Homology, Homotopy and Applications, vol. 23(1), 2021, pp.367–386

THE LOCAL HYPERBOLICITY OF $\mathbf{A}_n^2$-COMPLEXES

ZHONGJIAN ZHU AND JIANZHONG PAN

(*communicated by John R. Klein*)

Abstract

In this paper, as an application of results about the decomposability of smash product of $\mathbf{A}_n^2$ complexes [**34**], we obtain the local hyperbolicity of $\mathbf{A}_n^2$-complexes by an analysis of decomposition of loop suspension.

1. Introduction

Describing completely the homotopy groups of finite complexes seems to be intractable. Progress has been made on the study of the asymptotic growth of homotopy groups. In [**11**], Felix–Halperin–Thomas proved that the rational homotopy groups of 1-connected finite complexes are either finite dimensional or grow exponentially. In [**18, 28, 30**], the authors showed that there are infinitely many p-torsion classes in homotopy groups of any simply connected finite complexes with nontrivial mod-p homology. The families of infinitely many higher order torsions in homotopy groups of Moore spaces were extensively studied in [**4, 5, 6, 7, 20**] and so on. Then in [**14**], Huang and Wu study the asymptotic behavior of the p-primary part of the homotopy groups of simply connected finite p-local complexes.

Definition 1.1. A p-local complex X is called $\mathbb{Z}/p^i$-hyperbolic, if the number of $\mathbb{Z}/p^i$-summands in π_*X has exponential growth, i.e.,

$$\varinjlim_{n} \frac{\ln t_n}{n} > 0, \qquad t_n = \sharp\{\mathbb{Z}/p^i\text{-summands in } \bigoplus_{j\leqslant n} \pi_j X\}.$$

A p-local complex X is called p-hyperbolic (hyperbolic mod p), if the p primary torsion part of π_*X has exponential growth, i.e.,

$$\varinjlim_{n} \frac{\ln T_n}{n} > 0, \qquad T_n = \sharp\{\mathbb{Z}/p^i\text{-summands in } \bigoplus_{j\leqslant n} \pi_j X, r \geqslant 1\}.$$

Henn [**13**] considered a different asymptotic problem, which can be measured by the radius of the convergence of the power series $P_\pi(X) := \sum \dim_{\mathbb{Z}/p}(\pi_n(X)\otimes \mathbb{Z}/p)x^n$. Henn conjectured that the radius of $P_\pi(X)$ equals the radius of another power series $P_H(X) := \sum \dim_{\mathbb{Z}/p}(H_n(\Omega X;\mathbb{Z}/p))x^n$ and Iriye gave a partial answer to this in [**15**].

Received March 23, 2020, revised July 12, 2020; published on December 9, 2020.
2010 Mathematics Subject Classification: 55P10, 55P35, 55Q05.
Key words and phrases: $\mathbf{A}_n^2$-complex, hyperbolicity, loop suspension, decomposition, homotopy group.
Article available at `http://dx.doi.org/10.4310/HHA.2021.v23.n1.a19`

Given any simply connected finite complex X which is a suspension, the following questions are concerned.

1) if there are infinitely many $\mathbb{Z}/p^i$-summands in the homotopy groups π_*X, is then X $\mathbb{Z}/p^i$-hyperbolic?

2) if there are infinitely many nontrivial summands in the p-primary torsion of π_*X, is then X p-hyperbolic?

In [**14**], the authors show that for any prime p, the Moore space $M(\mathbb{Z}/p^r; n-1)$, $n \geqslant 3$, is $\mathbb{Z}/p^i$-hyperbolic for $i = r, r+1$, while $M(\mathbb{Z}/2; n-1)$, $n \geqslant 3$, is $\mathbb{Z}/8$-hyperbolic. They gave two main conditions for a space X to be hyperbolic (see Theorem 5.3 and Theorem 5.4 in [**14**]):

1) $\bar{Q}_k^{\max}(X)$ contains a wedge summand $\Sigma^* X$ for infinitely many k;

2) there exist k such that $X^{\wedge k} \simeq \Sigma^* X \vee \Sigma^* X \vee \text{other space}$,

where the functors $\bar{Q}_k^{\max}$, together with $\bar{A}^{\min}$, are introduced by Selick and Wu to establish a functorial homotopy decomposition of the loop suspension of any path-connected CW complex [**25, 26**]:

$$\Omega\Sigma X \simeq \Omega(\bigvee_{n=2}^{\infty} \bar{Q}_n^{\max}(X)) \times \bar{A}^{\min}(X).$$

However, the hyperbolicity of general Moore space is proved by previously known explicit product decomposition of loop space of Moore space. Since it is difficult to compute $\bar{Q}_k^{\max}(X)$ for large number k, in Theorem 3.3 of [**14**], the hyperbolicity of $M(\mathbb{Z}/2; n-1)$ is obtained by studying $L_{2k+1}(M(\mathbb{Z}/2; n-1))$ for infinitely many k, where the functor L_{2k+1} is defined in Section 2.

In the following, we denote the Moore space $M(\mathbb{Z}/p^r; n)$ with the only nontrivial homology group $H_n(M(\mathbb{Z}/p^r, n); \mathbb{Z}) \cong \mathbb{Z}/p^r$ by $M_{p^r}^n$.

In this article, we will prove the local hyperbolicity for more examples, namely, elementary Chang-complexes (described below) which were found by S C Chang when he classified indecomposable homotopy types in $\mathbf{A}_n^2 (n \geqslant 3)$ [**3**], where $\mathbf{A}_n^k$ is the homotopy category consisting of $(n-1)$-connected finite CW-complexes with dimension less than or equal to $n+k$ $(n \geqslant k+1)$. S C Chang showed that all indecomposable homotopy types in $\mathbf{A}_n^2 (n \geqslant 3)$ are spheres S^n, S^{n+1}, S^{n+2}, elementary Moore spaces $M_{p^r}^n$, $M_{p^r}^{n+1}$ (p is a prime, $r \in \mathbb{Z}^+$) and elementary Chang complexes listed below.

Elementary Chang complexes:

- $C_\eta^{n+2} = S^n \bigcup_\eta \mathbf{C}S^{n+1}$,
- $C^{n+2,s} = (S^n \vee S^{n+1}) \bigcup_{\binom{\eta}{2^s}} \mathbf{C}S^{n+1}$;
- $C_r^{n+2} = S^n \bigcup_{(2^r, \eta)} \mathbf{C}(S^n \vee S^{n+1})$;
- $C_r^{n+2,s} = (S^n \vee S^{n+1}) \bigcup_{\begin{pmatrix} 2^r, & \eta \\ 0, & 2^s \end{pmatrix}} \mathbf{C}(S^n \vee S^{n+1})$,

where η is the suspended Hopf map, i and q are canonical inclusion and projection respectively, $n, r, s \in \mathbb{Z}^+$, $n \geqslant 3$ and for finite CW-complexes X_i, Y_j $(i = 1, \dots, t, j = 1, \dots, s)$, the matrix

$$f := (f_{ij}) = \begin{pmatrix} f_{11} & \cdots & f_{1t} \\ \vdots & \vdots & \vdots \\ f_{s1} & \cdots & f_{st} \end{pmatrix} : \bigvee_{i=1}^{t} X_i \to \bigvee_{j=1}^{s} Y_j, \qquad f_{kl} \in [X_l, Y_k]$$

represents a map such that $p_{Y_i} f j_{X_j} \simeq f_{ij}$, with canonical inclusions and projections being denoted by j_{X_j} and p_{Y_i} respectively.

Theorem 1.2 (Main theorem I)**.** *The Chang complexes $C_r^{n+2,r}$ $(n \geqslant 4)$ are $\mathbb{Z}/2^i$-hyperbolic for $i = 1, r, r+1$.*

Theorem 1.3 (Main theorem II)**.** *Let C be one of Chang complexes C_η^{n+2}, C_r^{n+2}, $C^{n+2,s}$, $C_r^{n+2,s} (s \neq r)$ with $n \geqslant 4$; and let*

$$u_C = \begin{cases} 1, & C = C_\eta^{n+2}, \\ r, & C = C_r^{n+2}, \\ s, & C = C^{n+2,s}, \\ \min\{r, s\}, & C = C_r^{n+2,s}. \end{cases}$$

Then C is $\mathbb{Z}/2$-hyperbolic for $u_C = 1$ and $\mathbb{Z}/2^i$-hyperbolic $(i = 1, u_C, u_C + 1)$ for $u_C > 1$.

Theorem 1.2 and Theorem 1.3 have a few consequences for the hyperbolicity of an $\mathbf{A}_n^2$-complex. Next is one example:

Theorem 1.4 (Main theorem III)**.** *Let A be a complex in $\mathbf{A}_n^2$ $(n \geqslant 4)$, which is neither a sphere nor a contractible space under p-localization, then A is $\mathbb{Z}/p$-hyperbolic.*

Remark 1.5. Our results rely on the decomposition $\Omega\Sigma X \simeq \prod_j \Omega\Sigma L_{k_j}(X) \times A$ in Proposition 2.5, where X is a suspension. In other words, what we can obtain is the information of homotopy groups of double suspension. One can try to apply functorial decomposition where the condition on X is weakened, it is, however, difficult to compute the corresponding product factors.

The main observation of this paper is that the computation of $L_k(X)$ is often easier than the computation of $\bar{Q}_k^{\max}(X)$. We improve Theorem 3.3 of [**14**] on hyperbolicity of M_2^{n-1} by reducing the condition concerning $L_k(X)$ to the existence of some positive integer k such that $L_k(X)$ contains as wedge summand $\Sigma^{l_1} X \vee \Sigma^{l_2} X$, $l_1, l_2 \geqslant 1$. It is helpful to note that in our method, the exponential growth of the number of $\mathbb{Z}/p^i$-summands in $\pi_* X$ is obtained from the stable homotopy groups of the product factor $\prod_{W(t)} \Omega\Sigma^* X$ of $\Omega\Sigma X$ and the growth of the number $W(t)$ when the primes t increases, where $W(t) = \dim L_t(\bar{V})$ and $\bar{V}$ is a vector space with $\dim \bar{V} = 2$. However, the authors of [**14**] showed the exponential growth of the number of $\mathbb{Z}/p^i$-summands in $\pi_* X$ by studying the stable homotopy groups of the product factor $\prod_{2^s \cdot W(s)} \Omega\Sigma^* X$ of $\Omega\Sigma X$ and the growth of the number 2^s when s increases.

Conjecture 1.6. *Let A be a complex in $\mathbf{A}_n^k$ $(n \geqslant k+2)$, which is neither a sphere nor a contractible space under p-localization, then A is $\mathbb{Z}/p$-hyperbolic.*

In order to simplify the writing, we prove Theorem 1.2, Theorem 1.3 for $n = 4$, the case that $n > 4$ can be similarly obtained.

This article is organized as follows. Section 2 introduces some notations and basic properties. Section 3 gives some criteria for hyperbolicity of spaces. Section 4 proves some lemmas about the decomposition of smash products of Chang complexes. The proofs of Theorem 1.2, Theorem 1.3 and Theorem 1.4 are given in Section 5.

Acknowledgments

First author was partially supported by National Natural Science Foundation of China (Grant No. 11701430); second author were partially supported by National Natural Science Foundation of China (Grant No. 11971461).

2. Preliminaries

Let p be a prime, X be a path-connected, p-local CW-complex of finite type which is a suspension, nX and $X^{\wedge n}$ be the n-fold self-wedge product and smash product of X respectively. We say a space X is decomposable, if $X \simeq Y \vee Z$ with Y, Z which are not contractible. In the following, we will recall some general properties about the decomposition of $X^{\wedge n}$ from [**4**] and [**24**].

Let S_n denote the symmetric group on n letters. Since X is a suspension, the natural action of any element $\delta \in \mathbb{Z}_{(p)}[S_n]$ on $V^{\otimes n}$ with $V = \bar{H}_*(X; \mathbb{Z}/p)$ by permuting coordinates can be realized by a map

$$\delta\colon X^{\wedge n} \longrightarrow X^{\wedge n},$$

where $\mathbb{Z}_{(p)}[S_n]$ denote the group ring over the p-local integers $\mathbb{Z}_{(p)}$.

Start with $\beta_2 = 1 - (1, 2) \in \mathbb{Z}_{(p)}[S_n]$ and let

$$\beta_n = \beta_{n-1} \wedge id - (1, 2, 3, \dots, n)(\beta_{n-1} \wedge id).$$

From [**7**], if $(n, p) = 1$, then the elements $\frac{1}{n}\beta_n$ and $id - \frac{1}{n}\beta_n$ are orthogonal idempotents. Let $\mathrm{hocolim}_f\, X^{\wedge n}$ be the mapping telescope of the sequence of maps $X^{\wedge n} \xrightarrow{f} X^{\wedge n} \xrightarrow{f} \cdots$. Then $L_n(X) := \mathrm{hocolim}_{\frac{1}{n}\beta_n} X^{\wedge n}$ is a wedge summand of $X^{\wedge n}$.

Let $p_n\colon X^{\wedge n} \to L_n(X)$ be the projection and let $i_n\colon L_n(X) \hookrightarrow X^{\wedge n}$ be the canonical inclusion. We have

$$(\frac{1}{n}\beta_n)_* = i_{n*}p_{n*}\colon H_*(X^{\wedge n}; \mathbb{Z}/p) \to H_*(X^{\wedge n}; \mathbb{Z}/p).$$

Given a graded vector space V over a field $\mathbb{Z}/p$ where p is a prime. In the tensor algebra $T(V)$, let

$$L_n(V) := \mathrm{Span}_{\mathbb{Z}/p}\ \{\ [\dots[\ v_1, v_2], v_3] \dots, v_n]\ |\ v_i \in V\ \}.$$

Note that by the definition of graded Lie algebra in [**21**], at odd primes $[x, x] \neq 0$ in $L_2(V)$ if $|x|$ is odd. It is zero if $|x|$ is even.

Now let $V = \bar{H}_*(X; \mathbb{Z}/p)$ where X is a p-local suspended space, then $\bar{H}_*(\Omega\Sigma X; \mathbb{Z}/p) \cong T(V)$. From [**7**] and [**21**] we know that for $(n, p) = 1$,

$$\bar{H}_*(L_n(X); \mathbb{Z}/p) \cong L_n(\bar{H}_*(X; \mathbb{Z}/p)) = L_n(V).$$

Remark 2.1. If the generators of $\mathbb{Z}/2$-vector space V are given, then we can list all the generators of $L_n(V)$ by the "Lyndon words" [**23**]. For example, if V is a 4-dimensional $\mathbb{Z}/2$-vector space with generators a, b, c, d, then the generators of vector space $L_3(V)$ are the following 20 elements: $[a,[a,b]]$, $[a,[a,c]]$, $[a,[a,d]]$, $[a,[b,c]]$, $[a,[b,d]]$, $[a,[c,d]]$, $[b,[b,c]]$, $[b,[b,d]]$, $[b,[c,d]]$, $[c,[c,d]]$, $[[a,b],b]$, $[[a,c],b]$, $[[a,c],c]$, $[[a,d],b]$, $[[a,d],c]$, $[[a,d],d]$, $[[b,c],c]$, $[[b,d],c]$, $[[b,d],d]$, $[[c,d],d]$.

For an element $u \in \mathbb{Z}_{(p)}[S_n]$, let $\bar{u}$ denote the image of u under the natural quotient map from $\mathbb{Z}_{(p)}[S_n]$ to $\mathbb{Z}/p[S_n]$.

Let $1 = \Sigma_\alpha f_\alpha$ be an orthogonal decomposition of the identity element in $\mathbb{Z}/p[S_n]$ in terms of primitive idempotents. By Proposition 56.4 of [**1**], it can be lifted to an orthogonal decomposition of the identity element $1 = \Sigma_\alpha e_\alpha$ in $\mathbb{Z}_{(p)}[S_n]$ in terms of primitive idempotents $\{e_\alpha\}$ with $\bar{e}_\alpha = f_\alpha$. For each α, we take $e_\alpha(X) = \mathrm{hocolim}_{e_\alpha} X^{\wedge n}$. From [**24, 33**], we have

$$X^{\wedge n} \simeq \bigvee_\alpha e_\alpha(X). \tag{1}$$

Lemma 2.2. *Let e, e' be two idempotents in $\mathbb{Z}_{(p)}[S_n]$, $\bar{e}, \bar{e}'$ be conjugate to each other in $\mathbb{Z}/p[S_n]$. Then for any simply connected p-local space X, $e(X) \simeq e'(X)$.*

Proof. There is an invertible element $f \in \mathbb{Z}/p[S_n]$, such that $\bar{e} = f\bar{e}'f^{-1}$. Let $u, v \in \mathbb{Z}_{(p)}[S_n]$ such that $\bar{u} = f$, $\bar{v} = f^{-1}$. Consider the following maps

$$e(X) \xrightarrow{e'v} e'(X) \xrightarrow{eu} e(X) \xrightarrow{e'v} e'(X).$$

Then we get that $(eu)(e'v)$ (respectively $(e'v)(eu)$) induces identity endomorphism on $H_*(e(X);\mathbb{Z}/p)$ (respectively $H_*(e(X);\mathbb{Z}/p)$). By Proposition 2.7 of [**19**], $e'v$ and eu are homotopy equivalences. □

From Corollary 54.14 of [**2**], it is known that up to conjugation, the primitive idempotents f_λ in $\mathbb{Z}/p[S_n]$ are in one-to-one correspondence with isomorphism classes of irreducible left $\mathbb{Z}/p[S_n]$-modules, which are also in one-to-one correspondence with p-regular partitions $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_s)$ of n, where s is called the length of λ, by Theorem 7.1.14 of [**17**].

Let d_λ be the number of idempotents in the orthogonal decomposition of 1 in $\mathbb{Z}/p[S_n]$ which are conjugate to f_λ. Let $Q_\lambda(X) = \mathrm{hocolim}_{e_\lambda} X^{\wedge n}$, where $\bar{e_\lambda} = f_\lambda$. From Lemma 2.2, the decomposition (1) can be written as

$$X^{\wedge n} \simeq \bigvee_\lambda d_\lambda Q_\lambda(X),$$

where λ runs over all p-regular partitions of n.

Lemma 2.3. *d_λ equals the dimension of the corresponding simple $\mathbb{Z}/p$-module*

$$(\mathbb{Z}/p[S_n])f_\lambda \Big/ rad(\mathbb{Z}/p[S_n])f_\lambda\ .$$

Proof. From Theorem 4.12 and Theorem 11.5 of [**16**], any field is the splitting field for S_n, then above lemma is obtained by Theorem 62.1 and Theorem 61.13 of [**2**]. □

For $p=2$, $n=3$, the only 2-regular partitions of 3 are $(3,0)$, $(2,1)$. From [**22**], there is an orthogonal primitive idempotent decomposition $1=f_1+f_2+f_3$ of the identity element in $\mathbb{Z}/2[S_3]$, where $f_1=1+\sigma+\sigma^2$, $f_2=(1+\tau)(1+\tau\sigma)$, $f_3=(1+\tau\sigma)(1+\tau)$ are primitive elements and $\sigma=(123)$, $\tau=(12)\in S_3$. f_2, f_3 are related by conjugation in $\mathbb{Z}/2[S_3]$ since $f_2=(\sigma\tau)f_3(\sigma\tau)^{-1}$. We lift this decomposition up to $\mathbb{Z}_{(2)}[S_3]$, i.e., $1=e_1+e_2+e_3$, which is an orthogonal primitive idempotent decomposition of 1 in $\mathbb{Z}_{(2)}[S_3]$.

Hence

$$X^{\wedge 3}\simeq e_1(X)\vee e_2(X)\vee e_3(X)$$

with $e_2(X)\simeq e_3(X)$. As a result of this,

$$X^{\wedge 3}\simeq Q_{(3,0)}(X)\bigvee 2Q_{(2,1)}(X),$$

where 2-regular partitions $(3,0)$ and $(2,1)$ correspond to e_1 and e_2 respectively. Moreover, let $\beta_3'=(1+\tau)(1+\sigma)\in\mathbb{Z}/2[S_3]$, then $\frac{1}{3}\beta_3\in\mathbb{Z}_{(2)}[S_3]$ is a lift of β_3'. Since $\beta_3'=\sigma^{-1}f_2\sigma$, from Lemma 2.2, we have

$$L_3(X)\simeq e_2(X)\simeq Q_{(2,1)}(X). \tag{2}$$

For $p=2$, $n=5$, the only 2-regular partitions of 5 are $(5,0)$, $(4,1)$, $(3,2)$. From [**12**], there is an orthogonal primitive idempotent decomposition of the identity element in $\mathbb{Z}/2[S_5]$

$$1=f_0+\sum_{i=1}^{4}f_i'+\sum_{j=1}^{4}f_j'',$$

where $f_0=1+\varrho+\varrho^2+\varrho^3+\varrho^4$, $\varrho=(12345)\in S_5$, corresponds to the 2-regular partition (5,0) of 5; $f_i', i=1,2,3,4$ (resp. f_j'', $j=1,2,3,4$), are in the same conjugacy class which corresponds to the 2-regular partition (4,1) (resp. (3,2)) of 5. We lift this decomposition up to $\mathbb{Z}_{(2)}[S_5]$, i.e., $1=e_0+\sum\limits_{i=1}^{4}e_i'+\sum\limits_{j=1}^{4}e_j''$, which is an orthogonal primitive idempotent decomposition of 1 in $\mathbb{Z}_{(2)}[S_5]$ and then we get

$$X^{\wedge 5}\simeq Q_{(5,0)}(X)\bigvee 4Q_{(4,1)}(X)\bigvee 4Q_{(3,2)}(X), \tag{3}$$

where $Q_{(5,0)}(X)\simeq e_0(X)$, $Q_{(4,1)}(X)\simeq e_i'(X)$, $Q_{(3,2)}(X)\simeq e_j''(X)$, $i,j=1,2,3,4$.

Lemma 2.4.

$$L_5(X)\simeq Q_{(4,1)}(X)\bigvee Q_{(3,2)}(X).$$

Proof. For idempotent $\frac{1}{5}\beta_5\in\mathbb{Z}_{(2)}[S_5]$, from Theorem 3.4.1 of [**10**], in $\mathbb{Z}/2[S_5]$, the idempotent

$$\overline{\frac{1}{5}\beta_5}=\bar{\beta}_5=\delta_0 af_0a^{-1}+\sum_{i=1}^{4}\delta_i' af_i'a^{-1}+\sum_{j=1}^{4}\delta_j'' af_j''a^{-1},$$

where a is an invertible element in $\mathbb{Z}/2[S_5]$, $\delta_0,\delta_i',\delta_j''\in\{0,1\}$. It is easy to check that $\bar{\beta}_5$ is orthogonal to f_0 and f_0 is in the center of $\mathbb{Z}/2[S_5]$, which implies that $\delta_0=0$.

From [**33**], for Moore space M_2^k,

$$(M_2^k)^{\wedge 5} \simeq M_2^{5k-6} \wedge C_\eta^5 \wedge C_\eta^5 \bigvee 4(M_2^{5k-2} \wedge C_\eta^5) \bigvee 4M_2^{5k+2},$$
$$L_5(M_2^k) \simeq M_2^{5k-2} \wedge C_\eta^5 \bigvee M_2^{5k+2},$$
$$Q_{(4,1)}(M_2^k) \simeq M_2^{5k-2} \wedge C_\eta^5,$$

where all wedge summands in the above equations are indecomposable spaces. Thus $\overline{\frac{1}{5}\beta_5} = \bar{\beta}_5 = af'_{i_0}a^{-1} + af''_{j_0}a^{-1}$ for some $i_0, j_0 \in \{1,2,3,4\}$. Then by Lemma 2.2, we get $L_5(X) \simeq Q_{(4,1)}(X) \bigvee Q_{(3,2)}(X)$. □

The functors L_k find important applications in product decomposition of loop suspension in [**32**].

Proposition 2.5. *Let $X = \Sigma X'$ be a path-connected p-local CW-complex of finite type. Let $1 < k_1 < k_2 < \cdots$ be a sequence of integers such that*

(1) *k_j is not divided by p;*

(2) *k_j is not a multiple of any k_i else for each j.*

Then there exists a topological space A such that

$$\Omega\Sigma X \simeq \prod_j \Omega\Sigma L_{k_j}(X) \times A$$

localized at p.

In this paper, we denote the $\mathbb{Z}$-coefficient homology groups by $H_*(-;\mathbb{Z})$ and $\mathbb{Z}/2$-coefficient cohomology groups $H^*(-;\mathbb{Z}/2)$ by $H^*(-)$.

At the end of this section, two obvious lemmas which will be used later are given.

Lemma 2.6. *Let Y be a wedge summand of X.*

If $Sq^k\colon H^*(X) \to H^{*+k}(X)$ is a surjection (an injection), then so is $Sq^k\colon H^*(Y) \to H^{*+k}(Y)$.

Lemma 2.7. *Let X be a finite CW-complex with torsion homology groups and $X \in \mathbf{A}_n^k$ is a self dual under the Spanier–Whitehead Duality $D_{2n+k}\colon \mathbf{A}_n^k \to \mathbf{A}_n^k$. Then $\bar{H}_s(X;\mathbb{Z}) \cong \bar{H}_{n-1-s}(X;\mathbb{Z})$.*

3. Criteria for hyperbolicity of spaces

In this section, we give some criteria for hyperbolicity of spaces.

Let $\Omega X \hookleftarrow \Omega Y$ mean that ΩY is a product factor of space ΩX and $\Sigma X \hookleftarrow \Sigma Y$ mean that ΣY is a wedge summand of ΣX. If A and B are groups, then $A \hookleftarrow B$ means that B is a direct summand of A.

Lemma 3.1. *Let $X = \Sigma X'$ be a path-connected p-local CW-complex of finite type. Suppose that $\Sigma^{l_1} X_1 \vee \Sigma^{l_2} X_1$ is a wedge summand of $L_k(X)$ for some integers $l_2 \geqslant l_1 \geqslant 1$ and $k \geqslant 1$ with $p \nmid k$, where $L_1(X) := X$; $\Sigma^{l_0+Nt} X_1$ is a wedge summand of $X_1^{\wedge t}$ for any sufficiently large prime t where $N(\geqslant 1)$ and l_0 are some integers; and $\mathbb{Z}/p^i$ is a direct summand of $\pi_m^s(X_1)$ for some m. Then ΣX is $\mathbb{Z}/p^i$-hyperbolic.*

Proof. From Proposition 2.5,

$$\Omega\Sigma X \hookleftarrow \Omega\Sigma L_k(X) \hookleftarrow \Omega\Sigma(\Sigma^{l_1}X_1 \vee \Sigma^{l_2}X_1).$$

By the Hilton–Milnor theorem [**31**], for any fixed positive integer t we get

$$\begin{aligned}\Omega\Sigma X &\hookleftarrow \prod_{j=1}^{W(t)} \Omega\Sigma((\Sigma^{l_1}X_1)^{\wedge a_j} \wedge (\Sigma^{l_2}X_1)^{\wedge b_j}) \simeq \prod_{j=1}^{W(t)} \Omega\Sigma^{1+l_1a_j+l_2b_j}X_1^{\wedge t}\\ &\hookleftarrow \prod_{j=1}^{W(t)} \Omega\Sigma^{1+l_0+l_1a_j+l_2b_j+Nt}X_1 = \prod_{j=1}^{W(t)} \Omega\Sigma^{M_0+Mt-(l_2-l_1)a_j}X_1^{\wedge t},\end{aligned}$$

where $M_0 := 1 + l_0$; $M := N + l_2$; $W(t) := \dim L_t(\bar{V})$ and $\bar{V}$ is an $\mathbb{F}$-vector space with $ch\mathbb{F} = 0$ and $\dim_{\mathbb{F}} \bar{V} = 2$. So

$$\pi_{*+1}(\Sigma X) \hookleftarrow \bigoplus_{j=1}^{W(t)} \pi_{*+1}(\Sigma^{M_0+Mt-(l_2-l_1)a_j}X_1).$$

Note that $0 \leqslant a_j \leqslant t$ which implies that $Mt - (l_2 - l_1)a_j \geqslant [M - (l_2 - l_1)]t$. From $M > l_2 - l_1$, we get

$$\pi_{M_0+Mt-(l_2-l_1)a_j+m}(\Sigma^{M_0+Mt-(l_2-l_1)a_j}X_1) \cong \pi_m^s(X_1) \hookleftarrow \mathbb{Z}/p^i$$

for sufficiently large t. By the well-known Betrand–Chebyshev theorem in number theory, for any n which is large enough, there is a prime q_n such that

$$\frac{1}{2}\left[\frac{n - M_0 - m}{M}\right] < q_n < \left[\frac{n - M_0 - m}{M}\right],$$

hence $M_0 + Mq_n - (l_2 - l_1)a_j + m \leqslant M_0 + Mq_n + m \leqslant n$. Note that q_n is large enough if n is large enough, so

$$\begin{aligned}t_n &:= \sharp\{\mathbb{Z}/p^i \text{ -summands in } \bigoplus_{j\leqslant n} \pi_j(\Sigma X)\}\\ &\geqslant W(q_n) = \frac{2^{q_n} - 2}{q_n} \qquad \text{(by Witt dimension formula)},\end{aligned}$$

$$\varliminf_{n} \frac{\ln t_n}{n} \geqslant \varliminf_{n} \frac{\ln \frac{2^{q_n}-2}{q_n}}{n} = \ln 2 \varliminf_{n} \frac{q_n}{n} + \varliminf_{n} \frac{\ln \frac{2^{q_n}-2}{2^{q_n}}}{n} - \varliminf_{n} \frac{\ln q_n}{n} \geqslant \frac{\ln 2}{2M} > 0.$$

Thus ΣX is $\mathbb{Z}/p^i$-hyperbolic. □

Corollary 3.2. *Let $X = \Sigma X'$ be a path-connected p-local CW-complex of finite type. Suppose that*

$$\begin{aligned}L_{k_1}(X) &\simeq X_1 \vee A_1 \quad \textit{for some integer } k_1 \geqslant 1;\\ L_{k_2}(X_1) &\simeq X_2 \vee A_2 \quad \textit{for some integer } k_2 \geqslant 1;\\ &\vdots\\ L_{k_\xi}(X_{\xi-1}) &\simeq X_\xi \vee A_\xi \quad \textit{for some integer } k_\xi \geqslant 1,\end{aligned}$$

where $k_1 \geqslant 1, k_2 \geqslant 1, \ldots, k_\xi \geqslant 1$ are integers which are not divided by p. If $\Sigma^{l_0+Nt}X_\xi$ is a wedge summand of $X_\xi^{\wedge t}$ for any sufficiently large prime t and $\Sigma^{l_1}X_\xi \vee \Sigma^{l_2}X_\xi$ is a *wedge summand of $L_k(X_\xi)$ for some positive integer $k \geqslant 1$ with $p \nmid k$, where $N(\geqslant 1)$, $l_1, l_2(\geqslant 1)$, l_0 are some integers; and $\mathbb{Z}/p^i$ is a direct summand of $\pi_m^s(X_\xi)$ for some m, then ΣX is $\mathbb{Z}/p^i$-hyperbolic.*

Proof. Taking multiple applications of Proposition 2.5,

$$\begin{aligned} \Omega\Sigma X &\hookleftarrow \Omega\Sigma L_{k_1}(X) \hookleftarrow \Omega\Sigma X_1 \hookleftarrow \Omega\Sigma L_{k_2}(X_1) \\ &\hookleftarrow \Omega\Sigma X_2 \hookleftarrow \Omega\Sigma L_{k_3}(X_2) \hookleftarrow \cdots \hookleftarrow \Omega\Sigma L_{k_\xi}(X_{\xi-1}) \\ &\hookleftarrow \Omega\Sigma X_\xi \hookleftarrow \Omega\Sigma L_k(X_\xi) \hookleftarrow \Omega\Sigma(\Sigma^{l_1}X_\xi \vee \Sigma^{l_2}X_\xi). \end{aligned}$$

Then by the similar proof as that of Lemma 3.1, this completes the proof of Corollary 3.2. □

4. Decomposition of smash product of Chang complexes

Firstly, we fix generators of homology (cohomology) of Chang complexes.

Let w_3, w_5 be generators of $H^*(C_\eta^5)$. Take generators v_n, v_{n+1}, $\bar{v}_{n+1}$, v_{n+2} of $\bar{H}^*(C_r^{n+2,r})$ which are given in Lemma 3.2 of [**34**]. Note that $Sq^2\colon H^n(C_r^{n+2,r}) \to H^{n+2}(C_r^{n+2,r})$ is an isomorphism.

Lemma 4.1. *Let $T_n := (C_\eta^5)^{\wedge n} \wedge C_r^{5,r}$ $(n \geqslant 1)$. Then the following sequences are exact*

$$H^{k-2}(T_n) \xrightarrow{Sq^2} H^k(T_n) \xrightarrow{Sq^2} H^{k+2}(T_n) \quad \text{for } 3n+3 \leqslant k \leqslant 5n+5.$$

Proof. Note that $\bar{H}^k(T_n)$ is nontrivial if and only if $k = 3n+3, 3n+4, \ldots, 5n+5$. We will prove this lemma by induction on n.

We will show that when $3n+3 \leqslant k \leqslant 5n+5$,

$$\mathrm{Im}[Sq^2\colon H^{k-2}(T_n) \to H^k(T_n)] = \mathrm{Ker}[Sq^2\colon H^k(T_n) \to H^{k+2}(T_n)], \tag{4}$$

where $\mathrm{Im}\, f$ (resp. $\mathrm{Ker}\, f$) denotes the image (resp. kernel) of homomorphism f.

For $n = 1, 2$, it's easy to check that (4) holds.

Now suppose that (4) holds for T_m, $m \leqslant n-1$.

Let V be a $\mathbb{Z}/2$-vector space with basis $x_1, x_2, \ldots, x_m$. We also denote V by $\mathbb{Z}/2\langle x_1, x_2, \ldots, x_m\rangle$. The notation $y \otimes V$ means a vector space spanned linearly by $y \otimes x_1, y \otimes x_2, \ldots, y \otimes x_m$.

Note that $T_n = C_\eta^5 \wedge T_{n-1}$. For any k, we have

$$H^k(T_n) = w_3 \otimes H^{k-3}(T_{n-1}) \bigoplus w_5 \otimes H^{k-5}(T_{n-1}).$$

We only prove the case when $H^{k-5}(T_{n-1}) \neq 0$, $H^{k-3}(T_{n-1}) \neq 0$, since if one of them is trivial, then the proof is easier.

By the condition of the induction, we assume
$H^{k-3}(T_{n-1}) = \mathbb{Z}/2\langle x_1, x_2, \ldots, x_\lambda, y_1, y_2, \ldots, y_\mu\rangle$
$H^{k-5}(T_{n-1}) = \mathbb{Z}/2\left\langle x_1', x_2', \ldots, x_\xi', y_1', y_2', \ldots, y_\mu'\right\rangle$ such that

(I) $\mathbb{Z}/2\langle y_1, y_2, \ldots, y_\mu\rangle = \mathrm{Im}[Sq^2\colon H^{k-5}(T_{n-1}) \to H^{k-3}(T_{n-1})]$;

(II) Sq^2 restricts to $\mathbb{Z}/2\langle x_1, x_2, \ldots, x_\lambda\rangle$ a monomorphism;

(III) $Sq^2(y_i') = y_i$; $Sq^2(y_i) = 0$ $(i = 1, \ldots, \mu)$; $Sq^2(x_j') = 0$ $(j = 1, \ldots, \xi)$.

$$Sq^2(w_3\otimes x_1, w_3\otimes x_2,\dots,w_3\otimes x_\lambda)=$$
$$(w_5\otimes x_1+w_3\otimes Sq^2x_1, w_5\otimes x_2+w_3\otimes Sq^2x_2,\dots,w_5\otimes x_\lambda+w_3\otimes Sq^2x_\lambda);$$
$$Sq^2(w_3\otimes y_1, w_3\otimes y_2,\dots,w_3\otimes y_\mu)=(w_5\otimes y_1, w_5\otimes y_2,\dots,w_5\otimes y_\mu);$$
$$Sq^2(w_5\otimes y_1', w_5\otimes y_2',\dots,w_5\otimes y_\mu')=(w_5\otimes y_1, w_5\otimes y_2,\dots,w_5\otimes y_\mu);$$
$$Sq^2(w_5\otimes x_1', w_5\otimes x_2',\dots,w_5\otimes x_\xi')=0.$$

Hence
$$\mathrm{Ker}[Sq^2\colon H^k(T_n)\to H^{k+2}(T_n)]=\mathbb{Z}/2\left\langle Y_1,Y_2,\dots Y_\mu, w_5\otimes x_1', w_5\otimes x_2',\dots,w_5\otimes x_\xi'\right\rangle,$$
where $Y_i=w_5\otimes y_i'+w_3\otimes y_i$ $(i=1,\dots,\mu)$. By the assumption,
$$\mathbb{Z}/2\left\langle x_1',x_2',\dots,x_\xi'\right\rangle=\mathrm{Im}[Sq^2\colon H^{k-7}(T_{n-1})\to H^{k-5}(T_{n-1})].$$
Let $H^{k-7}(T_{n-1})=\mathbb{Z}/2\langle z_1,z_2,\dots,z_\xi,z_1',z_2',\dots,z_\theta',\rangle$ such that $Sq^2z_i=x_i'$ $(i=1,\dots,\xi)$; $Sq^2z_i'=0$ $(i=1,\dots,\theta)$.
$$H^{k-2}(T_n)=w_3\otimes H^{k-5}(T_{n-1})\bigoplus w_5\otimes H^{k-7}(T_{n-1}).$$
Then we easily obtain
$$\mathrm{Im}[Sq^2\colon H^{k-2}(T_n)\to H^k(T_n)]=\mathbb{Z}/2\left\langle Y_1,Y_2,\dots Y_\mu, w_5\otimes x_1', w_5\otimes x_2',\dots,w_5\otimes x_\xi'\right\rangle.$$
So (4) holds for n. This completes the proof of Lemma 4.1. □

Lemma 4.2.
(1) *Any Moore space is not a wedge summand of* $(C_r^{5,r})^{\wedge m}$ *for any* $m\geqslant 1$;
(2) *Chang complex* $C_r^{k+2,r}$ $k\geqslant 3$ *is not a wedge summand of* T_n *for any* $n\geqslant 1$.

Proof. Note that from Theorem 1.1 of [**34**], we get
$$(C_r^{5,r})^{\wedge m}\simeq 2^mC_r^{4m+1,r}\vee\bigvee_{i=1}^{m-1}a_i(C_\eta^5)^{\wedge i}\wedge C_r^{1+4(m-i),r}$$
$$\simeq 2^mC_r^{4m+1,r}\vee\bigvee_{i=1}^{m-1}a_i\Sigma^{4(m-i-1)}T_i\quad(a_i\in\mathbb{Z}^+).$$

For (1) of Lemma 4.2, it suffices to show that $M_{2^r}^k$ is not a wedge summand of T_n for any $n\geqslant 1$.

If not, then for some $n\geqslant 1$, there exist maps such that the composition
$$M_{2^r}^k\xrightarrow{j}T_n\xrightarrow{p}M_{2^r}^k$$
$pj\simeq id$. We have the following commutative diagram
$$\begin{array}{ccccc}
H^{k-1}(M_{2^r}^k) & \overset{p^*}{\hookrightarrow} & H^{k-1}(T_n) & \overset{j^*}{\twoheadrightarrow} & H^{k-1}(M_{2^r}^k)=0\\
\downarrow{\scriptstyle Sq^2} & & \downarrow{\scriptstyle Sq^2} & & \downarrow{\scriptstyle Sq^2}\\
H^{k+1}(M_{2^r}^k) & \overset{p^*}{\hookrightarrow} & H^{k+1}(T_n) & \overset{j^*}{\twoheadrightarrow} & H^{k+1}(M_{2^r}^k)\\
\downarrow{\scriptstyle Sq^2} & & \downarrow{\scriptstyle Sq^2} & & \downarrow{\scriptstyle Sq^2}\\
0=H^{k+3}(M_{2^r}^k) & \overset{p^*}{\hookrightarrow} & H^{k+3}(T_n) & \overset{j^*}{\twoheadrightarrow} & H^{k+3}(M_{2^r}^k),
\end{array}$$

where p^* are injective and j^* are surjective.

Take $0 \neq x_{k+1} \in H^{k+1}(M_{2^r}^k)$. From the square at the lower-left corner, $Sq^2(p^* x_{k+1}) = p^* Sq^2 x_{k+1} = 0$, which implies that $p^* x_{k+1} = Sq^2 y_{k-1}$ for $y_{k-1} \in H^{k-1}(T_n)$ (by Lemma 4.1). Then $x_{k+1} = j^* p^* x_{k+1} = j^* Sq^2 y_{k-1} = Sq^2 j^* y_{k-1} = 0$, which is a contradiction.

In order to prove (2) of Lemma 4.2, we suppose that

$$C_r^{k+2,r} \xrightarrow{j} T_n \xrightarrow{p} C_r^{k+2,r},$$

with $pj \simeq id$. If we replace $M_{2^r}^k$ by $C_r^{k+2,r}$ in above commutative diagram, we get a contradiction by a similar argument. □

Lemma 4.3. *Let $n \geqslant 4$. If spaces $X_i (i = 1, 2)$ satisfy*

1) $\bar{H}_k(X_i; \mathbb{Z}) \cong \begin{cases} \mathbb{Z}/2^r, & k = n, n+1, n+2, n+3; \\ 0, & \text{otherwise.} \end{cases}$
2) *$0 \to H^n(X_i) \xrightarrow{Sq^2} H^{n+2}(X_i) \xrightarrow{Sq^2} H^{n+4}(X_i) \to 0$ are exact sequences and $H^n(X_i) \xrightarrow{Sq^4} H^{n+4}(X_i)$ are isomorphisms for $i = 1, 2$.*
3) *There is a map $f : X_1 \to X_2$ such that $f^* : H^n(X_2) \to H^n(X_1)$ is nontrivial (i.e. isomorphic).*

Then f is a homotopy equivalence.

Proof. Firstly, it is easy to show that $f_* : H_n(X_1; \mathbb{Z}) \to H_n(X_2; \mathbb{Z})$ is an isomorphism. From the following commutative diagram

$$\begin{array}{ccc} H^n(X_2) & \xrightarrow[\cong]{f^*} & H^n(X_1) \\ {\scriptstyle Sq^4}\downarrow\cong & & \cong\downarrow{\scriptstyle Sq^4} \\ H^{n+4}(X_2) & \xrightarrow{f^*} & H^{n+4}(X_1) \end{array}$$

we get $f^* : H^{n+4}(X_2) \to H^{n+4}(X_1)$ is an isomorphism, then $H_{n+3}(f) : H_{n+3}(X_1; \mathbb{Z}) \to H_{n+3}(X_2; \mathbb{Z})$ is an isomorphism by the Universal Coefficient Theorem for $H^{n+4}(-)$.

We also have the following commutative diagram with exact rows.

$$\begin{array}{ccccc} H^n(X_2) & \xhookrightarrow{Sq^2} & H^{n+2}(X_2) & \xrightarrow{Sq^2} & H^{n+4}(X_2) \\ {\scriptstyle f^*}\downarrow\cong & & {\scriptstyle f^*}\downarrow & & \cong\downarrow{\scriptstyle f^*} \\ H^n(X_1) & \xhookrightarrow{Sq^2} & H^{n+2}(X_1) & \xrightarrow{Sq^2} & H^{n+4}(X_1). \end{array}$$

So $f^* : H^{n+2}(X_2) \to H^{n+2}(X_1)$ is also an isomorphism by Five Lemma. From the Universal Coefficient Theorem for $H^{n+2}(-)$, we get isomorphisms

$$H_k(f; \mathbb{Z}) : H_k(X_1; \mathbb{Z}) = \mathbb{Z}/2^r \xrightarrow{\cong} H_k(X_2; \mathbb{Z}) = \mathbb{Z}/2^r, \ k = n+1, n+2.$$

Hence f is a homotopy equivalence since it induces the isomorphisms between all nontrivial homology groups. □

Remark 4.4. $C_\eta^5 \wedge C_r^{n,r}$ $(n \geqslant 5)$ satisfies the conditions 1), 2) of Lemma 4.3.

Proposition 4.5.

$$(C_r^{5,r})^{\wedge 3} \simeq 4C_r^{13,r} \vee 4(C_\eta^5 \wedge C_r^{9,r}) \vee C_\eta^5 \wedge C_\eta^5 \wedge C_r^{5,r},$$

where all wedge summands are indecomposable.

Proof. From [**34**] we have $C_r^{5,r} \wedge C_r^{5,r} \simeq 2C_r^{9,r} \vee C_\eta^5 \wedge C_r^{5,r}$ and $C_\eta^5 \wedge C_r^{9,r}$ is indecomposable, so we easily get the decomposition of $(C_r^{5,r})^{\wedge 3}$ in this lemma. It remains to prove that $T_2 = C_\eta^5 \wedge C_\eta^5 \wedge C_r^{5,r}$ is indecomposable.

The nontrivial reduced homology groups of T_2 are given as follows:

k	9	10	11	12	13	14
$\bar{H}_k(C_\eta^5 \wedge C_\eta^5 \wedge C_r^{5,r};\mathbb{Z})$	$\mathbb{Z}/2^r$	$\mathbb{Z}/2^r$	$\mathbb{Z}/2^r \oplus \mathbb{Z}/2^r$	$\mathbb{Z}/2^r \oplus \mathbb{Z}/2^r$	$\mathbb{Z}/2^r$	$\mathbb{Z}/2^r$

Claim 1: If $T_2 \simeq X \vee Y$, where X, Y are not contractible and X is indecomposable with $H_9(X;\mathbb{Z}) = \mathbb{Z}/2^r$, then Y is indecomposable with following reduced homology groups

k	10	11	12	13
$\bar{H}_k(Y;\mathbb{Z})$	$\mathbb{Z}/2^r$	$\mathbb{Z}/2^r$	$\mathbb{Z}/2^r$	$\mathbb{Z}/2^r$

(5)

We will prove this claim later.

From [**33**], we have 2-local homotopy equivalence

$$C_\eta^5 \wedge C_\eta^5 \wedge C_\eta^5 \simeq 2C_\eta^{13} \vee C_\eta^5 \wedge C_\nu^{10}, \tag{6}$$

where $C_\nu^{10} = \Sigma^2\mathbb{H}P^2$ with $Sq^4\colon H^6(C_\nu^{10}) \xrightarrow{\cong} H^{10}(C_\nu^{10})$.

Thus

$$C_\eta^5 \wedge C_\eta^5 \wedge C_\eta^5 \wedge C_r^{5,r} \simeq 2(C_\eta^{13} \wedge C_r^{5,r}) \vee C_\eta^5 \wedge C_\nu^{10} \wedge C_r^{5,r}. \tag{7}$$

Suppose that T_2 is decomposable, then $T_2 \simeq X \vee Y$, where X, Y come from Claim 1. From (7) we get

$$C_\eta^5 \wedge T_2 \simeq C_\eta^5 \wedge X \vee C_\eta^5 \wedge Y \simeq 2(C_\eta^{13} \wedge C_r^{5,r}) \vee C_\eta^5 \wedge C_\nu^{10} \wedge C_r^{5,r}.$$

$C_\eta^5 \wedge C_\nu^{10} \wedge C_r^{5,r}$ is indecomposable with 16 cells by the proof of Lemma 3.4 of [**35**], which contains the bottom and top cells. Hence $C_\eta^5 \wedge X \simeq C_\eta^5 \wedge C_\nu^{10} \wedge C_r^{5,r}$ and $C_\eta^5 \wedge Y \simeq 2(C_\eta^{13} \wedge C_r^{5,r})$. This is impossible since $H^{13}(C_\eta^5 \wedge Y) = \mathbb{Z}/2 \neq H^{13}(2(C_\eta^{13} \wedge C_r^{5,r})) = 0$. □

Proof of Claim 1. Suppose that $T_2 \simeq X \vee Y$ as above, injectivity or surjectivity of Sq^k on $H^*(X)$ in the following argument comes from the corresponding property of Sq^k on $H^*(T_2)$ by Lemma 2.6.

Now $H^9(X) \xrightarrow{Sq^6(\cong)} H^{15}(X)$ is an isomorphism, which implies that $H_{14}(X;\mathbb{Z}) = \mathbb{Z}/2^r$. By the injection $Sq^2\colon H^9(X) \to H^{11}(X)$, we get $H_{10}(X;\mathbb{Z}) \oplus H_{11}(X;\mathbb{Z}) \neq 0$. Note that T_2 is self dual under the Spanier–Whitehead Duality $D_{24}\colon \mathbf{A}_9^6 \to \mathbf{A}_9^6$ and X is the indecomposable wedge summand with the bottom and top cells of T_2. So X and Y are also self dual.

If $H_{10}(X;\mathbb{Z}) \neq 0$, then $H_{13}(X;\mathbb{Z}) \neq 0$. By the injection $Sq^2\colon H^{10}(X) \to H^{12}(X)$, we get $H_{11}(X;\mathbb{Z}) \oplus H_{12}(X;\mathbb{Z}) \neq 0$. Since X is self dual under D_{24}, we have $H_{11}(X;\mathbb{Z}) \cong H_{12}(X;\mathbb{Z}) \neq 0$. In this case, Y may have the nontrivial reduced homology groups only at dimension 11 and 12, which implies that Y is $C_r^{13,r}$ or a wedge of Moore spaces. This is a contradiction by Lemma 4.2.

Thus $H_{10}(X;\mathbb{Z}) = 0$ and $H_{11}(X;\mathbb{Z}) \neq 0$. By self duality of X, $H_{13}(X;\mathbb{Z}) = 0$.

Suppose that $H_{11}(X;\mathbb{Z}) = \mathbb{Z}/2^r \oplus \mathbb{Z}/2^r$, so $H_{12}(X;\mathbb{Z}) = \mathbb{Z}/2^r \oplus \mathbb{Z}/2^r$. Then $H^{12}(X) \cong H^{12}(T_2)$, which implies that $\dim \mathrm{Im}(Sq^2\colon H^{12}(X) \to H^{14}(X)) = \dim \mathrm{Im}(Sq^2\colon H^{12}(T_2) \to H^{14}(T_2)) = 2$. Hence $H^{14}(X) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2$ and we get $H_{13}(X;\mathbb{Z}) = \mathbb{Z}/2^r$, a contradiction.

Hence $H_{11}(X;\mathbb{Z}) = \mathbb{Z}/2^r$ and $H_{12}(X;\mathbb{Z}) = \mathbb{Z}/2^r$. So the nontrivial reduce homology groups of Y are given by (5).

Next we show that Y is indecomposable. From the Steenrod operation on $H^*(T_2)$ and $H^*(X)$, we get exact sequences

(Sq1) $0 \to H^{10}(Y) \xrightarrow{Sq^2} H^{12}(Y) \xrightarrow{Sq^2} H^{14}(Y) \to 0;$

(Sq2) $0 \to H^{10}(Y) \xrightarrow{Sq^4} H^{14}(Y) \to 0.$

Suppose that $Y \simeq Z \vee W$ where Z is indecomposable with $H_{10}(Z;\mathbb{Z}) = \mathbb{Z}/2^r$. From (Sq2), we get $H_{13}(Z;\mathbb{Z}) = \mathbb{Z}/2^r$. (Sq1) implies that $H^{12}(Z) \neq 0$, and thus $H_{11}(Z;\mathbb{Z}) \oplus H_{12}(Z;\mathbb{Z}) \neq 0$. Since Y is self dual, so is Z. Thus $H_{11}(Z;\mathbb{Z}) \cong H_{12}(Z;\mathbb{Z}) = \mathbb{Z}/2^r$. Hence W is contractible.

This completes the proof of Claim 1. □

Proposition 4.6.

$$L_3(C_r^5) \simeq C_\eta^5 \wedge C_r^{9,r}; \qquad L_3(C^{5,s}) \simeq C_\eta^5 \wedge C_s^{9,s};$$

$$L_3(C_r^{5,s}) \simeq \begin{cases} C_r^5 \wedge C_r^{9,s} \vee C_\eta^5 \wedge C_r^{9,r}, & s > r, \\ C_r^{13,r} \vee 2C_\eta^5 \wedge C_r^{9,r} & s = r, \\ C^{5,s} \wedge C_r^{9,s} \vee C_\eta^5 \wedge C_s^{9,s}, & s < r. \end{cases}$$

C_r^5 and $C_r^{5,s}$ $(s > r)$ are dual to $C^{5,r}$ and $C_s^{5,r}$ respectively under the Spanier–Whitehead Duality $D_8\colon \mathbf{A}_3^2 \to \mathbf{A}_3^2$. The homotopy type of $L_3(C_r^{5,r})$ is given in Lemma 3.4 of [**35**]. Thus we only prove

(i) $L_3(C_r^5) \simeq C_\eta^5 \wedge C_r^{9,r}$; (ii) $L_3(C_r^{5,s}) \simeq C_r^5 \wedge C_r^{9,s} \vee C_\eta^5 \wedge C_r^{9,r}, (s > r)$, and the other cases can be proved similarly.

Proof of (i). There is a canonical quotient map $C_r^{5,r} \xrightarrow{q} C_r^5$ satisfying the following commutative diagram

$$\begin{array}{ccccccccc}
S^3 \vee S^4 & \xrightarrow{\begin{pmatrix} 2^r, & \eta \\ 0, & 2^r \end{pmatrix}} & S^3 \vee S^4 & \longrightarrow & C_r^{5,r} & \longrightarrow & S^4 \vee S^5 & \longrightarrow & S^4 \vee S^5 \\
\Vert & & {\scriptstyle q_1}\downarrow & & {\scriptstyle q}\downarrow & & \Vert & & \downarrow{\scriptstyle \Sigma q_1} \\
S^3 \vee S^4 & \xrightarrow{(2^r,\eta)} & S^3 & \longrightarrow & C_r^5 & \longrightarrow & S^4 \vee S^5 & \longrightarrow & S^4,
\end{array}$$

where q_1 is the canonical projection to the wedge summand S^3 of $S^3 \vee S^4$. Since the decomposition (1) is functorial, we get

$$\begin{array}{ccc} (C_r^{5,r})^{\wedge 3} & \xrightarrow{\simeq} & e_1(C_r^{5,r}) \vee e_2(C_r^{5,r}) \vee e_3(C_r^{5,r}) \\ \Big\downarrow{\scriptstyle q^{\wedge 3}} & & \Big\downarrow{\scriptstyle e_1(q)\vee e_2(q)\vee e_3(q)} \\ (C_r^5)^{\wedge 3} & \xrightarrow{\simeq} & e_1(C_r^5) \vee e_2(C_r^5) \vee e_3(C_r^5). \end{array}$$

From Proposition 4.5 and (2), we have

$$e_2(C_r^{5,r}) \simeq e_3(C_r^{5,r}) \simeq L_3(C_r^{5,r}) \simeq C_r^{13,r} \vee 2(C_\eta^5 \wedge C_r^{9,r});$$
$$e_1(C_r^{5,r}) \simeq C_\eta^5 \wedge C_\eta^5 \wedge C_r^{5,r} \vee 2C_r^{13,r}.$$

The action of e_i $(i = 1, 2, 3)$ on $H^*((C_r^5)^{\wedge 3})$ is given by the permuting coordinates by $\sigma = (123)$ and $\tau = (12)$. Hence we are able to calculate the cohomology and homology of $e_i(C_r^5)$ and we get

k	10	11	12	13
$\bar{H}_k(e_2(C_r^5);\mathbb{Z})$	$\mathbb{Z}/2^r$	$\mathbb{Z}/2^r$	$\mathbb{Z}/2^r$	$\mathbb{Z}/2^r$

For any k, $H^k((C_r^5)^{\wedge 3}) \xrightarrow{(q^{\wedge 3})^*} H^k((C_r^{5,r})^{\wedge 3})$ is an injection, so is $e_i(q)^*$ for $i = 1, 2, 3$. Thus there is an inclusion of the wedge summand $\bar{j}\colon C_\eta^5 \wedge C_r^{9,r} \hookrightarrow e_2(C_r^{5,r})$ such that the composition $\bar{q} := e_2(q)\bar{j}\colon C_\eta^5 \wedge C_r^{9,r} \xrightarrow{\bar{j}} e_2(C_r^{5,r}) \xrightarrow{e_2(q)} e_2(C_r^5)$ induces an isomorphism

$$\bar{q}^*\colon H^{10}(e_2(C_r^5)) = \mathbb{Z}/2 \xrightarrow{\cong} H^{10}(C_\eta^5 \wedge C_r^{9,r}) = \mathbb{Z}/2 \tag{8}$$

and $H_{10}(\bar{q})\colon H_{10}(C_\eta^5 \wedge C_r^{9,r};\mathbb{Z}) \to H_{10}(e_2(C_r^5);\mathbb{Z}) = \mathbb{Z}/2^r$ is also an isomorphism.

Claim 2: $C_\eta^5 \wedge C_r^{9,r} \xrightarrow{\bar{q}} e_2(C_r^5)$ is a homotopy equivalence.

Proof of Claim 2. By the Steenrod operations on $H^*((C_r^5)^{\wedge 3})$, we have $H^{10}((C_r^5)^{\wedge 3}) \xrightarrow{Sq^4} H^{14}((C_r^5)^{\wedge 3})$ is an isomorphism and

$$H^{10}((C_r^5)^{\wedge 3}) \xhookrightarrow{Sq^2} H^{12}((C_r^5)^{\wedge 3}) \xrightarrow{Sq^2} H^{14}((C_r^5)^{\wedge 3}),$$

with $Sq^2Sq^2 = 0$ and $\dim \frac{\mathrm{Ker}[Sq^2\colon H^{12}((C_r^5)^{\wedge 3}) \to H^{14}((C_r^5)^{\wedge 3})]}{\mathrm{Im}[Sq^2\colon H^{10}((C_r^5)^{\wedge 3}) \to H^{12}((C_r^5)^{\wedge 3})]} = 1$. So $e_2(C_r^5)$ satisfies the 1) and 2) of Lemma 4.3 for $n = 10$. From Remark 4.4 and (8), we get Claim 2 by Lemma 4.3. □

From (2), $L_3(C_r^5) \simeq e_2(C_r^5) \simeq C_\eta^5 \wedge C_r^{9,r}$. This completes the proof of (i). □

Proof of (ii). By the decomposition of the smash product of Chang complexes in [**34**], we get

$$(C_r^{5,s})^{\wedge 3} \simeq C_r^{13,s} \vee 2(C_r^5 \wedge C_r^{9,s}) \vee C_r^5 \wedge C_r^5 \wedge C_r^{5,s} \qquad (s > r),$$

where $C_r^5 \wedge C_r^{9,s}$ is indecomposable. The dimensions of cohomology groups of related spaces are given in the following table:

Table 1: $\dim H^k(-)$

k	9	10	11	12	13	14	15	total
$(C_r^{5,s})^{\wedge 3}$	1	6	15	20	15	6	1	64
$e_1(C_r^{5,s})$	1	2	5	8	5	2	1	24
$e_2(C_r^{5,s}) \simeq e_3(C_r^{5,s})$	0	2	5	6	5	2	0	20
$C_r^5 \wedge C_r^{9,s}$	0	1	3	4	3	1	0	12
$C_r^5 \wedge C_r^5 \wedge C_r^{5,s}$	1	4	8	10	8	4	1	36

It is impossible that $2(C_r^5 \wedge C_r^{9,s})$ splits off of $e_1(C_r^{5,s})$ since $\dim H^{13}(2(C_r^5 \wedge C_r^{9,s})) > \dim H^{13}(e_1(C_r^{5,s}))$. So $e_2(C_r^{5,s}) \simeq e_3(C_r^{5,s}) \simeq C_r^5 \wedge C_r^{9,s} \vee X$, where X is a wedge summand of $C_r^5 \wedge C_r^5 \wedge C_r^{5,s}$. By comparing the homology groups, we get

$$\bar{H}_*(X;\mathbb{Z}) = \begin{cases} \mathbb{Z}/2^r, & * = 10, 11, 12, 13, \\ 0, & \text{otherwise.} \end{cases}$$

From the Steenrod operations on $H^*(C_r^5 \wedge C_r^5 \wedge C_r^{5,s})$ and $H^*(C_r^5 \wedge C_r^{9,s})$, we get that X satisfies conditions 1) and 2) of Lemma 4.3.

There are canonical maps $C_r^{5,r} \xrightarrow{q'} C_r^{5,s}$ and $C_r^{5,s} \xrightarrow{q} C_r^5$ satisfying the following commutative diagram

$$\begin{array}{ccccccccc}
S^3 \vee S^4 & \xrightarrow{\left(\begin{smallmatrix} 2^r, & \eta \\ 0, & 2^r \end{smallmatrix}\right)} & S^3 \vee S^4 & \longrightarrow & C_r^{5,r} & \longrightarrow & S^4 \vee S^5 & \longrightarrow & S^4 \vee S^5 \\
\| & & \downarrow{\scriptstyle q_1'} & & \downarrow{\scriptstyle q'} & & \| & & \downarrow{\scriptstyle \Sigma q_1'} \\
S^3 \vee S^4 & \xrightarrow{\left(\begin{smallmatrix} 2^r, & \eta \\ 0, & 2^s \end{smallmatrix}\right)} & S^3 \vee S^4 & \longrightarrow & C_r^{5,s} & \longrightarrow & S^4 \vee S^5 & \longrightarrow & S^4 \\
\| & & \downarrow{\scriptstyle q_1} & & \downarrow{\scriptstyle q} & & \| & & \downarrow{\scriptstyle \Sigma q_1} \\
S^3 \vee S^4 & \xrightarrow{(2^r,\eta)} & S^3 & \longrightarrow & C_r^5 & \longrightarrow & S^4 \vee S^5 & \longrightarrow & S^4,
\end{array}$$

where $q_1' = \left(\begin{smallmatrix} 1, & 0 \\ 0, & 2^{s-r} \end{smallmatrix}\right)$. We get

$$\begin{array}{ccc}
(C_r^{5,r})^{\wedge 3} & \xrightarrow{\simeq} & e_1(C_r^{5,r}) \vee e_2(C_r^{5,r}) \vee e_3(C_r^{5,r}) \\
\downarrow{\scriptstyle (q')^{\wedge 3}} & & \downarrow{\scriptstyle e_1(q') \vee e_2(q') \vee e_3(q')} \\
(C_r^{5,s})^{\wedge 3} & \xrightarrow{\simeq} & e_1(C_r^{5,s}) \vee e_2(C_r^{5,s}) \vee e_3(C_r^{5,s}) \\
\downarrow{\scriptstyle (q)^{\wedge 3}} & & \downarrow{\scriptstyle e_1(q) \vee e_2(q) \vee e_3(q)} \\
(C_r^5)^{\wedge 3} & \xrightarrow{\simeq} & e_1(C_r^5) \vee e_2(C_r^5) \vee e_3(C_r^5).
\end{array}$$

Consider $C_r^{13,r} \vee 2(C_\eta^5 \wedge C_r^{9,r}) \xrightarrow{e_2(q')} C_r^5 \wedge C_r^{9,s} \vee X \xrightarrow{e_2(q)} C_\eta^5 \wedge C_r^{9,r}$. It is easy to get $\dim \mathrm{Ker}(H^{10}((q')^{\wedge 3})) = 3$, which implies that

$H^{10}(C_r^5 \wedge C_r^{9,s} \vee X) \xrightarrow{(e_2(q'))^*} H^{10}(C_r^{13,r} \vee 2(C_\eta^5 \wedge C_r^{9,r}))$ is nontrivial.

Similarly, $H^{10}(C_\eta^5 \wedge C_r^{9,r}) \xrightarrow{(e_2(q))^*} H^{10}(C_r^5 \wedge C_r^{9,s} \vee X)$ is nontrivial.

Let

$$C_r^5 \wedge C_r^{9,s} \underset{r_1}{\overset{j_1}{\rightleftarrows}} C_r^5 \wedge C_r^{9,s} \vee X \underset{r_2}{\overset{j_2}{\leftrightarrows}} X\ ,$$

where $j_1, j_2 (r_1, r_2)$ are the canonical inclusions (projections).

If both $C_r^{13,r} \vee 2(C_\eta^5 \wedge C_r^{9,r}) \xrightarrow{r_1 e_2(q')} C_r^5 \wedge C_r^{9,s}$ and $C_r^5 \wedge C_r^{9,s} \xrightarrow{e_2(q) j_1} C_\eta^5 \wedge C_r^{9,r}$ induce nontrivial maps on $H^{10}(-)$, then there is a map $C_\eta^5 \wedge C_r^{9,r} \xrightarrow{\omega} C_r^5 \wedge C_r^{9,s}$ such that the composition

$$e_2(q) j_1 \omega \colon C_\eta^5 \wedge C_r^{9,r} \xrightarrow{\omega} C_r^5 \wedge C_r^{9,s} \xrightarrow{e_2(q) j_1} C_\eta^5 \wedge C_r^{9,r}$$

induces nontrivial endomorphism on $H^{10}(C_\eta^5 \wedge C_r^{9,r})$. From Lemma 4.3, we get $e_2(q) j_1 \omega$ is a homotopy equivalence and it implies that $C_\eta^5 \wedge C_r^{9,r}$ is a retract of $C_r^5 \wedge C_r^{9,s}$. It contradicts the indecomposability of $C_r^5 \wedge C_r^{9,s}$.

So, we have one of the following two cases:

Case 1: $C_r^{13,r} \vee 2(C_\eta^5 \wedge C_r^{9,r}) \xrightarrow{r_2 e_2(q')} X$ induces nontrivial homomorphism on $H^{10}(-)$, which implies that there is a map $h \colon C_\eta^5 \wedge C_r^{9,r} \to X$ inducing a nontrivial homomorphism (i.e. an isomorphism) on $H^{10}(-)$.

Case 2: $X \xrightarrow{e_2(q) j_2} C_\eta^5 \wedge C_r^{9,r}$ induces a nontrivial homomorphism (i.e. an isomorphism) on $H^{10}(-)$.

From Lemma 4.3, we get that h or $e_2(q) j_2$ is a homotopy equivalence. Thus

$$X \simeq C_\eta^5 \wedge C_r^{9,r}; \qquad e_2(C_r^{5,s}) \simeq C_r^5 \wedge C_r^{9,s} \vee C_\eta^5 \wedge C_r^{9,r}.$$

We get $L_3(C_r^{5,s}) \simeq e_2(C_r^{5,s}) \simeq C_r^5 \wedge C_r^{9,s} \vee C_\eta^5 \wedge C_r^{9,r}$. This completes the proof of (ii). □

Proposition 4.7.

$$L_5(C_r^{5,r}) \simeq 3C_r^{21,r} \bigvee 10(C_\eta^5 \wedge C_r^{17,r}) \bigvee 5(C_\eta^5 \wedge C_\eta^5 \wedge C_r^{13,r}) \bigvee 2(C_\eta^5 \wedge C_\nu^{10} \wedge C_r^{9,r}).$$

Proof. By Lemma 2.4, we know that

$$L_5(C_r^{5,r}) \simeq Q_{(4,1)}(C_r^{5,r}) \bigvee Q_{(3,2)}(C_r^{5,r}). \tag{9}$$

Taking multiple applications of the decomposition of $C_r^{5,r} \wedge C_r^{5,r}$ given in Theorem 1.1 of [**34**], we get

$$\begin{aligned}(C_r^{5,r})^{\wedge 5} \simeq 16C_r^{21,r} \bigvee 48(C_\eta^5 \wedge C_r^{17,r}) \bigvee 26(C_\eta^5 \wedge C_\eta^5 \wedge C_r^{13,r}) \\ \bigvee 8(C_\eta^5 \wedge C_\nu^{10} \wedge C_r^{9,r}) \bigvee (C_\eta^5 \wedge C_\eta^5 \wedge C_\nu^{10} \wedge C_r^{5,r})\end{aligned} \tag{10}$$

From [**12**], $Q_{(5,0)}(C_r^{5,r})$ corresponds to the idempotent element $e_0 \in \mathbb{Z}_{(2)}[S_5]$, which is the lift of $f_0 = \sum_{i=0}^{4} \varrho^i \in \mathbb{Z}/2[S_5]$, where $\varrho = (12345) \in \mathbb{Z}/2[S_5]$. The action of e_0 on $H^*((C_r^{5,r})^{\wedge 5})$ is given by permuting coordinates on

$$H^*((C_r^{5,r})^{\wedge 5}) \cong V^{\otimes 5} \quad (V = \mathbb{Z}/2\,\langle v_3, v_4, \bar{v}_4, v_5 \rangle).$$

Hence we get the dimension of $\mathbb{Z}/2$-vector spaces $\bar{H}^k(Q_{(5,0)}(C_r^{5,r}))$ for $k = 15, 16,$

$\ldots, 25$ as follows:

k	15	16	17	18	19	20	21	22	23	24	25
$\dim \bar{H}^k(Q_{(5,0)}(C_r^{5,r}))$	1	2	9	24	42	52	42	24	9	2	1

Claim 3: $W := C_\eta^5 \wedge C_\eta^5 \wedge C_\nu^{10} \wedge C_r^{5,r}$ is a wedge summand of $Q_{(5,0)}(C_r^{5,r})$.

Proof of Claim 3. Since the cohomology groups $H^k(-)$ $(k = 15, 16, 24, 25)$ of all wedge summands of $(C_r^{5,r})^{\wedge 5}$ in (10) are trivial besides W, there is a wedge summand W' of W with $H^k(W') = H^k(W)$ for $k = 15, 16, 24, 25$ such that W' is also a wedge summand of $Q_{(5,0)}(C_r^{5,r})$. By the action of Steenrod operations Sq^t $(t = 2, 4, 6, 8, 10)$ on $H^*(W)$, we get that $H^k(W') = H^k(W)$ for all $k = 15, 16, \ldots, 25$. Hence $W' = W$. This completes the proof of Claim 3. □

We don't know whether W is decomposable or not, but from Proposition 4.5 we know that all other wedge summands of $(C_r^{5,r})^{\wedge 5}$ in (10) are indecomposable. Now by comparing the dimensions of $\mathbb{Z}/2$-cohomology groups, together with (10) and Claim 3, we get that $Q_{(5,0)}(C_r^{5,r})$ is homotopy equivalent to

$$4C_r^{21,r} \bigvee 8(C_\eta^5 \wedge C_r^{17,r}) \bigvee 6(C_\eta^5 \wedge C_\eta^5 \wedge C_r^{13,r}) \bigvee (C_\eta^5 \wedge C_\eta^5 \wedge C_\nu^{10} \wedge C_r^{5,r}).$$

Thus Proposition 4.7 can be easily obtained from (3) and (9). □

Proposition 4.8. *$2(C_\eta^5 \wedge C_r^{3n+12,r})$ is a wedge summand of $L_3(C_\eta^5 \wedge C_r^{n+2,r})$ $(n \geqslant 3)$.*

Proof. Without loss of generality, we prove it for $n = 3$. From Proposition 4.5 and (6), we get

$$\begin{aligned}(C_\eta^5 \wedge C_r^{5,r})^{\wedge 3} \simeq 12(C_\eta^5 \wedge C_r^{21,r}) \bigvee 8(C_\eta^5 \wedge C_\eta^5 \wedge C_r^{17,r}) \bigvee 8(C_\eta^5 \wedge C_\nu^{10} \wedge C_r^{5,r}) \\ \bigvee 4(C_\eta^5 \wedge C_\eta^5 \wedge C_\nu^{10} \wedge C_r^{9,r}) \bigvee C_\eta^5 \wedge (C_\nu^{10})^{\wedge 2} \wedge C_r^{5,r}.\end{aligned} \tag{11}$$

So $(C_\eta^5 \wedge C_r^{5,r})^{\wedge 3} \simeq 12(C_\eta^5 \wedge C_r^{21,r}) \vee Y$ for some spaces Y.

Similarly as before, we are able to get the dimension of $\mathbb{Z}/2$-vector space $H^k(e_i(C_\eta^5 \wedge C_r^{5,r}))$ $(i = 1, 2, 3)$, as a result, all reduced homology groups of the following spaces are obtained:

Table 2: $\bar{H}_k(-;\mathbb{Z})$

k	18	19	20	21	22	23	24	25	26	27	28	29
$(C_\eta^5 \wedge C_r^{5,r})^{\wedge 3}$	1	5	13	25	38	46	46	38	25	13	5	1
$e_1(C_\eta^5 \wedge C_r^{5,r})$	1	1	5	9	12	16	16	12	9	5	1	1
$L_3(C_\eta^5 \wedge C_r^{5,r})$	0	2	4	8	13	15	15	13	8	4	2	0
$12C_\eta^5 \wedge C_r^{21,r}$	0	0	0	0	12	12	12	12	0	0	0	0
Y	1	5	13	21	22	34	34	22	21	13	5	1

Note: $L_3(C_\eta^5 \wedge C_r^{5,r}) \simeq e_i(C_\eta^5 \wedge C_r^{5,r})$ $(i = 2, 3)$. Homology groups of spaces appeared in above table are direct sums of $\mathbb{Z}/2^r$ and the integers in the above table are the numbers of the $\mathbb{Z}/2^r$-direct summands in the corresponding homology groups.

Since $H_{22}(Y;\mathbb{Z}) = \bigoplus_{i=1}^{22} \mathbb{Z}/2^r$; $H_{22}(2L_3(C_\eta^5 \wedge C_r^{5,r});\mathbb{Z}) = \bigoplus_{i=1}^{26} \mathbb{Z}/2^r$, at least 2 copies of $C_\eta^5 \wedge C_r^{21,r}$ split out of $L_3(C_\eta^5 \wedge C_r^{5,r})$. This completes the proof of Proposition 4.8. □

5. Proof of Main theorems

In this section, we prove Main Theorem I, II, III introduced in Section 1.

Proof of Theorem 1.2 (Main theorem I). By Proposition 4.7, $L_5(C_r^{5,r})$ has $C_r^{21,r} \vee C_r^{21,r}$ as a retract. From [**34**], $\pi_n^s C_r^{n+2,r} = \mathbb{Z}/2^r$; $\pi_{n+1}^s C_r^{n+2,r} = \mathbb{Z}/2^{r+1}$; $\pi_{n+2}^s C_r^{n+2,r} = \mathbb{Z}/2 \oplus \mathbb{Z}/2$ for any $n \geqslant 3$. Then by Lemma 3.1, we get that $C_r^{6,r}$ is $\mathbb{Z}/2^i$-hyperbolic for $i = 1, r, r+1$ and so is $C_r^{n+2,r}$ $(n \geqslant 4)$. □

Proof of Theorem 1.3 (Main theorem II). For $C = C_\eta^{n+2}$, by the same method as that used in [**14**] to prove the hyperbolicity of M_2^n, we get $L_9(C_\eta^{n+1})$ has a wedge summand $C_\eta^{9n+1} \vee C_\eta^{9n+1}$. Clearly, $\Sigma^{nt-n} C_\eta^{n+1}$ is a wedge summand of $(C_\eta^{n+1})^{\wedge t}$ for any odd positive integer t. Then from $\pi_{n+6}^s(C_\eta^{n+2}) \cong \mathbb{Z}/2$, we get C_η^{n+2} is $\mathbb{Z}/2$-hyperbolic by Lemma 3.1.

For C, one of Chang complexes C_r^6, $C^{6,s}$, $C_r^{6,s} (s \neq r)$, we get that $C_\eta^5 \wedge C_{u_C}^{9,u_C}$ is a wedge summand of $L_3(C)$ by Proposition 4.6 and $2(C_\eta^5 \wedge C_{u_C}^{33,u_C})$ is a wedge summand of $L_3(C_\eta^5 \wedge C_{u_C}^{9,u_C})$ by Proposition 4.8. Similar to formula (11), we get that $C_\eta^5 \wedge C_{u_C}^{12t-3,u_C}$ is a wedge summand of $(C_\eta^5 \wedge C_{u_C}^{9,u_C})^{\wedge t}$ for any $r \geqslant 1$ by induction on t. From [**34**],

$$\pi_{n+3}^s(C_\eta^5 \wedge C_{u_C}^{n+2,u_C}) = \mathbb{Z}/2^{u_C};$$
$$\pi_{n+6}^s(C_\eta^5 \wedge C_{u_C}^{n+2,u_C}) = \begin{cases} \mathbb{Z}/2 \oplus \mathbb{Z}/2, & u_C = 1, \\ \mathbb{Z}/2^{u_C+1} \oplus \mathbb{Z}/2, & u_C \geqslant 2. \end{cases}$$

Hence from Corollary 3.2, we get C is $\mathbb{Z}/2$-hyperbolic for $u_C = 1$ and $\mathbb{Z}/2^i$-hyperbolic $(i = 1, u_C, u_C + 1)$ for $u_C > 1$.

This completes the proof of Theorem 1.3. □

Proof of Theorem 1.4 (Main theorem III). For $p = 2$, any non-contractable complex in $\mathbf{A}_n^2$ is homotopy equivalent to a wedge product of spaces from S^k $(k = n, n+1, n+2)$, $M_{2^r}^k$ $(r \geqslant 1, k = n, n+1)$, and Chang-complexes under 2-localization (or just one of them). Since the elementary Moore spaces $M_{2^r}^n$, $M_{2^r}^{n+1}$ $(r \geqslant 1)$ are $\mathbb{Z}/2$-hyperbolic by [**14**], from Theorem 1.2 and Theorem 1.3, it suffices to show the $\mathbb{Z}/2$-hyperbolicity of $S^n \vee S^{n'}$ $(n \leqslant n')$.

Since $\mathbb{Z}/2$ is a direct summand of $\pi_{n+1}^s(S^n)$, from Lemma 3.1, we get $S^n \vee S^{n'}$ is $\mathbb{Z}/2$-hyperbolic. Hence Theorem 1.4 holds for $p = 2$.

For odd prime p, any non-contractable complex in $\mathbf{A}_n^2$ is homotopy equivalent to a wedge product of spaces from S^k $(k = n, n+1, n+2)$ and $M_{p^r}^k$ $(r \geqslant 1, k = n, n+1)$ under p-localization (or just one of them). It suffices to show the p-hyperbolicity of $S^n \vee S^{n'}$ $(n \leqslant n')$ and $M_{p^r}^n$.

From [**29**], for odd integer $n \geqslant 3$, we get $\pi_{n+2p-3}^s(S^n) \hookleftarrow \mathbb{Z}/p$. Since $\pi_i(S^{2l}; p) \cong \pi_{i-1}(S^{2l-1}; p) \oplus \pi_i(S^{4l-1}; p)$ where $\pi_i(X; p)$ is the p-primary components of $\pi_i(X)$ [**27**], we get $\pi_{n+2p-4}^s(S^n) \hookleftarrow \mathbb{Z}/p$ for even integer $n \geqslant 4$. Hence for any integer $n \geqslant 3$, $\mathbb{Z}/p$ is a direct summand of $\pi_m^s(S^n)$ for some positive integer m. Hence from Lemma 3.1, we get $S^n \vee S^{n'}$ is $\mathbb{Z}/p$-hyperbolic for any odd prime p.

For $M_{p^r}^n (n \geqslant 4)$, $\Omega\Sigma M_{p^r}^n \hookleftarrow \Omega(\Sigma M_{p^r}^{n_1} \vee \Sigma M_{p^r}^{n_2})$ for some positive integers $n_1, n_2 > n$ from [**8, 6, 20**]. Since $(M_{p^r}^3)^{\wedge 3} \hookleftarrow 2M_{p^r}^{10}$ and $\mathbb{Z}/p$ is a direct summand of some stable homotopy group $\pi_*^s(M_{p^r}^n)$ by [**9**], we get $M_{p^r}^{n_1} \vee M_{p^r}^{n_2}$ is $\mathbb{Z}/p$-hyperbolic by Lemma 3.1. Hence $M_{p^r}^n$ is $\mathbb{Z}/p$-hyperbolic.

So Theorem 1.4 holds for odd prime p. □

References

[1] C.W. Curtis and I. Reiner, *Methods of Representation Theory Volume II*, John Wiley & Sons Inc, New York (2010).

[2] C.W. Curtis and I. Reiner, *Representation Theory of Finite Groups and Associative Algebras*, Amer. Math. Monthly **72** (1965) no. 2, 223

[3] S.C. Chang, *Homotopy invariants and continuous mappings*, Proc. Roy. Soc. London. Ser. A **202** (1950), 253–263.

[4] W.D. Chen and J. Wu, *Decomposition of loop spaces and periodic problem on* π_*, Algebr. Geom. Topol. **13** (2013), 3245–3260.

[5] F.R. Cohen, *A course in some aspects of classic homotopy theory*, In: Lecture Notes in Math. **1286**, Springer, Berlin (1986), 1–92.

[6] F. R. Cohen, J.C. Moore and J.A. Neisendorfer, *Torsion in homotopy groups*, Ann. of Math. **109** (1979), 121–168.

[7] F.R. Cohen and J. Wu, *A remark on the homotopy groups of* $\Sigma^n\mathbb{R}P^2$, Contemp. Math. **181** (1995), 65–81.

[8] F.R. Cohen, J.C. Moore and J.A. Neisendorfer, *Exponent in homotopy theory, from: Algebraic topology and algebraic K-theory* **113**, Princeton Univ. Press (1987).

[9] Y.A. Drozd and P. Kolesnyk, *Atoms in the p-localization of Stable Homotopy Category*, Ukrainian Math. J. **66** (2014) no. 4, 514–529.

[10] Y.A. Drozd and V.V. Kirichenko, *Finite Dimensional Algebras*, Springer, Berlin, Heidelberg (1994).

[11] Y. Félix and S. Halperin and J.C. Thomas, *Rational homotopy theory*, Grad. Texts in Math. **205**, Springer-Verlag, New York (2001).

[12] M. Feshbach and S. Priddy S, *Stable splittings associated with Chevalley groups, II*, Comment. Math. Helv. **64** (1989), 496–507.

[13] H.W. Henn, *On the growth of homotopy groups*, Manuscripta Math. **56** (1986), 235–245.

[14] R.Z. Huang and J. Wu, *Exponential growth of homotopy groups of suspended finite complexes*, Math. Z. **295** (2020), 1301–1321.

[15] K. Iriye, *On the ranks of homotopy groups of a space*, Publ. RIMS, Kyoto Univ. **23** (1987), 209–213.

[16] G. James, *The representation theory of the symmetric groups*, Springer Lecture Notes in Math. **682** (1978).

[17] G. James and A. Kerber, *The Representation Theory of the Symmetric Group*, Encyclopedia Math. Appl. **16**, Section: Algebra (1983).

[18] C.A. McGibbon and J.A. Neisendorfer, *On the homotopy groups of a finite dimensional space*, Comment. Math. Helv. **59** (1984), 253–257.

[19] J.A. Neisendorfer, *Primary homotopy theory*, Mem. Amer. Math. Soc. **232** (1980).

[20] J.A. Neisendorfer, *3-primary exponents*, Math. Proc. Cambridge Philos. Soc. **90** (1981), 63–83.

[**21**] J.A. Neisendorfer, *Algebraic methods in unstable homotopy theory*, Cambridge Univ. Press (2009).

[**22**] S. Priddy, *Lectures on the stable homotopy of BG*, Geom. Topol. Monogr. **11** (2007), 289–308.

[**23**] C. Reutenauer, *Free Lie Algebras*, Oxford University Press (1993).

[**24**] P. Selick and J. Wu, *On functorial decompositions of self-smash products*, Manuscripta Math. **111** (2003) no. 4, 435–457.

[**25**] P. Selick and J. Wu, *On natural coalgebra decompositions of tensor algebras and loop susupensions*, Mem. Amer. Math. Soc. **148** (2000)

[**26**] P. Selick and J. Wu, *The functor A^{min} on p-local spaces*, Math. Z. **253** (2006), 435–451.

[**27**] J.P. Serre, *Homologie singuliére des espaces fibrés*, Ann. of Math. (2) **54**, 425–505 (1951)

[**28**] J.P. Serre, *Cohomologie modulo 2 des complexes d'Eilenberg–MacLane*, Comment. Math. Helv. **27**(1953), 198–232.

[**29**] H. Toda, *Composition Methods in the Homotopy Groups of Spheres*, Ann. of Math. Stud. **49**, Princeton (1962).

[**30**] Y. Umeda Y, *A remark on a theorem of J.-P. Serre*, Proc. Japan Acad. **35**(1959), 563–566.

[**31**] G.W. Whitehead, *Elements of homotopy theory*, Springer-Verlag **97** (1978).

[**32**] J. Wu, *On combinatorial calculations for the James Hopf maps*, Topology **37** (1998), 1011–1023.

[**33**] J. Wu, *Homotopy theory of the suspensions of the projective plane*, Mem. Amer. Math. Soc. **162** (2003) no. 769.

[**34**] Z.J. Zhu and J.Z. Pan, *The decomposability of smash product of A_n^2 complexes*, Homology Homotopy Appl. **19** (2017), 293–318.

[**35**] Z.J. Zhu , C.P. Li and J.Z. Pan, *Periodic problem on homotopy groups of Chang complexes $C_r^{n+2,r}$*, Homology Homotopy Appl. **21(2)** (2019), 363–375.

Zhongjian Zhu `zhuzhongjian@amss.ac.cn`

College of Mathematics and Physics, Wenzhou University, Wenzhou, Zhejiang 325035, China

Jianzhong Pan `pjz@amss.ac.cn`

Hua Loo-Keng Key Mathematical Laboratory, Institute of Mathematics, Academy of Mathematics and Systems Science, Chinese Academy of Sciences; University of Chinese Academy of Sciences, Beijing, 100190, China