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GEOMETRIC APPROACH TO GRAPH MAGNITUDE HOMOLOGY

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Abstract

In this paper, we introduce a new method to compute the magnitude homology of general graphs. To each direct sum component of the magnitude chain complexes, we assign a pair of simplicial complexes whose simplicial chain complex is isomorphic to it. First we state our main theorem specialized to trees, which gives another proof for the known fact that trees are diagonal. After that, we consider general graphs, which may have cycles. We also demonstrate some computation as an application.

1. Introduction

Leinster [**6**] introduced the magnitude of finite metric spaces which measures “the number of efficient points”. Magnitude homology has been invented as a categoryfication of the magnitude of a graph which is equipped with a graph metric, by Hepworth–Willerton [**3**]. The magnitude homology $MH_{k,\ell}(G)$ of a graph G is defined by the k-th homology group of a chain complex $MC_{*,\ell}(G)$, whose chain groups are generated by tuples of vertices of length ℓ.

Several tools for computing the magnitude homology of a graph have been studied so far. For examples, Hepworth–Willerton [**3**] proves a Mayer–Vietoris type exact sequence and a Künneth type formula, and Gu [**2**] uses algebraic Morse theory for computation for some graphs. Although, in general, computation of magnitude homology remains a difficult problem.

In this paper, we introduce another method to compute the magnitude homology of general graphs. Our strategy is to replace the computation of the magnitude chain complex $MH_{k,\ell}(G)$ by that of simplicial homology. A similar method using order complexes is studied by Kaneta–Yoshinaga [**4**], whereas we assign simplicial complexes in another way. A subtle difference from Kaneta–Yoshinaga's method which restricts us to work within a range with no 4-cuts, is that our method can be applied to general graphs.

For a magnitude chain complex $MC_{*,\ell}(G)$, we denote by $MC_{*,\ell}(a,b)$ a direct sum component of it, which consists of tuples with ends a and b. Our main result is the following which appears as Theorem 4.3 in this paper. We assume that graphs are connected and contain no loops.

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Theorem 1.1. *Let a, b be vertices of a graph G, and fix an integer $\ell \geqslant 3$. Then we can construct a pair of simplicial complexes $(K_\ell(a,b), K'_\ell(a,b))$ which satisfies*

$$C_*(K_\ell(a,b), K'_\ell(a,b)) \cong s^{-2} MC_{*,\ell}(a,b),$$

where s denotes suspension of the complex detailed in section 2. In particular, we have

$$MH_{k,\ell}(a,b) \cong H_{k-2}(K_\ell(a,b), K'_\ell(a,b))$$

for $k, \ell \geqslant 3$. Moreover, for $k = 2$, we also have

$$MH_{2,\ell}(a,b) \cong \begin{cases} H_0(K_\ell(a,b), K'_\ell(a,b)) & \text{if } d(a,b) < \ell, \\ \tilde{H}_0(K_\ell(a,b)) & \text{if } d(a,b) = \ell, \end{cases}$$

where $\tilde{H}_*$ denotes the reduced homology group.

Our theorem yields an interpretation of magnitude homology groups as homology groups of a simplicial complex. Therefore, our method allows us to apply sophisticated tools of homotopy theory. In the special cases of $2 \leqslant \ell \leqslant 4$ we obtain a visualization of the magnitude chain complex since the dimensions of the corresponding simplical complexes are 0, 1, and 2, respectively.

The organization of this paper is the following: After giving some basic definitions and notation in section 2, we first give a new method for computing the magnitude homology of trees based on our simplicial strategy. The computational results thus obtained coincide with [**3**, Corollary 31]. The computation for a tree is simpler than that for the general graphs studied in the following section, because of the fact that the magnitude chain complex of a tree can be decomposed into simple ones. In section 4, we give a proof of our main theorem, and compute the magnitude homology of the graph Sq_2 introduced in [**7**] as an application.

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2. Preliminaries

In this section, we recall some basic definitions for graphs and their magnitude homology together with related notation. The main definitions are taken from [**3**].

2.1. Simplicial complexes

Definition 2.1. Let V be a set, and let $P(V)$ be its power set. A subset $S \subset P(V) \setminus \{\emptyset\}$ is called a *simplicial complex* if it satisfies that

$$B \in S \text{ for every } \emptyset \neq B \subset A \in S.$$

A subset S' of a simplicial complex S is called *a subcomplex of* S if S' itself is a simplicial complex.

Definition 2.2. For a simplicial complex $S \subset P(X) \setminus \{\emptyset\}$, we associate a chain complex $(C_*(S), \partial_*)$ defined as follows:

$$\begin{cases} C_n(S) = \mathbb{Z}\langle \sigma \in S \mid \#\sigma = n+1 \rangle, \\ \partial_n\{s_0, \dots, s_n\} := \sum_{i=0}^{n} (-1)^i \{s_0, \dots, \hat{s}_i, \dots, s_n\}, \end{cases}$$

where the index s_i is a fixed total order on X, the notation $\hat{s}_i$ means the removal of the vertex s_i, and $\#$ denotes the cardinality of a set. For a subcomplex S' of S, the associated chain complex $C_*(S')$ is obviously a subcomplex of $C_*(S)$. We define

$$C_*(S, S') := C_*(S)/C_*(S').$$

We suppose that any chain complex C_* has no negative component, that is, $C_i = 0$ for $i < 0$. For a chain complex (C_*, ∂_*), we denote by $s^{-N}C_*$ the chain complex (D_*, ∂'_*) defined as follows:

$$D_i = \begin{cases} C_{i+N} & (i \geqslant 0), \\ 0 & (i < 0), \end{cases} \qquad \partial'_i = \begin{cases} \partial_{i+N} & (i > 0), \\ 0 & (i \leqslant 0). \end{cases}$$

Note that our usage of the notation of suspension is unusual because we also truncate the chain complex.

2.2. Graphs

Definition 2.3. For a simplicial complex S, an element $A \in S$ with $\#A = 1$ is called *a vertex* of S, and an element $A \in S$ with $\#A = 2$ is called *an edge* of S.

Definition 2.4. A *graph* G is a simplicial complex with $\#A \leqslant 2$ for every $A \in S$. We denote by $V(G)$ the set of vertices of G, and denote by $E(G)$ the set of edges of G, which are called *a vertex set* and *an edge set* respectively.

Definition 2.5.

1. A graph G is *finite* if its vertex set $V(G)$ is a finite set.
2. A graph G is *connected* if any two vertices $a, b \in V(G)$ are connected, that is, there exists a finite sequence of edges
$$e_1, \dots, e_n \in E(G),$$
with
$$\begin{cases} a \in e_1, b \in e_n, \\ e_i \cap e_{i+1} \neq \emptyset & \text{for } 1 \leqslant i \leqslant n-1. \end{cases}$$
3. A *cycle* in a graph G is a finite sequence of edges
$$e_1, \dots, e_n \in E(G),$$
with
$$\begin{cases} n \geqslant 2 \\ e_1 \cap e_n \neq \emptyset, \\ e_i \cap e_{i+1} \neq \emptyset & \text{for } 1 \leqslant i \leqslant n-1, \\ e_i \neq e_j & \text{for } i \neq j. \end{cases}$$

4. A graph is a *tree* if it is finite, connected, and contains no cycles.

Throughout the paper, we assume that graphs are connected.

Definition 2.6. For a graph G, we define a distance function $d \colon V(G) \times V(G) \longrightarrow \mathbb{Z}_{\geqslant 0}$ as follows:

- In case $a \neq b$, we define $d(a, b) = n$ where n is the smallest integer such that there exists a sequence of edges

$$e_1, \dots, e_n \in E(G),$$

with

$$\begin{cases} a \in e_1, b \in e_n, \\ e_i \cap e_{i+1} \neq \emptyset & \text{for } 1 \leqslant i \leqslant n-1. \end{cases}$$

- In case $a = b$, we define $d(a, b) = 0$.

2.3. Magnitude homology of graphs

Definition 2.7. For a graph G, we call a tuple $(x_0, \dots, x_k) \in V(G)^{k+1}$ a $(k+1)$-*sequence* if it satisfies $x_j \neq x_{j+1}$ for every $0 \leqslant j \leqslant k-1$.

Definition 2.8. Let G be a graph and fix an integer $\ell \geqslant 0$. *The magnitude chain complex of length* ℓ of G is denoted by $MC_{*,\ell}(G)$ and defined as follows. The graded module $MC_{*,\ell}(G)$ is defined as the family of free $\mathbb{Z}$-modules $\{MC_{k,\ell}(G)\}_{k \geqslant 0}$ generated by all $(k+1)$-sequences

$$\mathbf{x} = (x_0, \dots, x_k) \in V(G)^{k+1}$$

satisfying

$$\sum_{i=0}^{k-1} d(x_i, x_{i+1}) = \ell.$$

The boundary map is defined by $\partial = \sum_{i=1}^{k-1} (-1)^i \partial_i$ with

$$\partial_i(\mathbf{x}) := \begin{cases} (x_0, \dots, \hat{x}_i, \dots, x_k) & \text{if } d(x_{i-1}, x_{i+1}) = d(x_{i-1}, x_i) + d(x_i, x_{i+1}), \\ 0 & \text{otherwise,} \end{cases}$$

where the notation $\hat{x}_i$ means the removal of the vertex x_i.

It has been proved that $\partial^2 = 0$ in [**3**, Lemma 2.11]. *The magnitude homology* $MH_{k,\ell}(G)$ of a graph G is defined as the homology group $H_k(MC_{*,\ell})$. For convenience, we define the *length function* L by

$$\mathrm{L}(\mathbf{x}) := \textstyle\sum_{i=0}^{k-1} d(x_i, x_{i+1})$$

for $\mathbf{x} \in V(G)^{k+1}$, and we call it the *length* of $\mathbf{x}$. The condition $d(x_{i-1}, x_{i+1}) = d(x_{i-1}, x_i) + d(x_i, x_{i+1})$ in Definition 2.8 is equivalent to the condition

$$\mathrm{L}(x_0, \dots, \hat{x}_i, \dots, x_k) = \mathrm{L}(x_0, \dots, x_k).$$

By definition, we have the following proposition.

Proposition 2.9. *For $\ell \geqslant 0$, we have the direct sum decomposition of a magnitude chain complex*

$$MC_{*,\ell}(G) = \bigoplus_{a,b \in V(G)} MC_{*,\ell}(a,b),$$

where $MC_{*,\ell}(a,b)$ is the subcomplex of $MC_{*,\ell}(G)$ generated by sequences which start at a and end at b.

Hence the computation of the magnitude homology of a graph G reduces to the computation of each (a,b)-component. We define a subsequence of a sequence as follows.

Definition 2.10. Let $\mathbf{x} = (x_0, \ldots, x_k)$ be a sequence, and $\mathbf{y} = (y_0, \ldots, y_{k'})$ a tuple. We call the tuple $\mathbf{y}$ a *subsequence* of $\mathbf{x}$ if there exist integers $0 = i_0 < \cdots < i_{k'} = k$ such that $x_{i_j} = y_j$ for each $0 \leqslant j \leqslant k'$. When $\mathbf{y}$ is a subsequence of $\mathbf{x}$, we denote this by $\mathbf{y} \prec \mathbf{x}$.

Note that a subsequence need not to be a sequence.

Definition 2.11. For a subsequence $\mathbf{y} = (x_0, x_{j_1}, \ldots, x_{j_{k'}}, x_k) \prec \mathbf{x}$, we call a set $\{j_1, \ldots, j_{k'}\}$ *the indices* of $\mathbf{y} \prec \mathbf{x}$.

3. Computation for trees

In this section, we compute the magnitude homology of a tree which is known in [**3**, Corollary 6.8] using simplicial homology.

Definition 3.1. Let $k \geqslant 1$. We call a sequence $(x_0, x_1, \ldots, x_k) \in V(G)^{k+1}$ *a path* in a graph G if it satisfies

$$d(x_i, x_{i+1}) = 1,$$

for every $0 \leqslant i \leqslant k-1$. For vertices $a, b \in V(G)$, we denote by $P_{\leqslant \ell}(a,b)$ the set of all paths $(x_0, \ldots, x_k)$ in G satisfying

$$\begin{cases} x_0 = a, x_k = b \\ \mathrm{L}(x_0, \ldots, x_k) = k \leqslant \ell. \end{cases}$$

Note that, for each sequence $\mathbf{x} = (x_0, \ldots, x_k)$, there exists a shortest path which passes through the vertices $x_0, \ldots, x_k$ in this order. We call such a shortest path *a path of* $\mathbf{x}$. If G is a tree, a path of $\mathbf{x}$ is unique for each sequence $\mathbf{x}$. Let G be a tree. For a path $\mathbf{x} = (a, x_1, \ldots, x_{k-1}, b) \in V(G)^{k+1}$, we denote by $MC_{k,\ell}(\mathbf{x})$ the submodule of $MC_{k,\ell}(a,b)$ generated by sequences whose paths coincide with $\mathbf{x}$. Clearly, $MC_{*,\ell}(\mathbf{x})$ is a subcomplex of $MC_{*,\ell}(a,b)$ since each $\partial_i(\mathbf{x})$ is 0 or has $\mathbf{x}$ as its path.

Proposition 3.2. *Let G be a tree. We have the following direct sum decomposition*

$$MC_{*,\ell}(a,b) = \bigoplus_{\mathbf{x} \in P_{\leqslant \ell}(a,b)} MC_{*,\ell}(\mathbf{x}),$$

for each $a, b \in V(G)$ and $\ell \geqslant 1$.

Proof. Since we have seen that each sequence belongs to the unique component of the decomposition, it is sufficient to see that $\partial\mathbf{y} \in MC_{k-1,\ell}(\mathbf{x})$ for $\mathbf{y} \in MC_{k,\ell}(\mathbf{x})$. Let $\mathbf{y} = (y_0, \dots, y_k) \in MC_{k,\ell}(\mathbf{x})$. Then we have

$$\partial\mathbf{y} = \sum_{\substack{\mathrm{L}(y_0,\dots,\hat{y}_i,\dots,y_k)=\ell \\ 1\leqslant i\leqslant k-1}} (-1)^i (y_0, \dots, \hat{y}_i, \dots, y_k).$$

Since the path of each sequence $(y_0, \dots, \hat{y}_i, \dots, y_k)$ is unique, it must coincide with $\mathbf{x}$ if the length $\mathrm{L}(y_0, \dots, \hat{y}_i, \dots, y_k)$ is preserved. Therefore we obtain that $\partial\mathbf{y} \in MC_{k-1,\ell}(\mathbf{x})$. □

In the following, we will construct a pair of simplicial complexes whose associated chain complex is isomorphic to the magnitude chain complex $MC_{*,\ell}(\mathbf{x})$ for each $\ell \geqslant 3$ and for each path $\mathbf{x}$ in G. For a path $\mathbf{x} = (x_0, \dots, x_\ell)$, we use the notion of the *frame* defined in [**4**, Definition 3.3], that is a subsequence $\varphi(\mathbf{x}) = (x_0, x_{i_1}, \dots, x_{i_m}, x_\ell) \prec \mathbf{x}$ defined as the maximal tuple satisfying

$$\begin{cases} 0 < i_s < \ell, \\ d(x_{i_s-1}, x_{i_s+1}) < d(x_{i_s-1}, x_{i_s}) + d(x_{i_s}, x_{i_s+1}), \end{cases}$$

for every $1 \leqslant s \leqslant m$. If G is a tree, it turns out that $\varphi(\mathbf{x})$ consists of all "turning points" of $\mathbf{x}$ and end points by the following lemma.

Lemma 3.3. *Let G be a tree, and let $\mathbf{x} = (x_0, \dots, x_\ell)$ be a path in G. For every $1 \leqslant i \leqslant \ell - 1$, we have $(x_0, x_i, x_\ell) \prec \varphi(\mathbf{x})$ if and only if $x_{i-1} = x_{i+1}$.*

Proof. Every consecutive three points of a path in a tree must have either of the configuration of Figure 1. Thus we have $x_{i-1} = x_{i+1}$ if and only if the triangle inequality

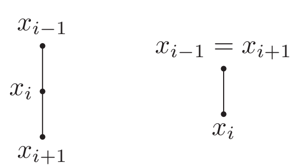

Figure 1: Consecutive three points in a tree.

is not an equality. □

Let $\Delta^{\ell-2}$ be the standard $(\ell-2)$-simplex $P(\{1, \dots, \ell-1\}) \setminus \emptyset$. For a path $\mathbf{x} = (x_0, \dots, x_\ell)$ and its subsequence $\varphi(\mathbf{x}) = (x_0, x_{i_1}, \dots, x_{i_m}, x_\ell)$, we define a subset $\Delta_{\mathbf{x}} \subset \Delta^{\ell-2}$ by

$$\Delta_{\mathbf{x}} = \{\sigma \mid \{i_1, \dots, i_m\} \not\subset \sigma \in \Delta^{\ell-2}\}.$$

For every $\sigma \in \Delta_{\mathbf{x}}$, any subset $\sigma' \subset \sigma$ is a simplex of $\Delta_{\mathbf{x}}$, which implies that $\Delta_{\mathbf{x}}$ is a subcomplex of $\Delta^{\ell-2}$.

Proposition 3.4. *There exists a chain isomorphism*

$$s^{-2}MC_{*,\ell}(\mathbf{x}) \cong (C_*(\Delta^{\ell-2}, \Delta_{\mathbf{x}}), -\partial).$$

Proof. Note that every sequence $\mathbf{y}$ belonging to $s^{-2}MC_{*,\ell}(\mathbf{x})$ is a subsequence of $\mathbf{x}$, that is

$$\mathbf{y} = (x_0, x_{j_1}, \dots, x_{j_k}, x_\ell) \prec \mathbf{x}.$$

We define a homomorphism

$$t\colon s^{-2}MC_{*,\ell}(\mathbf{x}) \longrightarrow C_*(\Delta^{\ell-2}, \Delta_{\mathbf{x}})$$

by sending each sequence to its indices (Definition 2.11)

$$(x_0, x_{j_1}, \dots, x_{j_k}, x_\ell) \longmapsto \{j_1, \dots, j_k\}$$

and extending it linearly. We can easily see that this is well defined by the definitions. We show that the homomorphism t is bijective and is a chain map. First we show its injectivity. Suppose that we have

$$t\left(\sum_{\alpha=1}^{N} c_\alpha (x_0, x_{j_1^\alpha}, \dots, x_{j_k^\alpha}, x_\ell)\right) = 0.$$

In the case that

$$\{j_1^\alpha, \dots, j_k^\alpha\} = \{j_1^{\alpha'}, \dots, j_k^{\alpha'}\}$$

for every $1 \leqslant \alpha, \alpha' \leqslant N$, it is clear that

$$\sum_{\alpha=1}^{N} c_\alpha (x_0, x_{j_1^\alpha}, \dots, x_{j_k^\alpha}, x_\ell) = 0.$$

In general, the index set $\{1, \dots, N\}$ can be decomposed into pairwise disjoint subsets $A_0, \dots, A_M$ such that $\alpha, \alpha' \in A_m$ implies that

$$\{j_1^\alpha, \dots, j_k^\alpha\} = \{j_1^{\alpha'}, \dots, j_k^{\alpha'}\},$$

for every $0 \leqslant m \leqslant M$. Hence we obtain

$$\sum_{\alpha=1}^{N} c_\alpha (x_0, x_{j_1^\alpha}, \dots, x_{j_k^\alpha}, x_\ell) = \sum_{m=0}^{M} \sum_{\alpha \in A_m} c_\alpha (x_0, x_{j_1^\alpha}, \dots, x_{j_k^\alpha}, x_\ell) = 0,$$

which implies that t is injective. Next we show its surjectivity. By definition, $[\{j_1, \dots, j_k\}] \neq 0 \in C_*(\Delta^{\ell-2}, \Delta_{\mathbf{x}})$ implies that

$$\varphi(\mathbf{x}) \prec (x_0, x_{j_1}, \dots, x_{j_k}, x_\ell).$$

By the definition of $\Delta_{\mathbf{x}}$, the paths of $\varphi(\mathbf{x})$ and $(x_0, x_{j_1}, \dots, x_{j_k}, x_\ell)$ coincide, which implies that

$$(x_0, x_{j_1}, \dots, x_{j_k}, x_\ell) \in MC_{k,\ell}(\mathbf{x}).$$

Hence we see that t is surjective. Finally, the following calculation shows that t is a chain map:

$$\begin{aligned}\partial t\left(\sum_{\alpha} c^{\alpha}(x_0, x_{j_1}^{\alpha}, \dots, x_{j_k}^{\alpha}, x_\ell)\right) &= \partial\left(\sum_{\alpha} c^{\alpha}\{j_1^{\alpha}, \dots, j_k^{\alpha}\}\right) \\ &= \sum_{\alpha} c^{\alpha} \sum_{\substack{0<i\leqslant k \\ \{i_1,\dots,i_m\}\subset\{j_1^{\alpha},\dots,\hat{j_i}^{\alpha},\dots,j_k^{\alpha}\}}} (-1)^{i-1}\{j_1^{\alpha}, \dots, \hat{j_i}^{\alpha}, \dots, j_k^{\alpha}\} \\ &= \sum_{\alpha} c^{\alpha} \sum_{\substack{0<i\leqslant k \\ d(x_{j_{i-1}}^{\alpha}, x_{j_i}^{\alpha})+d(x_{j_i}^{\alpha}, x_{j_{i+1}}^{\alpha})= \\ d(x_{j_{i-1}}^{\alpha}, x_{j_{i+1}}^{\alpha})}} (-1)^{i-1}\{j_1^{\alpha}, \dots, \hat{j_i}^{\alpha}, \dots, j_k^{\alpha}\} \\ &= t\left(-\partial\sum_{\alpha} c^{\alpha}(x_0, x_{j_1}^{\alpha}, \dots, x_{j_k}^{\alpha}, x_\ell)\right). \qquad \square\end{aligned}$$

To compute the homology of $C_*(\Delta^{\ell-2}, \Delta_{\mathbf{x}})$, we use the homology exact sequence

$$\cdots \longrightarrow H_k(\Delta^{\ell-2}) \longrightarrow H_k(\Delta^{\ell-2}, \Delta_{\mathbf{x}}) \longrightarrow H_{k-1}(\Delta_{\mathbf{x}}) \longrightarrow H_{k-1}(\Delta^{\ell-2}) \longrightarrow \cdots.$$

Since $\Delta^{\ell-2}$ is contractible, we have $H_k(\Delta^{\ell-2}, \Delta_{\mathbf{x}}) \cong H_{k-1}(\Delta_{\mathbf{x}})$ while $k > 1$.

Now we determine the homotopy type of $\Delta_{\mathbf{x}}$. When $m = 0$, we have $\Delta_{\mathbf{x}} = \emptyset$. When $m = \ell - 1$, we can see that $\Delta_{\mathbf{x}}$ is homotopy equivalent to the sphere $S^{(\ell-3)}$. The following proposition shows that it is contractible in the other cases.

Proposition 3.5. *For every path* $\mathbf{x}$*, the complex* $\Delta_{\mathbf{x}}$ *is contractible when* $0 < m < \ell - 1$ *where* $\varphi(\mathbf{x}) = (x_0, x_{i_1}, \dots, x_{i_m}, x_\ell)$*.*

Proof. Note that any union of two contractible simplicial complexes with contractible intersection is contractible by a standard argument. Hence we can deduce that any finite union of contractible simplicial complexes is contractible if every subunion of pairwise intersections is contractible. The complex $\Delta_{\mathbf{x}}$ is a union of maximal faces $\{1, \dots, \hat{i}_j, \dots, \ell-1\}$ for $1 \leqslant j \leqslant m$ (see an illustration for $m = 2, 3$ in Figure 2). Hence

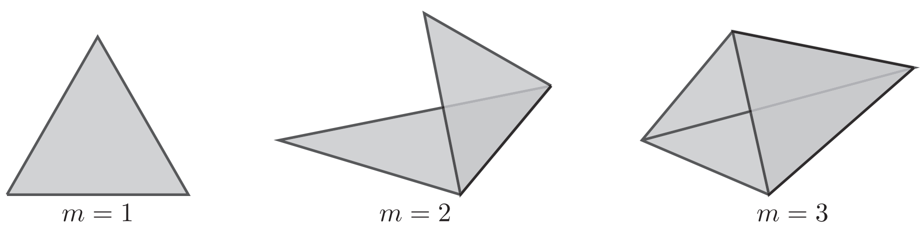

Figure 2: Case $l = 5$.

$\Delta_{\mathbf{x}}$ is contractible if every subunion of pairwise intersections $\{1, \dots, \hat{i}_j, \dots, \hat{i}_k, \dots, \ell-1\}$ for $1 \leqslant j < k \leqslant m$ is contractible. This is the case when every subunion of pairwise intersections $\{1, \dots, \hat{i}_j, \dots, \hat{i}_k, \dots, \hat{i}_{k'}, \dots, \hat{i}_{k''}, \dots, \ell-1\}$ for $1 \leqslant j < k < k' \leqslant k'' \leqslant m$ is contractible. By repeating this procedure, we can deduce that the complex $\Delta_{\mathbf{x}}$ is contractible when the simplex $\{1, \dots, \ell-1\} \setminus \{i_1, \dots, i_m\}$ is contractible, which is the case. $\square$

Now we can completely compute the magnitude homology of trees, which has been obtained in [**3**, Corollary 6.8].

Theorem 3.6. *Let G be a tree, and let $k, \ell \geqslant 3$ be integers. Then we have*

$$MH_{k,\ell}(G) = \begin{cases} \mathbb{Z}^{2\#E(G)}, & k = \ell, \\ 0, & k \neq \ell, \end{cases}$$

where $\#E(G)$ denotes the cardinality of the edge set $E(G)$.

Proof. By Proposition 2.9 and 3.2, we have

$$MH_{k,\ell}(G) = \bigoplus_{a,b \in V(G)} \bigoplus_{\mathbf{x} \in P_{\leqslant \ell}(a,b)} MH_{k,\ell}(\mathbf{x}).$$

If $\mathbf{x}$ is a path such that $\varphi(\mathbf{x})$ has less than $(\ell+1)$ points, then the $\mathbf{x}$-component $MH_{k,\ell}(\mathbf{x})$ is trivial by Proposition 3.4 and 3.5. In case $\varphi(\mathbf{x})$ has $(\ell+1)$ points, we have that

$$x_0 = x_2 = x_4 = \cdots \text{ and } x_1 = x_3 = x_5 = \cdots$$

by Lemma 3.3. We can easily see that the amount of such paths is twice the cardinality of the edge set $E(G)$. Then by Proposition 3.4, we have

$$\begin{aligned} MH_{\ell,\ell}(G) &= \bigoplus_{\{x_0,x_1\} \in E(G)} MH_{\ell,\ell}(x_0, x_1, x_0, \dots) \oplus MH_{\ell,\ell}(x_1, x_0, x_1, \dots) \\ &\cong \bigoplus_{\{x_0,x_1\} \in E(G)} H_{\ell-3}(S^{(\ell-3)}) \oplus H_{\ell-3}(S^{(\ell-3)}) \\ &\cong \mathbb{Z}^{2\#E(G)}. \end{aligned}$$

□

4. Theory for general graphs

For general graphs, we cannot decompose magnitude homology indexed by paths as in the case of trees, since sequences may have more than one shortest path. Hence we develop a method to compute each (a,b)-component in a similar way as in the tree case. Let G be a connected graph and let $a, b \in V(G)$. We fix an integer $\ell \geqslant 3$.

Definition 4.1. Let $K_\ell(a,b)$ be the set whose elements are subsets

$$\{(x_{i_1}, i_1), \dots, (x_{i_k}, i_k)\} \subset V(G) \times \{1, 2, \dots, \ell-1\}$$

with $1 \leqslant i_1 < \cdots < i_k \leqslant \ell - 1$ such that there exists a path

$$(a, x_1, \dots, x_{\ell'-1}, b) \in P_{\leqslant \ell}(a,b)$$

with $(a, x_{i_1}, \dots, x_{i_k}, b) \prec (a, x_1, \dots, x_{\ell'-1}, b)$.

For simplicity, we abbreviate $\{(x_{i_1}, i_1), \dots, (x_{i_k}, i_k)\}$ to $\{x_{i_1}, \dots, x_{i_k}\}$ if there is no confusion. The set $K_\ell(a,b)$ is a simplicial complex since

$$\{x_{j_1},\dots,x_{j_{k'}}\}\subset\{x_{i_1},\dots,x_{i_k}\}\in K_l(a,b)$$

implies that there exists a path $(a,x_1,\dots,x_{\ell'-1},b)\in P_{\leqslant\ell}(a,b)$ with

$$(a,x_{j_1},\dots,x_{j_{k'}},b)\prec(a,x_{i_1},\dots,x_{i_k},b)\prec(a,x_1,\dots,x_{\ell'-1},b),$$

that is

$$\{x_{j_1},\dots,x_{j_{k'}}\}\in K_\ell(a,b).$$

Clearly, the complex $K_{\ell-1}(a,b)$ is a subcomplex of $K_\ell(a,b)$. We also define a subcomplex $K'_\ell(a,b)\subset K_\ell(a,b)$ by

$$K'_\ell(a,b):=\{\{x_{i_1},\dots,x_{i_k}\}\in K_\ell(a,b)\mid \mathrm{L}(a,x_{i_1},\dots,x_{i_k},b)\leqslant\ell-1\}.$$

Our goal is to construct an isomorphism between $s^{-2}MC_{*,\ell}(a,b)$ and the quotient chain complex $C_*(K_\ell(a,b),K'_\ell(a,b))$. We assume that $d(a,b)\leqslant\ell$, since we have $K_\ell(a,b)=\emptyset$ for $d(a,b)>\ell$.

Lemma 4.2. *Let $\{x_{i_1},\dots,x_{i_k}\}$ and $\{y_{j_1},\dots,y_{j_k}\}$ be simplices of $K_\ell(a,b)$. If we have*

$$\mathrm{L}(a,x_{i_1},\dots,x_{i_k},b)=\mathrm{L}(a,y_{j_1},\dots,y_{j_k},b)=\ell$$

and $x_{i_s}=y_{j_s}$ for $1\leqslant s\leqslant k$, then we have

$$i_s=j_s,$$

for $1\leqslant s\leqslant k$.

Proof. By definition, there is a path

$$(a,x_1,\dots,x_{\ell-1},b)\in P_{\leqslant\ell}(a.b)$$

with

$$(a,x_{i_1},\dots,x_{i_k},b)\prec(a,x_1,\dots,x_{\ell-1},b).$$

For $1\leqslant s\leqslant k$, we have

$$i_s=i_{s-1}+d(x_{i_{s-1}},x_{i_s})=\sum_{n=0}^{s-1}d(x_{i_n},x_{i_{n+1}}).$$

Similarly we also have

$$j_s=\sum_{n=0}^{s-1}d(y_{j_n},y_{j_{n+1}}),$$

for $1\leqslant s\leqslant k$. Since we have $x_{i_s}=y_{j_s}$ for $1\leqslant s\leqslant k$, we obtain $i_s=j_s$. □

Next we define a homomorphism

$$t\colon (C_*(K_\ell(a,b),K'_\ell(a,b)),-\partial)\longrightarrow s^{-2}MC_{*+2,\ell}(a,b)$$

by

$$[\{x_{i_1},\dots,x_{i_k}\}]\longmapsto(a,x_{i_1},\dots,x_{i_k},b).$$

It is well defined since $\partial_i(a,x_{i_1},\dots,x_{i_k},b)$ vanishes exactly when ∂_i shortens the length of the sequence $(a,x_{i_1},\dots,x_{i_k},b)$, which is equivalent to saying that $\partial_i\{x_{i_1},\dots,x_{i_k}\}\in C_*(K'_\ell(a,b))$.

Theorem 4.3. *For $\ell \geqslant 3$, the above homomorphism*

$$t\colon (C_*(K_\ell(a,b), K'_\ell(a,b)), -\partial) \longrightarrow s^{-2}MC_{*+2,\ell}(a,b)$$

is a chain map. Furthermore, it is an isomorphism for $* \geqslant 0$.

Proof. We can show that the homomorphism t is a chain map by the same computation as in the proof of Proposition 3.4. Next we show that t is injective. Suppose that

$$t\left(\sum_{\alpha=1}^{N} c_\alpha[\{x^\alpha_{j_1}, \dots, x^\alpha_{j_k}\}]\right) = 0.$$

We can assume that

$$[(x_0, x^\alpha_{j_1}, \dots, x^\alpha_{j_k}, x_l)] = [(x_0, x^{\alpha'}_{j_1}, \dots, x^{\alpha'}_{j_k}, x_l)] \neq 0$$

for any $1 \leqslant \alpha, \alpha' \leqslant N$ as in the proof of Proposition 3.4. By Lemma 4.2, it turns out that their indices coincide. Hence we have

$$\sum_{\alpha=1}^{N} c_\alpha\{x^\alpha_{j_1}, \dots, x^\alpha_{j_k}\} = 0,$$

which implies that t is injective. To prove surjectivity, let $(a, x_{i_1}, \dots, x_{i_{k+1}}, b)$ be a base element of $MC_{k+2,l}(a,b)$. Since we have

$$\mathrm{L}(a, x_{i_1}, \dots, x_{i_{k+1}}, b) = \ell,$$

there exists a path

$$(a, x_1, \dots, x_{\ell-1}, b) \in P_{\leqslant \ell}(a,b)$$

such that

$$(a, x_{i_1}, \dots, x_{i_{k+1}}, b) \prec (a, x_1, \dots, x_{\ell-1}, b).$$

Hence we obtain

$$\{x_{i_1}, \dots, x_{i_{k+1}}\} \in K_\ell(a,b),$$

and we also have

$$\{x_{i_1}, \dots, x_{i_{k+1}}\} \notin K'_\ell(a,b).$$

Therefore t is an isomorphism on the module of each dimension. □

By Theorem 4.3, we can compute the most part of the magnitude homology for arbitrary graphs as follows.

Corollary 4.4. *We have*

$$MH_{k,\ell}(a,b) \cong H_{k-2}(K_\ell(a,b), K'_\ell(a,b))$$

for $k, \ell \geqslant 3$. Moreover, for $k = 2$, we also have

$$MH_{2,\ell}(a,b) \cong \begin{cases} H_0(K_\ell(a,b), K'_\ell(a,b)) & \text{if } d(a,b) < \ell, \\ \tilde{H}_0(K_\ell(a,b)) & \text{if } d(a,b) = \ell, \end{cases}$$

where $\tilde{H}_*$ denotes the reduced homology group.

Proof. Suppose that $\ell, k \geqslant 3$. Since the homology of $C_*(K_\ell(a,b), K'_\ell(a,b))$ does not depend on the sign of the boundary map, we have

$$MH_{k,\ell}(a,b) \cong H_{k-2}(K_\ell(a,b), K'_\ell(a,b))$$

from Theorem 4.3. Hence the statement follows. In case $k = 2$, we consider a sequence of submodules

$$\mathrm{Im}\ \partial_3 \subset \mathrm{Ker}\ \partial_2 \subset MC_{2,\ell}(a,b),$$

where ∂_* denotes differentials of $MC_{*,\ell}(a,b)$. Then we have a short exact sequence

$$0 \longrightarrow \mathrm{Ker}\ \partial_2/\mathrm{Im}\ \partial_3 \longrightarrow MC_{2,\ell}(a,b)/\mathrm{Im}\ \partial_3 \longrightarrow MC_{2,\ell}(a,b)/\mathrm{Ker}\ \partial_2 \longrightarrow 0,$$

which is isomorphic to

$$0 \longrightarrow MH_{2,\ell}(a,b) \longrightarrow H_0(K_\ell(a,b), K'_\ell(a,b)) \longrightarrow \mathrm{Im}\ \partial_2 \longrightarrow 0.$$

Then, by the definition of ∂_2, we have

$$\mathrm{Im}\ \partial_2 \cong \begin{cases} 0 & \text{if } d(a,b) < \ell, \\ \mathbb{Z} & \text{if } d(a,b) = \ell. \end{cases}$$

Hence we obtain

$$H_0(K_\ell(a,b), K'_\ell(a,b)) \cong \begin{cases} MH_{2,\ell}(a,b) & \text{if } d(a,b) < \ell, \\ MH_{2,\ell}(a,b) \oplus \mathbb{Z} & \text{if } d(a,b) = \ell. \end{cases}$$

Then we have

$$MH_{2,\ell}(a,b) \cong H_0(K_\ell(a,b), K'_\ell(a,b))$$

for $d(a,b) < \ell$. If $d(a,b) = \ell$, we see

$$H_0(K_\ell(a,b), K'_\ell(a,b)) = H_0(K_\ell(a,b))$$

since $K'_\ell(a,b) = \emptyset$. Therefore, it also holds that

$$MH_{2,\ell}(a,b) \cong \tilde{H}_0(K_\ell(a,b)).$$ □

Finally, we give an example of computation using the above method.

Example 4.5. The graph Sq_2 [**7**, pp. 14–15] is described in Figure 3. We compute the magnitude homology $MH_{*,4}(Sq_2)$. The magnitude chain complex can be decomposed

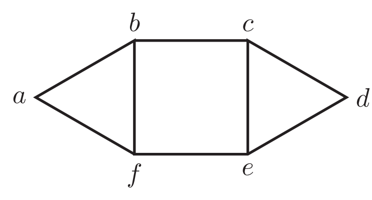

Figure 3: The graph Sq_2.

into 36 components which are classified into 8 types

$$(a,a),(a,b),(a,c),(a,d),(b,b),(b,c),(b,f),(b,e).$$

First, consider the (a,a)-type components. We have two (a,a)-type components, the (a,a)- and (d,d)-component. These components have the same chain complexes by symmetry. The paths of length 4 belonging to the (a,a)-component are

$$(a,b,a,b,a),(a,b,a,f,a),(a,b,c,b,a),(a,b,f,b,a),$$
$$(a,f,a,f,a),(a,f,a,b,a),(a,f,e,f,a),(a,f,b,f,a).$$

The paths whose length are less than 4 are not related to the homology since they vanish by the quotient operation. Assigning $(4-2)=2$-simplices for these 8 paths, we can construct the simplicial complex $K_4(a,a)$ and the subcomplex $K'_4(a,a)$ as shown in Figure 4. By Corollary 4.4, we have

$$MH_{k,4}(a,a) \cong \tilde{H}_{k-2}(|K_4(a,a)|/|K'_4(a,a)|) \text{ for } k \geqslant 2.$$

Since $|K_4(a,a)|/|K'_4(a,a)|$ is homotopy equivalent to $S^2 \vee S^2 \vee S^2 \vee S^2 \vee S^2 \vee S^2$, we have

$$MH_{k,4}(a,a) \cong \begin{cases} \mathbb{Z}^6 & (k=4), \\ 0 & (k \neq 4, k \geqslant 2). \end{cases}$$

In addition, we can also identify the generators of the magnitude homology from Figure 4.

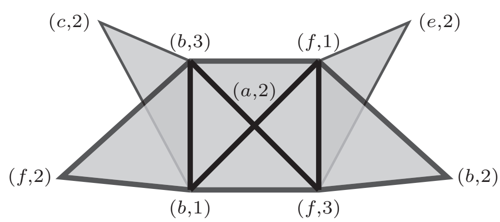

Figure 4: The 2-dimensional simplicial complex $K_4(a,a)$ and the subcomplex $K'_4(a,a)$ (bold part).

The (a,d)-type is one of the types which give non-trivial elements to the 3rd magnitude homology. There are two (a,d)-type components, the (a,d)-component and the (d,a)-component. The paths belonging to the (a,d)-component are

$$(a,b,c,e,d),(a,b,f,e,d),(a,f,b,c,d),(a,f,e,c,d).$$

The resulting simplicial complexes $K_4(a,d)$ and $K'_4(a,d)$ are shown in Figure 5. $|K_4(a,d)|/|K'_4(a,d)|$ is homotopy equivalent to $S^1 \vee S^1$, so we have

$$MH_{k,4}(a,d) \cong \begin{cases} \mathbb{Z}^2 & (k=3), \\ 0 & (k \neq 3, k \geqslant 2). \end{cases}$$

We can also compute the other components, and the result is following:

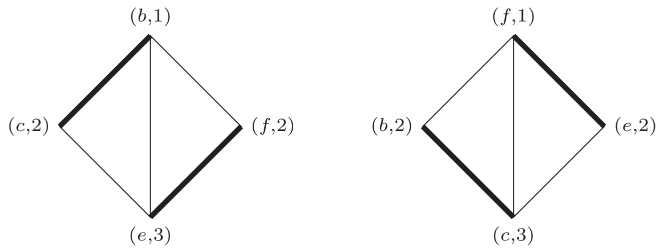

Figure 5: The simplicial complex $K_4(a,d)$ and the subcomplex $K_4'(a,d)$ (bold part).

rank	(a,a)	(a,b)	(a,c)	(a,d)	(b,b)	(b,c)	(b,f)	(b,e)
$k=2$	0	0	0	0	0	0	0	0
$k=3$	0	0	8	4	0	0	0	0
$k=4$	12	40	0	0	32	0	20	8

The rank of the 3rd magnitude homology is 12 and that of the 4th magnitude homology is 112, so these coincide with the result [**7**, Table 3].

References

[**1**] K. Gomi, Magnitude homology of geodesic space, arXiv:1902.07044, 2019.

[**2**] Y. Gu, Graph magnitude homology via algebraic Morse theory, arXiv:1809.07240, 2018.

[**3**] R. Hepworth and S. Willerton, Categorifying the magnitude of a graph, *Homology Homotopy Appl.* **19** (2017), 31–60.

[**4**] R. Kaneta and M. Yoshinaga, Magnitude homology of metric spaces and order complexes, arXiv:1803.04247, 2018.

[**5**] T. Leinster, The magnitude of a graph, *Math. Proc. Cambridge Philos. Soc.* **166** (2019), 247–264.

[**6**] T. Leinster, The magnitude of metric spaces, *Doc. Math.* **18** (2013), 857–905.

[**7**] R. Sazdanovic and V. Summers, Torsion in the magnitude homology of graphs, arXiv:1912.13483, 2019.

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