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LEFT BOUSFIELD LOCALIZATION AND EILENBERG–MOORE CATEGORIES

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Abstract

We prove the equivalence of several hypotheses that have appeared recently in the literature for studying left Bousfield localization and algebras over a monad. We find conditions so that there is a model structure for local algebras, so that localization preserves algebras, and so that localization lifts to the level of algebras. We include examples coming from the theory of colored operads, and applications to spaces, spectra, and chain complexes.

1. Introduction

Left Bousfield localization has become a fundamental tool in modern abstract homotopy theory. The ability to take a well-behaved model category and a prescribed set of maps, and then produce a new model structure where those maps are weak equivalences, has applications in a variety of settings. Left Bousfield localization is required to construct modern stable model structures for spectra, including equivariant and motivic spectra. Left Bousfield localization also provides a powerful computational device for studying spaces, spectra, homology theories, and numerous algebraic examples of interest (e.g., in the model categories $\mathrm{Ch}(R)$ and R-mod that arise when studying homological algebra and the stable module category, respectively, for a ring R).

In recent years, several groups of researchers have been applying the machinery of left Bousfield localization to better understand algebras over (colored) operads, especially results regarding when algebraic structure is preserved by localization, e.g., [**Bat17, BB17, BH15, HH14, HH16, CGMV10, CRT14, GRSO18, HW20, Whi14b, WY18**]. Unsurprisingly, different approaches have emerged. In this paper, we will prove that these approaches are equivalent, and will use this equivalence to provide new structural features that may be used in any of these settings.

Our setting will be a model category $\mathcal{M}$, with an action of a monad T, and with a prescribed set of maps $\mathcal{C}$ that we wish to invert. In our applications, $\mathcal{M}$ will be sufficiently nice that the left Bousfield localization $L_{\mathcal{C}}(\mathcal{M})$ exists, and so that the category of T-algebras, $\mathrm{Alg}_T(\mathcal{M})$, admits a transferred semi-model structure (reviewed in

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Section 2). Our main theorem follows (and will be proven as Theorem 5.6). For the sake of generality, we proved the theorem without requiring strong hypotheses on $\mathcal{M}$. All of the assumptions in the statement below are introduced in Section 2.

Theorem A. *Suppose that $\mathcal{M}$ is a model category, $\mathcal{C}$ is a set of morphisms in $\mathcal{M}$, and $L_{\mathcal{C}}(\mathcal{M})$ exists and has a set of generating cofibrations with cofibrant domains. Suppose that $\mathcal{M}$ contains a saturated class of morphisms $\mathcal{K}$ such that T is $\mathcal{K}$-semi-admissible (so $Alg_T(\mathcal{M})$ has a transferred semi-model structure), $\mathcal{M}$ is $\mathcal{K}$-compactly generated, and $L_{\mathcal{C}}(\mathcal{M})$ is $\mathcal{K}$-compactly generated. Then the following are equivalent:*

1. *$L_{\mathcal{C}}$ lifts to a left Bousfield localization $L_{F(\mathcal{C})}$ of semi-model categories on $Alg_T(\mathcal{M})$, where $F(\mathcal{C})$ is the set of free T-algebra maps on $\mathcal{C}$.*
2. *The forgetful functor $U\colon Alg_T(\mathcal{M}) \to \mathcal{M}$ preserves local equivalences.*
3. *$Alg_T(L_{\mathcal{C}}(\mathcal{M}))$ has a transferred semi-model structure.*
4. *$L_{\mathcal{C}}$ preserves T-algebras.*

Furthermore, any of the above implies that

(5) T preserves $\mathcal{C}$-local equivalences between cofibrant objects.

This theorem unifies all known approaches to studying the homotopy theory of localization for algebras over a monad. There is also a dual result [**WY16**], in the setting of right Bousfield localization, that also has numerous applications [**WY20**]. In general, all of the statements of Theorem A are hard to verify, but in any particular setting, usually at least one of them is approachable. Many examples are given in Example 5.7 where one (hence all) of the equivalent conditions hold. In particular, the second author has a series of papers [**WY18, WY20, WY19, WY17**] providing conditions on the model category $\mathcal{M}$, the monad T (usually given by a colored operad in these settings), and the left Bousfield localization L_C so that (3), and hence all of the equivalent statements, are satisfied. Theorem A implies that, in all these settings, the localizations can actually be constructed on the level of algebras (using (1)), a result that was previously unknown. These settings include simplicial sets and topological spaces (with the Quillen model structure), chain complexes (with the projective model structure), symmetric spectra, (equivariant) orthogonal spectra, the category of small categories, and the stable module category of a ring.

Similarly, in a series of papers [**CGMV10, CRT14, GRSO12, GRSO18, JN14**], Casacuberta, Gutiérrez, and others provide numerous examples of model categories, monads, and left Bousfield localizations where conditions (1) or (2) hold, and Theorem A now implies the existence of transferred semi-model structures $\mathrm{Alg}_T(L_{\mathcal{C}}(\mathcal{M}))$ that were not previously known to exist. These examples include symmetric spectra, S-modules, abelian localizations, and motivic spectra, with different conditions to check than in [**WY18**], that are easier for certain localizations or monads. Finally, several recent papers have focused on preservation of equivariant multiplicative norms (encoded by equivariant operads) under left Bousfield localizations of equivariant spaces and spectra [**HH14, HH16, GW18**]. In these settings, Theorem A now implies that localizations can be constructed on the level of algebras, and that transferred semi-model structures exist.

The authors proved Theorem A (and its partial converse, Theorem 6.2) as part of a larger research program, with applications to several well-studied conjectures in homotopy theory, higher category theory, and mathematical physics [**BW20a, BW20b**].

In Section 3, we discuss how to pass from the localized model structure $L_{\mathcal{C}}(\mathcal{M})$ to the category $\mathrm{Alg}_T(L_{\mathcal{C}}(\mathcal{M}))$, needed for item (3) in the theorem. In Section 4, we discuss lifting the localization $L_{\mathcal{C}}$ to a localization $L_{F(\mathcal{C})}$ of algebras, where $F(\mathcal{C})$ denotes the set of free T-algebra maps on $\mathcal{C}$. This is needed for items (1) and (2) of the theorem. In Section 5, we compare these two approaches to studying local algebras, and we also compare them to item (4) above regarding preservation of T-algebras by localization. After proving our main theorem in Section 5, we prove a partial converse in Theorem 6.2, i.e., we determine what can be said if we know that item (5) is true. In particular, we show that item (5), together with mild hypotheses on the monad T, implies that U reflects local equivalences. Numerous examples are given throughout the paper, and the motivating applications are explained in Section 6.

2. Preliminaries

We assume that the reader is familiar with the basics of model categories. For our entire paper, $\mathcal{M}$ will denote a cofibrantly generated model category (or, sometimes, semi-model category), and I (resp. J) will denote the generating cofibrations (resp. generating trivial cofibrations). Semi-model categories are recalled below, and familiarity with semi-model categories is not assumed.

2.1. Left Bousfield localization

Given a model category $\mathcal{M}$ and a set of morphisms $\mathcal{C}$, a left Bousfield localization of $\mathcal{M}$ with respect to $\mathcal{C}$ is, if it exists, a new model structure on $\mathcal{M}$, denoted $L_{\mathcal{C}}(\mathcal{M})$, satisfying the following universal property: the identity functor $\mathcal{M} \to L_{\mathcal{C}}(\mathcal{M})$ is a left Quillen functor taking the morphisms of $\mathcal{C}$ to weak equivalences, and for any model category $\mathcal{N}$ and any left Quillen functor $F\colon \mathcal{M} \to \mathcal{N}$ taking the maps in $\mathcal{C}$ to weak equivalences, there is a unique left Quillen functor $L_{\mathcal{C}}(\mathcal{M}) \to \mathcal{N}$ through which F factors.

For the sake of the following definition, we need to recall [**Hir03**, Notation 17.4.2]. Given a model category $\mathcal{M}$, and objects X and Y, we denote by $\mathrm{Map}_{\mathcal{M}}(X, Y)$ (or just $\mathrm{Map}(X, Y)$ if the context is clear) a simplicial set that is a functorial homotopy function complex (left, right, or two-sided). As shown in [**Hir03**, Theorem 17.4.13], the category of homotopy function complexes has contractible classifying space, and so the choice of a model for $\mathrm{Map}(X, Y)$ does not matter up to homotopy. Commonly, $\mathrm{Map}(X, Y)$ is constructed by taking a cofibrant replacement of X and a simplicial resolution of Y, or a cosimplicial resolution of X and a fibrant replacement of Y. Neither of these approaches requires (co)fibrancy assumptions on X or Y. Observe, furthermore, that $\mathrm{Map}(X, Y)$ is weakly equivalent to the simplicial set of morphisms in the hammock localization $L^H(\mathcal{M}, W)$ [**Bar10**, Scholium 3.64]. We will denote by $\mathrm{Hom}_{\mathcal{M}}(X, Y)$ the set of morphisms in $\mathcal{M}$ from X to Y.

Definition 2.1. Fix a model category $\mathcal{M}$ and a set of morphisms $\mathcal{C}$.

1. An object N is called *$\mathcal{C}$-local* if it is fibrant in $\mathcal{M}$ and for all $g\colon X \to Y$ in $\mathcal{C}$, the induced map on simplicial sets $\mathrm{Map}(g, N)\colon \mathrm{Map}(Y, N) \to \mathrm{Map}(X, N)$ is a weak equivalence.

2. A morphism $f\colon A \to B$ is a $\mathcal{C}$-*local equivalence* if for all N as above, the morphism $\mathrm{Map}(f, N)\colon \mathrm{Map}(B, N) \to \mathrm{Map}(A, N)$ is a weak equivalence.
3. The left Bousfield localization $L_{\mathcal{C}}\mathcal{M}$ of $\mathcal{M}$, if it exists, is a model structure on $\mathcal{M}$ defined to have weak equivalences the $\mathcal{C}$-local equivalences, to have the same cofibrations as $\mathcal{M}$, and to have the fibrations defined via the right lifting property.

Throughout this paper, we assume that $L_{\mathcal{C}}(\mathcal{M})$ exists, and that its fibrant objects are precisely the $\mathcal{C}$-local objects in $\mathcal{M}$. This can be guaranteed by assuming that $\mathcal{M}$ is left proper and either cellular [**Hir03**, Theorem 4.1.1] or combinatorial [**Bar10**, Theorem 4.7]. We do not need these conditions in the present paper, other than for the existence of $L_{\mathcal{C}}(\mathcal{M})$. Throughout this paper, we assume that $\mathcal{C}$ is a set of cofibrations between cofibrant objects. This can always be arranged by taking cofibrant replacements of the maps in $\mathcal{C}$.

2.2. Algebras over monads

We now discuss the theory required to transfer (semi-)model structures to categories of algebras.

Definition 2.2. 1. A *monad* (T, μ, η) on a category $\mathcal{M}$ [**Mac98**, VI.1] consists of a functor $T\colon \mathcal{M} \to \mathcal{M}$ and natural transformations $\mu\colon T \circ T \to T$ and $\eta\colon Id \to T$ satisfying associativity and unitality, i.e. $\mu \circ T\mu = \mu \circ \mu T\colon T^3 \to T$ and $\mu \circ T\eta = \mu \circ \eta T = Id_T\colon T \to T$.

2. A monad (T, μ, η) is *finitary* if T preserves filtered colimits.
3. A T-*algebra* in $\mathcal{M}$ is a pair (X, h) consisting of an object X of $\mathcal{M}$ and a morphism $h\colon TX \to X$ satisfying associativity ($h \circ Th = h \circ \mu_X\colon T^2X \to X$) and unitality ($h \circ \eta_X = Id_X\colon X \to X$). A morphism of T-algebras $(X, h) \to (X', h')$ is a morphism $f\colon X \to X' \in \mathcal{M}$ compatible with the structure morphisms h and h'.
4. With (T, μ, η) abbreviated to T, the category of T-algebras is denoted $\mathrm{Alg}_T(\mathcal{M})$. The corresponding free-forgetful adjunction is denoted by

$$\mathcal{M} \underset{U}{\overset{F}{\rightleftarrows}} \mathrm{Alg}_T(\mathcal{M}) .$$

For example, when $(\mathcal{M}, \otimes, 1)$ is a cocomplete monoidal category, the category of monoids is encoded as T-algebras for the monad $TX = \coprod_{n \geqslant 0} X^{\otimes n}$. Similarly, for any colored operad P in $\mathcal{M}$ (with color set $\mathfrak{C}$), the category of P-algebras is encoded as T-algebras in $\mathcal{M}^{\mathfrak{C}}$ for the free P-algebra monad $TX = P \circ X$, where $\circ$ is Kelly's colored circle product [**Kel05**] (also described in [**WY18**]).

Definition 2.3. A model structure on the category of T-algebras will be called *transferred* from $\mathcal{M}$ if a map f of T-algebras is a weak equivalence (resp. fibration) if and only if $U(f)$ is a weak equivalence (resp. fibration). We also call this structure on $\mathrm{Alg}_T(\mathcal{M})$ *projective.*

Most of the model category axioms for the transferred model structure are easy to check. The difficult one has to do with proving that the trivial cofibrations of

T-algebras are saturated (that is, closed under transfinite composition, pushout, and retracts) and contained in the weak equivalences [**SS00**]. These trivial cofibrations are generated by the set $F(J)$, and so pushouts in $\mathrm{Alg}_T(\mathcal{M})$ of the following sort must be considered:

$$\begin{array}{ccc} F(K) & \xrightarrow{F(u)} & F(L) \\ \downarrow & & \downarrow \\ A & \xrightarrow[h]{} & B \end{array} \tag{1}$$

Often, such pushouts are computed by filtering the map $h\colon A \to B$ as a transfinite composition of pushouts in $\mathcal{M}$. The following definition ensures this works, and the terminology is borrowed from the language of semi-model categories (recalled in Section 2.3).

Definition 2.4. Let $\mathcal{K}$ be a saturated class of morphisms in a model category $\mathcal{M}$. A monad on $\mathcal{M}$ will be called $\mathcal{K}$-*admissible* if, for each (trivial) cofibration $u\colon K \to L$ in $\mathcal{M}$, the pushout (1) has $U(h)$ in $\mathcal{K}$ (resp. in $\mathcal{K}$ and a weak equivalence). A monad T will be called $\mathcal{K}$-*semi admissible* if this holds for pushouts into cofibrant T-algebras A. A monad T will be called $\mathcal{K}$-*semi admissible over* $\mathcal{M}$ if this holds for pushouts into T-algebras A for which UA is cofibrant in $\mathcal{M}$.

The value of this definition is that it allows for transferred model structures and transferred semi-model structures (to be discussed below) on $\mathrm{Alg}_T(\mathcal{M})$. To build these (semi-)model structures, the small object argument [**Hov99**, Theorem 2.1.14] is required in $\mathrm{Alg}_T(\mathcal{M})$. In order to know that the small object argument converges, we need a smallness condition on the objects of $\mathcal{M}$. Recall that an object A is *small* relative to a class of morphisms $\mathcal{X}$ if the functor $\mathrm{Hom}_{\mathcal{M}}(A, -)$ takes κ-directed colimits of morphisms in $\mathcal{X}$ to κ-directed colimits of sets [**Hov99**, Definition 2.1.3].

We next recall a smallness condition on $\mathcal{M}$ (from [**BB17**]) that allows the transfer theory to work. This condition is not necessary if $\mathcal{M}$ is combinatorial.

Definition 2.5. Let $\mathcal{K}$ be a saturated class of morphisms in a model category $\mathcal{M}$. We say $\mathcal{M}$ is $\mathcal{K}$-*compactly generated* if the weak equivalences are closed under filtered colimits along morphisms in $\mathcal{K}$, and if all objects are small relative to $\mathcal{K}$-cell (the class of transfinite compositions of pushouts of maps in $\mathcal{K}$).

We illustrate with an example. One common source of applications is when the monad T arises from an operad. In this situation, $\mathcal{M}$ is often a monoidal model category (i.e., satisfies the pushout product axiom as in [**SS00**]), and $\mathcal{K}$ is often taken to be the class $(\mathcal{M} \otimes I)$-cell, i.e., the monoidal saturation of the cofibrations.

We now have all the ingredients required to state our main transfer theorem, which synthesizes three different transfer theorems already in the literature.

Theorem 2.6. *Let T be a finitary, $\mathcal{K}$-admissible (resp. $\mathcal{K}$-semi-admissible, resp. $\mathcal{K}$-semi-admissible over $\mathcal{M}$) monad on a $\mathcal{K}$-compactly generated model category $\mathcal{M}$ for some saturated class of morphisms $\mathcal{K}$. Then $\mathrm{Alg}_T(\mathcal{M})$ admits a transferred model structure (resp. semi-model structure, resp. semi-model structure over $\mathcal{M}$).*

This theorem is proven as Theorem 2.11 of [**BB17**] for model structures, and is a consequence of [**BW20b**, Theorem 2.2.1] for semi-model categories (see also [**Fre09**, Theorem 12.1.9] and [**Spi01**, Theorem 2] for the case of semi-model categories over $\mathcal{M}$).

This theorem has many applications. For example, if $\mathcal{M}$ is a cofibrantly generated monoidal model category, and is $(\mathcal{M} \otimes I)$-compactly generated (where I is the set of generating cofibrations), then algebras over any Σ-cofibrant colored operad inherit a transferred semi-model structure [**GRSO12**, Theorem A.8]. Under mild conditions, algebras over entrywise cofibrant colored operads inherit transferred semi-model structures [**WY18**, Theorem 6.2.3]. In the absence of the monoid axiom, monoids inherit a transferred semi-model structure [**Hov98**, Theorem 3.3]. In the presence of the commutative monoid axiom but the absence of the monoid axiom, commutative monoids inherit a transferred semi-model structure [**Whi17**, Corollary 3.8]. We will provide an example at the end of Section 2.3 of a category of algebras that only admits a semi-model structure, not a full model structure. So semi-admissibility for a monad T occurs much more frequently than admissibility.

2.3. Semi-model categories

There are many different definitions of semi-model categories in the literature, all involving a weakening of the model category axioms to require certain cofibrancy conditions on the domains of maps involved in the axioms. The first definition, called a J-semi model structure, was in [**Spi01**], and axiomatized precisely the structure obtained on $\mathrm{Alg}_T(\mathcal{M})$ from Theorem 2.6. What we call semi-model structures over $\mathcal{M}$ (following Spitzweck's terminology) are called relative semi-model structures in [**Fre09**], but note that [**Fre09**] has a weaker notation of semi-model structure, called an (I, J)-semi model structure in [**Spi01**].

The following definition is [**Bar10**, Definition 1.4] (taking $E = C$ there), and is a slight abstraction of Spitzweck's notion of a J-semi model category. The additional generality allows for semi-model categories to arise from situations other than a transfer, such as [**BW20a**].

Definition 2.7. A *semi-model structure* on a category $\mathcal{D}$ consists of classes of weak equivalences $\mathscr{W}$, fibrations $\mathscr{F}$, and cofibrations $\mathscr{Q}$ satisfying the following axioms:

- M1 Fibrations are closed under pullback.
- M2 The class $\mathscr{W}$ is closed under the two-out-of-three property.
- M3 $\mathscr{W}, \mathscr{F}, \mathscr{Q}$ are all closed under retracts.
- M4
  - i Cofibrations have the left lifting property with respect to trivial fibrations.
  - ii Trivial cofibrations whose domain is cofibrant have the left lifting property with respect to fibrations.
- M5
  - i Every map in $\mathcal{D}$ can be functorially factored into a cofibration followed by a trivial fibration.
  - ii Every map whose domain is cofibrant can be functorially factored into a trivial cofibration followed by a fibration.

If, in addition, $\mathcal{D}$ is bicomplete, then we call $\mathcal{D}$ a *semi-model category.*

In practice, there is often an adjunction $F : \mathcal{M} \rightleftarrows \mathcal{D} : U$ where $\mathcal{M}$ is a cofibrantly generated model category, and weak equivalences (resp. fibrations) in $\mathcal{D}$ are defined as in Definition 2.3. Note that (M1) is automatic in such a setting. We say an object X of $\mathcal{D}$ is *cofibrant in* $\mathcal{M}$ if $U(X)$ is cofibrant. In this setting, it is possible to modify Definition 2.7 to require a different cofibrancy assumption. On $\mathcal{D}$, a *semi-model* *structure over* $\mathcal{M}$ refers to a triple $(\mathscr{W}, \mathscr{F}, \mathscr{Q})$ satisfying the axioms on Definition 2.7 but where 'cofibrant' is replaced by 'cofibrant in $\mathcal{M}$' ([**Bar10**, Definition 1.4], [**Spi01**, Definition 1]). This affects (M4ii) and (M5ii). Furthermore, one must assume that the initial object of $\mathcal{D}$ is cofibrant in $\mathcal{M}$ (note that one does not need to assume that the initial object in $\mathcal{M}$ is cofibrant in Definition 2.7, as this can easily be deduced from a factorization, lifting, and retract argument applied to the identity morphism on the initial object). In practice, it often occurs that cofibrant objects in $\mathcal{D}$ are cofibrant in $\mathcal{M}$ [**BW20b**, Theorem 2.2.2], so that in a semi-model category over $\mathcal{M}$, strictly more morphisms follow the axioms of a model category. The main source of examples of semi-model structures over $\mathcal{M}$ is Theorem 2.6.

Note that, in a semi-model category $\mathcal{D}$, the axioms of a full model structure are satisfied on the subcategory of cofibrant objects. Furthermore, $\mathcal{D}$ has a cofibrant replacement functor defined on every object. Consequently, every result about model categories has a semi-model categorical analogue, usually obtained by cofibrantly replacing as needed. This includes the Fundamental Theorem of Model Categories (characterizing morphisms in the homotopy category), left and right Quillen functors, Ken Brown's lemma, path and cylinder objects, the retract argument, the cube lemma, simplicial mapping spaces, hammock localization, projective/injective/Reedy semi-model structures, latching and matching objects, cosimplicial and simplicial resolutions, computations of homotopy limits and colimits, and more. In practice, a semi-model structure is just as useful as a full model structure. In particular, Definition 2.1 can also be made for localizations of semi-model categories, as is done in [**BW20a**].

We often transfer a semi-model structure from $L_{\mathcal{C}}(\mathcal{M})$ to $\mathrm{Alg}_T(L_{\mathcal{C}}(\mathcal{M}))$. The most general setting for such a transfer to exist is that given by Theorem 2.6 applied to $L_{\mathcal{C}}(\mathcal{M})$. We work in this general setting, but the reader is encouraged to keep the following examples in mind.

Example 2.8. Suppose that $\mathcal{M}$ is a monoidal model category [**Hov99**, Definition 4.2.6], e.g., simplicial sets, compactly generated spaces, chain complexes, symmetric spectra, or (equivariant) orthogonal spectra. If $L_{\mathcal{C}}(\mathcal{M})$ satisfies the pushout product axiom, $L_{\mathcal{C}}$ is called a *monoidal Bousfield localization.* Conditions to guarantee this are given in [**Whi14a**] and worked out for the model categories just listed, as well as counterexamples demonstrating it does not come for free.

For monoidal Bousfield localizations, $\mathrm{Alg}_T(L_{\mathcal{C}}(\mathcal{M}))$ inherits a transferred semi-model structure whenever T comes from a Σ-cofibrant colored operad. Similar results hold for entrywise cofibrant colored operads [**WY18**]. Results of this nature have been proven for commutative monoids in [**Whi17**], and are recalled in Example 5.8. In [**WY20**], conditions are given so that such transfers exist in a right Bousfield localization $R_K(\mathcal{M})$.

Note that another source of semi-model categories is as left Bousfield localizations of other semi-model categories, or of non-left-proper model categories [**BW20a**].

We conclude this section with an example of a semi-model structure that is not a full model structure, to demonstrate that semi-model categories are inescapable when one studies algebras over operads.

Example 2.9. Consider the colored operad P whose algebras are non-reduced symmetric operads. Consider P-algebras in $\mathcal{M} = \mathrm{Ch}(\mathbb{F}_2)$ with the projective model structure, from [**Hov99**]. Because P is Σ-cofibrant and $\mathcal{M}$ is a cofibrantly generated monoidal model category, there is a transferred semi-model structure on P-alg, by [**WY18**, Theorem 6.3.1] (this result goes back to [**Spi01**]). This transferred structure does not form a full model structure, as we now show. Let Com be the operad for commutative differential graded algebras, so $\mathrm{Com}(n)$ is $\mathbb{F}_2$ with the trivial Σ_n-action for all n.

For any acyclic complex C, the inclusion morphism $0 \to C$ is a trivial cofibration. Define a collection K to have $K_0 = C$ and $K_i = 0$ otherwise. Then the inclusion from the 0 collection to K is a trivial cofibration. Thus, $P(0) \to P(K)$ is supposed to be a trivial cofibration, so pushouts of it would need to be weak equivalences if P-alg was a model category. Yet in the pushout

$$\begin{array}{ccc} P(0) & \longrightarrow & P(K) \\ \downarrow & & \downarrow \\ \mathrm{Com} & \longrightarrow & P(K)\coprod \mathrm{Com} \end{array}$$

the coproduct $P(K)\coprod \mathrm{Com}$ always contains a summand of $C\otimes C/\Sigma_2$, as explained in [**BB17**, Section 12.30]. Thus, whenever $C\otimes C/\Sigma_2$ is not contractible, the bottom morphism cannot be a weak equivalence. We now demonstrate an explicit example of C such that $C\otimes C/\Sigma_2$ is not contractible, because $H_1(C\otimes C/\Sigma_2)\cong \mathbb{F}_2$. To ease notation, let $k=\mathbb{F}_2$.

Let C be the complex $0\to k\to k\oplus k\to k\oplus k\to k\to 0$ where the differential d_3 takes a to (a,a), d_2 takes (a,b) to $(a+b,a+b)$, and d_1 takes (a,b) to $a+b$. Observe that

$$(C\otimes C)_n \cong \begin{cases} k\otimes_k k\cong k & \text{if } n=0\\ (k\otimes_k (k\oplus k))\oplus((k\oplus k)\otimes_k k)\cong k^4 & \text{if } n=1\\ (k\otimes_k (k\oplus k))\oplus((k\oplus k)\otimes_k (k\oplus k))\oplus((k\oplus k)\otimes_k k)\cong k^8 & \text{if } n=2\end{cases}$$

The differential d_1 takes $(u\otimes(a,b),(c,d)\otimes w)$ to $ua+ub+cw+dw$, while d_2 takes $(u\otimes(a,b),(e,f)\otimes(s,t),(c,d)\otimes w)$ to $(ua+ub+es+et+fs+ft, ua+ub+es+et+fs+ft, wc+wd+es+et+fs+ft, wc+wd+es+et+fs+ft)$.

Now consider $(C\otimes C)/\Sigma_2$, where the Σ_2 action is induced by swapping C_p and C_q in the formula $\bigoplus_{n=p+q} C_p\otimes_k C_q$. The action on the degree 1 part swaps the first two coordinates and swaps the second two coordinates. The differential d_1 is an epimorphism, while $\mathrm{im}(d_2)$ lies in the Σ_2-invariant subspace, hence goes to zero when we pass to coinvariants. It follows that $H_1(C\otimes C/\Sigma_2)\cong k$.

This example demonstrates that symmetric operads in $\mathrm{Ch}(\mathbb{F}_2)$ do not carry a full model structure, since pushouts of trivial cofibrations need not be weak equivalences. This example does not rule out the transferred semi-model structure because Com is not cofibrant, and only pushouts of trivial cofibrations into cofibrant objects need to be again trivial cofibrations for semi-model categories.

3. Algebras in a localized category

Suppose that $\mathcal{M}$ is a cofibrantly generated model category, $\mathcal{C}$ is a class of morphisms such that $L_{\mathcal{C}}(\mathcal{M})$ exists, and T is a monad on $\mathcal{M}$ such that $\mathrm{Alg}_T(\mathcal{M})$ inherits a transferred model structure from $\mathcal{M}$. In practice, T will often be finitary, and we will use Theorem 2.6 to verify the model category axioms on $\mathrm{Alg}_T(\mathcal{M})$, but our results do not rely on T being finitary. We also note that the theory of [**BW20a**] could be used to weaken the assumption on $\mathrm{Alg}_T(\mathcal{M})$ to only require a semi-model structure.

Definition 3.1. The *local projective (semi)-model structure*, $\mathrm{Alg}_T(L_{\mathcal{C}}(\mathcal{M}))$, on the category $\mathrm{Alg}_T(\mathcal{M})$ is the (semi-)model structure transferred from the model structure $L_{\mathcal{C}}(\mathcal{M})$ along the forgetful functor $U\colon \mathrm{Alg}_T(\mathcal{M}) \to \mathcal{M}$. A morphism f is a weak equivalence (resp. fibration) in $\mathrm{Alg}_T(L_{\mathcal{C}}(\mathcal{M}))$ if and only if $U(f)$ is a weak equivalence (resp. fibration) in $L_{\mathcal{C}}(\mathcal{M})$.

There are two ways to define what it means for an algebra Z to be local. We can define it based on whether $U(Z)$ is local in $\mathcal{M}$, or we can define it based on a localization of $\mathrm{Alg}_T(\mathcal{M})$. We now show these two notions agree. We recall that $F(\mathcal{C})$ denotes the set of free T-algebra morphisms on $\mathcal{C}$.

Definition 3.2. We will call an algebra $Z \in \mathrm{Alg}_T(\mathcal{M})$ *local* if $U(Z)$ is a local object in $\mathcal{M}$.

Lemma 3.3. *$Z \in Alg_T(\mathcal{M})$ is a local object in $Alg_T(\mathcal{M})$ with respect to $F(\mathcal{C})$ if and only if Z is a local algebra.*

Proof. An algebra Z is a local object with respect to $F(\mathcal{C})$ if and only if for any $f \in F(\mathcal{C})$, which we denote $f\colon X \to Y$ the morphism of homotopy function complexes

$$\mathrm{Map}_{\mathrm{Alg}_T(\mathcal{M})}(f, Z)\colon \mathrm{Map}_{\mathrm{Alg}_T(\mathcal{M})}(Y, Z) \to \mathrm{Map}_{\mathrm{Alg}_T(\mathcal{M})}(X, Z)$$

is a weak equivalence of simplicial sets. It suffices to prove this for any chosen model of Map [**Hir03**, Theorem 17.5.31]. Let $f = F(g)$ for some $g\colon V \to W$ in $\mathcal{C}$. Recalling that morphisms in $\mathcal{C}$ are cofibrations between cofibrant objects, and F is left Quillen, it is no loss of generality to assume that X and Y are cofibrant algebras. Thus, the homotopy function complexes $\mathrm{Map}_{\mathrm{Alg}_T(\mathcal{M})}(X, Z)$ and $\mathrm{Map}_{\mathrm{Alg}_T(\mathcal{M})}(Y, Z)$ can be constructed as fibrant simplicial sets $\mathrm{Hom}_{\mathrm{Alg}_T(\mathcal{M})}(X, Z_*)$ and $\mathrm{Hom}_{\mathrm{Alg}_T(\mathcal{M})}(Y, Z_*)$ where Z_* is a simplicial resolution of Z in the category of algebras [**Hir03**, Section 17.1]. Since U preserves limits, fibrations, and weak equivalences, $U(Z_*)$ is a simplicial resolution of $U(Z)$. Hence, a homotopy function complex $\mathrm{Map}_{\mathcal{M}}(V, U(Z))$ in $\mathcal{M}$ can be computed via $\mathrm{Hom}_{\mathcal{M}}(V, U(Z_*))$.

Finally, by adjointness we have that $\mathrm{Map}_{\mathrm{Alg}_T(\mathcal{M})}(f, Z)$ is a weak equivalence if and only if the morphism

$$\mathrm{Map}_{\mathcal{M}}(g, U(Z))\colon\ \mathrm{Map}_{\mathcal{M}}(W, U(Z)) \to \mathrm{Map}_{\mathcal{M}}(V, U(Z))$$

is a weak equivalence. This occurs if and only if $U(Z)$ is a local object in $\mathcal{M}$, as required. ☐

The proof above also works if $\mathrm{Alg}_T(\mathcal{M})$ is only a semi-model category, using [**Bar10**, Scholium 3.64] instead of the appeal to [**Hir03**].

Theorem 3.4. *Suppose that $Alg_T(\mathcal{M})$ and $Alg_T(L_{\mathcal{C}}(\mathcal{M}))$ have transferred model structures. Then $L_{F(\mathcal{C})}Alg_T(\mathcal{M})$ exists and coincides with $Alg_T(L_{\mathcal{C}}(\mathcal{M}))$. Furthermore, if $Alg_T(\mathcal{M})$ and $Alg_T(L_{\mathcal{C}}(\mathcal{M}))$ exist as semi-model categories then the local projective semi-model structure $L_{F(\mathcal{C})}Alg_T(\mathcal{M})$ exists and coincides with the semi-model structure $Alg_T(L_{\mathcal{C}}(\mathcal{M}))$.*

This result means that, of the two ways of going around the following diagram to study local algebras, the ability to go counterclockwise (localize then transfer) implies the ability to go clockwise (transfer then localize). The opposite is not true: there are examples where one can go clockwise, but not counterclockwise (see Remark 5.10 below).

$$\begin{array}{ccc} \mathrm{Alg}_T(\mathcal{M}) & \xmapsto{L_{F(\mathcal{C})}} & \mathrm{Alg}_T(L_{\mathcal{C}}(\mathcal{M})) = L_{F(\mathcal{C})}\mathrm{Alg}_T(\mathcal{M}) \\ \uparrow & & \uparrow \\ \mathcal{M} & \xmapsto[L_{\mathcal{C}}]{} & L_{\mathcal{C}}(\mathcal{M}) \end{array}$$

The importance of semi-model structures is that transfers to categories of algebras often only result in semi-model structures (see [**WY18**]), especially when one is transferring from $L_{\mathcal{C}}(\mathcal{M})$ where one has lost control over the trivial cofibrations. We have already seen in Example 2.9 that semi-model categories are unavoidable, and cannot necessarily be improved to full model structures.

Proof. We first focus on the situation of model structures, delaying discussion of semi-model structures until the end of the proof. We first show that the identity functor $Id\colon \mathrm{Alg}_T(\mathcal{M}) \to \mathrm{Alg}_T(L_{\mathcal{C}}(\mathcal{M}))$ is a left Quillen functor. It is sufficient to prove that its inverse $Id^{-1} : \mathrm{Alg}_T(L_{\mathcal{C}}(\mathcal{M})) \to \mathrm{Alg}_T(\mathcal{M})$ maps (trivial) fibrations to (trivial) fibrations. A fibration in $\mathrm{Alg}_T(L_{\mathcal{C}}(\mathcal{M}))$ is a morphism $f : X \to Y$ such that $U(f)$ is a fibration in $L_{\mathcal{C}}(\mathcal{M})$. Hence, $Id^{-1}(U(f))$ is also a fibration in $\mathcal{M}$ because $Id^{-1} : L_{\mathcal{C}}(\mathcal{M}) \to \mathcal{M}$ is a right Quillen functor. Therefore, $Id^{-1}(f)$ is also a fibration in $\mathrm{Alg}_T(\mathcal{M})$ because $\mathrm{Alg}_T(\mathcal{M})$ carries a transferred model structure. The same argument applies for trivial fibrations.

As a consequence, the class of cofibrations in $\mathrm{Alg}_T(\mathcal{M})$ coincides with the class of cofibrations in $\mathrm{Alg}_T(L_{\mathcal{C}}(\mathcal{M}))$. In addition, an adjunction argument shows that the fibrant objects in $\mathrm{Alg}_T(L_{\mathcal{C}}(\mathcal{M}))$ are exactly fibrant $F(\mathcal{C})$-local algebras. The machinery of homotopy function complexes may be applied in $\mathrm{Alg}_T(\mathcal{M})$, and if Z is a fibrant algebra, a model for $\mathrm{Map}_{\mathrm{Alg}_T(\mathcal{M})}(X, Z)$ is provided by $\mathrm{Hom}_{\mathrm{Alg}_T(\mathcal{M})}(X^*, Z)$, where X^* denotes a cosimplicial resolution in $\mathrm{Alg}_T(\mathcal{M})$. Recall that if a morphism induces weak equivalences on one model for Map, then it induces weak equivalences on all models for Map [**Hir03**, Theorem 17.5.31]. So when we write $\mathrm{Hom}_{\mathrm{Alg}_T(\mathcal{M})}(X^*, Z)$ below, it should be read as equivalent to $\mathrm{Map}_{\mathrm{Alg}_T(\mathcal{M})}(X, Z)$.

To complete the proof, note that $f : X \to Y$ is a $F(\mathcal{C})$-local equivalence in $\mathrm{Alg}_T(\mathcal{M})$ if and only if it induces a weak equivalence between simplicial sets:

$$\mathrm{Hom}_{\mathrm{Alg}_T(\mathcal{M})}(Y^*, Z) \to \mathrm{Hom}_{\mathrm{Alg}_T(\mathcal{M})}(X^*, Z)$$

for all fibrant local objects $Z \in \mathrm{Alg}_T(\mathcal{M})$, where X^*, Y^* are cosimplicial resolutions of X and Y [**Bar10**, Scholium 3.64]. But any cosimplicial resolution in $\mathrm{Alg}_T(\mathcal{M})$ is also a cosimplicial resolution in $\mathrm{Alg}_T(L_{\mathcal{C}}(\mathcal{M}))$ because they have the same class of cofibrations, and because any weak equivalence in $\mathrm{Alg}_T(\mathcal{M})$ is also a weak equivalence in $\mathrm{Alg}_T(L_{\mathcal{C}}(\mathcal{M}))$. Hence, we have

$$\mathrm{Hom}_{\mathrm{Alg}_T(\mathcal{M})}(X^*, Z) \cong \mathrm{Hom}_{\mathrm{Alg}_T(L_{\mathcal{C}}(\mathcal{M}))}(X^*, Z)$$

for any local fibrant object Z since it is also a fibrant object in $\mathrm{Alg}_T(L_{\mathcal{C}}(\mathcal{M}))$. So, f

induces a weak equivalence

$$\mathrm{Map}_{\mathrm{Alg}_T(L_{\mathcal{C}}(\mathcal{M}))}(Y,Z) \to \mathrm{Map}_{\mathrm{Alg}_T(L_{\mathcal{C}}(\mathcal{M}))}(X,Z)$$

for any fibrant object Z in $\mathrm{Alg}_T(L_{\mathcal{C}}(\mathcal{M}))$, so f is a weak equivalence in $\mathrm{Alg}_T(L_{\mathcal{C}}(\mathcal{M}))$, as required.

For semi-model categories, precisely the same proof works. As the first paragraph focuses entirely on fibrations and trivial fibrations, it applies verbatim to semi-model structures and proves that the classes of fibrations in these two semi-model structures on $\mathrm{Alg}_T(\mathcal{M})$ coincide. The second paragraph proves that cofibrations coincide, by lifting. This works for semi-model categories as well.

Lastly, to prove that weak equivalences coincide, note that Scholium 3.64 in [**Bar10**] is written to work for semi-model categories (there called left model categories), and that the machinery to carry out cosimplicial resolutions for cofibrant objects is also provided (indeed, it goes back to [**Spi01**]). An analogue of [**Hir03**, Theorem 17.5.31] is similarly easy to prove. Thus, $F(\mathcal{C})$-local equivalences $f\colon X \to Y$ between cofibrant objects are weak equivalences in $\mathrm{Alg}_T(L_{\mathcal{C}}(\mathcal{M}))$. We now remove the condition that X and Y are cofibrant. For a general $F(\mathcal{C})$-local equivalence $f\colon A \to B$, replace f by its cofibrant replacement Qf in $\mathrm{Alg}_T(L_{\mathcal{C}}(\mathcal{M}))$. Note that the morphisms $QA \to A$ and $QB \to B$ are trivial fibrations in $\mathrm{Alg}_T(L_{\mathcal{C}}(\mathcal{M}))$, hence in $L_{\mathcal{C}}(\mathcal{M})$, hence in $\mathcal{M}$. So the two-out-of-three property for $F(\mathcal{C})$-local equivalences (which holds independently of the existence of a semi-model structure on $L_{F(\mathcal{C})}\mathrm{Alg}_T(\mathcal{M})$) shows that Qf is a $F(\mathcal{C})$-local equivalence. The argument just given proves that Qf is a weak equivalence in $\mathrm{Alg}_T(L_{\mathcal{C}}(\mathcal{M}))$, as are the morphisms $QA \to A$ and $QB \to B$. Thus, by the two-out-of-three property, so is f, completing our proof. $\square$

The hypotheses of this Theorem are satisfied for the free monoid monad if $L_{\mathcal{C}}(\mathcal{M})$ satisfies the pushout product and monoid axioms. Conditions on $\mathcal{M}$ and $\mathcal{C}$ are provided in [**Whi14b**] so that this occurs. For $F = \mathrm{Sym}$ one must also prove that $L_{\mathcal{C}}(\mathcal{M})$ satisfies the commutative monoid axiom, but again conditions for this to occur are given in [**Whi14b**].

We now characterize when this lifted model structure exists. When we say U preserves local equivalences we mean that U takes $F(\mathcal{C})$-local equivalences to $\mathcal{C}$-local equivalences.

Theorem 3.5. *Let $\mathcal{M}$ be a $\mathcal{K}$-compactly generated model category where $\mathcal{K}$ is a saturated class in $\mathcal{M}$. Let T be a $\mathcal{K}$-admissible monad on $\mathcal{M}$. Assume that the localization $L_{\mathcal{C}}(\mathcal{M})$ is a $\mathcal{K}$-compactly generated model category and, moreover, the domains of generating trivial cofibrations in $L_{\mathcal{C}}(\mathcal{M})$ are cofibrant (for example, if the domains are cofibrant in $\mathcal{M}$). Assume also that the projective structure on $Alg_T(\mathcal{M})$ is left proper.*

Then the transferred model structure $Alg_T(L_{\mathcal{C}}(\mathcal{M}))$ exists if and only if U preserves local equivalences.

Note that conditions are given in [**Whi14b**] so that $L_{\mathcal{C}}(\mathcal{M})$ satisfies the hypothesis of being a $\mathcal{K}$-compactly generated monoidal model category, so that Theorem 2.6 can be used to put (semi-)model structures on categories of algebras over colored operads. Furthermore, [**Whi14b**, Corollary 4.15] provides conditions guaranteeing cofibrancy for the domains of morphisms in J. Lastly, [**BB17**] provides conditions guaranteeing that $\mathrm{Alg}_T(\mathcal{M})$ is left proper.

Below, we use U^{loc} to denote the forgetful functor from $\mathrm{Alg}_T(L_{\mathcal{C}}(\mathcal{M}))$ to $L_{\mathcal{C}}(\mathcal{M})$. As a functor, this is of course the same as U from $\mathrm{Alg}_T(\mathcal{M})$ to $\mathcal{M}$.

Proof. If the transfer from $L_{\mathcal{C}}(\mathcal{M})$ to $\mathrm{Alg}_T(L_{\mathcal{C}}(\mathcal{M}))$ exists, then by Theorem 3.4 the transferred model structure coincides with the lifted localization. So, U^{loc} preserves weak equivalences. Hence, U preserves local equivalences.

Let us prove the converse. The hypotheses of the theorem allow us to use the machinery from [**BB17**] to prove that the transfer exists if we can prove that T is $\mathcal{K}$-admissible for $L_{\mathcal{C}}(\mathcal{M})$. For this we need to analyze a pushout created from a generating trivial cofibration $f : K \to L$ in $L_{\mathcal{C}}(\mathcal{M})$, and a morphism of algebras $g : F(K) \to X$:

$$\begin{array}{ccc} F(K) & \xrightarrow{F(f)} & F(L) \\ {\scriptstyle g}\downarrow & & \downarrow \\ X & \xrightarrow{\widetilde{f}} & P \end{array} \tag{2}$$

We must prove that the underlying morphism of $\widetilde{f}$ belongs to $\mathcal{K}$ and is a local equivalence. Observe that $U(\widetilde{f})$ is in $\mathcal{K}$ because T is $\mathcal{K}$-admissible. We must prove that it is a local equivalence. Since the transferred model structure on $\mathrm{Alg}_T(\mathcal{M})$ is left proper, the pushout (2) is actually a homotopy pushout in $\mathrm{Alg}_T(\mathcal{M})$. Let Z be a local fibrant algebra, so that $U(Z)$ is a local fibrant object in $L_{\mathcal{C}}(\mathcal{M})$. Apply the simplicial mapping space functor $\mathrm{Map}_{\mathrm{Alg}_T(\mathcal{M})}(-, Z)$ to the pushout (2). Since $F(K)$ and $F(L)$ are cofibrant, $\mathrm{Map}_{\mathrm{Alg}_T(\mathcal{M})}(-, Z)$ can be constructed as a levelwise Hom functor $\mathrm{Hom}_{\mathrm{Alg}_T(\mathcal{M})}(-, Z_*)$, where Z_* is a simplicial resolution of Z. Note that $U(Z_*)$ is a simplicial resolution of $U(Z)$ in $L_{\mathcal{C}}(\mathcal{M})$, so adjunction yields a homotopy pullback in simplicial sets:

$$\begin{array}{ccc} \mathrm{Map}_{\mathrm{Alg}_T(\mathcal{M})}(P, Z) & \xrightarrow{\mathrm{Map}_{\mathrm{Alg}_T(\mathcal{M})}(\widetilde{f}, Z)} & \mathrm{Map}_{\mathrm{Alg}_T(\mathcal{M})}(X, Z) \\ \downarrow & & \downarrow \\ \mathrm{Map}_{\mathcal{M}}(L, U(Z)) & \xrightarrow{\mathrm{Map}_{\mathcal{M}}(f, U(Z))} & \mathrm{Map}_{\mathcal{M}}(K, U(Z)) \end{array}$$

Note that $\mathrm{Map}_{L_{\mathcal{C}}(\mathcal{M})}(f, U(Z))$ is a trivial fibration since f is a local trivial cofibration and $U(Z)$ is a fibrant local object. Therefore, $\mathrm{Map}_{\mathrm{Alg}_T(\mathcal{M})}(\widetilde{f}, Z)$ is a trivial fibration for all Z, so $\widetilde{f}$ is a local equivalence in $\mathrm{Alg}_T(\mathcal{M})$. Since U preserves local equivalences, $U(\widetilde{f})$ is a local equivalence and we have completed the proof. □

We now give the semi-model category version of Theorem 3.5. In the following, an algebra X is said to be *relatively cofibrant* if $U(X)$ is cofibrant (in either $\mathcal{M}$ or $L_{\mathcal{C}}(\mathcal{M})$, since the two model categories have the same cofibrant objects).

Corollary 3.6. *Let $\mathcal{M}$ be a $\mathcal{K}$-compactly generated model category, for some saturated class of morphisms $\mathcal{K}$. Let T be a $\mathcal{K}$-semi-admissible monad (so there is a transferred semi-model structure $Alg_T(\mathcal{M})$). Assume that $L_{\mathcal{C}}(\mathcal{M})$ exists as a semi-model category and that the domains of its generating trivial cofibrations are cofibrant. Then the transferred semi-model structure on $Alg_T(L_{\mathcal{C}}(\mathcal{M}))$ exists if and only if U*

preserves local equivalences. In addition, if cofibrant algebras are relatively cofibrant then we obtain on $Alg_T(L_{\mathcal{C}}(\mathcal{M}))$ *a transferred semi-model structure over* $L_{\mathcal{C}}(\mathcal{M})$.

Note that conditions such that cofibrant algebras are relatively cofibrant are given in [**WY18**] (Theorems 6.2.3 and 6.3.1).

Proof. We proceed as in the proof of the theorem, but assume that the algebra X is a cofibrant algebra. We form the pushout square

$$\begin{array}{ccc} F(K) & \xrightarrow{F(f)} & F(L) \\ {\scriptstyle g}\downarrow & & \downarrow \\ X & \xrightarrow{\widetilde{f}} & P \end{array}$$

and we note that it is a homotopy pushout square even if $\mathrm{Alg}_T(\mathcal{M})$ is not left proper, because all objects in the square are cofibrant and $F(f)$ is a cofibration of algebras. The rest of the proof follows precisely as in the theorem, using that $F(K)$ and $F(L)$ are cofibrant algebras. If cofibrant algebras are relatively cofibrant, then we can do the same proof starting with an algebra X cofibrant in $\mathcal{M}$ and using that $F(K)$ and $F(L)$ are also cofibrant in $\mathcal{M}$ to deduce that the pushout square is a homotopy pushout. □

4. Localizing a category of algebras

In the previous section we provided conditions so that we could first localize $\mathcal{M}$ at a set of morphisms $\mathcal{C}$ and then transfer the local (semi)-model structure to the category of T-algebras. An alternative way to study local T-algebras is to lift the localization $L_{\mathcal{C}}$ to a localization $L_{F(\mathcal{C})}$ on the category of algebras (where $F(\mathcal{C})$ is the set of free T-algebra morphisms on $\mathcal{C}$). Conditions for such lifts to exist have been found in [**CRT14**], along with an extensive discussion of colocalization and a comparison between $L_{\mathcal{C}}$ and $L_{U(F(\mathcal{C}))}$ with applications to classical localizations for spaces, abelian groups, and spectra. We will now summarize the main results of [**CRT14**] that are related to our results on localizations, and then we deduce new results regarding preservation of T-algebra structure by $L_{\mathcal{C}}$ in the next section, building both on our work above and on [**CRT14**].

In [**CRT14**], the localizations under consideration are *homotopical localizations*. Usually, such a setting would be a slight generalization of the notion of a left Bousfield localization, avoiding the need to assume that there is a *set* of morphisms being inverted. However, in [**CRT14**] the focus is on homotopical localizations that come from left Bousfield localizations L_f, so the setting matches ours. Borrowing from [**CRT14**, Section 6], we say that a $\mathcal{C}$-localization of an object X is a trivial fibration $\ell_X : X \to L_{\mathcal{C}}(X)$ in $L_{\mathcal{C}}(\mathcal{M})$. This can be taken to be fibrant replacement in $L_{\mathcal{C}}\mathcal{M}$, so the pair $(L_{\mathcal{C}}, \ell)$ induces a monad on $\mathcal{M}$ that is idempotent on the homotopy category $\mathrm{Ho}(\mathcal{M})$. A key concern of [**CRT14**] is determining when this endofunctor, which we also denote $L_{\mathcal{C}}$, lifts to a localization of algebras in the following sense:

Definition 4.1. We say that $L_{\mathcal{C}}$ *lifts to the homotopy category of* T*-algebras* if there is an endofunctor L^T on $\mathrm{Ho}(\mathrm{Alg}_T(\mathcal{M}))$, and a natural isomorphism $h : L_{\mathcal{C}}U \to UL^T$

in Ho($\mathcal{M}$) such that $h \circ l_{UE} = Ul_E^T$ in Ho($\mathcal{M}$) for all E, where $l_X \colon X \to L_{\mathcal{C}}(X)$ and $l_E^T \colon E \to L^T(E)$ are the localization morphisms.

The main results from [**CRT14**] that we will be interested in regard when forgetful functors preserve local equivalences (as in Theorem 3.4) and when localizations lift to the category of algebras. The following two results are consequences of Lemma 7.3 and Theorem 7.8 in [**CRT14**].

Lemma 4.2. *If a localization L lifts to the homotopy category of T-algebras, then U preserves and reflects local objects and equivalences.*

The following is a uniqueness theorem for the lift, proving that if the lift exists then it must be the lift we expect to exist, namely $L_{F(\mathcal{C})}$. Note, however, that $L_{F(\mathcal{C})}$ could exist and not be a lift of $L_{\mathcal{C}}$; see Example 5.11. Conditions to force $L_{F(\mathcal{C})}$ to be a lift of $L_{\mathcal{C}}$ are given in [**GRSO18**]. We have weakened the hypothesis in [**CRT14**] from requiring full transferred model structures to only requiring semi-model structures:

Theorem 4.3. *Suppose that $Alg_T(\mathcal{M})$ has a transferred semi-model structure, that $L_{F(\mathcal{C})}Alg_T(\mathcal{M})$ exists as a semi-model category, and that $L_{\mathcal{C}}$ lifts to $Ho(Alg_T(\mathcal{M}))$. Then:*

1. *T preserves $\mathcal{C}$-local equivalences between cofibrant objects,*
2. *there is a natural isomorphism $\beta_X : L_{\mathcal{C}}U(X) \cong UL_{F(\mathcal{C})}(X)$ in $Ho(\mathcal{M})$ for all algebras X, and*
3. *U preserves and reflects local equivalences, trivial fibrations, and fibrant objects.*

The reason we can weaken the hypothesis to only requiring a semi-model structure is that the proof in [**CRT14**] only ever works on the homotopy level or on the subcategory of cofibrant objects. The argument required to define β_X (recalled in the next section) requires a lift in $L_{\mathcal{C}}(\mathcal{M})$, not in a category of algebras. We will see some consequences of this generalization in Section 5, and the generalization allows us to apply this theory to the many situations where only semi-model structures are known.

5. Preservation of algebras by localization

The following definition has appeared in [**Whi14b**], where $\mathcal{M}$ is a model category and $\mathcal{C}$ is a set of morphisms in $\mathcal{M}$. As in the previous section, $L_{\mathcal{C}}$ is an endofunctor of Ho($\mathcal{M}$) induced by fibrant replacement in $L_{\mathcal{C}}(\mathcal{M})$.

Definition 5.1. $L_{\mathcal{C}}$ is said to *preserve T-algebras* if

1. When E is a T-algebra there is some T-algebra $\widetilde{E}$ that is weakly equivalent in $\mathcal{M}$ to $L_{\mathcal{C}}(UE)$.
2. In addition, when E is a cofibrant T-algebra, then there is a choice of $\widetilde{E}$ in $\mathrm{Alg}_T(\mathcal{M})$ with $U(\widetilde{E})$ local in $\mathcal{M}$, a T-algebra homomorphism $r_E \colon E \to \widetilde{E}$ that lifts the localization morphism $l_{UE} \colon UE \to L_{\mathcal{C}}(UE)$ up to homotopy, and a weak equivalence $\beta_E \colon L_{\mathcal{C}}(UE) \to U\widetilde{E}$ such that $\beta_E \circ l_{UE} \cong Ur_E$ in Ho($\mathcal{M}$).

Observe that, when $L_{\mathcal{C}}$ lifts to the homotopy category of T-algebras in the sense of Definition 4.1, it implies the preservation of Definition 5.1 on the homotopy category level, but does not necessarily imply there is an actual *morphism* from E to $\widetilde{E}$ in $\mathrm{Alg}_T(\mathcal{M})$. However, Definition 5.1 does imply that $L_{\mathcal{C}}$ lifts to the homotopy category of T-algebras, and naturality of β is deduced as part of Theorem 5.6.

The following is proven in [**Whi14b**], and provides a host of examples where preservation occurs, discussed in Example 5.7.

Theorem 5.2. *If $Alg_T(\mathcal{M})$ and $Alg_T(L_{\mathcal{C}}(\mathcal{M}))$ have transferred semi-model structures then $L_{\mathcal{C}}$ preserves T-algebras.*

Proof. This has already been proven in [**Whi14b**, Section 3] and [**WY18**, Theorem 7.2.3], but for the sake of being self-contained we recall the main points of the proof. We must verify the statements in Definition 5.1, and of course (1) follows from (2) after cofibrantly replacing in the category of T-algebras. So, let E be a cofibrant T-algebra, and define $\widetilde{E}$ to be $R_{\mathcal{C},T}E$, the fibrant replacement in $\mathrm{Alg}_T(L_{\mathcal{C}}(\mathcal{M}))$. That $\widetilde{E}$ is weakly equivalent to $L_{\mathcal{C}}(UE)$ is proven in [**Whi14b**] and [**WY18**] by constructing local equivalences $L_{\mathcal{C}}(UE) \simeq R_{\mathcal{C}}UE \to R_{\mathcal{C},T}E$ and then observing that a local equivalence between local objects (using that $\mathrm{Alg}_T(L_{\mathcal{C}}(\mathcal{M}))$ is transferred from $L_{\mathcal{C}}(\mathcal{M})$) is a weak equivalence. Observe that $U(\widetilde{E})$ is local because the semi-model structure on $\mathrm{Alg}_T(L_{\mathcal{C}}(\mathcal{M}))$ is transferred. The morphism $r_E \colon E \to \widetilde{E}$ is just the fibrant replacement morphism $R_{\mathcal{C},T}$, and the comparison β_E is the following lift in $L_{\mathcal{C}}(\mathcal{M})$:

$$\begin{array}{ccc} UE & \longrightarrow & U\widetilde{E} \\ {\scriptstyle \simeq_{\mathcal{C}}}\big\downarrow & \overset{\beta_E}{\nearrow} & \big\downarrow \\ L_{\mathcal{C}}(UE) & \longrightarrow & * \end{array}$$

□

We will see in Theorem 5.6 that, under mild hypotheses on $\mathcal{M}$ and T, if $L_{\mathcal{C}}$ preserves T-algebras, then the transferred semi-model structure $\mathrm{Alg}_T(L_{\mathcal{C}}(\mathcal{M}))$ exists. Furthermore, this implies the localization $L_{\mathcal{C}}$ lifts, as we now show.

Proposition 5.3. *Suppose that $Alg_T(\mathcal{M})$ and $Alg_T(L_{\mathcal{C}}(\mathcal{M}))$ have transferred semi-model structures. Then $L_{\mathcal{C}}$ lifts to the homotopy category of T-algebras in the sense of Definition 4.1.*

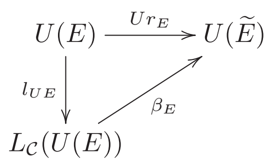
Proof. Define the localization L^T to be (the image in $\mathrm{Ho}(\mathrm{Alg}_T(\mathcal{M}))$ of) $R_{\mathcal{C},T}$. The natural isomorphism h will be the image in $\mathrm{Ho}(\mathcal{M})$ of β. We construct β via lifting in $L_{\mathcal{C}}(\mathcal{M})$, using Theorem 3.4 to realize that $\mathrm{Alg}_T(L_{\mathcal{C}}(\mathcal{M})) = L_{F(\mathcal{C})}\mathrm{Alg}_T(\mathcal{M})$ and so $R_{\mathcal{C},T} = R_{F(\mathcal{C})}$:

$$\begin{array}{ccc} UE & \longrightarrow & UR_{F(\mathcal{C})}E \\ {\scriptstyle \simeq_{\mathcal{C}}}\big\downarrow & \overset{\beta_E}{\nearrow} & \big\downarrow \\ R_{\mathcal{C}}UE & \longrightarrow & * \end{array}$$

This lift demonstrates immediately that $\beta \circ lU = Ul^T$ in $\mathrm{Ho}(\mathcal{M})$. Note that the top horizontal morphism is a $\mathcal{C}$-local equivalence because $\mathrm{Alg}_T(L_{\mathcal{C}}(\mathcal{M}))$ has the transferred model structure, so the lift is a $\mathcal{C}$-local equivalence by the two-out-of-three property. Note that the domain and codomain of the lift are $\mathcal{C}$-local objects (again using that the fibrations are transferred from $L_{\mathcal{C}}(\mathcal{M})$). Hence, the lift is a weak equivalence in $\mathcal{M}$. In addition, the lift is unique in $\mathrm{Ho}(\mathcal{M})$ by the universal property of localization, since any other lift would necessarily be a weak equivalence in $\mathcal{M}$ between the same two $\mathcal{C}$-local objects. Finally, the lift is natural in $\mathrm{Ho}(\mathcal{M})$ because if we began with a morphism $E \to F$ and constructed this lift on its domain and codomain then we could in addition construct a homotopy unique lift from $R_{\mathcal{C}}UE$ to $UR_{F(\mathcal{C})}F$, and so uniqueness tells us the relevant naturality square commutes in $\mathrm{Ho}(\mathcal{M})$. □

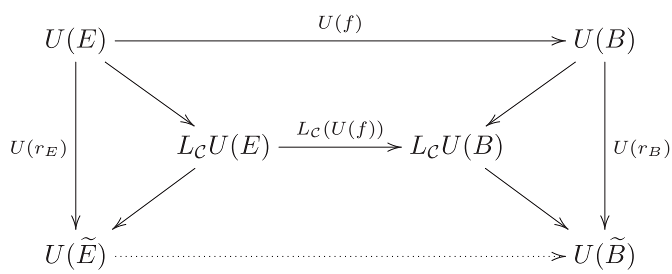
Proposition 5.4. *If $L_{\mathcal{C}}$ preserves T-algebras then U preserves local equivalences.*

Proof. Let E be a T-algebra, cofibrant in $\mathcal{M}$. Consider the following diagram, guaranteed to exist because $L_{\mathcal{C}}$ preserves T-algebras:



Since $l_E \colon U(E) \to L_{\mathcal{C}}(U(E))$ is a local equivalence and β_E is a weak equivalence, Ur_E is a local equivalence. So U preserves local equivalences of the form r_E.

Suppose that $f \colon E \to B$ is a $F(\mathcal{C})$-local equivalence in $\mathrm{Alg}_T(\mathcal{M})$. Consider the diagram



We proved above that the outside vertical morphisms are local equivalences. The morphisms in the lower trapezoid are all weak equivalences in $\mathcal{M}$ because they are local equivalences between local objects. Thus, in $\mathrm{Ho}(\mathcal{M})$, the dotted arrow exists and is an isomorphism. It follows that $U(f)$ is a local equivalence because its localization is an isomorphism in $\mathrm{Ho}(\mathcal{M})$. □

We turn now to the consequences we can deduce when preservation occurs. Recall that semi-admissibility is a weak hypothesis often satisfied in practice by monads encoding colored operads.

Theorem 5.5. *Assume that T is $\mathcal{K}$-semi-admissible, $\mathcal{M}$ is $\mathcal{K}$-compactly generated, and $L_{\mathcal{C}}(\mathcal{M})$ is $\mathcal{K}$-compactly generated, and has cofibrant domains of the generating trivial cofibrations, for some saturated class of morphisms $\mathcal{K}$. If $L_{\mathcal{C}}$ lifts to the homotopy category of T-algebras (as in Definition 4.1) then $Alg_T(L_{\mathcal{C}}(\mathcal{M}))$ has a transferred semi-model structure, which is a full model structure if T is $\mathcal{K}$-admissible and* *if* $Alg_T(\mathcal{M})$ *is left proper. In either case, the lifted homotopy localization* L^T *is induced by a left Bousfield localization* $L_{F(\mathcal{C})}Alg_T(\mathcal{M})$.

Proof. Lemma 4.2 proves that $U\colon \mathrm{Ho}(\mathrm{Alg}_T(\mathcal{M})) \to \mathrm{Ho}(\mathcal{M})$ preserves local equivalences. Note that $U\colon \mathrm{Alg}_T(\mathcal{M}) \to \mathcal{M}$ also preserves local equivalences. Theorem 3.5 and Corollary 3.6 imply that $\mathrm{Alg}_T(L_\mathcal{C}(\mathcal{M}))$ exists. Theorem 3.4 implies that $L_{F(\mathcal{C})}\mathrm{Alg}_T(\mathcal{M})$ exists and coincides with $\mathrm{Alg}_T(L_\mathcal{C}(\mathcal{M}))$. The homotopy uniqueness of β in Theorem 4.3 implies that $L_{F(\mathcal{C})}$ is a lift of $L_\mathcal{C}$ to the model category level, i.e., agrees with L^T. □

Note that the converse to this theorem also holds, i.e., we can deduce that $L_\mathcal{C}$ lifts to the homotopy category of T-algebras if we know that $\mathrm{Alg}_T(L_\mathcal{C}(\mathcal{M}))$ exists as a semi-model category, using Proposition 5.3.

We are finally ready for our omnibus theorem relating the notions considered in this paper and [**CRT14**].

Theorem 5.6. *Suppose that* $\mathcal{M}$ *contains a saturated class of morphisms* $\mathcal{K}$ *such that* T *is* $\mathcal{K}$*-semi-admissible,* $\mathcal{M}$ *is* $\mathcal{K}$*-compactly generated,* $\mathcal{C}$ *is a set of morphisms in* $\mathcal{M}$*, and* $L_\mathcal{C}(\mathcal{M})$ *is a* $\mathcal{K}$*-compactly generated model category with cofibrant domains of the generating trivial cofibrations. The following are equivalent:*

1. $L_\mathcal{C}$ *lifts to a left Bousfield localization* $L_{F(\mathcal{C})}$ *of semi-model categories on* $Alg_T(\mathcal{M})$*, where* $F(\mathcal{C})$ *is the set of free* T*-algebra maps on* $\mathcal{C}$.
2. $U\colon Alg_T(\mathcal{M}) \to \mathcal{M}$ *preserves local equivalences.*
3. $Alg_T(L_\mathcal{C}(\mathcal{M}))$ *has a transferred semi-model structure.*
4. $L_\mathcal{C}$ *preserves* T*-algebras.*

Furthermore, any of the above implies that

(5) T *preserves* $\mathcal{C}$*-local equivalences between cofibrant objects.*

Proof. That (1) implies (2) is part of Lemma 4.2. That (2) is equivalent to (3) is Corollary 3.6, and that (3) implies (1) is Theorem 3.4. That (3) implies (4) is Theorem 5.2. That (4) implies (2) is Proposition 5.4. That (1) implies (5) is part of Theorem 4.3. □

In [**CRT14**, Theorem 7.1], it is shown that (5) is equivalent to the statement that L_f lifts to TQ-algebras in $\mathrm{Ho}(\mathcal{M})$, where Q is cofibrant replacement in $\mathcal{M}$. In [**CRT14**, Theorem 7.8], it is shown that (2) is equivalent to the statement that L_f lifts to $\mathrm{Ho}(\mathrm{Alg}_T(\mathcal{M}))$. There are many examples demonstrating that the category of TQ-algebras in $\mathrm{Ho}(\mathcal{M})$ is not equivalent to $\mathrm{Ho}(\mathrm{Alg}_T(\mathcal{M}))$ (e.g., [**JN14**, Example 5.2]), and so there is no reason to expect that (5) would be equivalent to (1)-(4). We will discuss in Theorem 6.2 what can be deduced from (5).

Example 5.7. The hypotheses regarding $\mathcal{K}$-semi-admissibility (Definition 2.4) and $\mathcal{K}$-compact generation (Definition 2.5) are satisfied for monads T arising from *any* colored operad, when the class $\mathcal{K}$ is the class of trivial cofibrations, and when $\mathcal{M}$ is the Quillen model structure on simplicial sets or compactly generated spaces, any of the family model structures on G-equivariant spaces, the Quillen model structure on chain complexes over a field of characteristic zero, the model structure from Chapter 2 of [**Hov99**] for the stable module category of a quasi-Frobenius ring, or the folk model structure on the category of small categories. The hypotheses are satisfied for the positive and positive flat model structure on symmetric or orthogonal spectra, for any monad T arising from a colored operad, when $\mathcal{K}$ is the class of trivial h-cofibrations defined in [**BB17**]. These statements are proven in [**WY18**] and [**WY20**]. With these statements in hand, we see numerous applications of Theorem 5.6:

1. When $\mathcal{M}$ is the category $\mathrm{Ch}_{\geqslant 0}(k)$ of bounded chain complexes over a field of characteristic zero, then for any left Bousfield localization $L_{\mathcal{C}}(\mathcal{M})$, and any colored operad P, there is a transferred model structure on P-algebras in $L_{\mathcal{C}}(\mathcal{M})$ [**WY18**, Corollary 8.1.3], i.e., statement (3) holds. Hence, by Theorem 5.6, such localizations can be constructed on the level of P-algebras, and such localizations preserve P-algebras.
2. When $\mathcal{M}$ is the category of simplicial sets, then for any left Bousfield localization $L_{\mathcal{C}}(\mathcal{M})$, and any colored operad P, there is a transferred model structure on P-algebras in $L_{\mathcal{C}}(\mathcal{M})$ [**WY18**, Theorem 8.2.1], i.e., statement (3) holds. This includes the homology localizations that motivated Bousfield to develop homotopical localization theory [**Bou75**]. Theorem 5.6 implies that such localizations can be constructed on the level of P-algebras, and such localizations preserve P-algebras.
3. When $\mathcal{M}$ is the category of simplicial k-modules, for k a field of characteristic zero, then any left Bousfield localization preserves algebras over any colored operad [**WY20**, Example 11.19], i.e., statement (4) holds. This implies that there is a transferred model structure on local algebras, and that the localization lifts to the level of algebras.
4. When $\mathcal{M}$ is the category of symmetric spectra with the positive flat model structure [**Shi04**], E is any cofibrant ring spectrum, and f is any morphism of spectra between cofibrant objects, then L_f lifts to the category of E-modules (i.e., statement (2) holds) if either E is connective or if L_f commutes with suspension [**CRT14**, Theorem 8.3]. This includes all of the homology localization functors considered by Bousfield [**Bou79**], Miller's finite localizations [**Mil92**], or the localizations with respect to $K(n)$ or the n^{th} Morava E-theory [**Rav92**], so fundamental to chromatic homotopy theory.
5. When $\mathcal{M}$ is the category of motivic symmetric spectra with the positive flat model structure, then the slice spectral sequence gives rise to an infinite tower of left and right Bousfield localizations, and the left localizations preserve A_∞- and E_∞-algebra structure [**GRSO12**, Proposition 5.4, Theorem 5.17] (i.e., statement (4) holds). Indeed, [**GRSO12**, Theorem 4.2] is a preservation result for algebras over any Σ-cofibrant operad with respect to monoidal left Bousfield localizations [**Whi14b**, Definition 4.4], including the motivic analogues of the localizations listed for symmetric spectra above. By Theorem 5.6, such localizations can be constructed on the level of algebras, and we now have transferred semi-model structures with which to study the local algebras.
6. When $\mathcal{M}$ is the category of G-equivariant orthogonal spectra, the machinery of Theorem 5.6 can be applied to monads arising from equivariant operads. The main result of [**HH14**] verifies statement (1) of Theorem 5.6 for the localization of commutative ring spectra that arises in the Hill-Hopkins-Ravenel proof of

the Kervaire Invariant One Theorem [**HHR16**], by using group-theoretic properties of the group ($G = \mathbb{Z}/8$) in question to construct the localization at the level of algebras one cell at a time. This same method has been used in non-equivariant spectra to lift localizations to E_∞-algebras, e.g., in [**EKMM97**]. By Theorem 5.6, we now know that there is a transferred model structure on local commutative ring spectra, and this has been generalized in [**GW18**] to the setting of algebras over N_∞-operads. Such algebras have some, but not all, of the Hill-Hopkins-Ravenel multiplicative norms, and Theorem 5.6 allows us to lift localizations to such categories of algebras, in analogy to the approach of [**HH14**].

7. Algebras over Σ-cofibrant colored operads are preserved by any monoidal left Bousfield localization when $\mathcal{M}$ is the category of small categories with the folk model structure [**WY20**, Section 13] (including localizations obtained from simplicial sets, as discussed in [**CRT14**, Example 6.2]), when $\mathcal{M}$ is the category of R-modules with the stable model structure [**WY20**, Section 12] (including the smashing localizations much studied in representation theory, and characterized in [**BIK11**]), and when $\mathcal{M}$ is Rezk's category of Θ_k-spaces or the corresponding model of (m, k)-categories [**Rez10**]. Localizations in Rezk's setting are closely connected to the Baez-Dolan stabilization hypothesis, as shown in [**BW20b**].

Example 5.8. In [**Whi17**], the second author introduced the *commutative monoid axiom* and proved that it implies the existence of a transferred model structure on commutative monoids. Let Sym denote the free commutative monoid functor. In [**Whi14b**], it was proven that if a monoidal left Bousfield localization $L_\mathcal{C}$ has the property that $\mathrm{Sym}(\mathcal{C})$ is contained in the class of $\mathcal{C}$-local equivalences, then $L_\mathcal{C}(\mathcal{M})$ inherits the commutative monoid axiom from $\mathcal{M}$, and hence $L_\mathcal{C}$ preserves commutative monoids. Theorem 5.6 implies the converse, i.e., if $L_\mathcal{C}$ preserves commutative monoids then Sym must preserve $\mathcal{C}$-local equivalences. This answers a question posed to the second author by Nitu Kitchloo.

Remark 5.9. The main result of [**GRSO18**] complements Theorem 5.6. The result states that, if $\mathcal{M}$ is a simplicial monoidal model category, if T comes from a colored operad O, if $\mathrm{Alg}_T(\mathcal{M})$ inherits a transferred model structure from $\mathcal{M}$, if $L_\mathcal{C}$ is a monoidal left Bousfield localization and $O(c_1, \ldots, c_n; c) \otimes -$ preserves $\mathcal{C}$-local equivalences, and if the model category $L_{F(\mathcal{C})}\mathrm{Alg}_T(\mathcal{M})$ exists, then U preserves local equivalences (and hence, all (1)-(5) of the statements in Theorem 5.6 are true). The authors wonder if the results of this paper can be made to work with only semi-model structures. If so, the existence of $L_{F(\mathcal{C})}\mathrm{Alg}_T(\mathcal{M})$ appears to occur more frequently than the existence of $\mathrm{Alg}_T(L_\mathcal{C}(\mathcal{M}))$, so this result might give easier to check conditions such that the five equivalent statements in Theorem 5.6 hold. Note that [**GRSO18**] also considers colocalizations.

We conclude this section with an example that demonstrates how the conditions of Theorem A may fail to be satisfied.

Remark 5.10. In [**BW20a**], we prove that, for combinatorial semi-model categories $\mathcal{M}$ where the domains of the generating cofibrations are cofibrant, $L_{F(\mathcal{C})}\mathrm{Alg}_T(\mathcal{M})$ exists as a semi-model category whenever $\mathrm{Alg}_T(\mathcal{M})$ does. However, this does not mean that preservation comes for free, because one does not know that the resulting localization lifts $L_{\mathcal{C}}$ unless one also knows that U preserves local equivalences, as the next example shows.

Example 5.11. Let $\mathcal{M}$ be the category of symmetric spectra, with the stable model structure [**HSS00**], and let S denote the sphere spectrum. Recall that the n^{th} Postnikov section functor P_n is the Bousfield localization L_g corresponding to the morphism $g = \Sigma^\infty(f)$ where $f\colon S^{n+1} \to *$ is the unique map from the space S^{n+1} to the space $*$. It was shown in [**CGMV10**, Section 6] that P_{-1} does not preserve monoids, because if R is a monoid (i.e., a ring spectrum), $P_{-1}R$ cannot be a ring spectrum (not even up to homotopy), since $[S, P_{-1}R] = 0$, but the unit map of a ring spectrum cannot be nullhomotopic. Thus, none of the 5 conclusions of Theorem 5.6 can be satisfied. Let T denote the free monoid monad on $\mathcal{M}$, and let $f\colon S^0 \to *$ be the unique morphism (so L_g yields P_{-1} on $\mathrm{Ho}(\mathcal{M})$). There is a left Bousfield localization $L_{F(g)}$ on the category of ring spectra (which is left proper by [**BB17**]), yielding a homotopical localization on the homotopy category of ring spectra. This localization cannot be a lift of P_{-1}, because it would violate Theorem 5.6. In fact, $L_{F(g)}$ is homotopically trivial, since g is a retract of $F(g)$ and so $L_{F(g)}(S) \cong *$. This implies that the unit map $S \to L_{F(g)}R$ is nullhomotopic for every ring spectrum R, so $L_{F(g)}R$ is contractible for all R.

6. Reflection of local equivalences

We conclude the paper with a partial converse to the last implication in Theorem 5.6, i.e., we state what can be deduced from knowing that a monad $T\colon \mathcal{M} \to \mathcal{M}$ preserves $\mathcal{C}$-local equivalences. We use this result in our companion paper [**BW20b**], and we anticipate further applications in future work. Recall that, for any monad (T, μ, ϵ) on $\mathcal{M}$ and any $X \in \mathcal{M}$, there is an augmented cosimplicial object defined as follows:

$$T^*(X) := X \longrightarrow T(X) \underset{\epsilon_T}{\overset{T\epsilon}{\leftleftarrows\!\!\rightrightarrows}} T^2(X) \rightleftarrows\!\!\rightleftarrows T^3(X) \ldots$$

The morphisms going left are multiplication μ on T. The morphisms going right are induced by the unit ϵ.

Similarly we have a comonad $(\mathcal{T}, \lambda, \eta)$, on the category of T-algebras, where $\mathcal{T}(X) = FU(X)$, and we have the classical simplicial bar resolution in $\mathrm{Alg}_T(\mathcal{M})$:

$$F(T^{*-1}(U(X))) := X \longleftarrow \mathcal{T}(X) \leftleftarrows\!\!\rightarrow \mathcal{T}^2(X) \leftleftarrows\!\!\rightleftarrows \mathcal{T}^3(X) \ldots \tag{3}$$

For our partial converse to Theorem 5.6, we will need the following definition, which goes back to [**Bat98**].

Definition 6.1. We will say that the monad (T, μ, ϵ) on $\mathcal{M}$ is *pointwise Reedy cofibrant* if, for any cofibrant $X \in \mathcal{M}$, the augmented cosimplicial object $T^*(X)$ is Reedy cofibrant.

Many monads we come across in practice are pointwise Reedy cofibrant. For example, if O is any nonsymmetric operad acting on a monoidal model category $\mathcal{M}$, and if $1 \to O(1)$ is a cofibration (where 1 is the unit of $\mathcal{M}$), then the free O-algebra functor is pointwise Reedy cofibrant [**Bat98**]. If T is a polynomial monad and the unit of $\mathcal{M}$ is cofibrant then T is pointwise Reedy cofibrant [**BB17**]. Work in progress of Mark Johnson and Donald Yau provides even more examples. With this definition in hand, we are ready for our partial converse to Theorem 5.6. We do not expect a full converse to be true in general, as we intend to show in the future using the results of [**JN14**].

Theorem 6.2. *Suppose that $Alg_T(\mathcal{M})$ admits a transferred semi-model structure and that:*

- *T is a pointwise Reedy cofibrant monad,*
- *T preserves local equivalences between cofibrant objects, and*
- *U sends cofibrant algebras to cofibrant objects.*

Then U reflects local equivalences between cofibrant algebras.

Proof. Suppose that U preserves cofibrant objects. Let $f\colon X \to Y$ be a morphism between cofibrant algebras such that $U(f)$ is a local equivalence. We must show f is a local equivalence. Let Z be a local fibrant algebra. We need to show that the induced map $f^*\colon \mathrm{Map}_{\mathrm{Alg}_T(\mathcal{M})}(Y,Z) \to \mathrm{Map}_{\mathrm{Alg}_T(\mathcal{M})}(X,Z)$ is a weak equivalence. Since X is cofibrant, the mapping space $\mathrm{Map}_{\mathrm{Alg}_T(\mathcal{M})}(X,Z)$ can be constructed as $\mathrm{Hom}_{\mathrm{Alg}_T(\mathcal{M})}(X,Z_*)$ where Z_* is a simplicial resolution of Z, again using [**Bar10**] [3.64]. Observe that Z_n is a local algebra for each Z because Z is local. Observe that in the bar resolution (3) all terms $T^k(U(X))$ are cofibrant in $\mathcal{M}$ (since $U(X)$ is cofibrant). After applying $\mathrm{Map}_{\mathrm{Alg}_T(\mathcal{M})}(-,Z)$ to this resolution we have, therefore, an augmented cosimplicial simplicial set:

$$\mathrm{Map}_{\mathrm{Alg}_T(\mathcal{M})}(X,Z) \simeq \mathrm{Hom}_{\mathrm{Alg}_T(\mathcal{M})}(X,Z_*) \to \mathrm{Hom}_{\mathrm{Alg}_T(\mathcal{M})}(F(T^{*-1})(U(X)),Z_*).$$

Then the cosimplicial simplicial set

$$\mathrm{Hom}_{\mathrm{Alg}_T(\mathcal{M})}(F(T^{*-1})(U(X)),Z_*) \cong \mathrm{Hom}_{\mathcal{M}}(T^{*-1}(U(X)),U(Z_*))$$

is Reedy fibrant by the Homotopy Lifting Extension Theorem [**Hir03**, Corollary 16.5.14] applied in $\mathcal{M}$, using that T is pointwise Reedy cofibrant. Moreover, this cosimplicial simplicial set has an extra degeneracy (precomposing with $X \to T(X)$) which shows that the augmentation

$$\mathrm{Map}_{\mathrm{Alg}_T(\mathcal{M})}(X,Z) \simeq \mathrm{Hom}_{\mathrm{Alg}_T(\mathcal{M})}(X,Z_*) \to \mathrm{Hom}_{\mathcal{M}}(T^{*-1}(U(X)),U(Z_*))$$

induces a deformation retraction (Lemma 2.1, [**Bat93**]) of fibrant simplicial sets

$$\mathrm{Map}_{\mathrm{Alg}_T(\mathcal{M})}(X,Z) \to \mathrm{Tot}(\mathrm{Hom}_{\mathcal{M}}(T^{*-1}(U(X)),U(Z_*))).$$

Hence, it suffices to show that f induces a weak equivalence

$$\mathrm{Hom}_{\mathcal{M}}(T^k(U(Y)),U(Z_*)) \xrightarrow{g} \mathrm{Hom}_{\mathcal{M}}(T^k(U(X)),U(Z_*))$$

for all $k \geqslant 0$. As we have assumed that $U(f)$ is a local equivalence between cofibrant algebras, and that such local equivalences are preserved by T, the morphism g is a weak equivalence, since all Z_n are local fibrant algebras. Hence, f is a local equivalence as required. □

Our main interest in Theorems 5.6 and 6.2 is their application to localizations of categories of n-operads, which are connected to the Baez-Dolan stabilization hypothesis [**BD95**] as explained in [**Bat17**]. In a companion paper, we improve on the main result of [**Bat17**] and prove a stronger form of Baez-Dolan stabilization, using the results of the previous sections [**BW20b**]. We also have plans to apply the dual to Theorem 5.6 (proven in [**WY16**]) to prove the McClure-Smith conjecture [**MS04b**] regarding Quillen equivalences between E_n-algebras and certain structured cosimplicial spaces. We anticipate many more applications of Theorem 5.6 in the years to come.

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