(p))$ of Equation 5 is given by the class in $\mathcal{SH}(k(p))$ of the canonical $\mathbb{A}^1$-weak equivalence $\mathrm{Th}(T_{\widetilde{p}}\mathbb{A}^n_{k(p)}) \xrightarrow{\sim} \mathrm{Th}(T_p\mathbb{A}^n_k)$. This has the class of

$$(\mathbb{P}^n_{k(p)}/\mathbb{P}^{n-1}_{k(p)}) \wedge \mathrm{id}_{\mathbf{1}_{k(p)}}$$

in $\mathcal{SH}(k(p))$ under the identification given by the purity isomorphism. Finally, we push forward via the functor $\rho_*\colon \mathcal{SH}(k(p)) \to \mathcal{SH}(k)$ to get that the class of Equation 5 is $(\mathbb{P}^n_k/\mathbb{P}^{n-1}_k) \wedge \rho_* \mathrm{id}_{\mathbf{1}_{k(p)}}$. ∎

Lemma 3.2. *Let $k(p)/k$ be a finite separable field extension, let $q \in \mathbb{A}^n_k$ be any k-rational point, and let $\pi\colon \mathbb{A}^n_{k(p)} \to \mathbb{A}^n_k$ be the projection map induced by the structure map $\rho\colon \mathrm{Spec}\,k(p) \to \mathrm{Spec}\,k$. Denote by $\widetilde{q} \in \mathbb{A}^n_{k(p)}$ the canonical $k(p)$-rational point* *above q. Then for any open neighborhood U containing $\widetilde{q}$, the stable class in $\mathcal{SH}(k)$ of the map*

$$\frac{U}{U \setminus \{\widetilde{q}\}} \to \frac{\pi(U)}{\pi(U) \setminus \{q\}}$$

is given by $(\mathbb{P}^n_k/\mathbb{P}^{n-1}_k) \wedge \epsilon_{\mathbf{1}_k}$.

Proof. Since q is k-rational, $\widetilde{q}$ is the unique canonical $k(p)$-rational point above q, so we may assume that $\widetilde{q}$ and q are the origins of $\mathbb{A}^n_{k(p)}$ and $\mathbb{A}^n_k$, respectively, in which case we may take $U = \mathbb{A}^n_{k(p)}$. It thus suffices to consider the class in $\mathcal{SH}(k)$ of the map of Thom spaces $\mathrm{Th}_0\,\mathbb{A}^n_{k(p)} \to \mathrm{Th}_0\,\mathbb{A}^n_k$ induced by the canonical map $\pi\colon \mathbb{A}^n_{k(p)} \to \mathbb{A}^n_k$. By viewing affine space as a trivial vector bundle over the origin, [**Hoy15b**, p. 9] implies that this is the desired component of the counit of the exceptional adjunction. ☐

Corollary 3.3. *The map g in Diagram (4) is $\left(\mathbb{P}^n_k/\mathbb{P}^{n-1}_k\right) \wedge \epsilon_{\mathbf{1}_k}$, and the map r is $\left(\mathbb{P}^n_k/\mathbb{P}^{n-1}_k\right) \wedge \rho_*\,\mathrm{id}_{\mathbf{1}_{k(p)}}$.*

Remark 3.4. Lemmas 3.1 and 3.2 hold more generally for schemes that are locally isomorphic to affine space, as we rely only on local computations in their proofs.

3.2. Proof of Theorem 1.3

By Corollary 3.3, we may rewrite Diagram (4) as

$$\begin{tikzcd}
\left(\mathbb{P}^n_k/\mathbb{P}^{n-1}_k\right) \wedge \mathrm{Spec}\,(k(p))_+ \arrow[r, "\rho_*(f_{k(p)})_{\widetilde{p}}"] \arrow[d, "\left(\mathbb{P}^n_k/\mathbb{P}^{n-1}_k\right)\wedge\rho_*\,\mathrm{id}_{\mathbf{1}_{k(p)}}"', "\sim"] & \left(\mathbb{P}^n_k/\mathbb{P}^{n-1}_k\right) \wedge \mathrm{Spec}\,(k(p))_+ \arrow[d, "\left(\mathbb{P}^n_k/\mathbb{P}^{n-1}_k\right)\wedge\epsilon_{\mathbf{1}_k}"] \\
\left(\mathbb{P}^n_k/\mathbb{P}^{n-1}_k\right) \wedge \mathrm{Spec}\,(k(p))_+ \arrow[r] & \mathbb{P}^n_k/\mathbb{P}^{n-1}_k \\
\mathbb{P}^n_k/\mathbb{P}^{n-1}_k \arrow[u, "\left(\mathbb{P}^n_k/\mathbb{P}^{n-1}_k\right)\wedge\eta_{\mathbf{1}_k}"] \arrow[ur, bend right, "f_p"'] &
\end{tikzcd} \tag{6}$$

In the stable homotopy category $\mathcal{SH}(k)$, Diagram (6) is $\mathbb{P}^n_k/\mathbb{P}^{n-1}_k \wedge (-)$ applied to the diagram

$$\begin{tikzcd}
\rho_*\mathbf{1}_{k(p)} \arrow[r, "\rho_*(f_{k(p)})_{\widetilde{p}}"] \arrow[d, "\rho_*\,\mathrm{id}_{\mathbf{1}_{k(p)}}"', "\sim"] & \rho_*\mathbf{1}_{k(p)} \arrow[d, "\epsilon_{\mathbf{1}_k}"] \\
\rho_*\mathbf{1}_{k(p)} \arrow[r] & \mathbf{1}_k \\
\mathbf{1}_k \arrow[u, "\eta_{\mathbf{1}_k}"] \arrow[ur, bend right, "f_p"'] &
\end{tikzcd} \tag{7}$$

We remark that we are able to invert the weak equivalence $\rho_*\,\mathrm{id}_{\mathbf{1}_{k(p)}}$ of Diagram (7). Since Diagram (7) is commutative, we may express f_p as the composite

$$\mathbf{1}_k \xrightarrow{\eta_{\mathbf{1}_k}} \rho_*\rho^*\mathbf{1}_k \simeq \rho_*\mathbf{1}_{k(p)} \xrightarrow{\rho_*\left(f_{k(p)}\right)_{\widetilde{p}}} \rho_*\mathbf{1}_{k(p)} \simeq \rho_!\rho^!\mathbf{1}_k \xrightarrow{\epsilon_{\mathbf{1}_k}} \mathbf{1}_k.$$

Recall that in the setting of Theorem 1.3, the morphism ρ is finite and étale, which gives a canonical isomorphism $\rho_* \simeq \rho_\sharp$ (Remark 2.10). Thus by Definition 2.13, we have $f_p = \mathrm{Tr}_{k(p)/k}\left(f_{k(p)}\right)_{\widetilde{p}}$. Applying Lemma 2.14, we conclude that $\deg_p^{\mathbb{A}^1}(f) = \mathrm{Tr}_{k(p)/k}\deg_{\widetilde{p}}^{\mathbb{A}^1}(f_{k(p)})$, as desired.

3.3. A brief proof of Corollary 1.4

In [**KW17**, Proposition 32], the authors prove that the Scheja–Storch bilinear form, denoted $\mathrm{ind}_p(f)$, is computed by the trace $\mathrm{ind}_p(f) = \mathrm{Tr}_{k(p)/k}\,\mathrm{ind}_{\widetilde{p}}(f_{k(p)})$. Moreover in [**KW19**], the authors prove that at any rational point, the Scheja–Storch form agrees with the local $\mathbb{A}^1$-degree. Combining these two results with Theorem 1.3, for any isolated zero p with finite separable residue field we have that

$$\mathrm{ind}_p(f) = \mathrm{Tr}_{k(p)/k}\,\mathrm{ind}_{\widetilde{p}}(f_{k(p)}) = \mathrm{Tr}_{k(p)/k}\deg_{\widetilde{p}}^{\mathbb{A}^1}(f_{k(p)}) = \deg_p^{\mathbb{A}^1}(f).$$

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