Homology, Homotopy and Applications, vol. 23(2), 2021, pp.379–402

THE P^1_2 MARGOLIS HOMOLOGY OF CONNECTIVE TOPOLOGICAL MODULAR FORMS

PRASIT BHATTACHARYA, IRINA BOBKOVA AND BRIAN THOMAS

(*communicated by Donald M. Davis*)

Abstract

The element P^1_2 of the mod 2 Steenrod algebra $\mathcal{A}$ has the property $(\mathsf{P}^1_2)^2 = 0$. This property allows one to view P^1_2 as a differential on $H_*(X, \mathbb{F}_2)$ for any spectrum X. Homology with respect to this differential, $\mathcal{M}(X, \mathsf{P}^1_2)$, is called the P^1_2 Margolis homology of X. In this paper we give a complete calculation of the P^1_2 Margolis homology of the 2-local spectrum of topological modular forms *tmf* and identify its $\mathbb{F}_2$ basis via an iterated algorithm. We apply the same techniques to calculate P^1_2 Margolis homology for any smash power of *tmf*.

Convention. Throughout this paper we work in the stable homotopy category of spectra localized at the prime 2.

1. Introduction

The connective E_∞ ring *spectrum of topological modular forms tmf* has played a vital role in computational aspects of chromatic homotopy theory over the last two decades [**Goe10**], [**DFHH14**]. It is essential for detecting information about the chromatic height 2, and it has the rare quality of having rich Hurewicz image. There is a $K(2)$-local equivalence [**HM14**]

$$L_{K(2)} tmf \simeq E_2^{hG_{48}},$$

where E_2 is the second Morava E-theory at $p = 2$ and G_{48} is the maximal finite subgroup of the Morava stabilizer group $\mathbb{G}_2$. The spectrum $E_2^{hG_{48}}$ can be used to build the $K(2)$-local sphere spectrum (see [**BG18**]). The homotopy groups of *tmf* approximate both the stable homotopy groups of spheres and the ring of integral modular forms. In many senses, *tmf* is the chromatic height 2 analogue of connective real K-theory *ko*. Further, the homotopy groups of *tmf* are completely known [**Bau08**].

Let us now recall the definition of the element $\mathsf{P}^1_2 \in \mathcal{A}$. Milnor described the mod 2 dual Steenrod algebra $\mathcal{A}_*$ as the graded polynomial algebra [**Mil58**, App. 1]

$$\mathcal{A}_* \cong \mathbb{F}_2[\xi_1, \xi_2, \xi_3, \dots],$$

where $|\xi_i| = 2^i - 1$. The Steenrod algebra $\mathcal{A}$ has an $\mathbb{F}_2$-basis dual to the monomial

Received May 7, 2019, revised May 21, 2020, December 25, 2020; published on September 29, 2021.
2010 Mathematics Subject Classification: 55S10, 55S20, 55P42, 55N35.
Key words and phrases: Steenrod algebra, Margolis homology, topological modular forms.
Article available at `http://dx.doi.org/10.4310/HHA.2021.v23.n2.a21`
Copyright © 2021, Prasit Bhattacharya, Irina Bobkova and Brian Thomas. Permission to copy for private use granted.

basis of $\mathcal{A}_*$. The elements of the $\mathbb{F}_2$-basis of $\mathcal{A}$ which are dual to $\xi_t^{2^s}$ are denoted by P_t^s, and the elements P_t^0 are denoted by Q_{t-1}. When $s < t$, the elements P_t^s are exterior power generators, i.e. $(\mathsf{P}_t^s)^2 = 0$. Thus, any left $\mathcal{A}$-module K can be regarded as a complex with differential given by the left multiplication by P_t^s (for $s < t$). This leads to the following definition.

Definition 1.1 ([**Mar83**])**.** Let K be any left $\mathcal{A}$-module and $0 \leqslant s < t$. Let

$$^L\mathcal{P}_t^s \colon K \longrightarrow K$$

denote the left action by P_t^s. The *left* P_t^s *Margolis homology group* of K, $\mathcal{M}^L(K, \mathsf{P}_t^s)$, is defined as

$$\mathcal{M}^L(K, \mathsf{P}_t^s) := \frac{\operatorname{Ker} {}^L\mathcal{P}_t^s \colon K \to K}{\operatorname{Im} {}^L\mathcal{P}_t^s \colon K \to K}.$$

For a right $\mathcal{A}$-module K, one can similarly define the *right* P_t^s *Margolis homology group* of K as

$$\mathcal{M}^R(K, \mathsf{P}_t^s) := \frac{\operatorname{Ker} {}^R\mathcal{P}_t^s \colon K \to K}{\operatorname{Im} {}^R\mathcal{P}_t^s \colon K \to K},$$

where ${}^R\mathcal{P}_t^s$ is the right action by P_t^s on K.

Notation 1.2. For a spectrum X, $\mathcal{M}(X, \mathsf{P}_t^s)$ will denote $\mathcal{M}^L(H^*(X), \mathsf{P}_t^s)$ or equivalently $\mathcal{M}^R(H_*(X), \mathsf{P}_t^s)$.

Computations of Margolis homology underly many essential computations in homotopy theory. For example, Adams work on $BP\langle 1\rangle$ cooperations [**Ada74**] relies on the computations of $\mathcal{M}(BP\langle 1\rangle, \mathsf{Q}_i)$ for $i = 0, 1$. Calculations like $\mathcal{M}(bo, \mathsf{Q}_i)$ for $i = 0, 1$ are essential ingredients in the work of Mahowald on *bo*-resolutions [**Mah81**]. More recently, Culver described $BP\langle 2\rangle$ resolutions [**Cul19**] by understanding $\mathcal{M}(BP\langle 2\rangle, \mathsf{Q}_i)$ for $i = 0, 1, 2$. Computation of $\mathcal{M}(tmf^{\wedge n}, \mathsf{Q}_2)$ is an essential ingredient in [**BBB**$^+$].

The element Q_i is primitive for all $i \in \mathbb{N}$. In other words, the comultiplication map Δ on $\mathcal{A}$ sends Q_i to

$$\Delta(\mathsf{Q}_i) = \mathsf{Q}_i \otimes 1 + 1 \otimes \mathsf{Q}_i\,. \tag{1}$$

Consequently, Q_i acts on $H_*(X)$ as a derivation, namely it follows the Leibniz rule

$$\mathsf{Q}_i(xy) = \mathsf{Q}_i(x) \cdot y + x \cdot \mathsf{Q}_i(y),$$

whenever X is a ring spectrum. The Leibniz rule implies the Künneth isomorphism [**Mar83**, Proposition 17, p. 343]

$$\mathcal{M}(X \otimes Y, \mathsf{Q}_i) \cong \mathcal{M}(X, \mathsf{Q}_i) \otimes \mathcal{M}(Y, \mathsf{Q}_i)$$

and hence, $\mathcal{M}(X, \mathsf{Q}_i)$ is an $\mathbb{F}_2$ algebra whenever X is a ring spectrum. As a result, computation of Q_i Margolis homology and its description is often fairly straightforward.

On the other hand, for $s > 0$, P_t^s is not a primitive element of $\mathcal{A}$. In particular,

$$\Delta(\mathsf{P}_2^1) = \mathsf{P}_2^1 \otimes 1 + \mathsf{Q}_1 \otimes \mathsf{Q}_1 + 1 \otimes \mathsf{P}_2^1$$

and its action on $H_*(X)$ for a ring spectrum X, does not follow the Leibniz rule.

Instead, we have

$$\mathsf{P}_2^1(xy) = \mathsf{P}_2^1(x)y + \mathsf{Q}_1(x)\,\mathsf{Q}_1(y) + x\,\mathsf{P}_2^1(y). \tag{2}$$

As a result, the product of two P_2^1 cycles may not necessarily be a P_2^1 cycle, hence $\mathcal{M}(X, \mathsf{P}_2^1)$ may not admit any multiplicative structure even if X is a ring spectrum. This is the main reason why the P_2^1 Margolis homology calculations are significantly more complicated.

Let us now consider the spectrum *tmf*. It is well-known ([**HM14**], [**Mat16**]) that

$$H_*(tmf; \mathbb{F}_2) \cong \mathbb{F}_2[\zeta_1^8, \zeta_2^4, \zeta_3^2, \zeta_4, \zeta_5, \dots] \subset \mathcal{A}_*$$

is a subalgebra of $\mathcal{A}_*$. Here the elements ζ_i are the images of ξ_i under the antipode of the Hopf algebra $\mathcal{A}_*$ (see Section 2). The right action of Q_i is given by the formula (see [**Cul19**, §2] for details)

$$\mathsf{Q}_i(\zeta_n) = \zeta_{n-i-1}^{2^{i+1}}.$$

Then, since the Q_i are derivations, it can be easily seen that

$$\mathcal{M}(tmf, \mathsf{Q}_0) = \mathbb{F}_2[\zeta_1^8, \zeta_2^4], \qquad \mathcal{M}(tmf, \mathsf{Q}_1) = \frac{\mathbb{F}_2[\zeta_1^8, \zeta_3^2, \zeta_4^2, \dots]}{\langle \zeta_3^4, \zeta_4^4, \dots \rangle},$$

$$\text{and} \quad \mathcal{M}(tmf, \mathsf{Q}_2) = \frac{\mathbb{F}_2[\zeta_2^4, \zeta_3^2, \zeta_4^2, \dots]}{\langle \zeta_2^8, \zeta_3^8, \zeta_4^8, \dots \rangle}.$$

In this paper, we give a complete calculation of $\mathcal{M}(tmf^{\wedge r}, \mathsf{P}_2^1)$ for arbitrary $r \geqslant 1$. In fact, the calculation for $r > 1$ follows from the case $r = 1$, because after forgetting the internal grading one can construct a non-canonical isomorphism (see Section 4)

$$\mathcal{M}(tmf^{\wedge r}, \mathsf{P}_2^1) \cong \mathcal{M}(tmf, \mathsf{P}_2^1).$$

For the case $r = 1$, we give an iterated algorithm (see Definition 3.14) that constructs an $\mathbb{F}_2$-basis of $\mathcal{M}(tmf, \mathsf{P}_2^1)$. We give a complete description of $\mathcal{M}(tmf, \mathsf{P}_2^1)$ in Theorem 3.16 which is the main result of this paper. Although $\mathcal{M}(tmf, \mathsf{P}_2^1)$ is not an algebra, we notice that $\mathcal{M}(tmf, \mathsf{P}_2^1)$ is a module over an infinitely generated exterior algebra $\mathcal{S}$ (see Lemma 3.1 for a description of $\mathcal{S}$). Theorem 3.16 also describes $\mathcal{M}(tmf, \mathsf{P}_2^1)$ as an $\mathcal{S}$-module.

The key tool we use is the *length spectral sequence* (9), which we define in Section 2. The length spectral sequence admits a d_0 differential and a d_2 differential and collapses at the E_3 page. The Leibniz rule does hold for the d_0, but not for d_2. In order to work around this issue, we notice that the E_2 page admits an action of $\mathcal{S}$ (i.e. d_2 are $\mathcal{S}$ linear) and we use it to simplify the computation of $E_\infty = E_3$.

We also notice that almost identical calculations lead to a complete description of $\mathcal{M}(((B\mathbb{Z}/2)^{\times k})_+, \mathsf{P}_2^1)$. The methods developed in this paper can be considered as a blueprint for computations of P_t^1 Margolis homology of a variety of other $\mathcal{A}$-modules.

Our calculations of $\mathcal{M}(tmf^{\wedge r}, \mathsf{P}_2^1)$ have many applications, as the spectrum *tmf* has a wide range of applications, particularly in chromatic homotopy theory. First note that the cohomology of *tmf*, as a module over the Steenrod algebra $\mathcal{A}$, is isomorphic to (see [**HM14**], [**Mat16**])

$$H^*(tmf; \mathbb{F}_2) \cong \mathcal{A}/\!/\mathcal{A}(2), \tag{3}$$

where $\mathcal{A}(2)$ is the subalgebra of $\mathcal{A}$ generated by $\mathsf{Sq}^1, \mathsf{Sq}^2$ and Sq^4. This, and a change of rings isomorphism, imply that the E_2 page of the Adams spectral sequence converging to tmf_*X (for a spectrum X) is

$$E_2^{s,t} := Ext^{s,t}_{\mathcal{A}(2)}(H^*(X), \mathbb{F}_2). \tag{4}$$

One can detect infinite families in the E_2 page via the map

$$q\colon Ext^{s,t}_{\mathcal{A}(2)}(H^*(X), \mathbb{F}_2) \longrightarrow Ext^{s,t}_{\Lambda(\mathsf{P}^1_2)}(H^*(X), \mathbb{F}_2).$$

The codomain of q can be understood by calculating $\mathcal{M}(X, \mathsf{P}^1_2)$. Note that

$$Ext^{s,t}_{\Lambda(\mathsf{P}^1_2)}(\mathbb{F}_2, \mathbb{F}_2) \cong \mathbb{F}_2[h_{2,1}],$$

where $|h_{2,1}| = (1, 6)$ and

$$\mathbb{F}_2[h_{2,1}] \otimes \mathcal{M}(X, \mathsf{P}^1_2) \subset Ext^{s,t}_{\Lambda(\mathsf{P}^1_2)}(H^*(X), \mathbb{F}_2)$$

accounts for all the elements with positive s filtration. This shows that the knowledge of $\mathcal{M}(X, \mathsf{P}^1_2)$ is crucial in detecting patterns in the E_2-page of (4).

Motivation I — Towards homotopy groups of $K(2)$-local sphere

Computation of the homotopy groups of $L_{K(n)}S^0$ — the sphere spectrum localized with respect to Morava K-theories $K(n)$ at various primes p and heights n — is the central question of chromatic homotopy theory. It is sometimes easier to compute $\pi_*L_{K(n)}X$ for finite complexes other than the sphere, although very little data like this is known at $n = p = 2$ anyway. Recently, Bhattacharya and Egger introduced a family of finite spectra Z [**BE20a**], and $\pi_*L_{K(2)}Z$ has been computed [**BBB$^+$, BE20b**], the first example of a finite complex at $p = 2$ whose $K(2)$-local homotopy groups are completely determined. The finite complex Z can be constructed from the sphere spectrum, by a succession of cofiber sequences of self-maps (see [**BE20a**]), the last one of which is

$$\Sigma^5 A_1 \wedge C\nu \xrightarrow{w} A_1 \wedge C\nu \longrightarrow Z.$$

In a quest to leverage the knowledge of $\pi_*L_{K(2)}Z$ to $\pi_*L_{K(2)}S^0$, one must first attempt to compute the $K(2)$-local homotopy groups of $A_1 \wedge C\nu$. Very briefly, our strategy is to use the v_2-local tmf-based Adams spectral sequence

$$E_1^{r,t} = v_2^{-1}\pi_t(tmf \wedge \overline{tmf}^{\wedge r} \wedge A_1 \wedge C\nu) \Longrightarrow \pi_{t-r}(L_{K(2)}A_1 \wedge C\nu)$$

and compare it with that of Z. One can identify the E_1-page of the above spectral sequence using the classical Adams spectral sequence

$$E_2^{s,t} = \mathrm{Ext}^{s,t}_{\mathcal{A}}(H^*(tmf \wedge \overline{tmf}^{\wedge r} \wedge A_1 \wedge C\nu), \mathbb{F}_2) \Rightarrow \pi_{t-s}(tmf \wedge \overline{tmf}^{\wedge r} \wedge A_1 \wedge C\nu). \tag{5}$$

Because of (3) and the fact that $H^*(A_1 \wedge C\nu) \cong \mathcal{A}(2) /\!\!/ \Lambda(\mathsf{Q}_2, \mathsf{P}^1_2)$, and the change of rings isomorphism, the E_2-page of the spectral sequence (5) has the form

$$\mathrm{Ext}^{s,t}_{\Lambda(\mathsf{Q}_2, \mathsf{P}^1_2)}(H^*(\overline{tmf}^{\wedge r}), \mathbb{F}_2).$$

Hence, computation of $\mathcal{M}(tmf^{\wedge r}, \mathsf{P}^1_2)$ is essential for understanding the E_2-page of (5).

Motivation II — *tmf* resolution of the sphere spectrum

The connective spectrum *bo* is not a flat ring spectrum, hence the E_2 page of the *bo*-based Adams spectral sequence does not have a straightforward expression like the classical Adams spectral sequence. However, Lellmann and Mahowald [**LM87**] were able to calculate the d_1 differentials (also see [**BBB+20**]) and gave a description of the "v_1-periodic part" of the E_2-page. They identified the free Eilenberg–MacLane summand of $bo^{\wedge r}$. To identify this free summand one needs to identify the $\mathcal{A}(1)$ free summand of

$$H^*(bo^{\wedge r}) \cong \mathcal{A}/\!/\mathcal{A}(1)^{\otimes r}.$$

This can be done by calculating $\mathcal{M}(bo^{\wedge r}, \mathsf{Q}_0)$ and $\mathcal{M}(bo^{\wedge r}, \mathsf{Q}_1)$ and using the following theorem due to Margolis.

Theorem 1.3 ([**Mar83**, Chapter 19, Theorem 6])**.** *An $\mathcal{A}(n)$-module K is free if and only if $\mathcal{M}(K, \mathsf{P}^s_t) = 0$ whenever $s + t \leqslant n + 1$ with $s < t$.*

To emulate the strategy of Lellmann and Mahowald to understand the *tmf*-based Adams spectral sequence for S^0 one needs to first identify the $\mathcal{A}(2)$-free part of

$$H^*(tmf^{\wedge r}) \cong (\mathcal{A}/\!/\mathcal{A}(2))^{\otimes r}.$$

Potentially, this can be identified using the knowledge of $\mathcal{M}(tmf^{\wedge r}, \mathsf{Q}_i)$ for $i = 0, 1, 2$ and $\mathcal{M}(tmf^{\wedge r}, \mathsf{P}^1_2)$, along with Theorem 1.3.

Motivation III — Infinite loop space of *tmf*

There are $\mathcal{A}$-modules $J(k)$, called Brown–Gitler modules [**BG73**], which assemble into a doubly graded $\mathcal{A}$-algebra, denoted here by $J(*)^*$. Moreover, there is an $\mathcal{A}$-module isomorphism $J(*)^* \cong \mathbb{F}_2[x_1, x_2, \dots]$ where $x_i \in J(2^i)^1$ and the left $\mathcal{A}$ action on $J(*)^*$ is [**Sch94**]

$$\mathsf{Sq}(x_i) = x_i + x_{i-1}^2.$$

In fact, $J(k)^*$ can be thought of as inheriting this action by virtue of being a subobject of $\mathcal{A}$. Because of this, minor modifications to methods of this paper apply to the calculation of $\mathcal{M}(J(k), \mathsf{P}^1_2)$. By [**KM13**] there is a spectral sequence, obtained by studying Goodwillie towers, relating the knowledge of $H_*(tmf; \mathbb{F}_2)$ to that of $H_*(\Omega^\infty tmf; \mathbb{F}_2)$ (also see [**HM16**] which provides a spectral sequence relating the cohomology of *tmf* to the cohomology of its infinite loop-space $H^*(\Omega^\infty tmf; \mathbb{F}_2)$). Roughly speaking, this relies on computing certain derived functors, usually labeled Ω^∞_s, in the category of unstable modules over $\mathcal{A}$. It turns out that there is an isomorphism (see [**Goe86**] or [**HK00**])

$$\Omega^\infty_s \Sigma^{-t} (\mathcal{A}/\!/\mathcal{A}(2))_* \cong \operatorname{Ext}^{s,t}_{\mathcal{A}(2)}(\mathbb{F}_2, J(*)),$$

so that these computations require an understanding of the $J(k)$ as modules over $\mathcal{A}(2)$, the hardest part of which is understanding how P^1_2 acts.

Organization of the paper

In Section 2, we recall some facts about the Steenrod algebra and its dual. We introduce the spectral sequence (9), which computes the P^1_2 Margolis homology of *tmf*, and discuss the d_0 differentials in it.

In Section 3, we compute the $E_3 = E_\infty$ page of the spectral sequence (9). We do that by introducing building blocks M_J and computing $\mathcal{M}(M_J, \mathsf{P}^1_2)$. Then we establish the relationship between $\mathcal{M}(tmf, \mathsf{P}^1_2)$ and $\mathcal{M}(M_J, \mathsf{P}^1_2)$ in Theorem 3.16.

In Section 4, we show how to apply the same methods to calculate P^1_2 Margolis homology for $tmf^{\wedge r}$ and $((B\mathbb{Z}/2)^{\times k})_+$. Theorem 3.16 essentially gives the complete answer in these cases.

Acknowledgments

The authors are indebted to Nicolas Ricka for many insightful conversations. We are grateful to Nick Kuhn, Haynes Miller and John Rognes for their help and comments. We would like to thank the anonymous referee for many helpful comments and suggestions for improvement. This material is based upon work supported by the National Science Foundation Grant DMS-1440140, while the second author was in residence at the MSRI during the Spring 2019 semester, and Grants DMS-1638352 and DMS-2005627.

2. Action of P^1_2 and the length spectral sequence

The dual Steenrod algebra $\mathcal{A}_* = \pi_*(H\mathbb{F}_2 \wedge H\mathbb{F}_2)$ has the structure of a graded commutative algebra which Milnor [**Mil58**] showed to be a polynomial algebra

$$\mathcal{A}_* \cong \mathbb{F}_2[\xi_1, \xi_2, \xi_3, \dots],$$

where $|\xi_i| = 2^i - 1$. Milnor defined $\mathsf{Sq}(r_1, r_2, \dots) \in \mathcal{A}$ as the dual of $\xi_1^{r_1}\xi_2^{r_2}\dots$ and showed that they form an $\mathbb{F}_2$ basis of the Steenrod algebra $\mathcal{A}$, known as the Milnor basis. The P^s_t elements are defined as

$$\mathsf{P}^s_t = \mathsf{Sq}(r_1, \dots), \text{ where } r_i = \begin{cases} 0, & i \neq t, \\ 2^s, & i = t. \end{cases}$$

The action of an element $a \in \mathcal{A}$ on an $\mathcal{A}$-algebra follows the product rule given by the Cartan formula, i.e.

$$a(x \cdot y) = \Sigma_i a'_i(x) \cdot a''_i(y),$$

where $\Delta(a) = \Sigma_i a'_i \otimes a''_i$ is the comultiplication in the Hopf algebra $\mathcal{A}$.

Remark 2.1. We would like to note that standard commonly used notation for the generators of the dual Steenrod algebra at $p = 2$ differs from the notation in the original paper [**Mil58**], and we are grateful to John Rognes for explaining this to us. In [**Mil58**, Appendix 1], Milnor denotes the polynomial generators of the dual Steenod algebra at $p = 2$ by ζ_i, so that $\mathcal{A}_* \cong \mathbb{F}_2[\zeta_1, \zeta_2, \dots]$ and defines $\mathsf{Sq}(r_1, r_2, \dots)$ as dual to the element $\zeta_1^{r_1}\zeta_2^{r_2}\cdots$. It has since become standard in the literature [**MT68, Ada74, Mar83**] to use a different notation and to denote the polynomial generators which were denoted by ζ_i in [**Mil58**, Appendix 1] by ξ_i, in order to match the notation for the odd primary Steenrod algebra. Hence in current standard notation $\mathsf{Sq}(r_1, r_2, \dots)$ is dual to $\xi_1^{r_1}\xi_2^{r_2}\cdots$. The symbol ζ_i is now usually used to denote the image of ξ_i under the antipode of the Hopf algebra $\chi\colon \mathcal{A}_* \to \mathcal{A}_*$, induced by the 'flip map' on

$H\mathbb{F}_2 \wedge H\mathbb{F}_2$. The elements $\zeta_i = \chi(\xi_i)$ can be computed recursively using the formula $\sum_{i+j=k} \xi_i^{2^j} \chi(\xi_j) = 0$, together with the assumption that $\xi_0 = 1$ and $\xi_i = 0$ when $i < 0$.

The homology of *tmf* is the subalgebra of $\mathcal{A}_*$ ([**HM14**], [**Mat16**, Theorem 5.13])

$$\mathfrak{T} := H_*(tmf; \mathbb{F}_2) \cong (\mathcal{A}/\!/\mathcal{A}(2))_* = \mathbb{F}_2[\zeta_1^8, \zeta_2^4, \zeta_3^2, \zeta_4, \zeta_5, \dots].$$

Thus the action of $\mathcal{A}$ on $\mathfrak{T}$ is simply the restriction of the action of $\mathcal{A}$ on $\mathcal{A}_*$.

The right action of $\mathcal{A}$ on $\mathcal{A}_*$ is determined by the action of the total squaring operation $\mathsf{Sq} = 1 + \sum_{i>0} \mathsf{Sq}^i$ [**Pea14**, Lemma 3.6]

$$(\zeta_i)\,\mathsf{Sq} = \zeta_i + \zeta_{i-1}^2 + \zeta_{i-2}^4 + \cdots + \zeta_1^{2^{i-1}} + 1 \tag{6}$$

which is a ring homomorphism.

Remark 2.2 (Action of the total squaring operation)*.* There are multiple ways to define the action of $\mathcal{A}$ on $\mathcal{A}_*$. While we will be using the action defined by (6), we would like to collect other commonly used actions here. By [**Mah81**], the right and left actions of Sq on ξ_i are given by the formulas

$$\mathsf{Sq}(\xi_i) = \xi_i + \xi_{i-1}^2, \qquad (\xi_i)\,\mathsf{Sq} = \xi_i + \xi_{i-1},$$

while the left action on ζ_i is

$$\mathsf{Sq}(\zeta_i) = \zeta_i + \zeta_{i-1} + \cdots + \zeta_1 + 1.$$

From these formulas we can derive

$$\mathsf{Q}_{i-1}(\xi_n) = \xi_{n-i}^{2^i}, \qquad (\zeta_n)\,\mathsf{Q}_{i-1} = \zeta_{n-i}^{2^i};$$

the second equation can also be found in [**Cul19**].

Important Notation 2.3. Since we only work with the right action of Sq in this paper, we will write $a(x)$ to denote the *right* action of $a \in \mathcal{A}$ on $x \in H_*(tmf)$ for the rest of the paper. Thus, from now on

$$\boxed{a(x) := (x)a.}$$

We now focus on the action of $\mathsf{P}_2^1 = \mathsf{Sq}(0,2) = \mathsf{Sq}^2\,\mathsf{Sq}^4 + \mathsf{Sq}^4\,\mathsf{Sq}^2$ on $\mathfrak{T}$. From (6), one can easily see that Sq^{2i} acts trivially on ζ_n, when $i > 0$ and $n \neq 1$. It follows immediately that

$$\mathsf{P}_2^1(\zeta_i) = 0.$$

Beware! This *does not* mean that $\mathsf{P}_2^1(\zeta_i\zeta_j) = 0$, as the Leibniz rule does not hold. Since $\Delta(\mathsf{P}_2^1) = \mathsf{P}_2^1 \otimes 1 + \mathsf{Q}_1 \otimes \mathsf{Q}_1 + 1 \otimes \mathsf{P}_2^1$, we obtain the product formula

$$\mathsf{P}_2^1(xy) = \mathsf{P}_2^1(x)y + \mathsf{Q}_1(x)\,\mathsf{Q}_1(y) + x\,\mathsf{P}_2^1(y). \tag{7}$$

Using $\mathsf{Q}_1(\zeta_i) = \zeta_{i-2}^4$, we get

$$\mathsf{P}_2^1(\zeta_i\zeta_j) = \zeta_{i-2}^4\zeta_{j-2}^4, \qquad \mathsf{P}_2^1(\zeta_i^2) = \zeta_{i-2}^8. \tag{8}$$

Formulas become more complicated for triple products, e.g.

$$\mathsf{P}_2^1(\zeta_i\zeta_j\zeta_k) = \zeta_{i-2}^4\zeta_{j-2}^4\zeta_k + \zeta_{i-2}^4\zeta_j\zeta_{k-2}^4 + \zeta_i\zeta_{j-2}^4\zeta_{k-2}^4,$$

and in general we have the following result.

Lemma 2.4. *The action of* P^1_2 *on* $\mathfrak{T}$ *is given by the formula*

$$\mathsf{P}^1_2(\zeta_{i_1}\dots\zeta_{i_n}) = \sum_{1\leqslant j<k\leqslant n} \frac{\zeta_{i_1}\dots\zeta_{i_n}}{\zeta_{i_j}\zeta_{i_k}}\,\mathsf{Q}_1(\zeta_{i_j})\,\mathsf{Q}_1(\zeta_{i_k})$$
$$= \sum_{1\leqslant j<k\leqslant n} \zeta_{i_1}\dots\zeta_{i_{j-1}}\zeta^4_{i_j-2}\zeta_{i_{j+1}}\dots\zeta_{i_{k-1}}\zeta^4_{i_k-2}\zeta_{i_{k+1}}\dots\zeta_{i_n},$$

where indices are allowed to repeat.

Proof. Follows from an inductive argument on n, using (7) and the facts that $\mathsf{P}^1_2(\zeta_i) = 0$ and $\mathsf{Q}_1(\zeta_i) = \zeta^4_{i-2}$. □

The technique developed in this paper begins with the following observation. Consider the subalgebra

$$\mathcal{E} := \mathbb{F}_2[\zeta_1^8, \zeta_2^4, \zeta_3^2, \zeta_4^2, \zeta_5^2, \dots] \subset \mathfrak{T} = \mathbb{F}_2[\zeta_1^8, \zeta_2^4, \zeta_3^2, \zeta_4, \zeta_5, \dots]$$

which we will call the *even* subalgebra of $\mathfrak{T}$, as every element in $\mathcal{E}$ has even grading. Since $|\,\mathsf{Q}_1\,| = 3$ and every element in $\mathcal{E}$ has even grading, Q_1 must act trivially on $\mathcal{E}$. Thus, P^1_2 restricted to $\mathcal{E}^{\otimes r}$ follows the Leibniz rule, therefore $(\mathcal{E}^{\otimes r}, \mathsf{P}^1_2)$ is a differential graded algebra, and hence, $\mathcal{M}(\mathcal{E}^{\otimes r}, \mathsf{P}^1_2)$ is an algebra. Using (8) and the Künneth isomorphism, we can easily deduce the following result.

Lemma 2.5. *The* P^1_2 *Margolis homology of* $\mathcal{E}$ *is given by*

$$\mathcal{M}(\mathcal{E}, \mathsf{P}^1_2) \cong \Lambda(\zeta_2^4, \zeta_3^4, \zeta_4^4, \dots).$$

Moreover

$$\mathcal{M}(\mathcal{E}^{\otimes r}, \mathsf{P}^1_2) \cong \mathcal{M}(\mathcal{E}, \mathsf{P}^1_2)^{\otimes r} \cong (\Lambda(\zeta_2^4, \zeta_3^4, \zeta_4^4, \dots))^{\otimes r}.$$

Notation 2.6. For a set A, we let $\mathbb{F}_2\langle A\rangle$ denote the $\mathbb{F}_2$-vector space which has the generating set A.

Now consider the quotient $\mathcal{K} := \mathfrak{T}/\!/\mathcal{E} \cong \mathbb{F}_2 \otimes_{\mathcal{E}} \mathfrak{T}$. We have an isomorphism $\mathcal{K} \cong \Lambda(\zeta_4, \zeta_5, \dots)$, and the induced action of Q_1 and P^1_2 on $\mathcal{K}$ is trivial. The algebra $\mathcal{K}$ admits a natural increasing filtration

$$G^p(\mathcal{K}) := \mathbb{F}_2\langle \zeta_{i_1}\dots\zeta_{i_k} | k \leqslant p\rangle,$$

induced by the length of the monomials. We call it the *length filtration.*

This length filtration on $\mathcal{K}$ induces an increasing filtration $\{G^p(\mathfrak{T})\}_{p\geqslant 0}$ on $\mathfrak{T}$, where $G^p(\mathfrak{T})$ is the pullback of $G^p(\mathcal{K})$ (in vector spaces) along the quotient map $\mathfrak{T} \twoheadrightarrow \mathcal{K}$

$$\begin{array}{ccc} G^p(\mathfrak{T}) & \longrightarrow & \mathfrak{T} \\ \downarrow & & \downarrow\!\!\downarrow \\ G^p(\mathcal{K}) & \longrightarrow & \mathcal{K}. \end{array}$$

Definition 2.7. Let I be a finite tuple of natural numbers, and for $I = \{i_1, \dots, i_n\}$ let ζ^I denote the monomial $\zeta_1^{i_1}\dots\zeta_n^{i_n}$. Then the *length* L of ζ^I is defined by

$$L(\zeta^I) = \sum_{j=1}^{|I|}(i_j \bmod 2).$$

In other words, $L(\zeta^I)$ counts the number of odd exponents in ζ^I. Then $G^p(\mathfrak{T})$ is the span of monomials ζ^I of length less than or equal to p

$$G^p(\mathfrak{T}) \cong \mathbb{F}_2\langle \zeta^I | L(\zeta^I) \leqslant p\rangle.$$

The length function L measures "how far" a given monomial in $\mathfrak{T}$ is from the even subalgebra $\mathcal{E}$. Since there is an $\mathbb{F}_2$-vector space isomorphism

$$\mathfrak{T} \cong \mathcal{E} \otimes \mathfrak{T}/\!/\mathcal{E} = \mathcal{E} \otimes \mathcal{K}$$

any monomial $m \in \mathfrak{T}$ can be uniquely written as $e \cdot k$ where $e \in \mathcal{E}$ and $k \in \mathcal{K}$.

Example 2.8. If $m = \zeta_3^4\zeta_5^5\zeta_8^3$, then there is an unique expression $m = e \cdot k$, where $e = \zeta_3^4\zeta_5^4\zeta_8^2 \in \mathcal{E}$ and $k = \zeta_5\zeta_8 \in \mathcal{K}$.

The following lemma shows that the action of Q_1 and P_2^1 preserves the length filtration.

Lemma 2.9. *Let $m \in \mathfrak{T}$ be any monomial.*

(i) If $m \in \mathcal{E}$, then $\mathsf{Q}_1(m) = 0$ and $\mathsf{P}_2^1(m) \in \mathcal{E}$.

(ii) If $m \notin \mathcal{E}$, then $\mathsf{Q}_1(m)$ is a sum of monomials of length exactly $L(m) - 1$ and

$$\mathsf{P}_2^1(m) = m_L + m_{L-2},$$

where m_L is a sum of monomials of length exactly $L(m)$ and m_{L-2} is a sum of monomials of length exactly $L(m) - 2$.

Proof. When $m \in \mathcal{E}$, $\mathsf{Q}_1(m) = 0$ by the Leibniz rule. Using Lemma 2.4 we have $\mathsf{P}_2^1(m) \in \mathcal{E}$ and $L(\mathsf{P}_2^1(m)) = L(m) = 0$.

Now assume $m \notin \mathcal{E}$, which means $m = e \cdot k$ for some $e \in \mathcal{E}$ and some $1 \neq k \in \mathcal{K}$. Note that k is of the form $\zeta_{i_1} \dots \zeta_{i_n}$, where indices do not repeat.

The action of Q_1 is given by the formula

$$\mathsf{Q}_1(\zeta_{i_1} \dots \zeta_{i_n}) = \sum_{k=1}^{n} \zeta_{i_1} \dots \zeta_{i_{k-1}} \zeta_{i_k - 2}^4 \zeta_{i_{k+1}} \dots \zeta_{i_n},$$

where we allow repetition of indices. Since Q_1 acts trivially on $\mathcal{E}$, it follows that

$$\mathsf{Q}_1(e \cdot k) = e \cdot \mathsf{Q}_1(k).$$

From the formula above we see that $\mathsf{Q}_1(k) \neq 0$ and $L(\mathsf{Q}_1(k)) = L(k) - 1$. Hence,

$$L(\mathsf{Q}_1(m)) = L(e \cdot \mathsf{Q}_1(k)) = L(\mathsf{Q}_1(k)) = L(k) - 1 = L(e \cdot k) - 1 = L(m) - 1.$$

Next, note that

$$\mathsf{P}_2^1(m) = \mathsf{P}_2^1(e) \cdot k + \mathsf{Q}_1(e) \cdot \mathsf{Q}_1(k) + e \cdot \mathsf{P}_2^1(k) = \mathsf{P}_2^1(e) \cdot k + e \cdot \mathsf{P}_2^1(k).$$

From Lemma 2.4, we see that $L(\mathsf{P}_2^1(k)) = L(\mathsf{P}_2^1(k)) - 2$ assuming $\mathsf{P}_2^1(k) \neq 0$. Now set $m_L = \mathsf{P}_2^1(e) \cdot k$ and $m_{L-2} = e \cdot \mathsf{P}_2^1(k)$ □

Lemma 2.10. *The Hopf algebra $\Lambda(\mathsf{Q}_1, \mathsf{P}_2^1)$ is commutative and cocommutative.*

Proof. Commutativity follows from the fact that P_2^1 and Q_1 commute, see [**AM71**], Lemma 1.3(2) (in the notation of [**AM71**], $\mathsf{P}_2^1 = P_2(2)$ and $\mathsf{Q}_1 = P_2(1)$). Cocommutativity follows from the fact that the diagram

$$\begin{array}{ccc} \Lambda(\mathsf{Q}_1, \mathsf{P}_2^1) & \xrightarrow{\Delta} & \Lambda(\mathsf{Q}_1, \mathsf{P}_2^1) \otimes \Lambda(\mathsf{Q}_1, \mathsf{P}_2^1) \\ & \searrow^{\Delta} & \downarrow \text{flip} \\ & & \Lambda(\mathsf{Q}_1, \mathsf{P}_2^1) \otimes \Lambda(\mathsf{Q}_1, \mathsf{P}_2^1) \end{array}$$

commutes, because of (1) and (2). □

If M is a $\Lambda(\mathsf{Q}_1, \mathsf{P}_2^1)$-module then let $\mathcal{C}_M^\bullet$ denote the periodic chain complex

$$\dots \xrightarrow{\mathsf{P}_2^1} M \xrightarrow{\mathsf{P}_2^1} M \xrightarrow{\mathsf{P}_2^1} \dots.$$

Its homology groups are isomorphic in each degree, i.e.

$$H_i(\mathcal{C}_M^\bullet) \cong H_j(\mathcal{C}_M^\bullet)$$

for all $i, j \in \mathbb{Z}$. We use $\mathcal{M}(M, \mathsf{P}_2^1)$ to denote this common homology group. When $M = \mathfrak{T}$, the filtration $G^\bullet(\mathfrak{T})$ induces a filtration on $C_{\mathfrak{T}}^\bullet$. By Lemma 2.9, P_2^1 respects the length filtration. This means we have a short exact sequence of chain complexes

$$0 \longrightarrow \bigoplus_{p \in \mathbb{Z}} G^{p-1}(C_{\mathfrak{T}}^\bullet) \longrightarrow \bigoplus_{p \in \mathbb{Z}} G^p(C_{\mathfrak{T}}^\bullet) \longrightarrow \bigoplus_{p \in \mathbb{Z}} \frac{G^p(C_{\mathfrak{T}}^\bullet)}{G^{p-1}(C_{\mathfrak{T}}^\bullet)} \longrightarrow 0.$$

Upon taking the homology, this short exact sequence of chain complexes produces an exact couple, resulting in a spectral sequence

$$E_1^{p,q} := H^q\left(\frac{G^p(C_{\mathfrak{T}}^\bullet)}{G^{p-1}(C_{\mathfrak{T}}^\bullet)}\right) \Rightarrow H^q(C_{\mathfrak{T}}^\bullet).$$

We rewrite this spectral sequence as

$$E_1^p := \mathcal{M}\left(\frac{G^p(\mathfrak{T})}{G^{p-1}(\mathfrak{T})}, \mathsf{P}_2^1\right) \Rightarrow \mathcal{M}(tmf, \mathsf{P}_2^1), \tag{9}$$

and we call it the *length spectral sequence.*

The E_1 page of (9) is easy to calculate. Note that the length filtration $G^\bullet(\mathfrak{T})$ is multiplicative, i.e.

$$G^p(\mathfrak{T}) \cdot G^{p'}(\mathfrak{T}) \subset G^{p+p'}(\mathfrak{T}),$$

hence the associated graded

$$\bigoplus_{p \geqslant 0} \frac{G^p(\mathfrak{T})}{G^{p-1}(\mathfrak{T})} \cong \mathcal{E} \otimes \mathcal{K}$$

is an $\mathbb{F}_2$-algebra. The action of $\Lambda(\mathsf{Q}_1, \mathsf{P}_2^1)$ on $\mathcal{E} \otimes \mathcal{K}$ is defined using the Cartan formula as in the definition below.

Definition 2.11 ([**Mar83**], p. 186)**.** Let Γ be any Hopf algebra. For two Γ-modules M and N, the underlying $\mathbb{F}_2$ vector space of $M \otimes N$ is simply $M \otimes_{\mathbb{F}_2} N$, and Γ acts via the diagonal map, i.e.

$$a(m \otimes n) = \sum_i a_i(m) \otimes a_i'(n),$$

where $a \in \Gamma$ and $\Delta(a) = \sum_i a_i \otimes a_i'$, where Δ is the coproduct of the Hopf algebra.

Now we describe the action of P_2^1 on a monomial $m \in \bigoplus_{p \geqslant -1} \frac{G^p(\mathfrak{T})}{G^{p-1}(\mathfrak{T})}$. Write $m = e \otimes k$ for some $e \in \mathcal{E}$ and $k \in \mathcal{K}$. By Definition 2.11

$$\mathsf{P}_2^1(m) = \mathsf{P}_2^1(e \otimes k) = \mathsf{P}_2^1(e) \otimes k.$$

Since P_2^1 restricted to $\mathcal{E}$ follows the Leibniz rule, the E_1 page of (9) is also an $\mathbb{F}_2$-algebra and isomorphic to

$$E_1^* \cong \mathcal{M}(\mathcal{E} \otimes \mathcal{K}, \mathsf{P}_2^1) \cong \mathcal{M}(\mathcal{E}, \mathsf{P}_2^1) \otimes \mathcal{K} \cong \Lambda(\zeta_2^4, \zeta_3^4, \dots) \otimes \Lambda(\zeta_4, \zeta_5, \dots).$$

In order to avoid confusion regarding the multiplicative structure of E_1^*, it is convenient to rename the generators.

Notation 2.12. We set $x_i := \zeta_{i+3}$ and $t_i := \zeta_{i+1}^4$. Further, for finite subsets $I = \{i_1, \dots, i_n\} \subset \mathbb{N}$ and $J = \{j_1, \dots, j_m\} \subset \mathbb{N}$, we let t_I and x_I denote the monomials $t_{i_1} \dots t_{i_n}$ and $x_{j_1} \dots x_{j_n}$ respectively. We use $t_I x_J$ to denote the tensor product $t_I \otimes x_J$.

Lemma 2.9 and Lemma 2.10 imply that we have a commutative diagram of chain complexes

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \bigoplus_p G^p(C_{\mathfrak{T}}^\bullet) & \longrightarrow & \bigoplus_p G^{p+1}(C_{\mathfrak{T}}^\bullet) & \longrightarrow & \bigoplus_p \frac{G^{p+1}(C_{\mathfrak{T}}^\bullet)}{G^p(C_{\mathfrak{T}}^\bullet)} & \longrightarrow & 0 \\
\| & & \downarrow{\scriptstyle \mathsf{Q}_1(\ )} & & \downarrow{\scriptstyle \mathsf{Q}_1(\ )} & & \downarrow{\scriptstyle \mathsf{Q}_1(\ )} & & \| \\
0 & \longrightarrow & \bigoplus_p G^{p-1}(C_{\mathfrak{T}}^\bullet) & \longrightarrow & \bigoplus_p G^p(C_{\mathfrak{T}}^\bullet) & \longrightarrow & \bigoplus_p \frac{G^p(C_{\mathfrak{T}}^\bullet)}{G^{p-1}(C_{\mathfrak{T}}^\bullet)} & \longrightarrow & 0.
\end{array}$$

Consequently there is an action of Q_1 on each page of (9), which shifts the length filtration by -1. In particular, we note $\mathsf{Q}_1(x_i) = t_i$ and in general

$$\mathsf{Q}_1(t_I x_J) = \sum_{j \in J} t_j t_I x_{J - \{j\}}. \tag{10}$$

Let $m \in \mathfrak{T}$ be any monomial, m_L and m_{L-2} be as in Lemma 2.9, and let $[m]$ denote the equivalence class in the E_1 page of (9) represented by m. Lemma 2.9 implies that the d_1 differential of (9) is trivial,

$$d_2([m]) = [m_{L-2}]$$

for the class of the monomial $m \in \mathfrak{T}$ in the E_1 page, and the spectral sequence (9) collapses at the E_3 page. If we write $m \in \mathfrak{T}$ as $m = e \cdot k$, where $e \in \mathcal{E}$ and $k \in \mathcal{K}$, then

$$d_2([m]) = [e \cdot \mathsf{P}_2^1(k)] = [e] \cdot [\mathsf{P}_2^1(k)].$$

This means that the d_2 differential of (9) is $\mathcal{M}(\mathcal{E}, \mathsf{P}_2^1)$-linear. It follows from the formula of Lemma 2.4 that

$$d_2(t_I x_J) = \sum_{K \in J[2]} t_K t_I x_{J-K}, \tag{11}$$

where $J[2]$ is the set of subsets of J which contain two elements.

The formula for the d_2 differentials is intimately related to the action of Q_1 on the E_2 page of (9). The $\Lambda(\mathsf{Q}_1)$-module structure on $E_2^\bullet$ (see (10)) can be extended to the $\Lambda(\mathsf{Q}_1, \mathsf{P}_2^1)$-module structure using the algebra structure of $E_2^\bullet$ and the product formula (7), together with

$$\mathsf{P}_2^1(x_i) = \mathsf{P}_2^1(t_i) = 0.$$

The action of P_2^1 that results from this procedure is

$$\mathsf{P}_2^1(t_I x_J) = \sum_{K \in J[2]} t_K t_I x_{J-K} \tag{12}$$

on the monomial basis, which can be extended to all of $E_2^\bullet$ using $\mathbb{F}_2$-linearity. Notice that the action we obtain through this process coincides with the formula for the d_2 differentials (11).

3. The reduced length

For convenience, we denote the E_2-page of (9) by

$$\mathcal{R} = \Lambda(t_i : i \geqslant 1) \otimes \Lambda(x_i : i \geqslant 1),$$

which is an $\mathbb{F}_2$-algebra, as well as a $\Lambda(\mathsf{Q}_1, \mathsf{P}_2^1)$-module, where the actions of Q_1 and P_2^1 are given by (10) and (12) respectively. In this section we analyze the $\Lambda(\mathsf{Q}_1, \mathsf{P}_2^1)$-module structure of $\mathcal{R}$, which leads us to a description of

$$E_\infty^\bullet \cong \ldots \cong E_3^\bullet \cong H(E_2^\bullet, d_2) \cong \mathcal{M}(\mathcal{R}, \mathsf{P}_2^1).$$

The main idea here is to notice (this will be shown in Lemma 3.3) that the action of P_2^1 is linear with respect to the subalgebra

$$\mathcal{S} := \Lambda(t_i x_i | i \in \mathbb{N}_+) \subset \mathcal{R},$$

which implies that $\mathcal{M}(\mathcal{R}, \mathsf{P}_2^1)$ admits an $\mathcal{S}$-module structure.

Lemma 3.1. *The subalgebra $\mathcal{S} \subset \mathcal{R}$ is a trivial $\Lambda(\mathsf{Q}_1, \mathsf{P}_2^1)$-submodule which splits off as a $\Lambda(\mathsf{Q}_1, \mathsf{P}_2^1)$-module.*

Proof. For any element $t_I x_I \in \mathcal{S}$, it is clear from (10) and (11) that

$$\mathsf{Q}_1(t_I x_I) = 0 = \mathsf{P}_2^1(t_I x_I).$$

Thus $\mathcal{S}$ is a trivial submodule.

Now observe from (10) and (11) that none of the monomials $t_I x_I \in \mathcal{S}$ is a summand of $\mathsf{Q}_1(t_{I'} x_{J'})$ or $\mathsf{P}_2^1(t_{I'} x_{J'})$ for any choice of I' and J'. Hence, $\mathcal{S}$ is a split summand. □

Corollary 3.2. *Every element of $\mathcal{S}$ is a nonzero cycle in the $\mathcal{M}(\mathcal{R}, \mathsf{P}_2^1)$.*

Lemma 3.3. *The action of P_2^1 on $\mathcal{R}$ is $\mathcal{S}$-linear.*

Proof. It is enough to show that

$$\mathsf{P}_2^1(t_i x_i \cdot t_I x_J) = (t_i x_i) \cdot \mathsf{P}_2^1(t_I x_J). \tag{13}$$

If $i \in I$, then $t_i t_I = 0$. Hence both the LHS and the RHS of (13) are zero.

If $i \in J$, then $x_i x_J = 0$, hence LHS of (13) is zero. On the other hand,

$$\begin{aligned} RHS &= t_i x_i \cdot \sum_{K \in J[2]} t_K t_I x_{J-K} \\ &= \sum_{i \in K \in J[2]} t_i t_K t_I x_i x_{J-K} + t_i \cdot \left(\sum_{i \notin K \in J[2]} t_i t_I x_i x_{J-K} \right) = 0, \end{aligned}$$

as $t_i t_K = 0$ when $i \in K$ and $x_i x_{J-K} = 0$ when $i \notin K$.

Now consider the case when $i \notin I \cup J$. Let $I' = I \cup \{i\}$ and $J' = J \cup \{i\}$. Then,

$$\begin{aligned} \mathsf{P}_2^1(t_i x_i \cdot t_I x_J) = \mathsf{P}_2^1(t_{I'} x_{J'}) &= \sum_{K \in J'[2]} t_K t_{I'} x_{J'-K} \\ &= \sum_{i \in K \in J'[2]} t_K t_{I'} x_{J'-K} + \sum_{i \notin K \in J'[2]} t_K t_{I'} x_{J'-K} \\ &= \sum_{i \notin K \in J'[2]} t_K t_{I'} x_{J'-K} \\ &= t_i x_i \cdot \sum_{K \in J[2]} t_K t_I x_{J-K} \\ &= t_i x_i \cdot \mathsf{P}_2^1(t_I x_J). \end{aligned}$$

□

Remark 3.4. While the E_2 page of (9) admits an $\mathbb{F}_2$-algebra structure, the E_3 page does not admit any multiplicative structure. This is because the d_2 differentials do not follow the Leibniz rule and the product of d_2 cycles may not be a cycle. For example, x_i for all $i \in \mathbb{N}_+$, is a d_2-cycle, whereas $x_i x_j$ for $i \neq j$ supports a differential $d_2(x_i x_j) = t_i t_j$ by (11). Even if α, β and $\alpha \cdot \beta$ are P_2^1 cycles it is unclear that the pairing $[\alpha] \cdot [\beta] = [\alpha \cdot \beta]$ is well-defined in the E_3 page.

Corollary 3.5. *$\mathcal{M}(\mathcal{R}, \mathsf{P}_2^1)$ is a module over the ring $\mathcal{S}$.*

Proof. By Lemma 3.3, there exists a pairing $\mu \colon \mathcal{S} \otimes \mathcal{R} \to \mathcal{R}$ such that the diagram

$$\begin{array}{ccc} \mathcal{S} \otimes \mathcal{R} & \xrightarrow{\mu} & \mathcal{R} \\ {\scriptstyle 1 \otimes \mathsf{P}_2^1} \downarrow & & \downarrow {\scriptstyle \mathsf{P}_2^1} \\ \mathcal{S} \otimes \mathcal{R} & \xrightarrow[\mu]{} & \mathcal{R} \end{array}$$

commutes. It follows that $\mathcal{M}(\mathcal{R}, \mathsf{P}_2^1)$ is an $\mathcal{S}$ module. □

As a result, we only need to understand the action of P_2^1 on the generators of $\mathcal{R}$ when viewed as an $\mathcal{S}$-module. In order to approach this problem we introduce the notion of *reduced length*.

Definition 3.6. For any monomial $t_I x_J \in \mathcal{R}$ the *reduced length* ℓ is

$$\ell(t_I x_J) = |J - I| = |J \cap I^c| = |J| - |J \cap I|,$$

where I^c denotes the complement of I.

Note that the length of $t_I x_J \in \mathcal{R}$ is given by the formula $L(t_I x_J) = |J|$; in other words, it is counting the number of factors of x_J. Whereas, $\ell(t_I x_J)$ counts only those factors x_j in x_J for which t_j is not a factor of t_I. For example,

$$\ell(x_1) = \ell(t_1 x_1 x_2) = \ell(t_1 t_2 x_1 x_2 x_3) = \ell(t_1 t_2 t_3 x_4) = 1,$$
$$\ell(x_1 x_2) = \ell(t_1 x_1 x_2 x_3) = \ell(t_1 t_2 t_3 t_4 x_5 x_6) = 2.$$

Remark 3.7. The reduced length function ℓ measures "how far" a given monomial in $\mathcal{R}$ is from the subalgebra $\mathcal{S}$.

For each $i \in \mathbb{N}_+$, let $M_i := \Lambda(\mathsf{Q}_1)\{x_i\} \subset \mathcal{R}$ denote the $\Lambda(\mathsf{Q}_1, \mathsf{P}^1_2)$-submodule isomorphic to $\Lambda(\mathsf{Q}_1)$ and generated by x_i. For an indexing set $K \underset{\text{finite}}{\subset} \mathbb{N}_+$, let

$$M_K := \bigotimes_{j \in K} M_j \subset \mathcal{R}$$

with the convention that $M_\emptyset := \mathbb{F}_2$. If the indexing set is $[n] = \{1, \ldots, n\} \subset \mathbb{N}_+$, then we write $M_{[n]}$ to denote $M_{\{1,\ldots,n\}}$.

In Figure 1, Figure 2 and Figure 3 we present M_i, $M_{\{1,2\}}$ and $M_{\{1,2,3\}}$ respectively as a $\Lambda(\mathsf{Q}_1, \mathsf{P}^1_2)$-module. In these figures the dotted curved lines depict the action of Q_1 and dashed straight lines depict the action of P^1_2.

Figure 1: M_i as a module over $\Lambda(\mathsf{Q}_1, \mathsf{P}^1_2)$

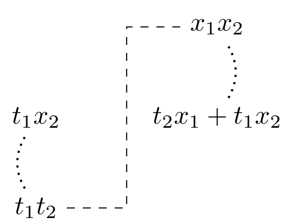

Figure 2: $M_{[2]}$ as a module over $\Lambda(\mathsf{Q}_1, \mathsf{P}^1_2)$, where $[2] = \{1, 2\}$

Note that the set $\mathcal{W} := \{t_I x_J \in \mathcal{R} | I \cap J = \emptyset\}$ forms a generating set for $\mathcal{R}$ as an $\mathcal{S}$-module as any monomial $t_I x_J \in \mathcal{R}$ can be uniquely written as a product of an element of $\mathcal{W}$ and a monomial in $\mathcal{S}$:

$$t_I x_J = t_{I \cap J} x_{I \cap J} \cdot t_{I-(I \cap J)} x_{J-(I \cap J)}.$$

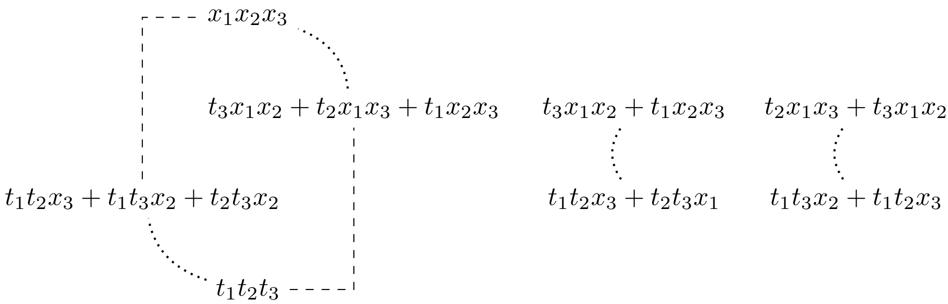

Figure 3: $M_{[3]}$ as a module over $\Lambda(\mathsf{Q}_1, \mathsf{P}_2^1)$, where $[3] = \{1, 2, 3\}$

For any finite subset $K \subset \mathbb{N}_+$,

$$\mathcal{W}_K := \{t_I x_J | I \cup J = K, I \cap J = \emptyset\} \subset \mathcal{W}$$

forms an $\mathbb{F}_2$-basis for M_K, i.e. $\mathbb{F}_2\langle \mathcal{W}_K \rangle = M_K$. Since

$$\mathcal{W} = \bigsqcup_{K \underset{\text{finite}}{\subset} \mathbb{N}_+} \mathcal{W}_K$$

and $\mathbb{F}_2\langle \mathcal{W}_K \rangle = M_K$ is closed under the action of Q_1 and P_2^1 (these actions preserve the total indexing set K, by (10) and (11)), we learn that

$$\mathcal{R}/\!\!/\mathcal{S} \cong \mathbb{F}_2 \otimes_{\mathcal{S}} \mathcal{R} \cong \mathcal{R} \otimes_{\mathcal{S}} \mathbb{F}_2 \cong \bigoplus_K M_K$$

is an isomorphism of $\Lambda(\mathsf{Q}_1, \mathsf{P}_2^1)$-modules. Consequently, we have the following lemma.

Lemma 3.8. *Let $\mathcal{S}_K \subset \mathcal{S}$ denote the subalgebra $\Lambda(t_I x_I | I \subset \mathbb{N}_+ - K)$. There is a $\Lambda(\mathsf{Q}_1, \mathsf{P}_2^1)$-module isomorphism*

$$\bigoplus_{K \underset{\text{finite}}{\subset} \mathbb{N}_+} \mathcal{S}_K \otimes M_K \cong \mathcal{R}.$$

Proof. Consider the $\mathbb{F}_2$-vector space isomorphism

$$\iota \colon \mathcal{R} \longrightarrow \bigoplus_{K \underset{\text{finite}}{\subset} \mathbb{N}_+} \mathcal{S}_K \otimes M_K$$

which sends

$$t_I x_J \mapsto t_{I\cap J} x_{I \cap J} \otimes t_{I-(I\cap J)} x_{J-(I\cap J)} \in \mathcal{S}_K \otimes M_K,$$

where $K = I \cup J - I \cap J$. The map ι^{-1} sends

$$t_B x_B \otimes t_{K-A} x_A \mapsto t_{B \cup (K-A)} \cdot x_{B \cup A},$$

where $A \subset K$. This map is also a $\Lambda(\mathsf{Q}_1, \mathsf{P}_2^1)$-module isomorphism as $\mathcal{S}_K \subset \mathcal{S}$ is a trivial $\Lambda(\mathsf{Q}_1, \mathsf{P}_2^1)$-module by Lemma 3.1. □

Hence we can reduce the problem of computing $\mathcal{M}(\mathcal{R}, \mathsf{P}_2^1)$ to computing $\mathcal{M}(M_K, \mathsf{P}_2^1)$ for various finite subsets K of $\mathbb{N}_+$. Thus, we first need to understand the structure of M_K as a $\Lambda(\mathsf{Q}_1, \mathsf{P}_2^1)$-module.

Remark 3.9. Let $[n]$ denote the indexing set $\{1,\dots,n\} \subset \mathbb{N}_+$. If $|K| = n$, then there exists the unique order preserving bijection

$$\iota\colon [n] \longrightarrow K$$

and it induces an isomorphism $\iota\colon M_{[n]} \xrightarrow{\cong} M_K$. Thus it is enough to understand $\Lambda(\mathsf{Q}_1, \mathsf{P}_2^1)$-module structure of $M_{[n]}$ for all $n \in \mathbb{N}_+$.

As depicted in Figure 3, $M_{[3]}$ splits as a $\Lambda(\mathsf{Q}_1, \mathsf{P}_2^1)$-module

$$M_{[3]} \cong \Lambda(\mathsf{Q}_1, \mathsf{P}_2^1)\{x_1x_2x_3\} \oplus \Lambda(\mathsf{Q}_1)\{t_3x_1x_2 + t_1x_2x_3\} \oplus \Lambda(\mathsf{Q}_1)\{t_2x_1x_3 + t_3x_1x_2\} \tag{14}$$

as a sum of a free $\Lambda(\mathsf{Q}_1, \mathsf{P}_2^1)$-module and two copies of $\Lambda(\mathsf{Q}_1)$.

Remark 3.10. The splitting of (14) is a consequence of Lemma 2.10. Since $\Lambda(\mathsf{Q}_1, \mathsf{P}_2^1)$ is cocommutative, for any $\Lambda(\mathsf{Q}_1, \mathsf{P}_2^1)$-module M and $\sigma \in \mathbb{F}_2[\Sigma_n]$, the induced map

$$\sigma\colon M^{\otimes n} \longrightarrow M^{\otimes n}$$

is a map of $\Lambda(\mathsf{Q}_1, \mathsf{P}_2^1)$-modules. Note that in the group ring $\mathbb{F}_2[\Sigma_3]$, the identity element can be written as a sum of idempotent elements

$$\mathbb{1} = e + f_1 + f_2.$$

For example, one can choose $e = \mathbb{1} + (1\ 2\ 3) + (1\ 2\ 3)$, $f_1 = \mathbb{1} + (1\ 2) + (1\ 3) + (1\ 3\ 2)$ and $f_2 = \mathbb{1} + (1\ 2) + (1\ 3) + (1\ 2\ 3)$. Then we have

$$M^{\otimes 3} \cong e(M^{\otimes 3}) \oplus f_1(M^{\otimes 3}) \oplus f_2(M^{\otimes 3}).$$

When $M \cong \Lambda(\mathsf{Q}_1)$, we get the decomposition of (14).

The splitting of (14), along with the following fact about finite dimensional Hopf algebras, is the key to understanding the structure of M_K.

Theorem 3.11 ([NZ89])**.** *If $\mathcal{H}$ is a finite dimensional connected Hopf algebra over a field $\mathbb{F}$, then for any $\mathcal{H}$-module M, $\mathcal{H} \otimes M$ is a free $\mathcal{H}$-module.*

Let us denote by A the $\Lambda(\mathsf{Q}_1, \mathsf{P}_2^1)$-module isomorphic to $\Lambda(\mathsf{Q}_1)$ and let $B := A \otimes A$. Then using (14) and Theorem 3.11, we notice that

$$M_{[3]} \cong B \otimes A \cong \{\text{Free}\} \oplus A^{\oplus 2}, \qquad M_{[4]} \cong \{\text{Free}\} \oplus B^{\oplus 2}, \qquad M_{[5]} \cong \{\text{Free}\} \oplus A^{\oplus 4},$$

where $\{\text{Free}\}$ denotes a free $\Lambda(\mathsf{Q}_1, \mathsf{P}_2^1)$ module. This iterative process can be continued as described in Lemma 3.12 below. We use $A\{y\}$, resp. $B\{y\}$, to specify that y generates A, resp. B, as a $\Lambda(\mathsf{Q}_1, \mathsf{P}_2^1)$ module. For example, $M_i \cong A\{x_i\}$.

Lemma 3.12. *There exist elements $h_{2r+1,i} \in M_{[2r+1]}$ with $\ell(h_{2r+1,i}) = r+1$ such that, as a $\Lambda(\mathsf{Q}_1, \mathsf{P}_2^1)$-module,*

$$M_{[2r+1]} \cong \{\mathit{Free}\} \oplus \left(\bigoplus_{i=1}^{2^r} A\{h_{2r+1,i}\} \right).$$

There exist elements $h_{2r,i} \in M_{[2r]}$ with $\ell(h_{2r+1,i}) = r+1$ such that, as a $\Lambda(\mathsf{Q}_1, \mathsf{P}_2^1)$-module

$$M_{[2r]} \cong \{\mathit{Free}\} \oplus \bigoplus_{i=1}^{2^{r-1}} B\{h_{2r,i}\}.$$

Proof. Our proof is by induction on r. From Figure 1, Figure 2 and Figure 3, the claim is true for $k = 1, 2, 3$. Note that

$$h_{1,1} = x_1, \qquad h_{2,1} = x_1x_2, \qquad h_{3,1} = (t_3x_1 + x_3t_1)x_2, \qquad h_{3,2} = (t_2x_3 + t_3x_2)x_1.$$

Now assume that the result is true for $2r - 1$, i.e.

$$M_{[2r-1]} \cong \{\text{Free}\} \oplus \bigoplus_{1\leqslant i\leqslant 2^{r-1}} A\{h_{2r-1,i}\},$$

where $\ell(h_{2r-1,i}) = r$ and $\{\text{Free}\}$ is a free $\Lambda(\mathsf{Q}_1, \mathsf{P}_2^1)$-module. It follows that

$$M_{[2r]} \cong M_{[2r-1]} \otimes M_{2r} \cong (\{\text{Free}\} \otimes A\{x_{2r}\}) \oplus \bigoplus_{1\leqslant i\leqslant 2^{r-1}} B\{h_{2r-1,i} \cdot x_{2r}\}.$$

By Theorem 3.11, the first summand is, again, a free module. Set

$$h_{2r,i} = h_{2r-1,i} \cdot x_{2r}$$

and notice $\ell(h_{2r-1,i}x_{2r}) = \ell(h_{2r-1,i}) + \ell(x_{2r}) = r + 1$.

To complete the inductive argument, observe

$$\begin{aligned} M_{[2r+1]} &\cong M_{[2r-1]} \otimes B\{x_{2r}x_{2r+1}\} \\ &\cong \left(\{\text{Free}\} \oplus \bigoplus_{1\leqslant i\leqslant 2^{r-1}} A\{h_{2r-1,i}\}\right) \otimes B\{x_{2r}x_{2r+1}\} \\ &\cong \{\text{Free}\} \oplus \bigoplus_{1\leqslant i\leqslant 2^{r-1}} (A\{h_{2r+1,2i-1}\} \oplus A\{h_{2r+1,2i}\}), \end{aligned}$$

where one can define the generators $h_{2r-1,j}$ from Figure 3 by replacing x_1, x_2, x_3 with $h_{2r-1,i}$, x_{2r} and x_{2r+1} respectively. More specifically, one can define

$$h_{2r+1,2i-1} = \mathsf{Q}_1(h_{2r-1,i} \cdot x_{2r+1}) \cdot x_{2r}, \qquad h_{2r+1,2i} = h_{2r-1,i} \cdot \mathsf{Q}_1(x_{2r}x_{2r+1}).$$

It is easy to check that $\ell(h_{2r+1,j}) = r + 1$. □

Following the proof of Lemma 3.12, we can provide an explicit basis of $\mathcal{M}(M_K, \mathsf{P}_2^1)$. By Remark 3.9 it suffices to provide a basis for $\mathcal{M}(M_{[n]}, \mathsf{P}_2^1)$ for all $n \geqslant 1$. We do so inductively (see Definition 3.14), however we must treat the odd and the even case separately, essentially because of Lemma 3.12. Since A is a trivial $\Lambda(\mathsf{P}_2^1)$-module, $\mathcal{M}(A, \mathsf{P}_2^1) \cong A$, and we get

$$\mathcal{M}(M_{[2r+1]}, \mathsf{P}_2^1) \cong \mathcal{M}\left(\bigoplus_{i=1}^{2^r} A\{h_{2r+1,i}\}, \mathsf{P}_2^1\right) \cong \bigoplus_{i=1}^{2^r} A\{h_{2r+1,i}\}.$$

Thus the collection

$$\{h_{2r+1,i} : 1 \leqslant i \leqslant 2^r\} \cup \{\mathsf{Q}_1(h_{2r+1,i}) : 1 \leqslant i \leqslant 2^r\}$$

is an $\mathbb{F}_2$-basis of $\mathcal{M}(M_{[2r+1]}, \mathsf{P}_2^1)$. When n is even, say $n = 2r$, then

$$\mathcal{M}(M_{[2r]}, \mathsf{P}_2^1) \cong \bigoplus_{i=1}^{2^{r-1}} \mathcal{M}(B\{h_{2r,i}\}, \mathsf{P}_2^1).$$

Now note that, if $B\{x \otimes y\} = A\{x\} \otimes A\{y\}$ (where x and y are generators), then

$$\{\mathsf{Q}_1(x) \otimes y, x \otimes \mathsf{Q}_1(y)\}$$

is an $\mathbb{F}_2$-basis of $\mathcal{M}(B\{x\otimes y\},\mathsf{P}^1_2)$. Using the fact that

$$h_{2r,i} = h_{2r-1,i}\cdot x_{2r},$$

we get Corollary 3.13 and Definition 3.14 thereafter.

Corollary 3.13. *Let $\mathcal{M}(M_K,\mathsf{P}^1_2)_l = \{x\in\mathcal{M}(M_K,\mathsf{P}^1_2)|\ell(x)=l\}$.*

If $|K| = 2r+1$, then

$$\dim\mathcal{M}(M_K,\mathsf{P}^1_2)_l = \begin{cases} 2^r, & \text{if } l = r, r+1,\\ 0, & \text{otherwise.}\end{cases}$$

If $|K| = 2r$, then

$$\dim\mathcal{M}(M_K,\mathsf{P}^1_2)_l = \begin{cases} 2^r, & \text{if } l = r,\\ 0, & \text{otherwise.}\end{cases}$$

Proof. Lemma 3.12 implies

$$M_{[2r+1]} \cong \{\text{Free}\}\oplus\bigoplus_{1\leqslant i\leqslant 2^r} A\{h_{2r+1,i}\},$$

where $\ell(h_{2r+1,i}) = r+1$. By Lemma 2.9 we have $\ell(\mathsf{Q}_1(h_{2r+1,i})) = r$. Thus $\{h_{2r+1,i}\}$ is the basis for $\mathcal{M}(M_{[2r+1]},\mathsf{P}^1_2)_{r+1}$ and $\{\mathsf{Q}_1(h_{2r+1,i})\}$ is the basis for $\mathcal{M}(M_{[2r+1]},\mathsf{P}^1_2)_r$. Applying Remark 3.9 we deduce the statement about dimension for any M_K with $|K| = 2r+1$.

For the even case we have from Lemma 3.12

$$M_{[2r]} \cong \{\text{Free}\}\oplus\bigoplus_{1\leqslant i\leqslant 2^{r-1}} B\{h_{2r,i}\},$$

where $\ell(h_{2r,i}) = r+1$. Then for each i, $\mathcal{M}(B\{h_{2r,i}\},\mathsf{P}^1_2) = \mathcal{M}(B\{h_{2r,i}\},\mathsf{P}^1_2)_r$ is an $\mathbb{F}_2$ vector space of dimension 2 generated by $\{h_{2r-1,i}\cdot t_{2r},\mathsf{Q}_1(h_{2r-1,i})\cdot x_{2r}\}$. ☐

Definition 3.14. We define the basis $\mathcal{B}_{[n],l}$ of $\mathcal{M}(M_{[n]},\mathsf{P}^1_2)_l$ for $0\leqslant l\leqslant n$ inductively starting with $\mathcal{B}_{[1],0} = \{t_1\}$ and $\mathcal{B}_{[1],1} = \{x_1\}$. Suppose we have defined

$$\mathcal{B}_{[2r-1],l} := \begin{cases} \{h_{2r-1,1},\dots,h_{2r-1,2^{r-1}}\} & \text{if } l = r,\\ \{\mathsf{Q}_1(h_{2r-1,1}),\dots,\mathsf{Q}_1(h_{2r-1,2^{r-1}})\} & \text{if } l = r-1,\\ \emptyset & \text{otherwise.}\end{cases}$$

Then define:

$$\mathcal{B}_{[2r],r} := \{h_{2r-1,1}\cdot t_{2r},\dots,h_{2r-1,2^{r-2}}\cdot t_{2r}\}\cup\{\mathsf{Q}_1(h_{2r-1,1})\cdot x_{2r},\dots,\mathsf{Q}_1(h_{2r-1,2^{r-2}})\cdot x_{2r}\}$$

and set $\mathcal{B}_{[2r],l} := \emptyset$ if $l\neq r$.

Now define $h_{2r+1,2i-1} = \mathsf{Q}_1(h_{2r-1,i})\cdot(x_{2r+1}\cdot x_{2r})$ and $h_{2r+1,2i} = h_{2r-1,i}\cdot\mathsf{Q}_1(x_{2r}x_{2r+1})$ and set

$$\mathcal{B}_{[2r+1],l} := \begin{cases} \{h_{2r+1,1},\dots,h_{2r+1,2^{r-2}}\} & \text{if } l = r+1,\\ \{\mathsf{Q}_1(h_{2r+1,1}),\dots,\mathsf{Q}_1(h_{2r+1,2^{r-2}})\} & \text{if } l = r,\\ \emptyset & \text{otherwise.}\end{cases}$$

We let $\mathcal{B}_{[n]}$ denote the union $\bigcup_l\mathcal{B}_{[n],l}$. Let $\mathcal{B}_K$ denote the image of the $\mathcal{B}_{[n]}$ under the isomorphism $\iota\colon M_{[n]}\to M_K$ of Remark 3.9.

Example 3.15 (Examples of $\mathcal{B}_K$). We explicitly identify $\mathcal{B}_{[n]}$ using Definition 3.14 for $n \leqslant 4$, and for $n = 1, 2, 3$ we can compare to Figures 1, 2 and 3, to see that $\mathcal{B}_{[n]}$ is indeed the basis for $\mathcal{M}(M_{[n]}, \mathsf{P}_2^1)$.

- $\mathcal{B}_{[1]} = \{t_1, x_1\}$,
- $\mathcal{B}_{[2]} = \{t_1x_2, t_2x_1\}$,
- $\mathcal{B}_{[3]} = \{t_1t_2x_3 + t_1t_3x_2, t_1t_2x_3 + t_2t_3x_1\} \cup \{t_3x_1x_2 + t_2x_1x_3, t_3x_1x_2 + t_1x_2x_3\}$,
- $\mathcal{B}_{[4]} = \{t_1t_2x_3x_4 + t_1t_3x_2x_4,\ t_1t_2x_3x_4 + t_2t_3x_1x_4,\ t_3t_4x_1x_2 + t_2t_4x_1x_3,\ t_3t_4x_1x_2 + t_1t_4x_2x_3\}$.

Note that $P_K := \mathbb{F}_2\langle\mathcal{B}_K\rangle \subset M_K$ is a split summand. This is because the inclusion map $P_K \hookrightarrow M_K$ induces $\mathcal{M}(-, \mathsf{P}_2^1)$-isomorphism, or equivalently, the quotient M_K/P_K is a free $\Lambda(\mathsf{P}_2^1)$-module.

Theorem 3.16. *Let K be a finite subset of $\mathbb{N}_+$. Let*

$$\mathcal{SB}_K := \{t_Ix_I \cdot b \mid I \cap K = \emptyset \text{ and } b \in \mathcal{B}_K\} \subset \mathcal{R}.$$

Then

$$\mathcal{B} := \bigsqcup_{K \underset{\text{finite}}{\subset} \mathbb{N}_+} \mathcal{SB}_K$$

forms a basis of the $\mathbb{F}_2$-vector space $\mathcal{M}(tmf, \mathsf{P}_2^1)$ and

$$\mathcal{M}(tmf, \mathsf{P}_2^1) \cong \bigoplus_{K \underset{\text{finite}}{\subset} \mathbb{N}_+} \mathbb{F}_2\langle\mathcal{SB}_K\rangle \cong \bigoplus_{K \underset{\text{finite}}{\subset} \mathbb{N}_+} \mathcal{S}_K \otimes \mathcal{M}(M_K, \mathsf{P}_2^1)$$

is an isomorphism of $\mathbb{F}_2$-vector spaces.

Proof. By Lemma 3.8, we have a $\Lambda(\mathsf{Q}_1, \mathsf{P}_2^1)$ module isomorphism

$$\mathcal{R} \cong \bigoplus_{K \underset{\text{finite}}{\subset} \mathbb{N}_+} \mathcal{S}_K \otimes M_K.$$

Therefore, the linearity of the action of P_2^1 (see Corollary 3.5) with respect to elements in $\mathcal{S}$ gives us

$$\begin{aligned}\mathcal{M}(tmf, \mathsf{P}_2^1) &\cong \mathcal{M}(\mathcal{R}, \mathsf{P}_2^1) \cong \mathcal{M}\left(\bigoplus\nolimits_{K \underset{\text{finite}}{\subset} \mathbb{N}_+} \mathcal{S}_K \otimes M_K, \mathsf{P}_2^1\right) \\ &\cong \bigoplus\nolimits_{K \underset{\text{finite}}{\subset} \mathbb{N}_+} \mathcal{S}_K \otimes \mathcal{M}(M_K, \mathsf{P}_2^1) \cong \bigoplus\nolimits_{K \underset{\text{finite}}{\subset} \mathbb{N}_+} \mathcal{S}_K \otimes P_K \\ &\cong \bigoplus\nolimits_{K \underset{\text{finite}}{\subset} \mathbb{N}_+} \mathbb{F}_2\langle\mathcal{SB}_K\rangle.\end{aligned}$$

□

Remark 3.17. Let e denote the exchange map $e\colon \mathcal{R} \to \mathcal{R}$ which sends

$$e\colon t_Ix_J \mapsto t_Jx_I.$$

It seems to be the case that $[m] \in \mathcal{M}(tmf, \mathsf{P}_2^1)$ if and only if $[e(m)] \in \mathcal{M}(tmf, \mathsf{P}_2^1)$. The source of such symmetry is unclear to the authors, although it might be related to Spanier–Whitehead duality.

Finally, we would like to say a word about the module structure of $\mathcal{M}(tmf, \mathsf{P}_2^1)$ over $\mathcal{S}$. Note that the collection of elements

$$\mathcal{B}_{\mathcal{S}} := \{t_I x_I | I \underset{\text{finite}}{\subset} \mathbb{N}_+\}$$

forms an $\mathbb{F}_2$-basis of $\mathcal{S}$. The $\mathcal{S}$-module structure on $\mathcal{M}(tmf, \mathsf{P}_2^1)$ is extended from a pairing at the level of bases

$$\mathcal{B}_{\mathcal{S}} \otimes \mathcal{S}\mathcal{B}_K \xrightarrow{\mu} \mathcal{S}\mathcal{B}_K$$
$$s \otimes (s' \cdot b) \mapsto \begin{cases} (s \cdot s') \cdot b, \text{ if } I \cap K = \emptyset, \\ 0, \text{ if } I \cap K \neq \emptyset. \end{cases}$$

Remark 3.18. Recall that $H_*(tmf)$ was described in terms of ζ_i. We can convert an element of the Margolis homology expressed in terms of t_i and x_i back to an expression involving ζ_i using the identifications of Notation 2.12. For example,

$$t_4 t_9 x_2 x_6 + t_2 t_9 x_4 x_6$$

can be identified with the class represented by element $\zeta_5^5 \zeta_{10}^4 \zeta_9 + \zeta_3^4 \zeta_{10}^4 \zeta_7 \zeta_9 \in \mathfrak{T}$.

4. P_2^1 Margolis homology of $tmf^{\wedge r}$ and $((B\mathbb{Z}/2)^{\times k})_+$

4.1. P_2^1 Margolis homology of $tmf^{\wedge r}$

Note that

$$H_*(tmf^{\wedge r}) \cong H_*(tmf)^{\otimes r} \cong \mathfrak{T}^{\otimes r}.$$

We first extend the notion of length to $\mathfrak{T}^{\otimes r}$. For a monomial $\zeta^{I_1}|\dots|\zeta^{I_r}$ for $\zeta^{I_i} \in \mathfrak{T}^{\otimes r}$, which is a tensor product of monomials in $\mathfrak{T}$, we define

$$L(\zeta^{I_1}|\dots|\zeta^{I_r}) = L(\zeta^{I_1}) + \dots + L(\zeta^{I_r}).$$

We define the even subalgebra $\mathbb{E}_r$ of $\mathfrak{T}^{\otimes r}$ as the span of those monomials in $\mathfrak{T}^{\otimes r}$ whose lengths are zero. Observe that,

$$\mathbb{E}_r \cong \mathcal{E}^{\otimes r}.$$

The notion of length leads to an increasing filtration on $\mathfrak{T}^{\otimes r}$, called the length filtration, by setting

$$G^p(\mathfrak{T}^{\otimes r}) = \{(\zeta^{I_1}|\dots|\zeta^{I_r}) \,|\, L(\zeta^{I_1}|\dots|\zeta^{I_r}) \leqslant p\}.$$

Let $\mathbb{K}_r = \mathcal{K}^{\otimes r}$, where $\mathcal{K}$ is as defined in Section 2. Just like in the case $r = 1$, we get a length spectral sequence and its E_1 page is

$$E_1^\bullet \cong \mathcal{M}(\mathbb{E}_r, \mathsf{P}_2^1) \otimes \mathbb{K}_r \Rightarrow \mathcal{M}(tmf^{\wedge r}, \mathsf{P}_2^1). \tag{15}$$

Since the action of P_2^1 follows the Leibniz rule when restricted to $\mathcal{E}$, we get

$$\mathcal{M}(\mathbb{E}_r, \mathsf{P}_2^1) \cong \mathcal{M}(\mathcal{E}, \mathsf{P}_2^1)^{\otimes r}.$$

Notation 4.1. For shorthand, we denote

$$x_{i,j} = (\underbrace{1|\dots|1}_{j-1}|\zeta_{i+3}|\underbrace{1|\dots|1}_{r-j}), \qquad t_{i,j} = (\underbrace{1|\dots|1}_{j-1}|\zeta_{i+1}^4|\underbrace{|\dots|1}_{r-j}).$$

With this notation we have

$$\mathsf{Q}_1(x_{i,j}) = t_{i,j}.$$

Using Notation 4.1, we see that the E_1 page of the length spectral sequence (15), as an algebra, is isomorphic to

$$\mathcal{R}_r := \Lambda(t_{i,j} : i \in \mathbb{N} - \{0\}, 1 \leqslant j \leqslant r) \otimes \Lambda(x_{i,j} : i \in \mathbb{N} - \{0\}, 1 \leqslant j \leqslant r).$$

It is easy to see that the map induced by the reindexing map

$$\iota\colon (i,j) \mapsto r(i-1)+j,$$

produces a (non-canonical) isomorphism of algebras between $\mathcal{R}_r$ (the E_2 page of (15)) and $\mathcal{R}$ (the E_2 page of (9)), after forgetting the internal grading. This is also an isomorphism of $\Lambda(\mathsf{Q}_1, \mathsf{P}_2^1)$-modules. Thus we have an isomorphism

$$\iota_*\colon \mathcal{M}(tmf, \mathsf{P}_2^1) \xrightarrow{\cong} \mathcal{M}(tmf^{\wedge r}, \mathsf{P}_2^1)$$

induced by the ι. Therefore, Theorem 3.16 essentially gives a complete calculation of $\mathcal{M}(tmf^{\wedge r}, \mathsf{P}_2^1)$.

Example 4.2. For example, let us assume $r = 3$. Then the element $t_2t_4x_6x_9 + t_2t_6x_4x_9 \in \mathcal{M}(tmf, \mathsf{P}_2^1)$ (see Example 3.15) corresponds to the element

$$t_{1,2}t_{2,1}x_{2,3}x_{3,3} + t_{1,2}t_{2,3}x_{2,1}x_{3,3} \in \mathcal{M}(tmf^{\wedge 3}, \mathsf{P}_2^1)$$

under the bijection obtained from the above reindexing. When expressed in terms of ζ_is (see Notation 4.1), the same element can be expressed as

$$\zeta_3^4|\zeta_2^4|\zeta_5\zeta_6|1 + \zeta_5|\zeta_2^4|\zeta_3^4\zeta_6|1.$$

Remark 4.3 (P_2^1 Margolis homology of Brown–Gitler spectra)*.* It is well-known that

$$H_*(tmf) \cong \bigoplus_{i \geqslant 0} H_*(\Sigma^{8i} bo_i),$$

where bo_i are certain Brown–Gitler spectra associated with bo. In [**Mah81**] Mahowald defined a multiplicative weight function, which is given by $w(\zeta_i) = 2^{i-1}$. $H_*(\Sigma^{8i}bo_i)$ is the summand of $H_*(tmf)$ which consists of elements of Mahowald weight exactly equal to $8i$. We assign Mahowald weight of $t_{i,j}$ and $x_{i,j}$ as

$$w(t_{i,j}) = w(x_{i,j}) = 2^{i+1}.$$

It follows that the Margolis homology $\mathcal{M}(bo_{q_1} \wedge \dots \wedge bo_{q_r}, \mathsf{P}_2^1)$ is a summand of $\mathcal{M}(tmf^{\wedge r}, \mathsf{P}_2^1)$. It consists of all polynomials of $\mathcal{M}(tmf^{\wedge r}, \mathsf{P}_2^1)$ expressed in terms of $x_{i,j}$ and $t_{i,j}$ such that $w(x_{i,j}) = w(t_{i,j}) = 4q_j$.

Remark 4.4. While it is true that $\mathcal{R}_r \cong \mathcal{R}^{\otimes r}$, as an $\mathbb{F}_2$-algebra as well as a $\Lambda(\mathsf{Q}_1, \mathsf{P}_2^1)$-module, it is not useful for the purposes of calculating $\mathcal{M}(\mathcal{R}_r, \mathsf{P}_2^1)$. This is because

P^1_2 does not obey the Leibniz rule and

$$\mathcal{M}(\mathcal{R}_r, \mathsf{P}^1_2) \ncong \mathcal{M}(\mathcal{R}, \mathsf{P}^1_2)^{\otimes r}.$$

However we overcome this difficulty by producing a $\Lambda(\mathsf{Q}_1, \mathsf{P}^1_2)$-module isomorphism ι_* at the expense of forgetting the internal grading.

4.2. P^1_2 Margolis homology of $((B\mathbb{Z}/2)^{\times k})_+$

The space $B\mathbb{Z}/2$ is also known as $\mathbb{RP}^\infty$, the real infinite-dimensional projective space. It is well-known that

$$H^*((B\mathbb{Z}/2)_+, \mathbb{F}_2) \cong \mathbb{F}_2[x]$$

and therefore

$$H^*(((B\mathbb{Z}/2)^{\times k})_+, \mathbb{F}_2) \cong \mathbb{F}_2[x_1, \dots x_k].$$

It can be seen that $\mathsf{P}^1_2(x_i) = 0$ and $\mathsf{Q}_1(x_i) = x_i^4$. We again define the length function on the monomials in the usual way

$$L(x_1^{i_1} \dots x_k^{i_k}) = (i_1 \bmod 2) + \cdots + (i_k \bmod 2).$$

The even complex $\mathcal{E}$, which is the span of elements of length zero, is isomorphic to

$$\mathcal{E} = \mathbb{F}_2[x_1^2, \dots, x_k^2].$$

It can be seen that $\mathsf{P}^1_2(x_i^2) = x_i^8$. Now observe that Q_1 acts trivially on $\mathcal{E}$, hence P^1_2 acts as a derivation and, therefore,

$$\mathcal{M}(\mathcal{E}, \mathsf{P}^1_2) \cong \Lambda(x_1^4, \dots, x_k^4).$$

Now the length function gives us an increasing length filtration

$$G^p(\mathbb{F}_2[x_1, \dots, x_k]) = \mathbb{F}_2\langle x_1^{i_1} \dots x_k^{i_k} : L(x_1^{i_1} \dots x_k^{i_k}) \leqslant p\rangle.$$

This results in a length spectral sequence which only has d_0 and d_2 differentials. If we denote x_i^4 by t_i for convenience, we can see that the action of Q_1 on the E_1-page of the length spectral sequence

$$E_1^\bullet = \Lambda(t_1, \dots, t_k) \otimes \Lambda(x_1, \dots, x_k) \Rightarrow \mathcal{M}(((B\mathbb{Z}/2)^{\times k})_+, \mathsf{P}^1_2)$$

is determined by the formula $\mathsf{Q}_1(x_i) = t_i$ and the Leibniz rule. Since the d_2-differentials are determined by the Q_1-action on the E_1-page, we conclude that the length spectral sequence above is a sub spectral sequence of (9), in fact, isomorphic to it when $k = \infty$. Thus, when k is finite, we can recover a complete description of $\mathcal{M}(((B\mathbb{Z}/2)^{\times k})_+, \mathsf{P}^1_2)$ from Theorem 3.16. More precisely, we obtain

$$\mathcal{M}(((B\mathbb{Z}/2)^{\times k})_+, \mathsf{P}^1_2) \cong \bigoplus_{K \subset [k]} S_K \otimes \mathcal{M}(M_K, \mathsf{P}^1_2),$$

where $S_K = \Lambda(t_i x_i \,|\, i \in [k] - K)$ and $\mathcal{M}(((B\mathbb{Z}/2)^{\times k})_+, \mathsf{P}^1_2)$ is a module over $\mathcal{S}_{[k]}$.

Example 4.5. $\mathcal{M}(\mathbb{RP}^\infty_+, \mathsf{P}^1_2) \cong \mathbb{F}_2\langle x_1, t_1, t_1 x_1\rangle$, where the internal degrees of x_1 and t_1

are 1 and 4 respectively and $\mathcal{S}_{[1]} = \Lambda(t_1 x_1)$. Similarly,

$$\mathcal{M}((\mathbb{RP}^\infty \times \mathbb{RP}^\infty)_+, \mathsf{P}_2^1) \cong \mathbb{F}_2\langle x_1, x_2, t_1, t_2, t_1x_1, t_2x_2, t_1x_2, t_2x_1, t_1x_1x_2, t_2x_2x_1, t_1t_2x_2, t_1t_2x_1\rangle,$$

where the internal degrees of x_i and t_i are 1 and 4 respectively. Here $\mathcal{S}_{[2]} = \Lambda(t_1x_1, t_2x_2)$. If we denote

$$H^*((\mathbb{RP}^\infty \times \mathbb{RP}^\infty)_+) \cong \mathbb{F}_2[y, z],$$

where $|y| = |z| = 1$, then one may choose $x_1 = [x]$, $x_2 = [y]$, $t_1 = [x^4]$ and $t_2 = [y^4]$.

References

[Ada74] J.F. Adams. *Stable homotopy and generalised homology.* Chicago Lectures in Math., University of Chicago Press, Chicago, IL, 1974.

[AM71] J.F. Adams and H.R. Margolis. Modules over the Steenrod algebra. *Topology*, 10:271–282, 1971.

[Bau08] T. Bauer. Computation of the homotopy of the spectrum `tmf`. In *Groups, homotopy and configuration spaces*, volume 13 of *Geom. Topol. Monogr.*, pages 11–40. Geometry & Topology Publications, Coventry, 2008.

[BBB$^+$] A. Beaudry, M. Behrens, P. Bhattacharya, D. Culver, and Z. Xu. The telescope conjecture at height 2 and the tmf resolution. arXiv preprint 1909.13379.

[BBB$^+$20] A. Beaudry, M. Behrens, P. Bhattacharya, D. Culver, and Z. Xu. On the E_2-term of the bo-Adams spectral sequence. *J. Topol.*, 13(1):356–415, 2020.

[BE20a] P. Bhattacharya and P. Egger. A class of 2-local finite spectra which admit a v_2^1-self-map. *Adv. Math.*, 360:106895, 40, 2020.

[BE20b] P. Bhattacharya and P. Egger. Towards the $K(2)$-local homotopy groups of Z. *Algebr. Geom. Topol.*, 20(3):1235–1277, 2020.

[BG73] E.H. Brown, Jr. and S. Gitler. A spectrum whose cohomology is a certain cyclic module over the Steenrod algebra. *Topology*, 12:283–295, 1973.

[BG18] I. Bobkova and P.G. Goerss. Topological resolutions in $K(2)$-local homotopy theory at the prime 2. *J. Topol.*, 11(4):917–956, 2018.

[Cul19] D.L. Culver. On BP⟨2⟩-cooperations. *Algebr. Geom. Topol.*, 19(2):807–862, 2019.

[DFHH14] C.L. Douglas, J. Francis, A.G. Henriques, and M.A. Hill, editors. *Topological modular forms*, volume 201 of *Math. Surv. Monogr.*, American Mathematical Society, Providence, RI, 2014.

[Goe86] P.G. Goerss. Unstable projectives and stable Ext: with applications. *Proc. London Math. Soc. (3)*, 53(3):539–561, 1986.

[Goe10] P.G. Goerss. Topological modular forms [after Hopkins, Miller and Lurie]. *Astérisque*, (332):Exp. No. 1005, viii, 221–255, 2010. Séminaire Bourbaki. Volume 2008/2009. Exposés 997–1011.

[HK00] D.J. Hunter and N.J. Kuhn. Characterizations of spectra with U-injective cohomology which satisfy the Brown-Gitler property. *Trans. Amer. Math. Soc.*, 352(3):1171–1190, 2000.

[HM14] M.J. Hopkins and M. Mahowald. From elliptic curves to homotopy theory. In *Topological modular forms*, volume 201 of *Math. Surveys Monogr.*, pages 261–285. American Mathematical Society, Providence, RI, 2014.

[HM16] R. Haugseng and H. Miller. On a spectral sequence for the cohomology of infinite loop spaces. *Algebr. Geom. Topol.*, 16(5):2911–2947, 2016.

[KM13] N.J. Kuhn and J. McCarty. The mod 2 homology of infinite loopspaces. *Algebr. Geom. Topol.*, 13(2):687–745, 2013.

[LM87] W. Lellmann and M. Mahowald. The *bo*-Adams spectral sequence. *Trans. Amer. Math. Soc.*, 300(2):593–623, 1987.

[Mah81] M. Mahowald. *bo*-resolutions. *Pacific J. Math.*, 92(2):365–383, 1981.

[Mar83] H.R. Margolis. *Spectra and the Steenrod algebra*, volume 29 of *N.-Holl. Math. Libr.*, North-Holland Publishing Co., Amsterdam, 1983. Modules over the Steenrod algebra and the stable homotopy category.

[Mat16] A. Mathew. The homology of tmf. *Homology Homotopy Appl.*, 18(2):1–29, 2016.

[Mil58] J. Milnor. The Steenrod algebra and its dual. *Ann. of Math. (2)*, 67:150–171, 1958.

[MT68] R.E. Mosher and M.C. Tangora. *Cohomology operations and applications in homotopy theory.* Harper & Row, Publishers, New York-London, 1968.

[NZ89] W.D. Nichols and M.B. Zoeller. A Hopf algebra freeness theorem. *Amer. J. Math.*, 111(2):381–385, 1989.

[Pea14] P.T. Pearson. The connective real K-theory of Brown-Gitler spectra. *Algebr. Geom. Topol.*, 14(1):597–625, 2014.

[Sch94] L. Schwartz. *Unstable modules over the Steenrod algebra and Sullivan's fixed point set conjecture.* Chicago Lectures in Math., University of Chicago Press, Chicago, IL, 1994.

Prasit Bhattacharya pbhattac@nd.edu

Department of Mathematics, University of Notre Dame, Notre Dame, IN 46556 USA

Irina Bobkova ibobkova@math.tamu.edu

Department of Mathematics, Texas A&M University, College Station, TX 77843 USA

Brian Thomas bt3hy@virginia.edu

Department of Mathematics, University of Virginia, Kerchoff hall, Charlottesville, VA 22904 USA