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# THE OPERAD THAT CO-REPRESENTS ENRICHMENT

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*Abstract*

I show that the theories of enrichment in a monoidal infinity-category defined by Hinich and by Gepner–Haugseng agree, and that the identification is unique. Among other things, this makes the Yoneda lemma available in the latter model.

## 1. Introduction

The notion of an enrichment of a 1-category C in a monoidal category $(\mathcal{V},\otimes)$ as a bifunctor

$$\mathrm{C}_{\mathcal{V}}(-,-)\colon \mathrm{C}^{\mathrm{op}}\times\mathrm{C}\to\mathcal{V},$$

together with a composition law satisfying an associativity constraint, goes back almost as far as category theory itself.[1] Trying to transplant this notion into higher category theory, one encounters the same difficulties controlling 'coherent associativity' as one does when defining algebra objects; that is, one-object enriched categories.

Recently, two approaches have appeared that provide a framework of $\mathcal{V}$-enrichments for any monoidal $\infty$-category $(\mathcal{V},\otimes)$: the *categorical algebras in* $\mathcal{V}$ of [4] and the $\mathcal{V}$*-enriched precategories* of [6]. While the former work provides fundamental facts about the category of enriched categories — such as presentability in the case $\mathcal{V}$ is presentable[2] — the latter adds essential methods for enriched $\infty$-category theory in practice, not least of which a $\mathcal{V}$-Yoneda lemma. Naturally, we want to know that these papers describe one and the same theory.

As in the better-known setting of coherently associative algebras, the trick to defining enrichments is in selecting a gadget that *indexes operations* — in other words, a *co-representing* object. In this case, the relevant object is a planar $\infty$-operad. It is claimed in [6] that its own corepresenting operad $\mathtt{Ass}_X$ is 'the same' as a simplicial multicategory $\mathcal{O}_X$ defined in [4], and therefore that the attendant theories of $\mathcal{V}$-enrichments agree. However, although the author suggests the premise for the comparison, no complete justification of the claim appears. This note fills that gap. We show that:

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[1] For a potted history, see the introduction to [7].

[2] Proposition 4.3.5 from *op. cit.*

- The Gepner–Haugseng theory of categorical algebras is naturally (in $\mathcal{V}$) equivalent to Hinich's $\mathcal{V}$-enriched precategories;
- that the equivalence is unique — the theory has no natural autoequivalences.

I now sketch the broad strokes of the argument. In what follows, **Cat** denotes the $\infty$-category of $\infty$-categories and `Ass` the associative operad.

**Theorem 1.1.** *There is a unique equivalence, universal in* $(\mathcal{V}, \otimes)$*, between the Gepner–Haugseng category of categorical algebras in* $\mathcal{V}$ *and the Hinich category of* $\mathcal{V}$*-enriched precategories.*

*Proof.* The corepresenting planar operads in the two cases are as follows (cf. §2):

- In [4, §4.2] we have the planar operad $L_{\mathrm{gen}}\Delta^{\mathrm{op}}_X$; we will use the fact [4, Cor. 4.2.8] that it is presented by the simplicial multicategory $\mathcal{O}_X$.
- Hinich's operad $\mathtt{Ass}(X)$ is defined [6, §3.2][3] by specifying its spaces of simplices

  $$\sigma : \Delta_{/\mathtt{Ass}^{\mathrm{op}}},\ X : \mathbf{Cat} \qquad \mathrm{Fun}_{\mathtt{Ass}}(\sigma, \mathtt{Ass}(X)) \cong \mathrm{Fun}(\mathcal{F}(\sigma), X)$$

  in terms of an explicit, combinatorially defined functor $\mathcal{F}\colon \Delta_{/\mathtt{Ass}} \to \mathbf{Cat}$ on the category of simplices of the associative operad.[4]

In light of the definition of $\mathtt{Ass}(X)$, it is enough to exhibit the $\infty$-functor associated to $\mathcal{O}$ as a right adjoint to a left Kan extension of $\mathcal{F}$:

$$\hat{\mathcal{F}}\colon \mathbf{Cat}_{/\mathtt{Ass}} \rightleftarrows \mathbf{Cat} : \mathcal{O}.$$

This follows from proposition 4.1 and corollary 3.4.

Proposition 5.1 tells us that $\mathrm{Aut}(\mathcal{F})$ is trivial, whence the adjunction data is unique. ☐

### 1.1. Perspective

The uniqueness statement in theorem 1.1 echoes the main results of [2] concerning $(\infty, n)$-categories — in fact, it is even more satisfying, because it is not required to fix the inclusion of a generating subcategory to 'orient' the theory. We have not proved that the uniqueness statement applies also to the subcategory of *complete* objects — that is, to enriched category theory proper — but it seems likely that it holds without modification.

A more direct connection to $(\infty, n)$-category theory exists in the form of results that compare the $n$-fold Segal space model with iterated enriched categories: see [5, §7] for the Gepner–Haugseng model and [6, §5] for Hinich. Using the result of this paper, it also makes sense to compare these comparisons: in other words, we could

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[3] Warning: section number citations to [6] are based on the linked preprint version and not the final published version.

[4] Note that we consider here $\mathrm{Fun}(-,-)$ as a space, rather than as a category.

ask if the diagram

$$\begin{tikzcd}
\mathrm{Alg}^{\mathrm{GH}}_{\mathrm{cat}}(n\mathbf{Cat}) \arrow[rr, leftrightarrow, "\text{Theorem 1.1}"] \arrow[dr, leftrightarrow, "{[5, \text{Thm. } 7.5]}"'] & & \mathrm{Alg}(\mathrm{Quiv}_{\_}(n\mathbf{Cat})) \\
& \mathrm{Seg}(n\mathbf{Cat}) \arrow[ur, leftrightarrow, "{[6, \text{Cor. } 5.6.1]}"'] &
\end{tikzcd}$$

commutes. Here the authors' approaches diverge significantly, and it would be interesting to obtain an isomorphism between these two functors.

The uniqueness statement becomes false if we restrict to $\mathcal{V}$ *symmetric* monoidal: there is a non-trivial symmetry given by formation of the opposite category. It seems likely that this is the entire symmetry group; however, a proof of this statement does not seem to be immediately accessible to the methods of this paper.

### 1.2. Bordism description of $\mathtt{Ass}(X)$

The definition of $\mathcal{O}_X$ makes use of simplicial category model in an essential way; meanwhile, although $\mathcal{F}$ can be defined without reference to a set-theoretic model, its definition uses laborious explicit combinatorics. Only $L_{\mathrm{gen}}\Delta^{\mathrm{op}}_X$ has a truly universal feel, though this too is unsatisfactory as it only gives the correct object when $X$ is a space. Is there a holistic approach to constructing the adjunction $\mathcal{F} \dashv \mathcal{O}$?

Here is a sketch of how $\mathtt{Ass}(X)$ may be defined using bordism theory. First, we recall that the associative operad $\mathtt{Ass} = \Delta^{\mathrm{op}}$ has a realisation as a bordism category in which:

- objects are finite disjoint unions of embedded intervals in the line $\mathbb{R}_x$;
- morphisms from $X_0$ to $X_1$ are surfaces $\Sigma$ (with corners) embedded in the plane $\mathbb{R}_x \times [0,1]_t$, transverse at $\{0,1\}$, with identifications $\partial_i\Sigma := \Sigma \cap \mathbb{R}_x \times \{i\} \cong X_i$. The surfaces must be simply-connected and $\pi_0(\partial_1\Sigma) \to \pi_0(\Sigma)$ bijective.

As usual, composition is defined by glueing surfaces at marked ends.

For a given such surface $\Sigma$, let us call *horizontal* the part of the boundary not contained in $\mathbb{R}_x \times \{0,1\}$. Then $\mathtt{Ass}(X)$ will be the operad whose objects (colours) are 1-manifolds as above whose boundary points are labelled with objects of $X$, and whose morphisms are embedded surfaces with horizontal boundary marked with morphisms of $X$ (oriented according to a fixed orientation of the ambient plane and with suitable source and target). Its structural morphism $\mathtt{Ass}(X) \to \mathtt{Ass}$ obtained by forgetting the labelling.

A look at the pictures in [**6**], or in §4 of this paper, will confirm the equivalence of this description. I leave a fuller development of this approach for another day.

### 1.3. Outline of the paper

In §2 we go over general conventions and review the relevant material from [**4, 6**]. In §3 we make some reductions to the case of 1-categories; this section addresses the matter of extensions of adjunctions, and model-categorical issues (especially those concerning $s\mathbf{Cat}_{/\mathtt{Ass}}$). In §4 we carry out the main argument — this is conceptually straightforward, but tedious in practice. Finally, §5 addresses the symmetries of $\mathcal{F}$.

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## 2. Preliminaries

### 2.1. Category theory

This paper is based on the $(\infty, 1)$-category theory developed in [9, 8]. Hence categories, by default, are $(\infty, 1)$-categories, while classical $(1, 1)$-categories whose mapping objects are sets are said to be *1-truncated* or *classical* categories. The $(\infty, 1)$-category of $(\infty, 1)$-categories is denoted **Cat**. The $(\infty, 1)$-category of $(\infty, 1)$-functors between two objects $X$, $Y$ : **Cat** is denoted $\mathrm{Fun}(X, Y)$; where necessary, we will also consider this as a space (by taking its maximal sub-groupoid), indicating when we do so in the text. The category of monoidal $\infty$-categories is denoted $\mathbf{Cat}^{\otimes}$, and the category of (planar) $\infty$-operads is **Op**.

Since one of the objects being compared is defined as a simplicial category, it was not possible to entirely avoid model category techniques. The methods we actually use are quite restricted:

- The notion of Quillen adjunction and the fact, proved in [11], that it induces an adjunction between the associated $(\infty, 1)$-categories.
- The Bergner model structure on the category $s\mathbf{Cat}$ of simplicially enriched categories.
  We also need some facts about the slice model structure; see §3.

### 2.2. The 1-category of 1-categories

The category of 1-categories is naturally organised into a $(2, 1)$-category, with paths in the mapping types given by natural isomorphism of functors. However, we will need to work in $\mathbf{Cat}^1$ as a subcategory of the model category $s\mathbf{Cat}$, which is usually formulated as a $(1, 1)$-category (since this is the domain of Quillen's model category theory). We will therefore work in the $(1, 1)$-category $\mathbf{Cat}^1$ of *strict* 1-categories up to isomorphism, rather than up to categorical equivalence. Beware that the natural functor $\mathbf{Cat}^1 \to \mathbf{Cat}$ is not fully faithful.

We should also fix a strict model of the simplex category $\Delta$ (although it does not matter which one). For sake of argument, let us say we chose a model whose objects are natural numbers, so that there is a unique object in each isomorphism class. The reader will see that Hinich's functor $\mathcal{F}$, sketched in §4.3, takes values in strict categories by definition.

*Remark 2.1* (The 2-category of 1-categories). Alternatively, using the theory of [10] it is also possible to consider $s\mathbf{Cat}$ as a model $(2, 1)$-category, in which case we may also treat $\mathbf{Cat}^1$ as a $(2, 1)$-category. This approach entails additional complication for our combinatorial construction of the main adjunction §4.4: we should provide commutativity *isomorphisms* for the naturality squares inspected in §4.5 and §4.6, which themselves should satisfy a further identity.

In fact, the arguments as written essentially provide the required isomorphisms. The additional compatibility, on the other hand, hardly seems worth the effort to formulate.

### 2.3. Membership declaration

Objects of categories (either variables or constants) are declared with a *colon*, i.e. $a : A$ states that $a$ is an object of $A$ (hence is equivalent to the usual notation $a \in A$). As usual, we may also write $f\colon A \to B$ or $g\colon C \cong D$ to declare a function or isomorphism; the category in which this morphism or isomorphism lives, if unclear, is underset.

### 2.4. Approaches to enriched categories

There have been various attempts to get enriched category theory working using model categories. The earliest attempts used classical enrichment compatible with a model structure; this approach continues to suffer from the disadvantages encountered in the special case of simplicially enriched categories, and it is impractical for addressing $(\infty, n)$-category theory. More recently, a more general notion of 'weak' enrichment was developed in [**12**]. For an overview, see the introduction to [**4**].

The notion of a $\mathcal{V}$-enriched category, for $(\mathcal{V}, \otimes)$ a general monoidal $\infty$-category, was introduced in [**4**]. The more flexible language of [**6**], although it appeared later, was apparently developed concurrently. It is worth mentioning that Hinich proves a Yoneda lemma [**6**, Cor. 6.2.7].

In both cases the definition of the 1-category $\mathcal{V}\mathbf{Cat}$ of $\mathcal{V}$-enriched categories proceeds in two stages:

1. First we define a notion which in this paper we will call an *algebroid* in $\mathcal{V}$ over a space $X$, encoding the $\mathcal{V}$-valued Hom-bifunctor $X \times X \to \mathcal{V}$ with its associative composition law. (These are the objects called 'categorical algebras' in [**4**] and 'enriched precategory' in [**6**].)
2. Second, localise to a full subcategory of *complete* (or *univalent*) algebroids. These are those algebroids for which the underlying space $X$ *classifies objects* [**4**, §5.2].

In this paper, we will not go into details on the localisation stage.

### 2.5. Corepresenting algebroids

To a monoidal category $\mathcal{V}$ and space $X$ each of the references [**4, 6**] defines the category of algebroids in $\mathcal{V}$ over $X$ by constructing a corepresenting object. Hinich also gives a definition when $X$ is a category.

- To $X$ [**4**] functorially attaches a certain generalised planar operad $\Delta_X^{\mathrm{op}} \to \Delta^{\mathrm{op}}$ [**4**, §4.1], which will corepresent algebroids with space of objects $X$. Note: $\Delta_X^{\mathrm{op}} = (\Delta_{/X})^{\mathrm{op}}$.
  The authors then define *categorical algebras*:
  $$\mathrm{Alg}_{\mathrm{cat}}(\mathcal{V}) = \mathrm{Alg}^{\mathrm{GH}}_{\mathrm{cat}}(\mathcal{V}) := \mathbf{Op}^{\mathrm{gen}}_{\mathrm{planar}}(\Delta_X^{\mathrm{op}}, \mathcal{V}^{\otimes})$$
  as a category of morphisms between generalised planar operads.
  Since the target is a monoidal category, it makes no difference if we replace $\Delta_X^{\mathrm{op}}$ with its nearest (non-generalised) operad quotient $L_{\mathrm{gen}}\Delta_X^{\mathrm{op}}$. When $X$ is

presented as a simplicial groupoid, this is modelled by a simplicial multicategory $\mathcal{O}_X$ [4, §4.2] (itself actually defined for any simplicial *category* $X$).

- The associative operad is denoted $\mathtt{Ass}(=\Delta^{\mathrm{op}})$. Hinich associates to $X$ the planar operad $\mathtt{Ass}(X)$ [6, (3.2.8)]. The category of $\mathcal{V}$-enriched precategories is then defined as the category of operad morphisms $\mathbf{Op}(\mathtt{Ass}(X), \mathcal{V}^{\otimes})$.

It is reasonable to regard $\mathtt{Ass}(X)$ as *always* corepresenting algebroids over $X$, even for $X$ a general $(\infty,1)$-category, while $\Delta_X^{\mathrm{op}}$ is correct only when $X$ is a space (some of the arrows end up pointing the wrong way in general).

Being a mapping category, as $X$ and $\mathcal{V}$ varies, it defines a bifunctor [6, (3.5.3)]

$$\mathrm{Algbrd}\colon \mathbf{Cat}^{\mathrm{op}} \times \mathbf{Cat}^{\otimes} \to \mathbf{Cat}.$$

Fixing $\mathcal{V} : \mathbf{Cat}^{\otimes}$ and integrating over $\mathbf{Cat}$ yields a Cartesian fibration

$$\mathrm{Algd}(\mathcal{V}) \to \mathbf{Cat}$$

whose total space is by definition the category of all algebroids in $\mathcal{V}$.

### 2.6. A clarification

A couple of remarks are required to fully equate the language of [6] with the form presented in §2.5.

- The paper is written in a generalised setting of operads over a 'strong approximation' to a (symmetric) operad [6, (2.5.2)]. In this language, $\mathtt{Ass} = \Delta^{\mathrm{op}}$ is a strong approximation to the associative operad, and $\mathbf{Op}_{/\mathtt{Ass}}$ is (definitionally) the same as the category of non-symmetric operads appearing in [4, §3].
- Rather than defining them directly as a mapping category, Hinich actually defines $\mathcal{V}$-enriched precategories as algebras in the 'Day convolution' internal mapping operad

  $$\mathrm{Quiv}_X(\mathcal{V}) = \mathrm{Funop}(\mathtt{Ass}(X), \mathcal{V}^{\otimes})$$

  from $\mathtt{Ass}(X)$ to $(\mathcal{V}, \otimes)$. This object is defined for so-called *flat* operads, a class which includes $\mathtt{Ass}(X)$ [6, §3.3]. By [6, Cor. 2.6.5], algebras in this category compute operad morphisms, so this agrees with the formula written here.

## 3. A Quillen adjunction

Before formulating the main statement 3.3, we must address some technical matters: since $\mathcal{F}$ is defined only on $\Delta_{/\mathtt{Ass}}$ and not $\mathbf{Cat}_{/\mathtt{Ass}}$, we should address the issue of extending adjunction data on this subcategory to a genuine adjunction with a left Kan extension.

**Definition 3.1** (Formal adjunction)**.** Let $\mathrm{C}^\wedge$ and $\mathrm{D}$ be categories with $\mathrm{C} \subseteq \mathrm{C}^\wedge$ a dense full subcategory, and let $\mathrm{C} \overset{L}{\to} \mathrm{D} \overset{R}{\to} \mathrm{C}^\wedge$ be two functors. A *formal adjunction* $L \dashv R$ between $L$ and $R$ is the data of a natural isomorphism

$$c : \mathrm{C},\ d : \mathrm{D} \qquad \mathrm{C}^\wedge(c, Rd) \cong \mathrm{D}(Lc, d)$$

(i.e. an isomorphism of functors of $c, d$). By the Yoneda lemma, $L$ is uniquely determined by $R$ as the pullback $\mathrm{C} \to \mathrm{Fun}(\mathrm{D}, \mathbf{S})^{\mathrm{op}}$, $c \mapsto \mathrm{D}(c, R-)$. The converse holds by density of $\mathrm{C}$ in $\mathrm{C}^\wedge$.

If $\mathrm{C} = \mathrm{C}^\wedge$, then using [1, Lemma 4.1] to exchange the bifunctor $\mathrm{C}(c, Rd)$ with a correspondence $\mathrm{C} \sqcup \mathrm{D} \subseteq \mathrm{M} \to \Delta^1$, the data of a formal adjunction is equivalent to that of a usual adjunction of $\infty$-categories [9, Def. 5.2.2.1].

**Lemma 3.2** (Extending a formal adjunction)**.** *Let* $\mathrm{adj} : L \dashv R,\ \mathrm{C} \to \mathrm{D} \to \mathrm{C}^\wedge$ *be a formal adjunction. Suppose that $L$ admits a (pointwise) left Kan extension* $L^\wedge \colon \mathrm{C}^\wedge \to \mathrm{D}$*. Then there is a unique adjunction $L^\wedge \dashv R$ extending the given formal adjunction. In particular,* $\mathrm{Aut}(L) = \mathrm{Aut}(L^\wedge)$*.*

*Proof.* By the definition of the Kan extension, for $c : \mathrm{C}^\wedge$ we have

$$\mathrm{D}(L^\wedge c, d) \cong \lim_{c' : \mathrm{C}_{/c}} \mathrm{D}(Lc', d) \cong \lim_{c' : \mathrm{C}_{/c}} \mathrm{C}(c', Rd) \cong \mathrm{C}^\wedge(c, Rd)$$

where the last line follows from $\operatorname{colim}_{c' : \mathrm{C}_{/c}} c' \cong c$ because $\mathrm{C}$ is dense in $\mathrm{C}^\wedge$. ☐

### 3.1. Hinich's functor $\mathcal{F}$

We recall that Hinich's functor $\mathcal{F}$ is, by definition, formally left adjoint to `Ass`, as in:

$$\begin{tikzcd}
\Delta_{/\mathtt{Ass}} \arrow[r, "\mathcal{F}"] \arrow[d, tail] & \mathbf{Cat} \arrow[dl, "\mathtt{Ass}"] \\
\mathbf{Cat}_{/\mathtt{Ass}}. &
\end{tikzcd}$$

Moreover, $\mathcal{F}$ actually lifts to a functor into $\mathbf{Cat}^1$.

### 3.2. Model structure on a slice

The slice category $s\mathbf{Cat}_{/\mathtt{Ass}}$ inherits sets of fibrations, cofibrations, and weak equivalences from the corresponding sets in the Bergner model structure by inverse image along the forgetful functor $s\mathbf{Cat}_{/\mathtt{Ass}} \to s\mathbf{Cat}$. It is well-known that because `Ass` is fibrant, this is also a model structure. By [3, Cor. 7.6.13], its localisation is the $\infty$-category $\mathbf{Cat}_{/\mathtt{Ass}}$.

### 3.3. Gepner–Haugseng's $\mathcal{O}_X$

Gepner–Haugseng's $\mathcal{O}_X$ is a functor of a simplicial category $X : s\mathbf{Cat}$ valued in simplicial multicategories. Throughout, we tacitly replace this multicategory with its operad of operators (reviewed in [4, §2.2]), which is a simplicial category over `Ass`.

It follows from the definition, and the fact that weak equivalences are preserved by products, that $\mathcal{O}_X$ descends to a functor $\mathbf{Cat} \to \mathbf{Cat}_{/\mathtt{Ass}}$.

We are now ready to formulate our main reduction step in the proof of Theorem 1.1.

**Proposition 3.3** ($\hat{\mathcal{F}}_\bullet \dashv \mathcal{O}_\bullet$)**.** *Suppose given a formal adjunction* $\mathrm{adj} \colon \mathcal{F} \dashv \mathcal{O}$ *between $\mathcal{F}$ and the restriction of $\mathcal{O}$ to the category* $\mathbf{Cat}^1$ *of 1-categories.*

*Then there is an extension of* $\mathrm{adj}$ *to a Quillen adjunction*

$$\hat{\mathcal{F}}_\bullet \colon s\mathbf{Cat}_{/\mathtt{Ass}} \rightleftarrows s\mathbf{Cat} : \mathcal{O}$$

*between the Bergner model structure on* $s\mathbf{Cat}$ *and the slice structure on* $s\mathbf{Cat}_{/\mathtt{Ass}}$*.*

*Proof.* We define the functor $\hat{\mathcal{F}}_\bullet \colon s\mathbf{Cat}_{/\mathtt{Ass}} \to s\mathbf{Cat}$ by applying the left Kan extension $\hat{\mathcal{F}} \colon \mathbf{Cat}^1_{\mathtt{Ass}} \to \mathbf{Cat}^1$ in each dimension. (Note that $\mathcal{O}$ is also defined by applying the same operation in each dimension, as products of simplicial sets are calculated dimension-by-dimension.) Then mapping sets are computed using an end:

$$\begin{aligned}\mathrm{Fun}_{\mathtt{Ass}}(S_\bullet, (\mathcal{O}_X)_\bullet) &= \mathrm{end}_{n,m}\mathrm{Fun}_{\mathtt{Ass}}(S_m, \mathcal{O}_{X_n})\\ &= \mathrm{end}_{n,m}\mathrm{Fun}(\hat{\mathcal{F}}(S_m), X_n)\\ &= \mathrm{Fun}(\hat{\mathcal{F}}_\bullet S_\bullet, X_\bullet)\end{aligned}$$

so that $\hat{\mathcal{F}}_\bullet \dashv \mathcal{O}_\bullet$.

Moreover, $\mathcal{O}$ preserves weak equivalences and fibrations because weak equivalences and fibrations of simplicial sets are preserved by products and the mapping spaces in $\mathcal{O}_X$ are products of those in $X$. In other words, it is a right Quillen functor, so $\hat{\mathcal{F}} \dashv \mathcal{O}$ is a Quillen adjunction. □

**Corollary 3.4** ($\mathcal{F} \dashv \mathcal{O}$)**.** *Suppose given a formal adjunction* adj: $\mathcal{F} \dashv \mathcal{O}$ *between* $\mathcal{F}$ *and the restriction of* $\mathcal{O}$ *to the category* $\mathbf{Cat}^1$ *of classical 1-categories. There is an extension to an adjunction*

$$\hat{\mathcal{F}}\colon \mathbf{Cat}_{/\Delta^{\mathrm{op}}} \rightleftarrows \mathbf{Cat} : \mathcal{O}$$

*between* $\mathcal{O}$ *and a left Kan extension* $\hat{\mathcal{F}}$ *of* $\mathcal{F}$*.*

*Proof.* By [11], the derived functors of a Quillen adjunction yield an adjunction of the localised $\infty$-categories. □

## 4. The technical bit

In this section, we will construct the adjunction at the level of the $(1,1)$-category $\mathbf{Cat}^1$ of 1-categories (cf. §2.2). This comparison is straightforward, and it is foreshadowed in [6, (3.2.9)], but sadly the specifics are rather fiddly.

**Proposition 4.1** ($\mathcal{F} \dashv \mathcal{O}$)**.** *There is a formal adjunction*

$$\begin{tikzcd} \Delta_{/\mathtt{Ass}} \arrow[r, "\mathcal{F}"] \arrow[d, tail] & \mathbf{Cat}^1 \arrow[dl, "\mathcal{O}"] \\ \mathbf{Cat}^1_{/\mathtt{Ass}} & \end{tikzcd}$$

*between Hinich's functor* $\mathcal{F}$ *and the restriction of Gepner–Haugseng's* $\mathcal{O}$ *to classical 1-categories.*

### 4.1. Notation

In what follows, we denote objects of $\mathtt{Ass} = \Delta^{\mathrm{op}}$ in the format $[n]$, corresponding to the $n$-simplex $\Delta^n : \Delta$ in the opposite category. If $\sigma\colon \Delta^k \to \mathtt{Ass}$ is a $k$-simplex of $\mathtt{Ass}$, we write $\sigma_i$ for its $i$th vertex, so $\sigma_i = [n]$ for some $n : \mathbb{N}$.

If $\mathcal{O} \to \mathtt{Ass}$ is a category over $\mathtt{Ass}$, such as a planar operad, then write $\mathcal{O}[n]$ for the fibre of the structure functor over $[n] : \mathtt{Ass}$.

*Inert.* Let us write $[k] \subseteq [n]$ when $[k]$ is a convex subset of the totally ordered set $[n]$ (dual to an *inert* morphism $[n] \to [k]$ in $\mathtt{Ass}$ [4, Def. 3.1.1]). If $f\colon [m] \to [n]$ is a morphism in $\mathtt{Ass}$, there is an induced pullback operation $f^{-1}$ on the poset of convex subsets. It is constructed by taking the convex hull of the image under the dual map $\Delta^n \to \Delta^m$.

*Decomposition.* In this case, the $n$ convex inclusions $i : [1] \subseteq [n]$ induce a decomposition of $[m]$ as a concatenation $\mathrm{cat}_{i:[1]\subseteq[n]}[m_i] = f^{-1}i$. This decomposition is used to formulate the key axiom in the definition of a planar operad [**4**, Def. 3.1.3, (ii)].

### 4.2. Graphs

Denote by $\Delta^{[0,1]} \subset \Delta$ the full subcategory spanned by $\Delta^0$ and $\Delta^1$. An *oriented graph* $\Gamma$ defines a presheaf of sets on $\Delta^{[0,1]}$ by defining $\mathrm{Map}(\Delta^k, \Gamma)$ to be the set of vertices of $\Gamma$ if $k = 0$ and the union of the sets of (oriented) edges and of vertices for $k = 1$. The graph may be recovered from its associated presheaf: the vertices are the 0-simplices, the edges are the non-degenerate 1-simplices, and the face maps yield the incidence relation.

Accordingly, we define the *category of oriented graphs* to be the category $\mathbf{Grf} = \mathrm{PSh}^0(\Delta^{[0,1]})$ of presheaves of sets on $\Delta^{[0,1]}$. Left Kan extension of the inclusion $\Delta^{[0,1]} \subset \mathbf{Cat}$ yields a colimit-preserving functor

$$\langle - \rangle \colon \mathbf{Grf} \to \mathbf{Cat}, \tag{1}$$

*free category* functor. It factors through $\mathbf{Cat}^1$ (whether we consider the latter as a $(1,1)$-category or a $(2,1)$-category).

The finite inhabited totally ordered sets may be thought of as oriented graphs with vertices the elements of the set and edges the nearest-neighbour order relations. Under this correspondence, the $n$-simplex corresponds to an $A_{n+1}$ graph. Let $\Delta^{\mathrm{Grf}} \subset \mathbf{Grf}$ be the full subcategory spanned by these objects. The free category functor fits into a square

$$\begin{tikzcd}
\Delta^{\mathrm{Grf}} \arrow[r] \arrow[d] & \Delta \arrow[d] \\
\mathbf{Grf} \arrow[r] & \mathbf{Cat}.
\end{tikzcd}$$

The map $\Delta^{\mathrm{Grf}} \to \Delta$ is the inclusion of a subcategory whose morphisms, in terms of the usual terminology for morphisms in $\Delta$, are generated by the degeneracy and *outer* face maps.

We denote by $\Delta^{[0,1]}_{/\mathtt{Ass}} \subset \Delta^{\mathrm{Grf}}_{/\mathtt{Ass}} \subset \Delta_{/\mathtt{Ass}}$ the corresponding comma categories.

### 4.3. Summary of the definition of $\mathcal{F}$

The definition of the functor $\mathcal{F}$ is rather involved — it occupies 5 pages of [**6**, §3.2] — and since the proof of 4.1 will follow the same lines it will be helpful to have a summary of it here.

1. First, $\mathcal{F}^{[0,1]}$ is defined explicitly on the full subcategory $\Delta^{[0,1]} \subseteq \Delta$ spanned by the zero and one-simplices of `Ass`:
   (a) It is defined as a map on the set $\{\Delta^0\}_{/\mathtt{Ass}}$ of zero-simplices by the formula:

$$\mathcal{F}^{[0,1]}(\sigma \colon \Delta^0 \to \mathtt{Ass}) := \{[1] \subseteq \sigma_0\} \times \{\mathbf{x}, \mathbf{y}\}, \tag{2}$$

   where $\{[1] \subseteq \sigma_0\}$ is the set (of cardinality $\#\sigma_0 - 1$) of edges of $\sigma_0$.
   Let us denote by $\mathcal{F}_0 \colon \Delta_{/\mathtt{Ass}} \to \mathbf{Set}$ the functor that associates to $\sigma \colon \Delta^k \to \mathtt{Ass}$ the disjoint union of $\mathcal{F}^{[0,1]}(\tau)$, where $\tau$ ranges over all vertices of $\Delta^k$.

Note that this is a left Kan extension of the restriction of $\mathcal{F}$ to the domain $\{\Delta^0\}_{\mathtt{Ass}}$ of (2). In general, $\mathcal{F}(\sigma)$ will be a category with underlying set of objects $\mathcal{F}_0(\sigma)$.

(b) For the set of 1-simplices I will not reproduce the full definition [**6**, (3.2.3)]; suffice it to say that for the generating case $\sigma_1 = [1]$, $\mathcal{F}^{[0,1]}(\sigma)$ is described by the diagram

$$\begin{tikzcd}
\mathbf{x}_{01} & \mathbf{y}_{01} \arrow[r] & \mathbf{x}_{12} & \cdots & \mathbf{y}_{(n-2)(n-1)} \arrow[r] & \mathbf{x}_{(n-1)n} & \mathbf{y}_{(n-1)n} \arrow[dll] \\
 & & \mathbf{x}_{01} \arrow[ull] & & \mathbf{y}_{01} & &
\end{tikzcd}$$

where the vertices of the two rows are $\mathcal{F}^{[0,1]}(\sigma_0)$ and $\mathcal{F}^{[0,1]}(\sigma_1)$, respectively. This diagram appears as [**6**, (3.2.4), fig. (30)]. More generally, for any $\sigma\colon \Delta^1 \to \mathtt{Ass}$, the result is a finite disjoint union of categories equivalent to $\Delta^1$.

(c) The action of $\mathcal{F}^{[0,1]}$ on the two face maps of $\Delta^{[0,1]}$ is such that their coproduct is the inclusion

$$\mathcal{F}_0(\sigma) = \mathcal{F}^{[0,1]}(\sigma_0) \sqcup \mathcal{F}^{[0,1]}(\sigma_1) \hookrightarrow \mathcal{F}^{[0,1]}(\sigma)$$

for any $\sigma\colon \Delta^1 \to \mathtt{Ass}$ that was implicit in the preceding diagram. For the degeneracy map, note only that the image of a degenerate edge on the vertex $[n]$ is the graph

$$\begin{tikzcd}
\mathbf{x}_{01} & \mathbf{y}_{01} \arrow[d] & \cdots & \mathbf{x}_{(n-1)n} & \mathbf{y}_{(n-1)n} \arrow[d] \\
\mathbf{x}_{01} \arrow[u] & \mathbf{y}_{01} & \cdots & \mathbf{x}_{(n-1)n} \arrow[u] & \mathbf{y}_{(n-1)n}
\end{tikzcd}$$

which maps in a unique way to $\mathcal{F}^{[0,1]}([n])$ preserving the labelling.

By illustration, the face maps are sections of the degeneracy map, that is, they obey the simplicial identities; hence these formulae define a functor $\mathcal{F}^{[0,1]}\colon \Delta^{[0,1]}_{/\mathtt{Ass}} \to \mathbf{Grf}$.

2. By left Kan extension we may immediately extend $\mathcal{F}^{[0,1]}$ to a functor of oriented graphs $\mathcal{F}^{\mathrm{Grf}}\colon \mathbf{Grf}_{/\mathtt{Ass}} \to \mathbf{Grf}$, and in particular, of $\Delta^{\mathrm{Grf}}_{/\mathtt{Ass}}$. The canonical map $\mathcal{F}_0^{\mathrm{Grf}} \to \mathcal{F}^{\mathrm{Grf}}$ is the inclusion of the set of objects.
   (Actually, Hinich's definition uses an explicit colimit [**6**, (3.2.5)] over the poset of cells of the graph $\Delta^k$. This poset is a reflective subcategory of the full comma category $\Delta^{[0,1]} \downarrow_{\mathbf{Grf}} \Delta^k$ with reflector given by taking the image. In particular, it is cofinal, and so this colimit does indeed compute the left Kan extension.)
3. Finally, one defines functoriality for the *inner* face maps, which encode composition in $\mathtt{Ass}$ and hence in the multicategory $\mathtt{Ass}_X$. Since $\mathcal{F}(\sigma) = \mathcal{F}^{\mathrm{Grf}}(\sigma)$ is a poset, and we already have the action of inner face maps on the underlying set $\mathcal{F}_0(\sigma)$, it is merely a condition for them to be functors. This is checked in [**6**, Lemma 3.2.7].

### 4.4. Vertices and edges

We will define the natural isomorphism $\mathrm{Fun}_{\mathtt{Ass}}(-, \mathcal{O}_-) \cong \mathrm{Fun}(\mathcal{F}(-), -)$ separately for each object and morphism in $\Delta^{[0,1]}$. (We are allowed to do this because we are now working in $\mathbf{Cat}^1$.) All isomorphisms constructed are natural in $X$.

($|\sigma| = 0$). The operad $\mathcal{O}_X$ has set of objects $\mathrm{Ob}(X) \times \mathrm{Ob}(X)$ which we identify with $\mathrm{Map}(\{\mathbf{x}, \mathbf{y}\}, X)$ in the order implied by the orthography.

$$\begin{aligned} \mathrm{adj}_0 : \mathrm{Fun}_{\mathtt{Ass}}(\sigma, \mathcal{O}_X) &= \mathrm{Ob}(\mathcal{O}_X(\sigma_0)) \\ &= (\mathrm{Ob}(X) \times \mathrm{Ob}(X))^{\{[1] \subseteq \sigma_0\}} \\ &= \mathrm{Fun}(\{[1] \subseteq \sigma_0\} \times \{\mathbf{x}, \mathbf{y}\}, X) \end{aligned}$$

(Note that there is exactly one other way to identify these spaces naturally in $X$, which simply reverses the factors. The arguments of §5 show that it is not possible to make this identification compatible with edges.)

($|\sigma| = 1$). A 1-simplex $\sigma \colon \Delta^1 \to \mathtt{Ass}$ is the data of a morphism $\sigma_0 \to \sigma_1$ in $\mathtt{Ass}$. As in Definition 4.3-i)-(b) of $\mathcal{F}$, we separate the case that $\sigma_1 = [1]$.

- ($\sigma_1 = [1]$). Write $[n] = \sigma_0$. The equivalence of $\mathrm{Map}(\mathcal{F}(\sigma), X)$ with [4, Def. 4.2.4] is defined by the labelling

$$\begin{tikzcd} x_0 & y_1 \arrow[r] & x_1 & \cdots & y_{n-1} \arrow[r] & x_{n-1} & y_n \arrow[dlll] \\ & & y_0 \arrow[ull] & x_n & & & \end{tikzcd} \tag{3}$$

of Hinich's diagram with Gepner–Haugseng's variables: the set of such diagrams is precisely

$$\mathcal{O}_X(\underbrace{(x_0, y_1)}_{u_1}, \ldots, \underbrace{(x_{n-1}, y_n)}_{u_n}; \underbrace{(y_0, x_n)}_{v}) = X(y_0, x_0) \times \cdots \times X(y_n, x_n) \tag{4}$$

as a functor of $(x_i, y_i)_{i=0}^n = (u_j, v)_{j=1}^n$, that is, it is the set of $n$-ary operations of $\mathcal{O}_X$.

Now integrating over $(X^{\mathrm{op}} \times X)^n$,

$$\begin{aligned} \mathrm{adj}_1 : \mathrm{Fun}_{\mathtt{Ass}}(\sigma, \mathcal{O}_X) &= \int_{u:\mathcal{O}_X[n]} \int_{v:\mathcal{O}_X[1]} \mathcal{O}_X(u, v) && (5) \\ &= \int_{u:(X^{\mathrm{op}} \times X)^n} \int_{v:X^{\mathrm{op}} \times X} \mathcal{O}_X(u, v) \\ &= \int_{u:(X^{\mathrm{op}} \times X)^n} \int_{v:X^{\mathrm{op}} \times X} \prod_{i=0}^n X(y_i, x_i) && \text{by def. (4)} \\ &= \mathrm{Fun}(\mathcal{F}(\sigma), X). && \text{by labelling (3)} \end{aligned}$$

For reasons of formatting I have found it necessary to mix Gepner–Haugseng's variables $x, y$ with my own variables $u, v$, underset in (4). Fully expanded, the definition of the polymorphism set in this notation is as follows:

$$\mathcal{O}_X((u_1^{\mathbf{x}}, u_1^{\mathbf{y}}), \ldots, (u_n^{\mathbf{x}}, u_n^{\mathbf{y}}); (v^{\mathbf{x}}, v^{\mathbf{y}})) = X(v^{\mathbf{x}}, u_1^{\mathbf{x}}) \times \prod_{i=1}^n X\left(u_{i-1}^{\mathbf{y}}, u_i^{\mathbf{x}}\right) \times X(u_n^{\mathbf{y}}, v^{\mathbf{y}}) . \tag{6}$$

The mapping from my variables to Hinich's is indicated by the superscripts. This also Beware that Gepner–Haugseng $x$s and $y$s do not occur in quite the same order as Hinich's $\mathbf{x}$s and $\mathbf{y}$s as described in Definition 4.3-i)-(c).

- (General case). As explained in §4.1, to each segment $i : [1] \subseteq \sigma_1 \simeq [m]$ there corresponds by pullback a segment $[n_i] \simeq \sigma_0^i \subseteq \sigma_0$, and $\sigma_0 = \mathrm{cat}_{i:[1]\subseteq\sigma_1}\sigma_0^i$. Denote by $\sigma^i$ the corresponding arrow $[n_i] \to [1]$ so that $\sigma = \mathrm{cat}_{i:[1]\subseteq\sigma_1}\sigma^i$.
  (Potential confusion: this concatenation is postcomposition of maps into $\mathtt{Ass}$ with the concatenation operation there, *not* composition of 1-simplices.)
  By definition,
  $$\mathcal{F}(\sigma) := \coprod_{i:[1]\subseteq\sigma_1} \mathcal{F}(\sigma^i) \tag{7}$$
  and so we calculate:
  $$\begin{aligned}
  \mathrm{Fun}_{\mathtt{Ass}}(\sigma, \mathcal{O}_X) &= \int_{u:\mathcal{O}_X[n]} \int_{v:\mathcal{O}_X[m]} \mathcal{O}_X(u,v) \\
  &= \int_{u:\mathcal{O}_X[n]} \int_{v:\mathcal{O}_X[m]} \prod_{i:[1]\subseteq\sigma_1} \mathcal{O}_X(u_i,v_i) && \mathcal{O}_X \text{ is an operad} \\
  &= \prod_{i:[1]\subseteq\sigma_1} \int_{u:\mathcal{O}_X[n_i]} \int_{v:\mathcal{O}_X[1]} \mathcal{O}_X(u,v) && \textstyle\int \leftrightarrow \prod \text{ (cf. 4.2)} \quad (8) \\
  &= \prod_{i=1}^{n} \mathrm{Fun}(\mathcal{F}(\sigma_i), X) && \text{via adj (5)} \\
  &= \mathrm{Fun}(\mathcal{F}(\sigma), X). && \text{by def. (7)}
  \end{aligned}$$

## 4.5. Naturality

*Face maps.* Compatibility with face maps is the commutativity of

$$\begin{tikzcd}
\iint_{u,v} \mathcal{O}_X(u,v) \arrow[r, leftrightarrow, "\mathrm{adj}_1"] \arrow[d] & \mathrm{Fun}(\mathcal{F}(\sigma), X) \arrow[d] \\
(\mathrm{Ob}(X) \times \mathrm{Ob}(X))^{n+m} \arrow[r, leftrightarrow, "\mathrm{adj}_0"] & \mathrm{Fun}(\mathcal{F}(\sigma_0), X) \times \mathrm{Fun}(\mathcal{F}(\sigma_1), X)
\end{tikzcd}$$

for any edge $\sigma : \Delta^{[0,1]}_{/\mathtt{Ass}}$. Now by definition, $\mathrm{adj}_1$ covers an identification between the bottom two terms defined by the labelling (3), which we must check is indeed given by $\mathrm{adj}_0$, after relabelling variables. The appropriate relabelling is explained in (6).

*Degeneracy.* Since the operad structure gives us a decomposition

$$\begin{tikzcd}
\mathrm{Fun}_{\mathtt{Ass}}(\sigma, \mathcal{O}_X) \arrow[r, equal] \arrow[d] & (X^{\mathrm{op}} \times X)^{\{[1]\subseteq\sigma_0\}} \arrow[d] \\
\mathrm{Fun}_{\mathtt{Ass}}(\delta\sigma, \mathcal{O}_X) \arrow[r, equal] & (X^{\mathrm{op}} \times X)^{\{[1]\subseteq\sigma_0\}\times\Delta^1}
\end{tikzcd}$$

for any vertex $\sigma$, it suffices to check that the square

$$\begin{tikzcd}
X^{\mathrm{op}} \times X \arrow[r, leftrightarrow, "\mathrm{adj}_0"] \arrow[d] & \mathrm{Fun}(\mathcal{F}(\sigma), X) \arrow[d] \\
(X^{\mathrm{op}} \times X)^{\Delta^1} \arrow[r, leftrightarrow, "\mathrm{adj}_1"] & \mathrm{Fun}(\mathcal{F}(\delta\sigma), X)
\end{tikzcd}$$

commutes in the case $\sigma_0 = [1]$. But then, applying $\mathrm{adj}_0$ to $u = (u^{\mathbf{x}}, u^{\mathbf{y}})$ yields the diagram

$$\begin{tikzcd}
u^{\mathbf{x}} & u^{\mathbf{y}} \arrow[d, "\mathrm{id}"] \\
u^{\mathbf{x}} \arrow[u, "\mathrm{id}"] & u^{\mathbf{y}}
\end{tikzcd}$$

which the reader will readily see is a special case of the labelling (3) applied to $\mathcal{O}_X(u,u) = \mathcal{O}_X((u^{\mathbf{x}}, u^{\mathbf{y}}), (u^{\mathbf{x}}, u^{\mathbf{y}}))$.

*Remark 4.2.* I comment here on the exchange of products with Grothendieck integral that occurs in line (8). We have used here the commutativity of a diagram

$$\begin{tikzcd}
\prod_i \mathrm{Fun}(\mathrm{C}_i, \mathbf{S}) \arrow[r] \arrow[d] & \prod_i \mathrm{RFib}(\mathrm{C}_i) \arrow[d] \\
\mathrm{Fun}(\prod_i \mathrm{C}_i, \mathbf{S}) \arrow[r] & \mathrm{RFib}(\mathrm{C}_i)
\end{tikzcd}$$

I don't know a reference for this fact, but it can be deduced rather easily from the naturality of the construction in C and the fact that $\int$ preserves products (and indeed, all limits).

### 4.6. Inner face maps

For our adjunct isomorphism adj to be natural in $\Delta_{\mathtt{Ass}}$, rather than merely $\Delta_{\mathtt{Ass}}^{\mathrm{Grf}}$, is yet another condition: that the square

$$\begin{tikzcd}
\mathrm{Fun}_{\mathtt{Ass}}(\sigma, \mathcal{O}_X) \arrow[r, leftrightarrow] \arrow[d, "\circ"'] & \mathrm{Fun}(\mathcal{F}(\sigma), X) \arrow[d] \\
\mathrm{Fun}_{\mathtt{Ass}}(\tau, \mathcal{O}_X) \arrow[r, leftrightarrow] & \mathrm{Fun}(\mathcal{F}(\tau), X)
\end{tikzcd}$$

induced by the inner face map $\Delta^1 \to \Delta^2$ is commutative. Here $\tau$ is the inner edge of the 2-simplex $\sigma : \Delta_{\mathtt{Ass}}$, whereby the left-hand vertical arrow is the composition law in $\mathcal{O}_X$. Since [**4**, Def. 4.2.4] is not absolutely explicit about the definition of this composition law, the most I can show here is that what they say corresponds intuitively to the behaviour of the right-hand vertical arrow.

The functor $\mathcal{F}(\tau) \to \langle \mathcal{F}(\sigma) \rangle$ (see (1) for notation) sends each morphism of $\mathcal{F}(\tau)$ to a (unique) composition of edges of the graph $\mathcal{F}(\sigma)$. Hence, the right-hand vertical arrow above takes a representation of the graph $\mathcal{F}(\sigma)$ in $X$ to the representation of $\mathcal{F}(\tau)$ obtained by composing the corresponding morphisms in $X$. It is reasonable to suppose that this is what Gepner–Haugseng meant by defining composition "in the obvious way, using composition in $X$".

The compositions that actually occur are those which, in terms of the bordism interpretation sketched in the introduction, lie along 'horizontal' boundary components of the surface with boundary $\mathcal{F}(\sigma)$. For more information, consult the pictures in [**6**, (3.2.6)].

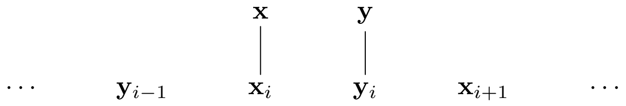
## 5. Unicity

In this section, we will show that the automorphism group of $\mathcal{F}$ is trivial. We continue to use the conventions explained in §§4.1, 4.3.

**Proposition 5.1.** *The automorphism groups of* $\mathcal{F}\colon \Delta_{/\mathtt{Ass}} \to \mathbf{Cat}^1$ *and of its composite with groupoid completion* $B\mathcal{F}\colon \Delta_{/\mathtt{Ass}} \to \mathbf{S}$ *are trivial.*

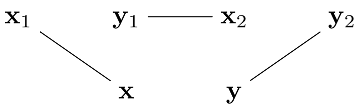
*Proof.* We will argue for $B\mathcal{F}$; the argument for $\mathcal{F}$ is the same, except a bit shorter. So let $\phi : \mathrm{Aut}(B\mathcal{F})$. By naturality with respect to the face maps $\Delta^0 \to \Delta^k$, the action of $\phi$ on $\mathcal{F}(\sigma)$ must preserve the decomposition of the set of objects as $\coprod_{i:\Delta^k} \mathcal{F}(\sigma_i)$. Since by [6, Lemma 3.2.5] each $\mathcal{F}(\sigma)$ is simply-laced, if $\phi$ is the identity on each 0-simplex, then it is the identity. Thus, it suffices to show that $\phi$ acts trivially on $B\mathcal{F}$ of 0-simplices.

*Decomposition into segments.* Each segment $[1] \subseteq \sigma_0$ yields a graph



from which it may be seen that $\phi$ must preserve each pair $\{\mathbf{x}_i, \mathbf{y}_i\}$, and moreover, act on that pair through $\phi \mid B\mathcal{F}([1])$.

($\phi \mid B\mathcal{F}([1])$). It remains to show that $\phi \mid B\mathcal{F}([1])$ is necessarily trivial. This follows from the diagram



associated by $B\mathcal{F}$ to the 1-simplex $[2] \to [1]$ given by the inner face map: no symmetry of this diagram preserves both the decomposition into top and bottom rows and of the top row into two pairs. ☐

*Remark 5.2* (Opposite). The formation of opposites as discussed in [6, §2.9] has the following expression in the variables of the present section:

$$\begin{aligned}
\mathcal{O}_{X^{\mathrm{op}}}\left[(u_1^{\mathbf{x}},u_1^{\mathbf{y}}),\dots,(u_n^{\mathbf{x}},u_n^{\mathbf{y}});(v^{\mathbf{x}},v^{\mathbf{y}})\right] &= X^{\mathrm{op}}(v^{\mathbf{x}},u_1^{\mathbf{x}})\times\prod_{i=1}^{n-1}X^{\mathrm{op}}(u_i^{\mathbf{y}},u_{i+1}^{\mathbf{x}})\times X^{\mathrm{op}}(u_n^{\mathbf{y}},v^{\mathbf{y}})\\
&= X(v^{\mathbf{y}},u_n^{\mathbf{y}})\times\prod_{i=1}^{n-1}X(u_{n-i+1}^{\mathbf{x}},u_{n-i}^{\mathbf{y}})\times X(u_1^{\mathbf{x}},v^{\mathbf{x}})\\
&= \mathcal{O}_X\left[(u_n^{\mathbf{y}},u_n^{\mathbf{x}}),\ \dots,\ (u_1^{\mathbf{y}},u_1^{\mathbf{x}});\ (v^{\mathbf{y}},v^{\mathbf{x}})\right]\\
&= \mathcal{O}_X^{\mathrm{rev}}\left[(u_1^{\mathbf{y}},u_1^{\mathbf{x}}),\ \dots,\ (u_n^{\mathbf{y}},u_n^{\mathbf{x}});\ (v^{\mathbf{y}},v^{\mathbf{x}})\right],
\end{aligned}$$

where $\mathcal{O}^{\mathrm{rev}}$ denotes the reversed operad. In other words, the right Quillen functor $\mathcal{O}$ can be made $\mathbb{Z}/2\mathbb{Z}$-equivariant with respect to the action by opposite on $s\mathbf{Cat}$ and reversal on $s\mathbf{Cat}_{/\mathtt{Ass}}$. Correspondingly, the $\infty$-adjunction $\mathcal{F} \dashv \mathtt{Ass}$ is also $\mathbb{Z}/2\mathbb{Z}$-equivariant. Since the adjunction has no automorphisms, this equivariance is even unique.

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