Homology, Homotopy and Applications, vol. 23(2), 2021, pp.349–372

RIGIDIFICATION OF DENDROIDAL INFINITY-OPERADS

PETER BONVENTRE AND LUÍS A. PEREIRA

(*communicated by Brooke Shipley*)

Abstract

We give an explicit description of the rigidification of an ∞-operad as a simplicial operad. This description is based on the notion of dendroidal necklace, extending work of Dugger and Spivak from the categorical context to the operadic context, although with a different framework, which relates constructions involving necklaces to a standard factorization of maps in the category of trees.

1. Introduction

The notion of ∞-*operad* is a generalization of the notion of *(colored) operad* (also sometimes called *multicategories*), introduced by Moerdijk–Weiss [**MW07**], where composition of operations is only defined "up to a contractible space of choices", in the same way that quasi-categories generalize categories. Moreover, just as quasi-categories are defined as those simplicial sets $X \in \mathsf{sSet} = \mathsf{Set}^{\Delta^{op}}$ (for Δ the simplicial category) satisfying a lifting condition against inner horn inclusions, so too are ∞-operads defined as those *dendroidal sets* $X \in \mathsf{dSet} = \mathsf{Set}^{\Omega^{op}}$ (for Ω the category of trees) satisfying a lifting condition against dendroidal inner horn inclusions [**CM11**, §2.1].

There are two main procedures for converting a presheaf $X \in \mathsf{dSet}$ into a (strict) operad, given by the left adjoints $W_!$, τ in two adjunctions as below, where Op (resp. sOp) denotes operads of sets (resp. of simplicial sets).

$$W_!\colon \mathsf{dSet} \rightleftarrows \mathsf{sOp}\colon hcN \qquad \tau\colon \mathsf{dSet} \rightleftarrows \mathsf{Op}\colon N \tag{1}$$

Before recalling how these adjunctions are defined, we discuss their importance. First, in the (τ, N)-adjunction, the right adjoint, the *nerve* N, is a fully faithful inclusion whose image consists of (certain) ∞-operads, cf. Remark 2.18, thus making precise the idea that ∞-operads generalize operads. On the other hand, the $(W_!, hcN)$-adjunction is central for the homotopy theory of ∞-operads, as it was shown to be a Quillen equivalence [**CM13**] between the model structure on dSet (with fibrant objects the ∞-operads) and the canonical model structure on sOp. Moreover, in [**BPb**] the authors established an equivariant version of the Quillen equivalence in [**CM13**], modeling the homotopy theory of *equivariant operads with norm maps*. In particular, our work here plays a minor but necessary role in the proofs in [**BPb**], cf. [**BPb**, Lemma 4.52],

Received April 26, 2020; published on September 15, 2021.
2020 Mathematics Subject Classification: 55P91, 18M75.
Key words and phrases: operad, dendroidal set.
Article available at `http://dx.doi.org/10.4310/HHA.2021.v23.n2.a18`

by giving an explicit description[1] of the simplicial operads $W_!(\partial\Omega[T]), W_!(\Lambda^E[T])$, cf. Examples 4.26, 4.27.

Common to both adjunctions in (1) is that the right adjoints hcN, N are straightforward to describe (cf. (2)), while the left adjoints $W_!$, τ are more mysterious, as they involve colimits in operads (cf. (3)). The main goal of this paper, which is an offshoot of our work in [**BPb**], is to give an explicit description of $W_!$, cf. Theorem 1.2, generalizing work of Dugger and Spivak [**DS11**] in the context of quasi-categories. Additionally, a variation of our main constructions gives a description of the simpler functor τ (cf. Remark 4.14).

We now recall the definitions of the functors in (1). First, each tree $T \in \Omega$ has an associated colored operad $\Omega(T) \in \mathsf{Op}$ with colors the edges of the tree and operations generated by the nodes ([**MW07**, §3]; see also Example 2.16). Moreover, there is a "fattened" replacement $W(T) \in \mathsf{sOp}$ for $\Omega(T)$, which can be built [**MW09**, Rem. 7.3] as the Boardman–Vogt construction on $\Omega(T)$ (though here we use a novel description of $W(T)$, cf. Proposition 1.1), which replaces the non-empty mapping sets of $\Omega(T)$, which are all singletons $*$, with larger *contractible spaces*. The functors hcN and N, which are called, respectively, the *homotopy coherent nerve* and the *nerve*, are then given by (where $\mathcal{O}$ is in sOp or Op as appropriate)

$$hcN\mathcal{O}(T) = \mathsf{sOp}(W(T), \mathcal{O}), \qquad N\mathcal{O}(T) = \mathsf{Op}(\Omega(T), \mathcal{O}). \tag{2}$$

Loosely speaking, hcN can thus be regarded as a variant of N obtained by replacing the notion of strict equality with that of homotopy. Writing $\Omega[T] \in \mathsf{dSet} = \mathsf{Set}^{\Omega^{op}}$ for the representable functor $\Omega[T](-) = \mathsf{dSet}(-, T)$ associated to $T \in \Omega$, by abstract nonsense one then has the formulas

$$W_!X = \operatorname{colim}_{\Omega[T]\to X} W(T), \qquad \tau = \operatorname{colim}_{\Omega[T]\to X} \Omega(T). \tag{3}$$

However, as previously noted, the colimits in (3) take place in sOp, Op, making these formulas rather opaque. Just as in the work in [**DS11**] in the categorical context, the key to obtaining explicit formulas for $W_!$, τ will be the notion of (dendroidal) necklace, which we now introduce (the reason why necklaces are useful in this process is explained following (6)).

In the work of Dugger and Spivak in the categorical context [**DS11**], a *necklace* is a simplicial set of the form $\Delta^{n_1} \vee \Delta^{n_2} \vee \cdots \vee \Delta^{n_k}$ where each Δ^{n_i} is glued along its terminal vertex to the initial vertex of $\Delta^{n_{i+1}}$. Moreover, we demand $n_i > 0$ except for the necklace Δ^0 consisting of a single point. On a terminological note, the initial and terminal vertices of the Δ^{n_i} are called the *joints* of the necklace, while the Δ^{n_i} with $n_i > 0$ are called beads[2]. Since the Δ^n are simply the representable presheaves in sSet, their role in the operadic context is naturally played by the representable presheaves $\Omega[T]$ of dSet for T a tree. However, formulating the notion of necklace in the dendroidal context requires some care. This is because, while each $\Omega[T]$ does have a terminal vertex, corresponding to the root of T, it in general has *multiple* initial

[1]It is worth noting that the description of these specific operads is well known, yet the extant references we are aware of seem to leave this description as an exercise to the reader.

[2]In particular, we consider the exceptional necklace Δ^0 to have no beads. This differs slightly from the convention in [**DS11**], which regards Δ^0 as a bead of the necklace Δ^0. This ultimately makes little difference, but in our convention beads are always in bijection with vertices of the *tree of joints*, cf. Figure 1, as discussed below.

vertices, corresponding to the leaves of T. As such, when specifying a dendroidal necklace one must also specify the leaves along which to glue. As an example, the tree arrangement of the trees T_1, T_2, T_3, T_4, T_5 on the left in Figure 1 gives rise to a

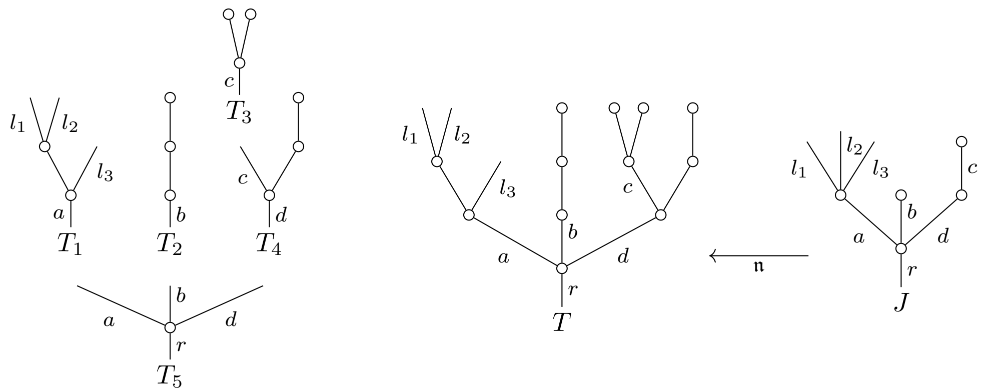


Figure 1: Encoding a dendroidal necklace

dendroidal necklace (where $\amalg_e$ denotes gluing along the edge e)

$$\Omega[T_1] \amalg_a (\Omega[T_2] \amalg_b (\Omega[T_3] \amalg_c \Omega[T_4] \amalg_d \Omega[T_5])) \tag{4}$$

However, in practice (4) is rather awkward to work with due to the need to include brackets, as well as the existence of distinct bracketing orders. To address this, we will prefer a different presentation of dendroidal necklaces. First, note that by gluing (also known as *grafting*) the trees T_i in Figure 1 as suggested by the arrangement, one obtains the tree T therein. In addition, the tree J encodes the *arrangement* of the T_i itself. To make this more precise, note that J can be obtained by replacing each T_i in the left arrangement with the corolla (i.e., tree with a single vertex) with the same number of leaves, and then performing the grafting. Moreover, this procedure gives rise to the indicated map $\mathfrak{n}\colon J \to T$ in Figure 1, which completely encodes the left arrangement of the T_i: inner edges of J encode the gluing edges; the vertices of J are in natural bijection with the set of the T_i; the T_i themselves are the (outer) subtrees of T whose outer edges (i.e., leaves and root) are the image under $\mathfrak{n}$ of the corresponding vertex of J. As such, we will regard such maps $\mathfrak{n}\colon J \to T$ themselves as our description of a dendroidal necklace, cf. Definition 3.1 (more precisely, necklaces are then the *planar inner face maps* in Ω). We note that, should all T_i be linear trees, so that the dendroidal necklace is one of the simplicial necklaces $\Delta^{n_1} \vee \cdots \vee \Delta^{n_k}$, the edges of J (which is then also linear) correspond to the joints of the necklace. As such, we refer to the tree J in a necklace as the *tree of joints*. Similarly, we call the T_i the *beads* of the necklace, where we require that beads T_i always have at least one vertex (generalizing the $n_i > 0$ requirement in the simplicial context).

We end this introduction by observing that our presentation of necklaces as maps $\mathfrak{n}\colon J \to T$ foreshadows our approach throughout the paper. More precisely, all our main constructions and proofs (e.g. Definition 4.1 and Proposition 4.4) are formal consequences of a standard factorization of maps in the category Ω of trees, cf. Proposition 2.6. Notably, this is rather different from the approach in [**DS11**], despite our approach broadly paralleling theirs, and we believe this more formal approach is of intrinsic value, as it may prove easier to generalize to other contexts.

1.1. Main result

As noted following (1), the nerve $N\colon \mathsf{Op} \to \mathsf{dSet}$ is fully faithful. Moreover, its (essential) image can be characterized as those $X \in \mathsf{dSet}$ satisfying a strict Segal condition, recalled in (10), (12). As such, we will throughout make use of the following trick: rather than describe an operad $\mathcal{O} \in \mathsf{Op}$, we directly build its nerve $N\mathcal{O} \in \mathsf{dSet}$ as a presheaf, then check that the described $N\mathcal{O}$ satisfies the required strict Segal condition. The advantage of this trick is that it provides rather compact descriptions of the main operads we care about (cf. Definition 4.1). For instance, the operad $\Omega(T)$ appearing in (2), (3) is characterized by the identification $N\Omega(T) = \Omega[T]$ (see also Example 2.16), where we recall that $\Omega[T] \in \mathsf{dSet}$ is simply the representable $\Omega[T](-) = \Omega(-, T)$.

In addition, recalling that sOp can be viewed as the subcategory of $\mathsf{Op}^{\Delta^{op}}$ such that the set of objects is constant in the simplicial direction, one likewise has a fully faithful inclusion $N\colon \mathsf{sOp} \to \mathsf{sdSet} = \mathsf{dSet}^{\Delta^{op}}$ with essential image those $X \in \mathsf{sdSet}$ which both satisfy the strict Segal condition on each simplicial level and have constant object set, cf. Remark 2.19. Using the trick above, one has the following compact description of the simplicial operads $W(T) \in \mathsf{sOp}$ in (3).

Proposition 1.1. *The simplicial operad $W(T) \in \mathsf{sOp}$ (cf. [**CM13**, (4.1)]) has nerve given by*

$$(NW(T))_n(S) = \left\{ \textit{composable strings } S \xrightarrow{t} J_0 \xrightarrow{i,p} J_1 \xrightarrow{i,p} \cdots \xrightarrow{i,p} J_n \xrightarrow{f,p} T \textit{ of arrows in } \Omega \right\} \tag{5}$$

where we label maps in Ω as $t/i/f/p$ to indicate they are tall/inner faces/faces/planar[3] *(cf. §2.1).*

The description in (5) makes heavy use of the standard factorization of maps in Ω, recalled in Proposition 2.6. As usual, the simplicial operators simply forget or replace the J_i. Functoriality of (5) on both S and T is a consequence of the properties of said factorization, and is described in (18), (20). Likewise, the properties that $NW(T)$ is levelwise Segal and has constant object set, also following from properties of the factorization, are discussed in Remark 4.3.

For how (5) recovers the original description of $W(T)$ in [**CM13**, (4.1)], see Example 4.25.

By combining Proposition 1.1 and (3) we now have a full definition of the functor $W_!\colon \mathsf{dSet} \to \mathsf{sOp}$, the explicit description of which is the goal of our main result, Theorem 1.2.

Before stating that result, we need additional notation. For a necklace $\mathfrak{n}\colon J \to T$ as in Figure 1, we write $\Omega[\mathfrak{n}] \in \mathsf{dSet}$ for the dendroidal set in (4) (cf. Definitions 3.1

[3] We expect most readers will be familiar with inner faces, faces, and planar maps. As for tall maps, they are defined as those maps in Ω that send the root to the root and leaves to leaves.

and 3.3), and let $\mathsf{Nec} \subset \mathsf{dSet}$ be the full subcategory spanned by the $\Omega[\mathfrak{n}]$. The description of $W_!$ in Theorem 1.2 will rely on a description of $W_!(\Omega[\mathfrak{n}])$ for $\mathfrak{n}$ a necklace (this is elaborated on after (6)). The real appeal of (5) is then that it can easily be modified to describe $W_!(\Omega[\mathfrak{n}])$.

Specifically, $NW_!(\Omega[\mathfrak{n}])$ is the subpresheaf of $NW(T)$ in (5) obtained by imposing an additional condition which is closely related to the characterization of maps between necklaces given in Proposition 3.12(ii). Should the map $S \to T$ in (5) be a tall map, this additional condition is that $J_0 \supseteq J$. Otherwise, one needs a more complex condition $J_0 \supseteq J_{\overline{\phi S}}$ (cf. Remark 4.2), related to "outer faces" of the necklace $\mathfrak{n}\colon J \to T$. See (14) for a depiction of this notion of outer face.

The following is our main result, giving an explicit description of the functor $W_!\colon \mathsf{dSet} \to \mathsf{sOp}$ based on (5).

Theorem 1.2 (cf. [**DS11**, Thm. 1.3 and Cor. 4.4])**.** *Let $X \in \mathsf{dSet}$. Then $W_!(X) \in \mathsf{sOp}$ is the simplicial operad whose nerve is described as follows. The simplices in the n-th level $NW_!(X)_n(S)$ are equivalence classes of quadruples $(\mathfrak{n}, S \xrightarrow{\phi} T, \Omega[\mathfrak{n}] \xrightarrow{x} X, J_\bullet)$ where:*

- *(i)* $(\mathfrak{n}\colon J \to T) \in \mathsf{Nec}$ *is a necklace;*
- *(ii)* $S \xrightarrow{\phi} T$ *is a tall map in Ω such that $J \supseteq \phi(S)$;*
- *(iii)* $\Omega[\mathfrak{n}] \xrightarrow{x} X$ *is a map in* dSet*;*
- *(iv)* $J_\bullet$ *denotes a factorization of ϕ as below, and for which $J_0 \supseteq J$. Arrow labels have the same meaning as in Proposition 1.1 (note that the label of the last map differs from (5)).*

$$S \xrightarrow{t} J_0 \xrightarrow{i,p} J_1 \xrightarrow{i,p} \cdots \xrightarrow{i,p} J_n \xrightarrow{i,p} T$$

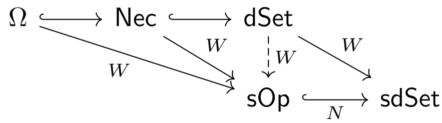
The equivalence relation is generated by considering $(\mathfrak{n}, \phi, x, J_\bullet)$ and $(\mathfrak{n}', \phi', x', J'_\bullet)$ to be equivalent if there is a map $\varphi\colon \Omega[\mathfrak{n}] \to \Omega[\mathfrak{n}']$ (encoded by a map $\varphi\colon T \to T'$, cf. Proposition 3.12(i)) such that $\phi' = \varphi\phi$, $x = x'\varphi$ and $J'_k = \varphi J_k$ (i.e. $J'_\bullet$ is obtained by pushing $J_\bullet$ along φ in the sense of (20)).

Moreover, all such data have a representative, unique up to isomorphism*, for which: $J_\bullet$ is* flanked*, i.e., $J_0 = J$ and $J_n = T$, and; x is* totally non-degenerate*, i.e., for all beads $T_b, b \in \boldsymbol{V}(J)$ of $\mathfrak{n}$ the dendrex $\Omega[T_b] \to \Omega[\mathfrak{n}] \to X$ is non-degenerate.*

In the following, $\eta \in \Omega$ denotes the *stick tree* with one edge and no vertices.

Remark 1.3. The set of objects of $W_!X$ is simply the set $NW_!X(\eta) = X(\eta)$. Moreover, for each $X(\eta)$-signature, i.e., tuple $(x_1, \cdots, x_n; x_0)$ with $x_i \in X(\eta)$, the space of maps $(W_!X)(x_1, \cdots, x_n; x_0) \in \mathsf{sSet}$ is read off of Theorem 1.2 by setting $S = C_n$ to be the n-corolla (i.e. the tree with n leaves and exactly one vertex) and restricting to those quadruples where the composite $\coprod_{\{0,1,\cdots,n\}} \Omega[\eta] \to \mathsf{Sc}[C_n] \to \Omega[\mathfrak{n}] \to X$ is the signature $(x_1, \cdots, x_n; x_0)$.

We now summarize the proof strategy for Theorem 1.2, which can be visualized by the following diagram.



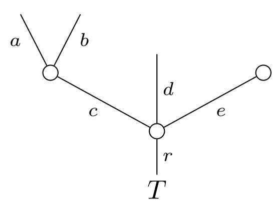
(6)

First, we extend (5) to a functor $W\colon \mathsf{Nec} \to \mathsf{sOp}$ via direct construction in Definition 4.1, and then show that this functor is the left Kan extension of its restriction to $\Omega \hookrightarrow \mathsf{Nec}$, cf. Proposition 4.9. The point of this is then as follows. Defining $W\colon \mathsf{dSet} \to \mathsf{sdSet}$ by making the right rhombus above into a left Kan extension diagram, one has that: (i) $W\colon \mathsf{dSet} \to \mathsf{sdSet}$ actually lands in the essential image of $N\colon \mathsf{sOp} \to \mathsf{sdSet}$, cf. Proposition 4.11, implicitly defining $W\colon \mathsf{dSet} \to \mathsf{sOp}$ and ensuring that the middle triangle is also a left Kan extension; (ii) the functor $W\colon \mathsf{dSet} \to \mathsf{sdSet}$ is easy to compute, due to being a left Kan extension onto a presheaf category, so that the description in Theorem 1.2 is then mostly a matter of unpacking notation, as done in Corollary 4.16. Crucially, we note that (i) would fail if left Kan extending directly from Ω to dSet. Lastly, the choice of the preferred flanked and totally non-degenerate representatives is addressed in Corollary 4.24.

2. Preliminaries

2.1. The category of trees

We begin by recalling the Moerdijk–Weiss category Ω of trees [**MW07**]. First, each object of Ω can be encoded by a (rooted) tree diagram T as below.



(7)

Edges with no vertices $\circ$ above them are called *leaves*, the unique bottom edge is called the *root*, and edges that are neither are called *inner edges*. In the example above, a, b and d are leaves, r is the root, and c and e are inner edges. The sets of edges, inner edges, and vertices of a tree T are denoted $\boldsymbol{E}(T)$, $\boldsymbol{E}^{\mathsf{i}}(T)$, and $\boldsymbol{V}(T)$, respectively.

While the tree diagram description above is helpful for visualizing objects in Ω, in order to describe the arrows, we will use the algebraic notion of a *broad poset*, originally due to Weiss [**Wei12**] and further developed in [**Per18**], which we now briefly recall. For each edge t in a tree topped by a vertex $\circ$, we write $t^{\uparrow}$ for the tuple of edges immediately above t. In (7) one has $r^{\uparrow} = cde$, $c^{\uparrow} = ab$, and $e^{\uparrow} = \epsilon$, where ϵ denotes the empty tuple. Each vertex can then be encoded symbolically as $t^{\uparrow} \leqslant t$, which we call a *generating broad relation*. This notation is motivated by a form of transitivity. For example, in (7) the relations $cde \leqslant r$ and $ab \leqslant c$ generate, under *broad transitivity*, the relation $abde \leqslant r$, and one may similarly obtain relations $cd \leqslant r$ and $abd \leqslant r$. These relations, together with identity relations $t \leqslant t$, then form the *broad poset associated with* T.

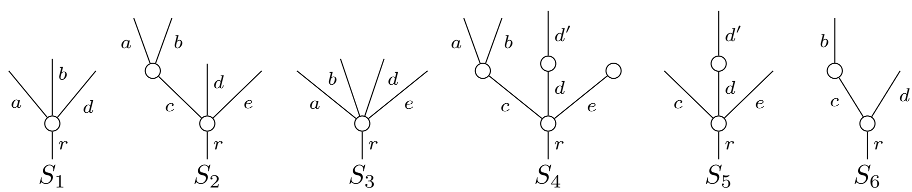
A map of trees $\varphi\colon S \to T$ in Ω is then an underlying map of edge sets $\varphi\colon \boldsymbol{E}(S) \to \boldsymbol{E}(T)$ which preserves broad relations.

If an edge t is pictorially above (or equal to) an edge s, we write $t \leqslant_d s$. Equivalently, $t \leqslant_d s$ if there exists a broad relation $s_1 \dots s_n \leqslant s$ such that $t = s_i$ for some i.

Our discussion will be simplified by assuming that Ω has exactly one representative of each *planarized tree*, by which we mean a tree together with a planar representation as in (7). Any map $\varphi\colon S \to T$ in Ω then has a unique factorization $S \xrightarrow{\simeq} S' \to T$ as an isomorphism followed by a *planar map* [**BP21**, Prop. 3.24]. In particular, the wide subcategory of Ω spanned by planar maps is skeletal, i.e., the only planar isomorphisms are the identities.

Notation 2.1. We write η for the *stick* tree, the unique tree with a single edge and no vertices.

Example 2.2. The edge labels in each tree S_i below determine maps $\boldsymbol{E}(S_i) \to \boldsymbol{E}(T)$, where T is as in (7). For $i \leqslant 5$ this encodes maps $S_i \to T$ in Ω, but not for $i = 6$.



Definition 2.3. A map of trees $\varphi\colon S \to T$ is called:

- a *tall map* if $\varphi(\underline{l}_S) = \underline{l}_T$ and $\varphi(r_S) = r_T$, with $\underline{l}_{(-)}$ and $r_{(-)}$ denoting the tuple of leaf edges and the root edge;
- a *face map* if it is injective on edges; an *inner face* if it is also tall; and an *outer face* if, for any factorization $\varphi \simeq \varphi_1\varphi_2$ with φ_1, φ_2 face maps and φ_2 inner, φ_2 is an isomorphism;
- a *degeneracy* if it is surjective on edges and preserves leaves (and is thus tall);
- a *convex map* if, whenever $e <_d e' <_d e''$ in T and e, e'' are in the image of φ, then so is e'.

Pictorially, inner face maps $S \to T$ remove some edges in T (and merge the vertices adjacent to those edges), outer face maps remove some vertices of T, and degeneracies collapse some of the unary vertices of S. Face maps combine inner and outer faces, tall maps combine inner faces and degeneracies, and convex maps combine outer faces and degeneracies (cf. Remark 2.8).

Example 2.4. In Example 2.2, $S_1 \to T$ is an inner face, $S_2 \to T$ is an outer face, $S_3 \to T$ is a face that is neither inner nor outer, $S_4 \to T$ is a degeneracy, and $S_5 \to T$ is a convex map.

Notation 2.5. Throughout the remainder of the paper, we will label a map in Ω by the letters d/i/o/t/f/p to indicate that the map is a degeneracy/inner face/outer face/tall/face/planar.

Proposition 2.6 ([**BP20**, Prop. 2.2])**.** *A map of trees* $\varphi\colon S \to T$ *has a strictly unique factorization*

$$S \xrightarrow{\simeq} S_p \xrightarrow{pd} \varphi S \xrightarrow{pi} \overline{\varphi S} \xrightarrow{po} T \tag{8}$$

as an isomorphism followed by a planar degeneracy, a planar inner face, and a planar outer face.

Notation 2.7. The notation φS is motivated by the fact that this tree has edge set $\boldsymbol{E}(\varphi S) = \varphi(\boldsymbol{E}(S))$, while the notation $\overline{\varphi S}$ is an instance of the *outer closure of a face* notation in [**BP20**, Not. 2.14] which, for a face F, defines $\overline{F}$ as the smallest (planar) outer face containing F.

Remark 2.8 (cf. [**BPb**, Remarks 2.8, 2.9, 2.13])*.* For any subset $\mathcal{S} \subseteq \{\simeq, pd, pi, po\}$ of the arrow labels in (8), the type of maps whose factors labeled by $\mathcal{S}$ are identities is closed under composition.

In particular, as (non-planar) tall maps (resp. face maps, convex maps) are characterized as those maps such that the component labeled po (resp. pd, pi) in (8) is the identity, we have that these types of maps (and their planar analogues) are closed under composition.

Remark 2.9. Modifying (8) by ignoring planarity gives a factorization $S \xrightarrow{d} U \xrightarrow{i} V \xrightarrow{o} T$, unique up to unique isomorphisms. Moreover, combining the i and o arrows recovers the usual degeneracy-face decomposition in [**MW07**, Lemma 3.1], while combining the d and i arrows recovers the tall-outer decomposition in [**BP21**, Prop. 3.36].

Notation 2.10. A *corolla* is a tree with a single vertex. For each $n \geqslant 0$, one has a corolla C_n with n leaves, and we write Σ for the category of corollas and isomorphisms, which is naturally identified with the category of standard finite sets $\{1, 2, \cdots, n\}$ and isomorphisms.

For any tree T with n leaves, we write $\mathsf{lr}(T)$, which we call the *leaf-root of* T, for the corolla C_n, which comes together with a unique planar tall map $\mathsf{lr}(T) \to T$.

Example 2.11. For the tree T in (7), the corolla $\mathsf{lr}(T)$ is S_1 in Example 2.2.

Notation 2.12. For a tree T and $v \in \boldsymbol{V}(T)$, we write $T_v \to T$ for the planar outer face consisting of only this vertex and its adjacent edges. Further, for a map $\varphi\colon J \to T$ and $b \in \boldsymbol{V}(J)$, we write $T_b = \overline{\varphi J_b}$. Compare with the notion of *bead* in Definition 3.1(ii) and Figure 1.

2.2. Dendroidal sets and operads

This subsection recalls the definitions of the key categories appearing in the main adjunctions (1). First, the category of *dendroidal sets* is the category $\mathsf{dSet} = \mathsf{Set}^{\Omega^{op}}$ of presheaves on Ω.

There are a number of presheaves that play a key role in the theory of dendroidal sets. First, for each tree $T \in \Omega$, one has the representable presheaf $\Omega[T](S) = \Omega(S, T)$. Moreover, one has the following subpresheaves of $\Omega[T]$, called the *boundary*, *inner* *horn*, and *Segal core*

$$\partial\Omega[T] = \bigcup_{\substack{U\in\mathsf{Face}(T),\\ U\neq T}} \Omega[U], \qquad \Lambda^E[T] = \bigcup_{\substack{U\in\mathsf{Face}(T),\\ U\not\to T-E}} \Omega[U], \qquad Sc[T] = \bigcup_{U\in\mathsf{Face}_{sc}(T)} \Omega[U], \tag{9}$$

where $\mathsf{Face}(T)$ is the poset of planar faces, $\emptyset \neq E \subseteq \boldsymbol{E}^{\mathsf{i}}(T)$ is a non-empty set of inner edges, and $\mathsf{Face}_{sc}(T)$ is the poset of planar outer faces with no inner edges (i.e., U with either a single edge or a single vertex). Typically $\partial\Omega[T]$ and $\Lambda^E[T]$ are the main objects of interest (see, e.g. [**BP20**, §4], for further discussion), but in this paper the $Sc[T]$ play the central role, partly due to $Sc[T]$ being a necklace, cf. Remark 3.5, and partly since they appear in the Segal condition below.

Given $X, A \in \mathsf{dSet}$, let us abbreviate $X(A) = \mathsf{dSet}(A, X)$. We then say that X satisfies the *strict Segal condition* if, for any tree $T \in \Omega$, the natural map below is an isomorphism.

$$X(T) = X(\Omega[T]) \xrightarrow{\simeq} X(Sc[T]) \tag{10}$$

As noted at the start of §1.1, we will identify the category Op of (colored) operads with its essential image under the nerve $N\colon \mathsf{Op} \to \mathsf{dSet}$, which consists of the objects satisfying the strict Segal condition (10) (more precisely, this follows from [**MW09**, Prop. 5.3 and Thm. 6.1] together with Remark 2.18 below). For the usual description of Op, see [**CM13**, §1] or [**BPa**, Def. 3.44].

Some of our arguments in §4 will be simplified by using an alternative formulation of (10), which is motivated by the fact that colored operads Op are most commonly defined using colored trees. As such, we first recall the following, cf. [**BPa**, Def. 3.21].

Definition 2.13. Let $\mathfrak{C}$ be a set of colors. The category $\Omega_{\mathfrak{C}}$ of $\mathfrak{C}$-*trees* has objects pairs $(T, \mathfrak{c})$ with $T \in \Omega$ a tree and $\mathfrak{c}\colon \boldsymbol{E}(T) \to \mathfrak{C}$ a coloring of its edges, and arrows $(S, \mathfrak{d}) \to (T, \mathfrak{c})$ given by maps $\varphi\colon S \to T$ in Ω such that $\mathfrak{d} = \mathfrak{c}\varphi$.

Moreover, any a map of color sets $f\colon \mathfrak{C} \to \mathfrak{D}$ induces a functor $f\colon \Omega_{\mathfrak{C}} \to \Omega_{\mathfrak{D}}$ via $(T, \mathfrak{c}) \mapsto (T, f\mathfrak{c})$.

Notation 2.14. Given $X \in \mathsf{dSet}$, tree $T \in \Omega$, and coloring $\mathfrak{c}\colon \boldsymbol{E}(T) \to X(\eta)$, we write $X_{\mathfrak{c}}(T) \in \mathsf{Set}$ for the pullback below.

$$\begin{array}{ccc} X_{\mathfrak{c}}(T) & \longrightarrow & X(T) \\ \downarrow & & \downarrow \\ * & \xrightarrow{\ \mathfrak{c}\ } & \prod_{\boldsymbol{E}(T)} X(\eta) \end{array}$$

Remark 2.15. The notation above gives a decomposition $X(T) \simeq \coprod_{\{\mathfrak{c}\colon \boldsymbol{E}(T)\to X(\eta)\}} X_{\mathfrak{c}}(T)$ for any $X \in \mathsf{dSet}$. Moreover, the assignment $(T, \mathfrak{c}) \mapsto X_{\mathfrak{c}}(T)$ is functorial on $X(\eta)$-trees $(T, \mathfrak{c}) \in \Omega^{op}_{X(\eta)}$, so that X has an equivalent description as a presheaf on $\Omega_{X(\eta)}$. In fact, a little more is true. If one writes $\mathsf{dSet}_{\mathfrak{C}} \subset \mathsf{dSet}$ for the subcategory of those X such that $X(\eta) = \mathfrak{C}$ and maps that are the identity on $X(\eta)$, there is an equivalence of categories (cf. [**BPb**, Eq. (3.21)])

$$\mathsf{dSet}_{\mathfrak{C}} \xrightarrow{\simeq} \mathsf{Fun}_*(\Omega^{op}_{\mathfrak{C}}, \mathsf{Set}), \qquad \big(T \mapsto X(T)\big) \mapsto \big((T, \mathfrak{c}) \mapsto X_{\mathfrak{c}}(T)\big) \tag{11}$$

where Fun_* denotes *pointed functors*, i.e., functors X such that $X_{\mathfrak{c}}(\eta) = *$ for any $\mathfrak{c} \in \mathfrak{C}$.

Using the $X_{\mathfrak{c}}(T)$ notation, the Segal condition in (10) then decomposes into isomorphisms

$$X_{\mathfrak{c}}(T) \xrightarrow{\simeq} \prod_{v \in \boldsymbol{V}(T)} X_{\mathfrak{c}_v}(T_v) \tag{12}$$

for any $X(\eta)$-tree $(T, \mathfrak{c})$, and with $\mathfrak{c}_v$ the restricted coloring $\boldsymbol{E}(T_v) \to \boldsymbol{E}(T) \to X(\eta)$.

Example 2.16. Representables $\Omega[S]$ satisfy the strict Segal condition (12). Explicitly, this Segal condition says that a map $T \to S$ in Ω is determined by maps $T_v \to S$, which is the content of [**Per18**, Prop. 5.11]. The operad $\Omega(S)$ such that $\Omega[S] = N\Omega(S)$ is defined in [**MW07**, §3].

Remark 2.17. Given a map of colors $f\colon \mathfrak{C} \to \mathfrak{D}$, the identification (11) and precomposition with $f\colon \Omega_{\mathfrak{C}} \to \Omega_{\mathfrak{D}}$ yield the left functor f^* below. Moreover, f^* is clearly compatible with the Segal condition (12) so that, writing $\mathsf{Op}_{\mathfrak{C}} = \mathsf{dSet}_{\mathfrak{C}} \cap \mathsf{Op}$, one has the restricted f^* functor on the right.

$$f^*\colon \mathsf{dSet}_{\mathfrak{D}} \to \mathsf{dSet}_{\mathfrak{C}} \qquad\qquad f^*\colon \mathsf{Op}_{\mathfrak{D}} \to \mathsf{Op}_{\mathfrak{C}}$$

Note that maps $X \to Y$ in either dSet or Op over a color map f are then in bijection with fixed color maps $X \to f^*Y$.

Remark 2.18. Condition (10) has other formulations. Indeed, by [**BP20**, Props. 3.22, 3.31] one may replace the Segal cores $Sc[T]$ in (10) with the inner horns $\Lambda^E[T]$, thus saying that X has the strict right lifting property against the maps $\Lambda^E[T] \to \Omega[T]$. This shows that ∞-operads generalize operads, as the first are defined by the non-strict version of this property [**MW09**, §5].

Remark 2.19. When dealing with simplicial operads sOp, we will also have need to discuss simplicial dendroidal sets $\mathsf{sdSet} = \mathsf{Set}^{\Delta^{op} \times \Omega^{op}}$, whose levels we write as $X_n(T)$ for $T \in \Omega$ and $[n] \in \Delta$. As noted in §1.1, applying the nerve along each simplicial direction yields a fully faithful inclusion $N\colon \mathsf{sOp} \to \mathsf{sdSet}$ with essential image those $X \in \mathsf{sdSet}$ for which $X(\eta)$ is a discrete simplicial set and which satisfy the Segal condition (10), (12) on each simplicial level (or equivalently, which satisfy (10), (12) when regarded as an identification of simplicial sets).

3. Dendroidal necklaces

We now formalize the notion of dendroidal necklace discussed in the introduction, cf. Figure 1, thus generalizing the key notion in [**DS11**]. For the meaning of $\overline{\mathfrak{n}J_b}$, T_b, see Notations 2.7, 2.12.

Definition 3.1 (cf. [**DS11**, §3])**.** A *necklace* is a planar inner face map $\mathfrak{n}\colon J \to T$ in Ω. Moreover:

(i) J is called the *inner face of joints* of the necklace;
(ii) for each vertex $b \in \boldsymbol{V}(J)$, the outer face $\overline{\mathfrak{n}J_b} = T_b \hookrightarrow T$ is called a *bead* of the necklace, and we write $\boldsymbol{B}(\mathfrak{n}) \simeq \boldsymbol{V}(J)$ for the set of beads.

Example 3.2. For $\mathfrak{n}\colon J \to T$ the necklace in Figure 1 the beads are the trees T_i depicted therein.

We now formalize the idea behind (4), thus defining the presheaves $\Omega[\mathfrak{n}]$. Recall, cf. (9), that the Segal core poset $\mathsf{Face}_{sc}(J)$ consists of the planar outer faces of J with no inner edges.

Definition 3.3. Given a necklace $\mathfrak{n}\colon J \to T$ we define its representable presheaf $\Omega[\mathfrak{n}] \in \mathsf{dSet}$ by

$$\Omega[\mathfrak{n}] = \operatorname*{colim}_{U \in \mathsf{Face}_{sc}(J)} \Omega[\overline{\mathfrak{n}U}] = \bigcup_{U \in \mathsf{Face}_{sc}(J)} \Omega[\overline{\mathfrak{n}U}]$$

where the union formula is taken inside $\Omega[T]$.

The category Nec of necklaces is then the full subcategory of dSet spanned by the $\Omega[\mathfrak{n}]$.

Remark 3.4. For any tall map $S \to T$ in Ω, [**BP21**, Cor. 3.75] says that one has a decomposition

$$T = \operatorname*{colim}_{U \in \mathsf{Face}_{sc}(S)} \overline{\mathfrak{n}U}$$

as a colimit in Ω, which formalizes the grafting procedure in Figure 1. Crucially, the relevance of Definition 3.3 comes from the fact that the Yoneda $\Omega[-]$ does not preserve this decomposition.

Remark 3.5. The $\Omega[\mathfrak{n}]$ presheaves for necklaces $\mathfrak{n}\colon J \to T$ interpolate between the representable and Segal core presheaves $\Omega[T]$ and $Sc[T]$ in §2.2. More explicitly, each tree $T \in \Omega$ gives rise to necklaces $T \xrightarrow{=} T$ and $\mathsf{lr}(T) \to T$ (cf. Notation 2.10) for which

$$\Omega[T \xrightarrow{=} T] = Sc[T], \qquad \Omega[\mathsf{lr}(T) \to T] = \Omega[T].$$

In particular, one obtains a natural inclusion $\Omega \hookrightarrow \mathsf{Nec}$ given by $T \mapsto (\mathsf{lr}(T) \to T)$. However, we caution that the assignment $T \mapsto Sc[T]$ is not functorial on T (more precisely, it is functorial only with respect to *convex* maps of trees, in the sense of Definition 2.3).

Remark 3.6. If $X \in \mathsf{dSet}$ satisfies the Segal condition (12) and $\mathfrak{n}\colon J \to T$ is a necklace, then

$$X_{\mathfrak{c}}(T) \simeq \prod_{v \in \boldsymbol{V}(T)} X_{\mathfrak{c}_v}(T_v) \simeq \prod_{b \in \boldsymbol{B}(\mathfrak{n})} \prod_{v \in \boldsymbol{V}(T_b)} X_{\mathfrak{c}_v}(T_v) \simeq \prod_{b \in \boldsymbol{B}(\mathfrak{n})} X_{\mathfrak{c}_b}(T_b),$$

so that the natural maps $X(T) \xrightarrow{\simeq} X(\Omega[\mathfrak{n}])$ are isomorphisms, cf. (10).

Lemma 3.7. *Let $\mathfrak{n}\colon J \to T$ be a necklace. Then*

(i) a face $U \hookrightarrow T$ is in $\Omega[\mathfrak{n}]$ iff its outer closure $\overline{U}$ is;

(ii) an outer face $U = \overline{U} \hookrightarrow T$ is in $\Omega[\mathfrak{n}]$ iff $\boldsymbol{E}^{\mathsf{i}}(J) \cap \boldsymbol{E}^{\mathsf{i}}(U) = \emptyset$;

(iii) there is a decomposition $\boldsymbol{E}(T) \simeq \boldsymbol{E}(J) \amalg \coprod_{b \in \boldsymbol{V}(J)} \boldsymbol{E}^{\mathsf{i}}(T_b) = \boldsymbol{E}(J) \amalg \coprod_{b \in \boldsymbol{B}(\mathfrak{n})} \boldsymbol{E}^{\mathsf{i}}(T_b)$.

Proof. (i) follows since $\Omega[\mathfrak{n}]$ is an union of outer faces.

The arguments for (ii), (iii) are by induction on the number of inner edges $\boldsymbol{E}^{\mathsf{i}}(J)$, with the base case $\boldsymbol{E}^{\mathsf{i}}(J) = \emptyset$, so that $J = T = \eta$, being obvious. Otherwise, letting $e \in \boldsymbol{E}^{\mathsf{i}}(J)$, since e is an inner edge of both J and T one has grafting decompositions $J = J' \amalg_e J''$, $T = T' \amalg_e R''$ together with inner face maps $\mathfrak{n}'\colon J' \to T'$, $\mathfrak{n}''\colon J'' \to T''$. One then has that U is in $\Omega[\mathfrak{n}]$ iff it is in either $\Omega[\mathfrak{n}']$ or in $\Omega[\mathfrak{n}'']$, yielding the induction step for (ii). The induction step for (iii) likewise follows. $\square$

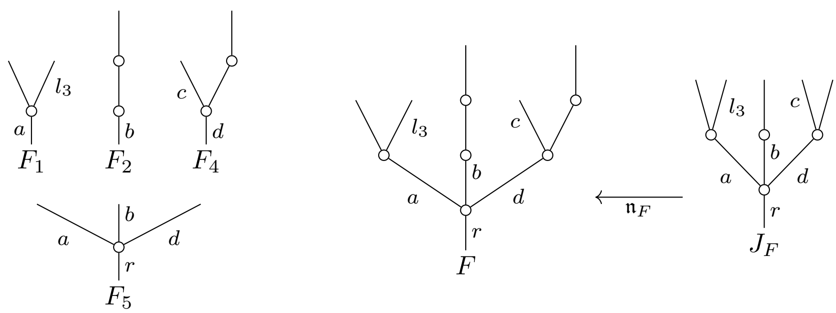
Remark 3.8. If $S \xrightarrow{d} S'$ is a degeneracy, the vertices of S' are identified with the vertices of S that are not collapsed to edges. Thus, by factoring a tall map $\varphi\colon S \xrightarrow{t} T$ as a degeneracy followed by inner face $S \xrightarrow{d} \varphi S \xrightarrow{i} T$, cf. Remark 2.8, the decomposition (iii) in Lemma 3.7 generalizes to

$$\boldsymbol{E}(T) = \boldsymbol{E}(\varphi S) \amalg \coprod_{v \in \boldsymbol{V}(S)} \boldsymbol{E}^{\mathsf{i}}(\overline{\varphi S_v}). \tag{13}$$

Notation 3.9. Given a necklace $\mathfrak{n}\colon J \to T$ and outer face $F \to T$ we write $\mathfrak{n}_F\colon J_F \to F$ for the necklace characterized by

$$\boldsymbol{E}^{\mathsf{i}}(J_F) = \boldsymbol{E}^{\mathsf{i}}(J) \cap \boldsymbol{E}^{\mathsf{i}}(F).$$

Example 3.10. Letting $\mathfrak{n}\colon J \to T$ be the necklace in Figure 1, and for $F \to T$ the outer face depicted in the middle below, the following represents $\mathfrak{n}_F\colon J_F \to F$, with the F_i its beads.



(14)

In general, the $\mathfrak{n}_F$ construction works as follows, where say a bead T_b is *outer* if it is connected to at most one other bead $T_{b'}$ (equivalently, if all outer edges of the bead T_b are outer edges of T itself, *except at most one*). First, $\mathfrak{n}_F$ removes some outer beads altogether. In this example, T_3 from Figure (1) is removed. Then, some of the resulting outer beads are replaced with outer faces of themselves. In this example, T_1, T_2, T_4 from Figure (1) are replaced with F_1, F_2, F_4 (note that T_4 was initially not an outer bead, but became so upon removal of T_3).

We caution that, just as in this example, one in general does not have a map $J_F \to J$, as $\boldsymbol{E}(J)$ needs not contain $\boldsymbol{E}(J_F)$. Instead, as will follow from Proposition 3.12(ii), one has a map of necklaces $\mathfrak{n}_F \to \mathfrak{n}$, which should be thought of as an outer face map in Nec.

Corollary 3.11. *Let* $\mathfrak{n}\colon J \to T$ *be a necklace and* $F \to T$ *be an outer face. Then*

$$\Omega[\mathfrak{n}_F] = \Omega[\mathfrak{n}] \cap \Omega[F]$$

where the intersection is taken as subpresheaves of $\Omega[T]$.

Proof. Combining (i), (ii) in Lemma 3.7 we see that a face $U \hookrightarrow F$ is in $\Omega[\mathfrak{n}]$ iff $\boldsymbol{E}(J) \cap \boldsymbol{E}^{\mathsf{i}}(\overline{U}) = \emptyset$, where (since F is outer) the outer closure $\overline{U}$ can be taken in either T or F. But, since $\overline{U} \hookrightarrow F$ implies $\boldsymbol{E}^{\mathsf{i}}(\overline{U}) \subseteq \boldsymbol{E}^{\mathsf{i}}(F)$, this is equivalent to $\boldsymbol{E}(J_F) \cap \boldsymbol{E}^{\mathsf{i}}(\overline{U}) = \emptyset$, i.e., to U being in $\Omega[\mathfrak{n}_F]$. ☐

We next characterize the maps in Nec. See Notations 2.7, 3.9 for the meaning of φJ, $J'_{\overline{\varphi J}}$.

Proposition 3.12. *Let $\mathfrak{n}\colon J \to T$ and $\mathfrak{n}'\colon J' \to T'$ be necklaces. Then:*

(i) A map $\mathfrak{n} \to \mathfrak{n}'$ in Nec is uniquely determined by some map $T \to T'$ in Ω. More precisely, there exists an unique dashed arrow making the following commute.

$$\begin{array}{ccc} \Omega[\mathfrak{n}] & \hookrightarrow & \Omega[T] \\ \downarrow & & \downarrow \exists! \\ \Omega[\mathfrak{n}'] & \hookrightarrow & \Omega[T'] \end{array} \tag{15}$$

(ii) A map of trees $\varphi\colon T \to T'$ in Ω induces a map $\mathfrak{n} \to \mathfrak{n}'$ in Nec iff $\varphi J \supseteq J'_{\overline{\varphi T}}$.

Proof. For $U \in \mathsf{Face}_{sc}(J)$ the composite $\Omega[\mathfrak{n}] \to \Omega[\mathfrak{n}'] \to \Omega[T']$ in (15) gives compatible maps $\Omega[\overline{\mathfrak{n}U}] \to \Omega[T']$ in dSet, and thus compatible maps $\overline{\mathfrak{n}U} \to T'$ in Ω, so (i) follows from Remark 3.4.

We now turn to (ii). The map φ defines a map of necklaces precisely if it induces maps $\Omega[T_b] \to \Omega[\mathfrak{n}']$ for each bead T_b, $b \in \boldsymbol{B}(\mathfrak{n})$ and, by Lemma 3.7, this is equivalent to

$$\emptyset = \boldsymbol{E}^{\mathsf{i}}(J') \cap \boldsymbol{E}^{\mathsf{i}}(\overline{\varphi T_b}) = \boldsymbol{E}^{\mathsf{i}}(J') \cap \boldsymbol{E}^{\mathsf{i}}(\overline{\varphi \mathfrak{n} J_b}). \tag{16}$$

Writing $\tilde{\varphi}$ for the composite $J \xrightarrow{\mathfrak{n}} T \xrightarrow{\varphi} \overline{\varphi T}$ and noting that $\tilde{\varphi}$ is tall, (13) becomes

$$\boldsymbol{E}(\overline{\varphi T}) = \boldsymbol{E}(\tilde{\varphi} J) \amalg \coprod_{b \in \boldsymbol{V}(J)} \boldsymbol{E}^{\mathsf{i}}(\overline{\tilde{\varphi} J_b}) = \boldsymbol{E}(\tilde{\varphi} J) \amalg \coprod_{b \in \boldsymbol{B}(\mathfrak{n})} \boldsymbol{E}^{\mathsf{i}}(\overline{\varphi T_b}).$$

Thus, (16) amounts to $\boldsymbol{E}^{\mathsf{i}}(J') \cap \boldsymbol{E}(\overline{\varphi T}) \subseteq \boldsymbol{E}(\varphi J)$, which is equivalent to the desired $\varphi J \supseteq J'_{\overline{\varphi T}}$ (as these trees have the same outer edges). ∎

Remark 3.13. Let $\mathfrak{n}, \mathfrak{n}', T, T'$ be as in Proposition 3.12 and suppose $\varphi\colon T \to T'$ defines a map $\mathfrak{n} \to \mathfrak{n}'$. Then for every outer face $F \to T$ it follows from Corollary 3.11 that the restriction $F \to \overline{\varphi F}$ likewise induces a restriction $\mathfrak{n}_F \to \mathfrak{n}'_{\overline{\varphi F}}$, from which it follows that $\varphi J_F \supseteq \left(J'_{\overline{\varphi T}}\right)_{\overline{\varphi F}} = J'_{\overline{\varphi F}}$.

Remark 3.14. Let $\mathfrak{n}, \mathfrak{n}', T, T', \varphi$ be as in the previous remark and suppose in addition that φ is a face map. Then, since different beads share at most one edge, for each bead $T_b \hookrightarrow T$ of $\mathfrak{n}$, there is a unique bead $T'_{\varphi_* b} \hookrightarrow T'$ of $\mathfrak{n}'$ such that $T_b \hookrightarrow T \to T'$ factors as $T_b \to T'_{\varphi_* b} \hookrightarrow T'$. In particular, this defines a map of bead sets $\varphi_*\colon \boldsymbol{B}(\mathfrak{n}) \to \boldsymbol{B}(\mathfrak{n}')$.

4. The dendroidal $W_!$-construction

This section will establish the description of the $W_!$-construction in (1) given in Theorem 1.2.

Throughout we make use of the factorizations in Ω given in Proposition 2.6, and follow Notation 2.5 by labeling a map by the letters d/i/o/t/f/p to indicate that the map is a degeneracy/inner face/outer face/tall/face/planar. Moreover, we implicitly use Remark 2.8, stating that for some types of maps the factorization (8) has only certain factors, as well as Remark 2.9, which combines factors in (8) to obtain simplified factorizations.

We first build $W(T)$ for a tree T, cf. Proposition 1.1.

Definition 4.1. Let $T \in \Omega$ be a tree. We define $W(T) \in \mathsf{sOp}$ to be the simplicial operad whose nerve is the simplicial dendroidal set $NW(T) \in \mathsf{sdSet}$ (cf. Remark 2.19) with n-simplices given by

$$NW(T)_n(S) = \Big\{\text{composable strings } S \xrightarrow{t} J_0 \xrightarrow{i,p} J_1 \xrightarrow{i,p} \cdots \xrightarrow{i,p} J_n \xrightarrow{i,p} F \xrightarrow{o,p} T \text{ of arrows in } \Omega\Big\}. \quad (17)$$

Equivalently, it suffices to require that the $S \to J_i$ are tall maps and the $J_i \to T$ are planar face maps. We note that F is superfluous, being determined by $J_n \xrightarrow{f,p} T$, but including it will make (19) below more readable. See also Remark 4.2.

Functoriality of $NW(T)$ with respect to a map $S^* \to S$ is described by the diagram

$$\begin{array}{ccccccccccccc} S^* & \xrightarrow{t} & J_0^* & \xrightarrow{i,p} & J_1^* & \xrightarrow{i,p} & \cdots & \xrightarrow{i,p} & J_n^* & \xrightarrow{i,p} & F^* & \xrightarrow{o,p} & T \\ \downarrow & & \downarrow o & & \downarrow o & & & & \downarrow o & & \downarrow o & & \| \\ S & \xrightarrow{t} & J_0 & \xrightarrow{i,p} & J_1 & \xrightarrow{i,p} & \cdots & \xrightarrow{i,p} & J_n & \xrightarrow{i,p} & F & \xrightarrow{o,p} & T \end{array} \quad (18)$$

where the maps $J_k^* \to J_k$ and $F^* \to F$ are inductively defined by taking $S^* \to J_0^* \to J_0$ (resp. $J_k^* \to J_{k+1}^* \to J_{k+1}$, $J_n^* \to F^* \to F$) to be the "tall followed by outer face" factorization of the composite $S^* \to S \to J_0$ (resp. $J_k^* \to J_k \to J_{k+1}$, $J_n^* \to J_n \to F$).

More generally, given a necklace $\mathfrak{n}\colon J \to T$, we define $NW(\mathfrak{n}) \subseteq NW(T)$ as the subpresheaf formed by those strings in (17) such that one has $J_0 \supseteq J_F$ (where J_F is as in Notation 3.9). The fact that this $NW(\mathfrak{n})$ is a presheaf follows since, for $S^* \to S$, J_1, J_i^*, F, F^* as in (18), it is

$$\boldsymbol{E}^{\mathsf{i}}(J_0^*) = \boldsymbol{E}^{\mathsf{i}}(J_0) \cap \boldsymbol{E}^{\mathsf{i}}(F^*) \supseteq \boldsymbol{E}^{\mathsf{i}}(J_F) \cap \boldsymbol{E}^{\mathsf{i}}(F^*) = \boldsymbol{E}^{\mathsf{i}}(J_{F^*}) \quad (19)$$

where the first step is [**BP20**, Lemma 2.5] applied to $J_0^* \xrightarrow{o} J_0 \xrightarrow{i} F$ and $J_0^* \xrightarrow{i} F^* \xrightarrow{o} F$, the second is the definition of $NW(\mathfrak{n})$, and the third follows from Notation 3.9 and the fact that $F^* \subseteq F$.

Remark 4.2. Writing $\phi\colon S \to T$ for the full composite in (17), one has that the F therein is $\overline{\phi S}$ (cf. Notation 2.7) In particular, the condition $J_0 \supseteq J_F$ defining $NW(\mathfrak{n})$ becomes $J_0 \supseteq J_{\overline{\phi S}}$.

Remark 4.3. The $NW(\mathfrak{n})$ given by Definition 4.1 are nerves of simplicial operads, cf. Remark 2.19. Indeed, to verify the Segal condition (12) note that, as the maps $S \to J_i$ in (18) are tall, they are uniquely determined by maps $S_v \to J_{i,v}$ for $v \in \boldsymbol{V}(S)$, cf. [**BP21**, Cor 3.75] (see also 3.4). Moreover, $NW(\mathfrak{n})(\eta)$ is a discrete simplicial set since for $S = \eta$ it must be $J_i = \eta$ in (17), due to only η receiving tall maps from η.

Next, we discuss the functoriality of $NW(T)$ with respect to $T \in \Omega$. For a map $T \to T'$ in Ω we define $NW(T)_n(S) \to NW(T')_n(S)$ via the diagram (where we drop the superfluous F in (17))

$$\begin{array}{ccccccccccc} S & \xrightarrow{t} & J_0 & \xrightarrow{i,p} & J_1 & \xrightarrow{i,p} & \cdots & \xrightarrow{i,p} & J_n & \xrightarrow{f,p} & T \\ \| & & \downarrow d & & \downarrow d & & & & \downarrow d & & \downarrow \\ S & \xrightarrow{t} & J_0' & \xrightarrow{i,p} & J_1' & \xrightarrow{i,p} & \cdots & \xrightarrow{i,p} & J_n' & \xrightarrow{f,p} & T' \end{array} \quad (20)$$

where the maps $J_k \to J_k'$ are (backwards) inductively defined by taking $J_n \to J_n' \to T'$

(resp. $J_{k-1} \to J'_{k-1} \to J'_k$) to be the "degeneracy followed by face" factorization of the composite $J_n \to T \to T'$ (resp. $J_{k-1} \to J_k \to J'_k$).

Proposition 4.4. *For any map $T \to T'$ in Ω, the induced map $NW(T)(S) \to NW(T')(S)$ in (20) is functorial on S.*

Proof. First, note that the composite $NW(T)(S) \to NW(T)(S^*) \to NW(T')(S^*)$ is computed by the left diagram below, where $S^* \to J_i^* \to J_i$ and $J_i^* \to (J_i^*)' \to T'$ are the unique factorizations with the indicated properties. On the other hand, the composite $NW(T)(S) \to NW(T')(S) \to NW(T')(S^*)$ is computed as on the right with $J_i \to J'_i \to T'$ and $S^* \to (J'_i)^* \to J'_i$ the unique indicated factorizations.

$$\begin{array}{ccccc} S & \xrightarrow{t} & J_i & \xrightarrow{f,p} & T \\ \uparrow & & \uparrow{\scriptstyle o,p} & & \| \\ S^* & \xrightarrow{t} & J_i^* & \longrightarrow & T \\ \| & & \downarrow{\scriptstyle d} & & \downarrow \\ S^* & \longrightarrow & (J_i^*)' & \xrightarrow{f,p} & T' \end{array} \qquad\qquad \begin{array}{ccccc} S & \xrightarrow{t} & J_i & \xrightarrow{f,p} & T \\ \| & & \downarrow{\scriptstyle d} & & \downarrow \\ S & \longrightarrow & J'_i & \xrightarrow{f,p} & T' \\ \uparrow & & \uparrow{\scriptstyle o,p} & & \| \\ S^* & \xrightarrow{t} & (J'_i)^* & \longrightarrow & T' \end{array} \tag{21}$$

The key to the proof is to show that the planar faces $(J_i^*)'$ and $(J'_i)^*$ of T' coincide, since it will then be automatic that all maps connecting the $(J_i^*)'$ and $(J'_i)^*$ and S^*, T' likewise match.

To see this, we consider the following diagram which combines the top halves in (21).

$$\begin{array}{ccccc} S^* & \xrightarrow{t} & J_i^* & \longrightarrow & T \\ \downarrow & & \downarrow{\scriptstyle o,p} & & \| \\ S & \xrightarrow{t} & J_i & \xrightarrow{f,p} & T \\ \| & & \downarrow{\scriptstyle d} & & \downarrow \\ S & \longrightarrow & J'_i & \xrightarrow{f,p} & T' \end{array}$$

Both faces $(J_i^*)'$ and $(J'_i)^*$ can be built by factoring the composite $J_i^* \to J_i \to J'_i$, with $(J_i^*)'$ coming from the degeneracy-face factorization and $(J'_i)^*$ coming from the tall-outer factorization. But since $J_i^* \to J_i \to J'_i$ is a composite of convex maps (cf. Definition 2.3) it is again convex (see Remark 2.8), so the two factorizations coincide, finishing the proof. □

Corollary 4.5. *Let $\mathfrak{n}\colon J \to T$ and $\mathfrak{n}'\colon J' \to T'$ be necklaces and suppose $\psi\colon T \to T'$ induces a map $\mathfrak{n} \to \mathfrak{n}'$. Then the induced map $NW(T) \to NW(T')$ restricts to a map $NW(\mathfrak{n}) \to NW(\mathfrak{n}')$.*

Proof. Following Remark 4.2, we need to show that the map $NW(T) \to NW(T')$ sends simplices (17) such that $J_0 \supseteq J_{\overline{\phi S}}$ to simplices such that $J'_0 \supseteq J'_{\overline{\phi' S}}$, where ϕ, ϕ' are the composites of each simplex. This follows since

$$J'_0 = \psi(J_0) \supseteq \psi(J_{\overline{\phi S}}) \supseteq J'_{\overline{\psi(\overline{\phi S})}} = J'_{\overline{\psi' S}}$$

where the third step is Remark 3.13. □

We now introduce a notation that plays an important role in two key technical results, Propositions 4.9 and 4.11. Recall that, for any tree $U \in \Omega$, the poset $\mathsf{Face}_{inn}(U)$ of planar inner faces is in fact a lattice, with the join $F \vee F'$ the characterized by $\boldsymbol{E}^{\mathsf{i}}(F \vee F') = \boldsymbol{E}^{\mathsf{i}}(F) \cup \boldsymbol{E}^{\mathsf{i}}(F')$.

Notation 4.6. Let $\mathfrak{n}\colon J \to T$ be a necklace, $\phi\colon S \to T$ a map in Ω and $S \xrightarrow{t} F \xrightarrow{p,o} T$ its tall-outer factorization. We then write $J^{\phi} = \phi S \vee J_F$, where the join is taken in $\mathsf{Face}_{inn}(F)$.

Remark 4.7. Following Notation 2.14, we write $NW(\mathfrak{n})_{\phi}(S) \subseteq NW(\mathfrak{n})(S) \in \mathsf{sSet}$ for the subsimplicial set over a coloring $\phi\colon \boldsymbol{E}(S) \to \boldsymbol{E}(T)$. Note that, by the description in (17), ϕ must in fact be the map of trees $\phi\colon S \to T$ given by the composite therein.

Remark 4.8. In the context of Notation 4.6, and writing $\bar{\phi}\colon S \to F$, $\iota\colon F \to T$, $\iota^{\phi}\colon J^{\phi} \to T$, for the natural maps, one has identifications

$$NW(\mathfrak{n})_{\iota^{\phi}}(J^{\phi}) \xrightarrow{\simeq} NW(\mathfrak{n})_{\phi}(S) \xleftarrow{\simeq} NW(\mathfrak{n}_F)_{\bar{\phi}}(S) \xleftarrow{\simeq} NW(J^{\phi} \to F)_{\bar{\phi}}(S)$$

induced by the natural maps $S \to J^{\phi}$ between trees and $(J^{\phi} \to F) \to \mathfrak{n}_F \to \mathfrak{n}$ between necklaces.

Proposition 4.9. *Let $\mathfrak{n}\colon J \to T$ be a necklace. Then one has an identification*

$$W(\mathfrak{n}) \simeq \underset{U \in \mathsf{Face}_{sc}(J)}{\operatorname{colim}} W(T_U) \tag{22}$$

where the colimit takes place in sOp.

Proof. We will verify (22) at the level of nerves. More explicitly, we will show that for any $X \in \mathsf{sdSet}$ with constant objects and satisfying the strict Segal condition (cf. Remark 2.19), giving a map $NW(\mathfrak{n}) \to X$ is the same as giving compatible maps $NW(T_U) \to X$.

Moreover, clearly both sides of (22) yield $\boldsymbol{E}(T)$ when evaluated at η. As such, we are free to fix a coloring $\mathfrak{c}\colon \boldsymbol{E}(T) \to X(\eta)$ and verify the universal property restricted to maps respecting this color assignment. And, by using the identification (11) and the last comment in Remark 2.17, we may evaluate $NW(\mathfrak{n}), X$ on $\boldsymbol{E}(T)$-colored trees (S, ϕ), rather than on uncolored trees.

Given maps $NW(T_U) \to X$ we now define the map $NW(\mathfrak{n}) \to X$ via (where J^{ϕ}, F, $\bar{\phi}$, ι, ι^{ϕ} are as in Remark 4.8, and $\iota_*\colon \boldsymbol{B}(\mathfrak{n}_F) \to \boldsymbol{B}(\mathfrak{n})$ is the map of bead sets in Remark 3.14)

$$\begin{array}{ccccc}
NW(\mathfrak{n})_{\phi}(S) & \underset{(I)}{\xleftarrow{\simeq}} & NW(\mathfrak{n})_{\iota^{\phi}}(J^{\phi}) & \underset{(II)}{\xrightarrow{\simeq}} & \prod_{b \in \boldsymbol{B}(\mathfrak{n}_F)} NW(\mathfrak{n})_{\iota^{\phi}_b}(J^{\phi}_b) \\
 & & & & \Big\uparrow {\scriptstyle (III)\ \simeq} \\
 & & & & \prod_{b \in \boldsymbol{B}(\mathfrak{n}_F)} NW(T_{\iota_* b})_{\iota^{\phi}_b}(J^{\phi}_b) \\
 & & & & \Big\downarrow {\scriptstyle (IV)} \\
X_{\mathfrak{c}\phi}(S) & \underset{(VI)}{\longleftarrow} & X_{\mathfrak{c}\iota^{\phi}}(J^{\phi}) & \underset{(V)}{\xrightarrow{\simeq}} & \prod_{b \in \boldsymbol{B}(\mathfrak{n}_F)} X_{\mathfrak{c}\iota^{\phi}_b}(J^{\phi}_b)
\end{array} \tag{23}$$

where the arrows (II) and (V) are isomorphisms by the Segal condition while (III) is an isomorphism since $J^{\phi}_b \to T$ factors through $T_{\iota_* b}$.

Moreover, the arrow (IV) in (23) is induced by the chosen maps $NW(T_U) \to X$, so clearly (23) denotes the only possible compatible map $NW(T) \to X$.

It only remains to check that (23) is indeed a map in sdSet, i.e., that it is natural on (S, ϕ). To see this, one first readily checks that a map $\psi\colon (S,\phi) \to (S^*, \phi^*)$ induces a compatible inclusion $\psi\colon J^{\phi} \hookrightarrow J^{\phi^*}$ showing the naturality of arrows (I), (VI) in the zigzag. Next, by Remark 3.14 one has a map of bead sets $\psi_*\colon \boldsymbol{B}(\mathfrak{n}_F) \to \boldsymbol{B}(\mathfrak{n}_{F^*})$ for which one has further compatible maps $J^{\phi}_b \to J^{\phi^*}_{\psi_* b}$, showing the naturality of the arrows (II), (V). Lastly, for any bead $b \in \boldsymbol{B}(\mathfrak{n}_F)$ one has $T_{\iota_* b} = T_{(\iota^*)_* \psi_* b}$, showing naturality of the arrows (III), (IV). ☐

Remark 4.10. Let $I \xrightarrow{A_\bullet} \mathsf{dSet}$ be a diagram of dendroidal sets and let $A = \mathrm{colim}_{i\in I} A_i$.

We will find it useful to describe A in light of the identification (11). For each $A(\eta)$-colored tree $\vec{S} = (S, \mathfrak{c})$ we write $I_{\vec{S}/}$ for the category with objects factorizations $\boldsymbol{E}(S) \to A_i(\eta) \to A(\eta)$ for some $i \in I$, which we represent by $\boldsymbol{E}(S) \to A_i(\eta)$, together with maps $i \to i'$ in I satisfying the obvious compatibility. Then

$$A_{\mathfrak{c}}(S) \simeq \underset{(\boldsymbol{E}(S)\to A_i(\eta))\in I_{\vec{S}/}}{\mathrm{colim}} A_{i,\mathfrak{c}_i}(S) \tag{24}$$

where $\mathfrak{c}_i$ in $A_{i,\mathfrak{c}_i}(S)$ denotes the coloring given by $\boldsymbol{E}(S) \to A_i(\eta)$.

Proposition 4.11. *Let $X \in \mathsf{dSet}$ and define $NW(X) \in \mathsf{sdSet}$ by*

$$NW(X) = \underset{(\Omega[\mathfrak{n}]\to X)\in \mathsf{Nec}_{/X}}{\mathrm{colim}} NW(\mathfrak{n}), \tag{25}$$

where $\mathsf{Nec}_{/X} = \mathsf{Nec} \downarrow X$ is the over category of maps $\Omega[\mathfrak{n}] \to X$, and the colimit is taken in sdSet.

Then $NW(X)$ satisfies the strict Segal condition (10), (12), and has constant objects, cf. Remark 2.19. In particular, since N is fully-faithful one has that $NW(X)$ is the nerve of the simplicial operad

$$W(X) = \underset{(\Omega[\mathfrak{n}]\to X)\in \mathsf{Nec}_{/X}}{\mathrm{colim}} W(\mathfrak{n}),$$

where the colimit is now taken in simplicial operads sOp.

Proof. We will evaluate $NW(X)$ at each $X(\eta)$-colored tree $\vec{S} = (S, \mathfrak{c})$ using Remark 4.10. We write $\mathsf{Nec}_{\vec{S}//X} = \left(\mathsf{Nec}_{/X}\right)_{\vec{S}/}$ for the category whose objects are pairs of arrows $\boldsymbol{E}(S) \xrightarrow{\phi_{\mathfrak{n}}} \Omega[\mathfrak{n}] \to X$ whose composite encodes the coloring $\mathfrak{c}\colon \boldsymbol{E}(T) \to X(\eta)$. Equation (24) then says that

$$NW(X)_{\mathfrak{c}}(S) \simeq \underset{(\boldsymbol{E}(S)\to\Omega[\mathfrak{n}]\to X)\in \mathsf{Nec}_{\vec{S}//X}}{\mathrm{colim}} NW(\mathfrak{n})_{\phi_{\mathfrak{n}}}(S). \tag{26}$$

To show that $NW(X)$ satisfies the strict Segal condition, we will rewrite (26) by identifying appropriate subcategories of $\mathsf{Nec}_{\vec{S}//X}$. First, write $\mathsf{Nec}^{\Omega}_{\vec{S}//X} \subset \mathsf{Nec}_{\vec{S}//X}$ for the full subcategory of those objects for which, writing $\mathfrak{n}\colon J \to T$, the map $\phi_{\mathfrak{n}}\colon \boldsymbol{E}(S) \to \boldsymbol{E}(T)$ gives a map $\phi_{\mathfrak{n}}\colon S \to T$ in Ω.

Next, for $\phi_{\mathfrak{n}}\colon S \to T$ as above, and as in Notation 4.6, we write $S \xrightarrow{\bar{\phi}_{\mathfrak{n}}} F_{\mathfrak{n}} \xrightarrow{\iota} T$ for the tall-outer factorization. We then write $\mathsf{Nec}^{\Omega,nor}_{\vec{S}//X} \subset \mathsf{Nec}^{\Omega}_{\vec{S}//X}$ for the full subcategory of "normalized factorizations", defined by the properties that $\phi_{\mathfrak{n}}\colon S \to T$ is a tall map, i.e., $F_{\mathfrak{n}} = T$, and $J \supseteq \phi_{\mathfrak{n}} S$.

Moreover, there is a retraction $\mathsf{Nec}^{\Omega}_{\vec{S}//X} \xrightarrow{n} \mathsf{Nec}^{\Omega,nor}_{\vec{S}//X}$ which sends $\boldsymbol{E}(S)\Omega[\eta] \to \Omega[J \xrightarrow{\mathfrak{n}} T] \to X$ to $n(\mathfrak{n}) = (\phi_{\mathfrak{n}} S \vee J_{F_{\mathfrak{n}}} \to F_{\mathfrak{n}}) = (J^{\phi_{\mathfrak{n}}} \to F_{\mathfrak{n}})$ (cf. Notation 4.6). Recall (cf. Remark 4.8) that the natural map $n(\mathfrak{n}) \to \mathfrak{n}$ in Nec induces isomorphisms $NW(n(\mathfrak{n}))_{\bar{\phi}_{\mathfrak{n}}}(S) \xrightarrow{\simeq} NW(\mathfrak{n})_{\phi_{\mathfrak{n}}}(S)$.

Since $\mathsf{Nec}^{\Omega}_{\vec{S}//X}$ is a cosieve[4] of $\mathsf{Nec}_{\vec{S}//X}$ and $NW(\mathfrak{n})_{\mathfrak{s}}(S) = \emptyset$ whenever $\mathfrak{n}$ is not in $\mathsf{Nec}^{\Omega}_{\vec{S}//X}$ one can replace $\mathsf{Nec}_{\vec{S}//X}$ with $\mathsf{Nec}^{\Omega}_{\vec{S}//X}$ in (26). Moreover, the existence of a retraction implies that the inclusion $\mathsf{Nec}^{\Omega,nor}_{\vec{S}//X} \subset \mathsf{Nec}^{\Omega}_{\vec{S}//X}$ is initial, so one can further replace $\mathsf{Nec}^{\Omega}_{\vec{S}//X}$ with $\mathsf{Nec}^{\Omega,nor}_{\vec{S}//X}$.

Lastly, note that the normalization conditions imply that $\mathsf{Nec}^{\Omega,nor}_{\vec{S}//X} \simeq \prod_{v \in \boldsymbol{V}(S)} \mathsf{Nec}^{\Omega,nor}_{\vec{S}_v//X}$ by a grafting argument. Putting everything together we now obtain that

$$\begin{aligned} NW(X)_{\mathfrak{c}}(S) &\simeq \operatorname*{colim}_{(\boldsymbol{E}(S)\to\Omega[\mathfrak{n}]\to X)\in\mathsf{Nec}^{\Omega,nor}_{\vec{S}//X}} NW(\mathfrak{n})_{\phi_{\mathfrak{n}}}(S) \\ &\simeq \operatorname*{colim}_{(\boldsymbol{E}(S_v)\to\Omega[\mathfrak{n}_{T_v}]\to X)\in\prod_{v\in\boldsymbol{V}(S)}\mathsf{Nec}^{\Omega,nor}_{\vec{S}_v//X}} \left(\prod_{v\in\boldsymbol{V}(S)} NW(\mathfrak{n})_{\phi_{\mathfrak{n},v}}(S_v)\right) \\ &\simeq \operatorname*{colim}_{(\boldsymbol{E}(S_v)\to\Omega[\mathfrak{n}_{T_v}]\to X)\in\prod_{v\in\boldsymbol{V}(S)}\mathsf{Nec}^{\Omega,nor}_{\vec{S}_v//X}} \left(\prod_{v\in\boldsymbol{V}(S)} NW(\mathfrak{n}_{T_v})_{\phi_{\mathfrak{n}},v}(S_v)\right) \\ &\simeq \prod_{v\in\boldsymbol{V}(S)} \left(\operatorname*{colim}_{(\boldsymbol{E}(S_v)\to\mathfrak{n}_{T_v}\to X)\in\mathsf{Nec}^{\Omega,nor}_{\vec{S}_v//X}} NW(\mathfrak{n}_{T_v})_{\phi_{\mathfrak{n},v}}(S_v)\right) \\ &\simeq \prod_{v\in\boldsymbol{V}(S)} NW(X)_{\mathfrak{c}_v}(S_v) \end{aligned} \tag{27}$$

where the first step follows from the previous paragraph, the second step is the identification of indexing categories above together with the strict Segal condition for $NW(\mathfrak{n})$, the third step uses the middle isomorphisms in Remark 4.6, the fourth step is the fact that products commute with colimits in each variable, and the last step simply specifies the first step for $\vec{S}_v = (S_v, \mathfrak{c}_v)$.

We have thus established the strict Segal condition for $NW(X)$ so that, as it is clear that the objects of $NW(X)$ are discrete (this is inherited from the $NW(\mathfrak{n})$), this finishes the proof. □

Remark 4.12. The normalization condition in the previous proof is equivalent to requiring that $\phi_{\mathfrak{n}}\colon S \to T$ is a tall map which induces a map $\mathsf{Sc}[S] \to \Omega[\mathfrak{n}]$.

Propositions 4.9 and 4.11 now combine to give the following, establishing (6).

[4]Recall that a subcategory $\mathcal{S} \subseteq \mathcal{C}$ is a *cosieve* if, for any map $s \to c$ with $s \in \mathcal{S}$, both c and $s \to c$ are also in $\mathcal{S}$.

Theorem 4.13 (cf. [**DS11**, Thm. 1.3])**.** *Consider the following diagram, where the functors labeled W are as defined by Definition 4.1 and Proposition 4.11.*

$$\begin{array}{ccccc} \Omega & \hookrightarrow & \mathsf{Nec} & \hookrightarrow & \mathsf{dSet} \\ & \searrow^{W} & & \searrow^{W} & \downarrow W \\ & & & & \mathsf{sOp} \end{array}$$

Then both triangles in this diagram are left Kan extensions. In particular, the functor $W\colon \mathsf{dSet} \to \mathsf{sOp}$ coincides with the functor $W_!\colon \mathsf{dSet} \to \mathsf{sOp}$ as defined in (1).

Remark 4.14. Our arguments so far can readily be modified to also describe τ in the $\tau\colon \mathsf{dSet} \rightleftarrows \mathsf{Op}\colon N$ adjunction in (1). First, Remark 3.6 implies that, for any necklace $\mathfrak{n}\colon J \to T$ it must be $\tau\Omega[\mathfrak{n}] \simeq \tau\Omega[T] = \Omega(T)$, i.e., the operad such that $N\Omega(T) = \Omega[T]$, cf. Example 2.16. We note that, in light of Remark 3.4, this last observation is the analogue of Proposition 22 for τ.

Adapting the proofs of Proposition 4.11 and Theorem 4.13, and writing $\mathfrak{c}, \phi$ as therein, one then has

$$\begin{aligned} (N\tau X)_{\mathfrak{c}}(S) &\simeq \operatorname*{colim}_{(\boldsymbol{E}(S)\to\Omega[J\to T]\to X)\in \mathsf{Nec}_{\vec{S}//X}} \Omega[T]_{\phi}(S) \\ &\simeq \operatorname*{colim}_{(\boldsymbol{E}(S)\to\Omega[J\to T]\to X)\in \mathsf{Nec}^{\Omega,nor}_{\vec{S}//X}} \Omega[T]_{\phi}(S). \end{aligned}$$

Moreover, the normalization conditions guarantee that one tautologically has $\Omega[T]_{\phi}(S) = *$ in the rightmost formula (as $\Omega[T]_{\phi}(S)$ consists of elements lifting the prescribed $\phi\colon S \to T$). Thus, by unpacking the left formula and specifying to the case of $S = C$ a corolla, one has that operations in $\tau X(C)$ are represented by data of the form $\Omega[C] \xrightarrow{t} \Omega[T] \leftarrow \Omega[J \to T] \to X$ subject to the equivalence relation generated by deeming two such data to be equivalent whenever there exists a map of necklaces $(J \to T) \to (J' \to T')$ making the diagram below commute.

$$\begin{array}{ccccccc} \Omega[C] & \xrightarrow{t} & \Omega[T] & \longleftarrow & \Omega[J \to T] & \longrightarrow & X \\ \| & & \downarrow & & \downarrow & & \| \\ \Omega[C] & \xrightarrow{t} & \Omega[T'] & \longleftarrow & \Omega[J' \to T'] & \longrightarrow & X \end{array}$$

Remark 4.15. The work in this paper can be adapted to the categories dSet_G and Op_G of *genuine equivariant dendroidal sets* and *genuine equivariant operads*, introduced in [**Per18**, §5.4] and [**BP21**], respectively. In particular, the "genuine operadification" functor $\tau_G\colon \mathsf{dSet}_G \to \mathsf{Op}_G$ from [**BPb**, Eq. (2.42)] can be described via an analogue of Remark 4.14.

Briefly, G-trees are defined as follows. First, the category Φ of forests has objects formal coproducts $\amalg_{i\in I} T_i$ of trees, with maps $\amalg_{i\in I} T_i \to \amalg_{j\in J} S_j$ given by a map of sets $\varphi\colon I \to J$ together with maps $T_i \to S_{\varphi(i)}$ of trees. Then, G-forests Φ^G are defined as the G-objects in Φ, while G-trees $\Omega_G \subset \Phi^G$ are the full subcategory of G-forests for which the G-action is transitive on tree components. Extending Proposition 2.6, maps in Ω_G likewise have a standard factorization [**BPb**, Cor. 2.27], which allows for a generalization of the work herein.

For instance, an *equivariant necklace* is a map $\mathfrak{n}\colon J \to T$ of G-trees that is a planar orbital inner face [**BPb**, Def. A.3]. Explicitly, this means that $\mathfrak{n}$ is an ordered isomorphism on roots/components which is a planar inner face on each tree component. Moreover, letting $\Omega[T] \in \mathsf{dSet}^G$ for $T \in \Omega_G$ be the representables in [**BPb**, §2.3], one may define $\Omega[J \to T]$ just as in (3.1). Altogether, adapting Remark 4.14, one has that for each G-corolla C (i.e., G-tree whose tree components are corollas) the operations in $\tau_G X(C) \in \mathsf{Op}_G$ can be represented by data $\Omega[C] \xrightarrow{t,r} \Omega[T] \leftarrow \Omega[J \to T] \to X$ (where the map labeled t, r induces an ordered isomorphism on roots which is tall in each component) subject to the equivalence relation generated by diagrams

$$\begin{array}{ccccccc} \Omega[C] & \xrightarrow{t,r} & \Omega[T] & \longleftarrow & \Omega[J \to T] & \longrightarrow & X \\ \| & & \downarrow & & \downarrow & & \| \\ \Omega[C] & \xrightarrow{t,r} & \Omega[T'] & \longleftarrow & \Omega[J' \to T'] & \longrightarrow & X. \end{array}$$

Theorem 4.13 established that $W_!\colon \mathsf{dSet} \to \mathsf{sOp}$ is computed by (25), which is the hard technical step in establishing Theorem 1.2. Thus, the remainder of the paper will mostly unpack (25) to obtain the description of $NW(X)$ for $X \in \mathsf{dSet}$ featured in Theorem 1.2, with the following establishing the non-unique description, reformulated using the spaces $X_{\mathfrak{c}}(T)$ in Notation 2.14.

Corollary 4.16 (cf. [**DS11**, Cor. 4.4])**.** *Let $X \in \mathsf{dSet}$. Then the simplices in $NW(X)_{n,\mathfrak{c}}(S)$ for a coloring $\mathfrak{c}\colon \boldsymbol{E}(T) \to X(\eta)$ are equivalence classes of quadruples $(\mathfrak{n}, S \xrightarrow{\phi} T, \Omega[\mathfrak{n}] \xrightarrow{x} X, J_\bullet)$ where:*

- *(i)* $(J \xrightarrow{\mathfrak{n}} T) \in \mathsf{Nec}$ *is a necklace;*
- *(ii)* $S \xrightarrow{\phi} T$ *is a tall map in Ω such that $J \supseteq \phi S$ (equivalently, ϕ induces a map $\mathsf{Sc}[S] \to \Omega[\mathfrak{n}]$);*
- *(iii)* $\Omega[\mathfrak{n}] \to X$ *is a map in* dSet *such that the induced composite $\boldsymbol{E}(S) \to \boldsymbol{E}(T) \to X(\eta)$ is the coloring $\mathfrak{c}$;*
- *(iv)* $J_\bullet$ *denotes a simplex in $NW(\mathfrak{n})_{n,\phi}$, i.e., a factorization of ϕ*

$$S \xrightarrow{t} J_0 \xrightarrow{i,p} J_1 \xrightarrow{i,p} \cdots \xrightarrow{i,p} J_n \xrightarrow{i,p} T \tag{28}$$

  such that $J_0 \supseteq J$.

The equivalence relation is generated by considering $(\mathfrak{n}, \phi, x, J_\bullet)$ and $(\mathfrak{n}', \phi', x', J'_\bullet)$ to be equivalent if there is a map $\varphi\colon \Omega[\mathfrak{n}] \to \Omega[\mathfrak{n}']$ such that $\phi' = \varphi\phi$, $x = x'\varphi$ and $J'_k = \varphi J_k$ (i.e. $J'_\bullet$ is obtained by pushing $J_\bullet$ along φ in the sense of (20)).

Proof. Conditions (i), (iii), (iv) follow by simply unpacking (26) in light of the construction of $NW(\mathfrak{n})$ in Definition 4.1 (except with ϕ then just a map $\phi\colon \boldsymbol{E}(S) \to \boldsymbol{E}(S)$ and the last map in (iv) only required to be tall rather than inner). The additional condition (ii) follows by replacing (26) with its reduction to "normalized factorizations" $\mathsf{Nec}^{\Omega,nor}_{\vec{S}//X}$, as in the first line of (27). □

Our last goal is to complete the proof of Theorem 1.2 by showing that, as claimed therein, the quadruples in Corollary 4.16 always have a nice suitably unique representative.

We first discuss uniqueness of the maps $\Omega[\mathfrak{n}] \xrightarrow{x} X$ up to degeneracy.

Definition 4.17. A map of necklaces $(J \to T) \to (J' \to T')$ is called a *necklace degeneracy* if the associated map $\varphi\colon T \to T'$ is a degeneracy in Ω and $\varphi J = J'$.

Definition 4.18 (cf. [**DS11**, §4])**.** Let $J \xrightarrow{\mathfrak{n}} T$ be a necklace and $X \in \mathsf{dSet}$. A map $\Omega[\mathfrak{n}] \to X$ is called *totally non-degenerate* if for all beads T_b, $b \in \boldsymbol{B}(\mathfrak{n})$ the induced dendrex $\Omega[T_b] \to X$ is non-degenerate (in the sense of, e.g. [**Per18**, Prop. 5.62]).

Lemma 4.19 (cf. [**DS11**, Prop. 4.7])**.** *Any map $\Omega[\mathfrak{n}] \to X$ has a factorization, unique up to unique isomorphism, as*

$$\Omega[J \xrightarrow{\mathfrak{n}} T] \to \Omega[J' \xrightarrow{\mathfrak{n}'} T'] \to X$$

where the first map is a degeneracy of necklaces and the second map is totally non-degenerate.

Proof. The proof is by induction on the size of $\boldsymbol{E}^{\mathsf{i}}(J)$. The base case is that of $\boldsymbol{E}^{\mathsf{i}}(J) = \emptyset$ (note that then it must also be $\boldsymbol{E}^{\mathsf{i}}(J') = \emptyset$), in which case the result reduces to [**CM11**, Prop. 6.9] or [**Per18**, Prop. 5.62].

Otherwise, let $e \in \boldsymbol{E}^{\mathsf{i}}(J)$ and consider the grafting decomposition $T = R \amalg_e S$. By the induction hypothesis, one has factorizations, unique up to unique isomorphism, $\Omega[\mathfrak{n}_R] \to \Omega[\mathfrak{n}'_R] \to X$, $\Omega[\mathfrak{n}_S] \to \Omega[\mathfrak{n}'_S] \to X$. Writing $\mathfrak{n}'_R = (J'_R \to R')$ and $\mathfrak{n}'_S = (J'_S \to S')$, we then set $\mathfrak{n}' = (J'_R \amalg_e J'_S \to T'_R \amalg_e T'_S)$. The uniqueness up to unique isomorphism property of $\mathfrak{n}'$ is readily seen to be inherited from the analogue property for $\mathfrak{n}'_R, \mathfrak{n}'_S$ (note that the "unique isomorphism" clause implies that there is no ambiguity concerning the grafting edge e), finishing the proof. □

Next, we also need a preferred form for the tall simplex data in (28).

Definition 4.20 (cf. [**DS11**, §4])**.** A tall simplex as in (28) is called *flanked* if $J_0 = J$ and $J_n = T$. Further, a quadruple $(\mathfrak{n}, \phi, x, J_\bullet)$ is called *flanked* if $J_\bullet$ is.

Remark 4.21. Suppose $(\mathfrak{n}, \phi, x, J_\bullet)$ is a flanked quadruple and set $\mathfrak{n}_k = (J_k \to T)$. Then the structure maps in (28) induce a diagram of maps of necklaces

$$\mathsf{Sc}[T] = \Omega[\mathfrak{n}_n] \longrightarrow \Omega[\mathfrak{n}_{n-1}] \longrightarrow \cdots \longrightarrow \Omega[\mathfrak{n}_0] = \Omega[\mathfrak{n}] \longleftarrow \mathsf{Sc}[S]$$

Remark 4.22. If both simplices $J_\bullet, J'_\bullet$ in a pushforward diagram (20) are flanked, then the associated map of necklaces $\mathfrak{n} \to \mathfrak{n}'$ is a degeneracy.

In what follows we say a quadruple $(\mathfrak{n}, \phi, x, J_\bullet)$ as in Corollary 4.16 is *flanked* if $J_\bullet$ is and *totally non-degenerate* if x is.

Lemma 4.23 (cf. [**DS11**, Lemma 4.5])**.** *(i) Any quadruple $(\mathfrak{n}, \phi, x, J_\bullet)$ as in Corollary 4.16 is equivalent a flanked one;*

(ii) if two flanked quadruples are equivalent, then the equivalence can be described via a zigzag involving only flanked quadruples.

Proof. The key to (i) is the fact that the map $J_n \to T$ induces a map of necklaces $(J_0 \to J_n) \to (J \to T)$. This map of necklaces induces a pushforward of simplices (i.e., a diagram as in (20))

$$\begin{array}{ccccccccccc} S & \xrightarrow{t} & J_0 & \xrightarrow{i,p} & J_1 & \xrightarrow{i,p} & \cdots & \xrightarrow{i,p} & J_n & = & J_n \\ \| & & \| & & \| & & & & \| & & \downarrow \\ S & \xrightarrow{t} & J_0 & \xrightarrow{i,p} & J_1 & \xrightarrow{i,p} & \cdots & \xrightarrow{i,p} & J_n & \xrightarrow{i,p} & T \end{array} \tag{29}$$

where the top simplex (and thus the associated quadruple) is now flanked, so (i) follows.

(ii) then follows by noting that the procedure above is natural. More precisely, an arbitrary pushforward of tall simplices (i.e., simplices whose composite map is a tall map) along the necklace map $(J,T) \to (J^* \to T^*)$ as in (20) induces a pushforward of flanked simplices

$$\begin{array}{ccccccccccc} S & \xrightarrow{t} & J_0 & \xrightarrow{i,p} & J_1 & \xrightarrow{i,p} & \cdots & \xrightarrow{i,p} & J_n & = & J_n \\ \| & & \downarrow d & & \downarrow d & & & & \downarrow d & & \downarrow \\ S & \xrightarrow{t} & J'_0 & \xrightarrow{i,p} & J'_1 & \xrightarrow{i,p} & \cdots & \xrightarrow{i,p} & J'_n & \xrightarrow{f,p} & T' \end{array}$$

along the necklace map $(J_0 \to J_n) \to (J'_0 \to J'_n)$. □

Corollary 4.24 (cf. [**DS11**, Cor. 4.8])**.** *Each quadruple* $(\mathfrak{n}, \phi, x, J_\bullet)$ *as in Corollary 4.16 has a representative, unique up to unique isomorphism, which is both flanked and totally non-degenerate.*

Proof. By Lemma 4.23(i) any quadruple is equivalent to a flanked quadruple and, by Lemma 4.19, any flanked quadruple is equivalent to a flanked quadruple that is also totally non-degenerate.

As for the uniqueness condition, by Lemma 4.23(ii) we need only consider zigzags of equivalences of flanked quadruples, which are induced by necklace degeneracies, in the sense of Definition 4.17, cf. Remark 4.22. Thus, arguing by induction on the size of the zigzag, Lemma 4.19 implies that all flanked quadruples in the zigzag have the same totally non-degenerate quotient, so the desired uniqueness claim reduces to the uniqueness claim in Lemma 4.19. □

We conclude the paper by using Theorem 1.2 to describe $W_!$ applied to the key dendroidal sets in §2.2. We first make some useful observations concerning $W_!$ applied to a representable $\Omega[U]$.

Example 4.25. For $X = \Omega[U] \in \mathsf{dSet}$, one can describe $W_!(\Omega[U])$ via either Proposition 1.1 or Theorem 1.2. In preparation for the next examples, which require Theorem 1.2, we will find it useful to work out how Theorem 1.2 recovers Proposition 1.1. Putting together all the data in the unique representative description in Theorem 1.2, a simplex of $W_!(\Omega[U])$ is strictly uniquely represented by

$$S \xrightarrow{t} J_0 = J \xrightarrow{i,p} J_1 \xrightarrow{i,p} \cdots \xrightarrow{i,p} J_n = T \xrightarrow[\phi]{f,p} U. \tag{30}$$

This requires some justification. First, note that the role of ϕ is to represent a map $\phi\colon \Omega[J \to T] \to \Omega[U]$. Then, the requirement in Theorem 1.2 that ϕ is totally non-degenerate as a map of necklaces reduces to the implied claim in (30) that ϕ is a face map of trees. The conditions $J_0 = J$ and $J_n = T$ are the flanked conditions.

Lastly, the assumption in (30) that ϕ is planar is a *choice*, which one is free to make, which turns the "uniqueness up to unique isomorphism" in Theorem 1.2 into strict uniqueness. We now see that (30) indeed recovers (5) (that this is also compatible with the simplicial structure follows from the flanking procedure in (29)).

We now apply Remark 1.3 to determine the mapping spaces of $W_!(\Omega[T])$. Since this operad has color set $\boldsymbol{E}(U)$, we consider signatures $(e_1, \cdots, e_n; e_0)$ with $e_i \in \boldsymbol{E}(U)$, which can be regarded as a map $e(-)\colon \boldsymbol{E}(C) \to \boldsymbol{E}(U)$ for C the n-corolla. We now claim it is (cf. [**CM13**, §4])

$$W_!(\Omega[U])(e_1, \cdots, e_n; e_0) = \begin{cases} \Delta[1]^{\times \boldsymbol{E}^{\mathsf{i}}(\overline{e(C)})} & \text{if } \boldsymbol{E}(C) \xrightarrow{e(-)} \boldsymbol{E}(U) \text{ defines a map in } \Omega \\ \varnothing & \text{otherwise} \end{cases} \tag{31}$$

where $\overline{e(C)}$ is the outer closure notation in Notation 2.7. In words, $\overline{e(C)}$ is the unique outer face of U with leaves $e_1, \cdots, e_n$ and root e_0, if such tree exists. The identification (31) now follows by setting $S = C$ in (30). Indeed, if $e(-)$ does not define a map $C \to U$, then no factorizations as in (30) exist. And, otherwise, the only restriction on the J_i therein is that they must be inner faces of $\overline{e(C)}$. But then (30) computes the nerve of the poset $\mathsf{Face}_{inn}(\overline{e(C)})$ of inner faces of $\overline{e(C)}$, which coincides with the poset $(0 \to 1)^{\times \boldsymbol{E}^{\mathsf{i}}(\overline{e(C)})}$ of subsets of its inner edges, establishing (31).

Example 4.26. We now apply Theorem 1.2 to compute $W_!(\partial\Omega[U])$. By the discussion in Example 4.25, its simplices are uniquely represented just as in (30), except with the caveat that ϕ now represents a map $\phi\colon \Omega[J \to T] \to \partial\Omega[U]$. This imposes the following restriction: U itself can not be a bead of the necklace $J \to T$, which amounts to either $J \neq \mathsf{lr}(U)$ or $T \neq U$.

As such, for any signature $(e_1, \cdots, e_n; e_0)$ of $\boldsymbol{E}(U)$ which is not the left-root signature, one has $W_!(\partial\Omega[U])(e_1, \cdots, e_n; e_0) = W_!(\Omega[U])(e_1, \cdots, e_n; e_0)$, since then T is a proper face of U.

And, for the leaf-root signature $(\underline{l}; r)$, this restriction amounts to excluding the boundary of the nerve of the poset $\mathsf{Face}_{inn}(U) \simeq (0 \to 1)^{\times \boldsymbol{E}^{\mathsf{i}}(U)}$, thus identifying $W_!(\partial\Omega[U])(\underline{l}, r)$ with the *domain* of the iterated pushout product

$$(\{0,1\} \to \Delta[1])^{\square \boldsymbol{E}^{\mathsf{i}}(U)}.$$

Example 4.27. Let $U \in \Omega$ and $\emptyset \neq E \subseteq \boldsymbol{E}^{\mathsf{i}}(U)$, and consider $W_!(\Lambda^E[U])$. As in Example 4.26, one now requires for ϕ in (30) to encode a map $\Omega[J \to V] \to \Lambda^E[U]$, which imposes the restriction that either $T \not\supseteq U - E$ or $J \neq \mathsf{lr}(U)$.

As in Example 4.26 one has $W_!(\Lambda^E[U])(e_1, \cdots, e_n; e_0) = W(\Omega[U])(e_1, \cdots, e_n; e_0)$ whenever $(e_1, \cdots, e_n; e_0) \not\simeq (\underline{l}; r)$, as then T can contain no inner faces.

Lastly, for the leaf-root signature $(\underline{l}, r)$, the given restrictions identify $W_!(\Lambda^E[U])(\underline{l}, r)$ with the *domain* of the iterated pushout product

$$(\{0,1\} \to \Delta[1])^{\square \boldsymbol{E}^{\mathsf{i}}(U) - E} \square (\{1\} \to \Delta[1])^{\square E}.$$

References

[BP20] P. Bonventre and L.A. Pereira, *Equivariant dendroidal Segal spaces and G-∞-operads*, Algebr. Geom. Topol. **20** (2020), no. 6, 2687–2778.

[BP21] ______, *Genuine equivariant operads*, Adv. Math. **381** (2021), 107502, 133 pp.

[BPa] ______, *Homotopy theory of equivariant operads with fixed colors.* To appear in *Tunis. J.Math.*, arXiv preprint 1908.05440v2.

[BPb] ______, *Equivariant dendroidal sets and simplicial operads*, arXiv preprint 1911.06399.

[CM11] D.-C. Cisinski and I. Moerdijk, *Dendroidal sets as models for homotopy operads*, J. Topol. **4** (2011), no. 2, 257–299.

[CM13] ______, *Dendroidal sets and simplicial operads*, J. Topol. **6** (2013), no. 3, 705–756.

[DS11] D. Dugger and D.I. Spivak *Rigidification of quasi-categories*, Algebr. Geom. Topol. **11** (2011), no. 1, 225–261.

[MW09] I. Moerdijk and I. Weiss, *On inner Kan complexes in the category of dendroidal sets*, Adv. Math. **221** (2009), no. 2, 343–389.

[MW07] I. Moerdijk and I. Weiss, *Dendroidal sets*, Algebr. Geom. Topol. **7** (2007), 1441–1470.

[Per18] L.A. Pereira, *Equivariant dendroidal sets*, Algebr. Geom. Topol. **18** (2018), no. 4, 2179–2244.

[Wei12] I. Weiss, *Broad posets, trees, and the dendroidal category*, arXiv preprint 1201.3987, 2012.

Peter Bonventre peterbonventre@gmail.com

University of Kentucky, 715 Patterson Office Tower, Lexington, KY 40506, USA

Luís A. Pereira luisalexandrepereira@outlook.com

Physics Building, Duke University, 120 Science Drive, Durham, NC 27708, USA