*Homology, Homotopy and Applications, vol. 24(1), 2022, pp.55–70*

# THE HOMOTOPY TYPES OF $SU(n)$-GAUGE GROUPS OVER $S^{2m}$

SAJJAD MOHAMMADI

(*communicated by Charles A. Weibel*)

*Abstract*

Let $m$ and $n$ be two positive integers such that $m \leqslant n$. Denote by $P_{n,k}$ the principal $SU(n)$-bundle over $S^{2m}$ with Chern class $c_m(P_{n,k}) = (m-1)!k$ and let $\mathcal{G}_{k,m}(SU(n))$ be the gauge group of $P_{n,k}$ classified by $k\varepsilon'$, where $\varepsilon'$ is a generator of $\pi_{2m}(B(SU(n)) \cong \mathbb{Z}$. In this article we partially classify the homotopy types of $\mathcal{G}_{k,m}(SU(n))$, by showing that if there is a homotopy equivalence $\mathcal{G}_{k,m}(SU(n)) \simeq \mathcal{G}_{k',m}(SU(n))$ then in case $m$ is odd and $m \geqslant 3$, $(\frac{2}{(m-1)!}p_2, k) = (\frac{2}{(m-1)!}p_2, k')$ and in case $m$ is even and $m \geqslant 4$, $(\frac{1}{2(m-1)!}p_2, k) = (\frac{1}{2(m-1)!}p_2, k')$, where $p_2 = (n+2)(n+1)n(n-1)\cdots(n-m+2)$. We study the group $[\Sigma^{2n}\mathbb{C}P^{n-1}, SU(n)]$. Also we discuss the order of the Samelson product $S^{2m-1} \wedge \Sigma\mathbb{C}P^{n-1} \to SU(n)$ when $m < n$.

*In memory of Professor Mohammad Ali Asadi-Golmankhaneh.*

## 1. Introduction

Let $G$ be a compact connected Lie group and let $P \to B$ be a principal $G$-bundle over a connected finite $CW$-complex $B$. The gauge group $\mathcal{G}(P)$ of $P$ is the group of $G$-equivariant automorphisms of $P$ which fix $B$. Crabb and Sutherland [**3**] have shown that if $B$ and $G$ are as above, then the number of homotopy types amongst all the gauge groups of principal $G$-bundles over $B$ is finite. This is in spite of the fact that the number of isomorphism classes of principal $G$-bundles over $B$ is often infinite.

It has been a subject of recent interest to determine the precise number of homotopy types in special cases. Precise enumerations of the homotopy types have been made in the following cases:
$SU(2)$-bundles over $S^4$ [**9**]; $SU(3)$-bundles over $S^4$ [**6**]; $SU(5)$-bundles over $S^4$ when localized at any prime $p$ or rationally [**15**]; $Sp(2)$-bundles over $S^4$ when localized at any prime $p$ or rationally [**16**]; $Sp(3)$-bundles over $S^4$ when localised at an odd prime [**2**]; $G_2$-bundles over $S^4$ [**11**] and in a non-simply-connected case, $SO(3)$-bundles over $S^4$ [**10**]; $SU(3)$-bundles over $S^6$ [**7**] and in the general case, $SU(n)$-bundles over $S^6$ [**14**]; $Sp(2)$-bundles over $S^8$ [**8**]; $SU(4)$-bundles over $S^8$ [**13**].

Received December 8, 2020, revised January 17, 2021, February 15, 2021; published on March 30, 2022.
2010 Mathematics Subject Classification: Primary 55P15, Secondary 54C35.
Key words and phrases: gauge group, homotopy type, Lie group, homotopy equivalence.
Article available at `http://dx.doi.org/10.4310/HHA.2022.v24.n1.a3`

In [**14**], we gave a lower bound for the number of homotopy types of the gauge groups of principal $SU(n)$-bundles over $S^6$ and showed that if there is a homotopy equivalence $\mathcal{G}_k \simeq \mathcal{G}_{k'}$ then $((n-1)n(n+1)(n+2), k) = ((n-1)n(n+1)(n+2), k')$. In this paper, in general case, we study the classification of the homotopy types of the gauge groups of principal $SU(n)$-bundles over $S^{2m}$ and we give a lower bound for the number of homotopy types and do not prove the converse, which is realistically out of reach. The method is the same as in [**13**, **14**] but pushed to its limit, and so is technically more demanding. The earlier papers avoided doing this as they were equally concerned with proving the converse direction. For integers $a$ and $b$ let $(a, b)$ be the greatest common divisor of $|a|$ and $|b|$, also $p_1 = (n+1)n(n-1)\cdots(n-m+3)$ and $p_2 = (n+2)(n+1)n(n-1)\cdots(n-m+2)$. Let $P_{n,k}$ be the principal $SU(n)$-bundles over $S^{2m}$ with Chern class $c_m(P_{n,k}) = (m-1)!k$ and $\mathcal{G}_{k,m}(SU(n))$ be the gauge group of $P_{n,k}$ classified by $k\varepsilon'$, where $\varepsilon'$ is a generator of $\pi_{2m}(B(SU(n)) \cong \mathbb{Z}$. We prove the following theorems.

**Theorem 1.1.** *If there is a homotopy equivalence $\mathcal{G}_{k,m}(SU(n)) \simeq \mathcal{G}_{k',m}(SU(n))$ then the following hold:*
*(a) if $m$ is an odd integer and $m \geqslant 3$, then*

$$(\frac{2}{(m-1)!}p_2, k) = (\frac{2}{(m-1)!}p_2, k'),$$

*(b) if $m$ is an even integer and $m \geqslant 4$, then*

$$(\frac{1}{2(m-1)!}p_2, k) = (\frac{1}{2(m-1)!}p_2, k').$$

**Theorem 1.2.** *Let $d$ be the order of the Samelson product $S^{2m-1} \wedge \Sigma\mathbb{C}P^{n-1} \to SU(n)$. Then the following hold:*

$$d = \begin{cases} 5! & \text{if } n = 5, m = 2, \\ \frac{1}{4!}7! & \text{if } n = 6, m = 2. \end{cases}$$

**Remark 1.3.** The result of gauge groups does not depend on the parity of $n$, despite the proof needing to be broken into $n$ odd and even cases.

Theorem 1.1 recovers the known case in [**14**] when $m = 3$.

## 2. Preliminaries

Let $G$ be a compact connected Lie group and suppose that $P \to S^{2m}$ is a principal $G$-bundle classified by a map $f\colon S^{2m} \to BG$. Let $Map_f(S^{2m}, BG)$ be the component of the space of continuous unbased maps from $S^{2m}$ to $BG$ which contains the map $f$, similarly let $Map^*_f(S^{2m}, BG)$ be the space of pointed continuous maps from $S^{2m}$ and $BG$ which contains the map $f$. We know that there is a fibration

$$Map^*_f(S^{2m}, BG) \to Map_f(S^{2m}, BG) \xrightarrow{ev} BG,$$

where the map $ev$ is evaluation map at the basepoint of $S^{2m}$. Let $B\mathcal{G}_f$ be the classifying space of $\mathcal{G}_f$. By Atiyah–Bott [**1**], there is a homotopy equivalence

$$B\mathcal{G}_f \simeq Map_f(S^{2m}, BG).$$

The evaluation fibration therefore determines a homotopy fibration sequence

$$G \xrightarrow{\alpha} Map^*_f(S^{2m}, BG) \to B\mathcal{G}_f \xrightarrow{ev} BG.$$

It is well known that there is a homotopy equivalence $Map^*_f(S^{2m}, BG) \simeq Map^*_0(S^{2m}, BG)$, we write $\Omega^{2m-1}_0 G$ for $Map^*_0(S^{2m}, BG)$. Let $\varepsilon_{m,n}\colon S^{2m-1} \to SU(n)$ be the represents the generator of $\pi_{2m-1}(SU(n))$ and $1\colon SU(n) \to SU(n)$ is the identity map on $SU(n)$. For an $H$-space $X$, let $k\colon X \to X$ be the $k^{th}$-power map. By [**12**] we have the following lemma.

**Lemma 2.1.** *The adjoint of the connecting map* $SU(n) \xrightarrow{\alpha_k} \Omega^{2m-1}_0 SU(n)$ *is homotopic to the Samelson product* $S^{2m-1} \wedge SU(n) \xrightarrow{\langle k\varepsilon_{m,n}, 1\rangle} SU(n)$.

The linearity of the Samelson product implies that $\langle k\varepsilon_{m,n}, 1\rangle \simeq k\langle \varepsilon_{m,n}, 1\rangle$. Taking adjoints therefore implies the following.

**Corollary 2.2.** *The connecting map* $\alpha_k$ *satisfies* $\alpha_k \simeq k \circ \alpha_1$.

By adjunction, we have

$$[\Sigma^{2n-(2m-1)}\mathbb{C}P^2, \Omega^{2m-1}_0 SU(n)] \cong [\Sigma^{2n}\mathbb{C}P^2, SU(n)],$$

and applying the functor $[\Sigma^{2n-(2m-1)}\mathbb{C}P^2, \quad]$ to the map $\alpha_k$ we get the following map:

$$(\alpha_k)_*\colon [\Sigma^{2n-(2m-1)}\mathbb{C}P^2, SU(n)] \to [\Sigma^{2n}\mathbb{C}P^2, SU(n)]$$
$$a \mapsto \langle a, k\varepsilon_{m,n}\rangle.$$

The organization of this article is as follows. In Section 3, in cases where $n$ is an odd integer and $n \geqslant 3$ and $n$ is an even integer and $n \geqslant 4$, we first calculate $[\Sigma^{2n}\mathbb{C}P^2, U(n+1)]$ and then, regarding $[\Sigma^{2n}\mathbb{C}P^2, SU(n)]$ as a subgroup of $[\Sigma^{2n}\mathbb{C}P^2, U(n+1)]$, we study the group $[\Sigma^{2n}\mathbb{C}P^2, SU(n)]$. In Section 4 we compute the order of the cokernel of $\alpha_{k*}$ and prove Theorem 1.1. In Section 5, we study the group $[\Sigma^{2n}\mathbb{C}P^{n-1}, SU(n)]$. In Section 6, we study the order of the Samelson product $S^{2m-1} \wedge \Sigma\mathbb{C}P^{n-1} \to SU(n)$ when $m < n$ and prove Theorem 1.2.

## 3. The group $[\Sigma^{2n}\mathbb{C}P^2, SU(n)]$

Put $X = \Sigma^{2n}\mathbb{C}P^2 \simeq S^{2n+2} \cup_\eta e^{2n+4}$, where $\eta$ is the generator of $\pi_{2n+3}(S^{2n+2}) \cong \mathbb{Z}$. We denote the infinite Stiefel manifold $U(\infty)/U(n)$ by $W_n$ and $[X, U(n)]$ by $U_n(X)$. By applying $[X, \quad]$ to the fibration sequence

$$\Omega U(\infty) \xrightarrow{\Omega p} \Omega W_n \xrightarrow{\delta} U(n) \xrightarrow{j} U(\infty) \xrightarrow{p} W_n,$$

we obtain the exact sequence

$$\tilde{K}^0(X) \xrightarrow{(\Omega p)_*} [X, \Omega W_n] \xrightarrow{\delta_*} U_n(X) \xrightarrow{j_*} \tilde{K}^1(X) \xrightarrow{p_*} [X, W_n],$$

where $\tilde{K}^0(X) \cong [X, \Omega U(\infty)]$ and $\tilde{K}^1(X) \cong [X, U(\infty)]$. Since $\tilde{K}^1(X) = 0$ thus we get the following exact sequence:

$$\tilde{K}^0(X) \xrightarrow{(\Omega p)_*} [X, \Omega W_n] \xrightarrow{\delta_*} U_n(X) \to 0.$$

Therefore for group $U(n+1)$ we get the following lemma.

**Lemma 3.1.** *The group* $[X, U(n+1)]$ *is isomorphic to* $\operatorname{Coker}(\Omega p)_*$.

It is well known that the cohomology of $BU(\infty)$ as an algebra is given by

$$H^*(BU(\infty)) = \mathbb{Z}[c_1, c_2, \ldots],$$

where $c_i$ is the universal $i$-th Chern class and

$$H^*(U(\infty)) = \Lambda(x_1, x_3, \ldots), \quad x_{2i-1} = \sigma(c_i),$$

where $\sigma$ is the cohomology suspension. The cohomology of $W_n$ is given by

$$H^*(W_n) = \Lambda(\bar{x}_{2n+1}, \bar{x}_{2n+3}, \ldots), \quad p^*(\bar{x}_{2i-1}) = x_{2i-1} \in H^*(U(\infty)).$$

We know that when $n$ is even then

$$W_{n+1} \simeq S^{2n+3} \cup_{\eta'} e^{2n+5} \cup e^{2n+7} \cup e^{2n+9} \cup \cdots,$$

where $\eta'$ is the generator of $\pi_{2n+4}(S^{2n+3})$ and

$$\Omega W_{n+1} \simeq S^{2n+2} \cup e^{2n+4} \cup e^{2n+6} \cup e^{2n+8} \cup \cdots.$$

Note that when $n$ is odd then

$$W_{n+1} \simeq (S^{2n+3} \vee S^{2n+5}) \cup e^{2n+7} \cup e^{2n+9} \cup \cdots.$$

Let $a_{2n+2}$ and $a_{2n+4}$ be generators of $H^{2n+2}(\Omega W_{n+1}) \cong \mathbb{Z}$ and $H^{2n+4}(\Omega W_{n+1}) \cong \mathbb{Z}$ respectively and $\alpha \in [X, \Omega W_{n+1}]$. We define a homomorphism

$$\lambda\colon [X, \Omega W_{n+1}] \to H^{2n+2}(X) \oplus H^{2n+4}(X)$$

by $\lambda(\alpha) = (\alpha^*(a_{2n+2}), \alpha^*(a_{2n+4}))$.

**Case one:** If $n$ is an odd integer and $n \geqslant 3$.

Since $W_{n+1}$ is $(2n+2)$-connected, for $i \leqslant 2n+2$ we have $\pi_i(W_{n+1}) = 0$. By the homotopy sequence of the fibration $U(n+1) \xrightarrow{j} U(\infty) \xrightarrow{p} W_{n+1}$ we have

$$\pi_{2n+3}(W_{n+1}) \cong \mathbb{Z}, \qquad \pi_{2n+4}(W_{n+1}) \cong \begin{cases} 0 & \text{if } n \text{ is even,} \\ \mathbb{Z}/2\mathbb{Z} & \text{if } n \text{ is odd.} \end{cases}$$

As in [**14**, Lemma 3.2], we have the following lemma.

**Lemma 3.2.** *The map of* $\lambda\colon [X, \Omega W_{n+1}] \to H^{2n+2}(X) \oplus H^{2n+4}(X)$ *is monic and* $\operatorname{Im}\lambda = \{(a,b) \mid a \equiv 0 \bmod 2\}$.

Put $u_1 = (2, n+2)$ and $u_2 = (0,1)$, then $u_1, u_2 \in \operatorname{Im}\lambda$ and generate $\operatorname{Im}\lambda$. The following lemma was proved in [**14**, Lemma 3.3].

**Lemma 3.3.** $\operatorname{Im}\lambda \circ (\Omega p)_*$ *is generated by* $\frac{1}{2}(n+1)!u_1$ *and* $(n+2)!u_2$.

Therefore according to the Lemmas 3.1 and 3.3 we get the following theorem.

**Theorem 3.4.** *The group* $[X, U(n+1)]$ *is isomorphic to* $\mathbb{Z}_{\frac{1}{2}(n+1)!} \oplus \mathbb{Z}_{(n+2)!}$.

Since we can consider the group $[X, SU(n)]$ as a subgroup of $[X, U(n+1)]$, we will study the group $[X, SU(n)]$.

Consider the map $\alpha_k\colon SU(n) \to \Omega_0^{2m-1} SU(n)$. By adjunction we have

$$[\Sigma^{2n-2m+1}\mathbb{C}P^2, \Omega_0^{2m-1} SU(n)] \cong [\Sigma^{2n}\mathbb{C}P^2, SU(n)].$$

Applying the functor $[\Sigma^{2n-2m+1}\mathbb{C}P^2, \quad]$ to the map $\alpha_k$ gives

$$(\alpha_k)_*\colon [\Sigma^{2n-2m+1}\mathbb{C}P^2, SU(n)] \to [\Sigma^{2n}\mathbb{C}P^2, SU(n)]$$
$$a \mapsto \langle a, k\varepsilon_{m,n}\rangle.$$

Recall that $H^*(\mathbb{C}P^2) = \mathbb{Z}[t]/(t^3)$, where $|t| = 2$ and $K(\mathbb{C}P^2) = \mathbb{Z}[x]/(x^3)$. Let $\zeta_n$ a generator of $\tilde{K}^0(S^{2n})$. According to the map $\alpha_{k*}$ we have

$$[\Sigma^{2n-2m+1}\mathbb{C}P^2, SU(n)] = \tilde{K}^0(\Sigma^{2(n-m)+2}\mathbb{C}P^2) = \mathbb{Z}\langle \zeta_{n-m+1}\hat{\otimes} x, \zeta_{n-m+1}\hat{\otimes} x^2\rangle,$$

where $\mathbb{Z}\langle a, b\rangle$ denote the free abelian group generated by $a$ and $b$. Let $H^1_{m,n}$ be the subgroup of $[X, U(n+1)]$ generated by $i \circ \langle \ell_1, \varepsilon_{m,n}\rangle$ and $i \circ \langle \ell_2, \varepsilon_{m,n}\rangle$, where $i\colon SU(n) \to U(n+1)$ is the inclusion, $\ell_1$ is the adjoint of $\zeta_{n-m+1} \otimes x$ and $\ell_2$ is the adjoint of $\zeta_{n-m+1} \otimes x^2$. Let $\gamma$ be the commutator map $U(n+1) \wedge U(n+1) \to U(n+1)$. Since $U(\infty)$ is an infinite loop space it is homotopy commutative. Therefore $\gamma$ composed to $U(\infty)$ is null homotopic, implying that there is a lift $\tilde{\gamma}\colon U(n+1) \wedge U(n+1) \to \Omega W_{n+1}$ such that $\delta \circ \tilde{\gamma} \simeq \gamma$.

Let $H^2_{m,n}$ be the subgroup of $[X, \Omega W_{n+1}]$ generated by $\tilde{\gamma}(i \circ \ell_1 \wedge i \circ \varepsilon_{m,n})$ and $\tilde{\gamma}(i \circ \ell_2 \wedge i \circ \varepsilon_{m,n})$. Then by Lemma 3.1 we have $H^1_{m,n} \cong H^2_{m,n}/(\mathrm{Im}\,(\Omega p)_* \cap H^2_{m,n})$. By using the method in [**5**] we have

$$\tilde{\gamma}^*(a_{2n+2}) = \sum_{i+j=n} x_{2i+1} \otimes x_{2j+1}, \qquad \tilde{\gamma}^*(a_{2n+4}) = \sum_{i+j=n+1} x_{2i+1} \otimes x_{2j+1}.$$

Note that

$$\begin{aligned}
c_{n-m+2}(\zeta_{n-m+1} \otimes x) &= (n-m+1)!\sigma^{2n-2m+2}t,\\
c_{n-m+3}(\zeta_{n-m+1} \otimes x) &= \frac{1}{2}(n-m+2)!\sigma^{2n-2m+2}t^2,\\
c_{n-m+2}(\zeta_{n-m+1} \otimes x^2) &= 0,\\
c_{n-m+3}(\zeta_{n-m+1} \otimes x^2) &= (n-m+2)!\sigma^{2n-2m+2}t^2.
\end{aligned}$$

Let $s$ be a generator of $H^{2m-1}(S^{2m-1})$. We have

$$\begin{aligned}
(\tilde{\gamma}(i \circ \ell_1 \wedge i \circ \varepsilon_{m,n}))^*(a_{2n+2}) &= (i \circ \ell_1 \wedge i \circ \varepsilon_{m,n})^*(\tilde{\gamma})^*(a_{2n+2})\\
&= (i \circ \ell_1 \wedge i \circ \varepsilon_{m,n})^*(x_{2m-1} \otimes x_{2n-2m+3})\\
&= (m-1)!s(n-m+1)!\sigma^{2n-2m+2}t,
\end{aligned}$$

and also

$$\begin{aligned}
(\tilde{\gamma}(i \circ \ell_1 \wedge i \circ \varepsilon_{m,n}))^*(a_{2n+4}) &= (i \circ \ell_1 \wedge i \circ \varepsilon_{m,n})^*(\tilde{\gamma})^*(a_{2n+4})\\
&= (i \circ \ell_1 \wedge i \circ \varepsilon_{m,n})^*(x_{2m-1} \otimes x_{2n-2m+5}))\\
&= \frac{1}{2}(m-1)!s(n-m+2)!\sigma^{2n-2m+2}t^2.
\end{aligned}$$

Therefore according to the map of $\lambda$ we have

$$\lambda(\tilde{\gamma}(i \circ \ell_1 \wedge i \circ \varepsilon_{m,n})) = ((m-1)!(n-m+1)!, \frac{1}{2}(m-1)!(n-m+2)!).$$

Similarly we can show that

$$\lambda(\tilde{\gamma}(i\circ\ell_2\wedge i\circ\varepsilon_{m,n}))=(0,(m-1)!(n-m+2)!).$$

Therefore $H^2_{m,n}$ is generated by

$$\begin{aligned}\alpha&=((m-1)!(n-m+1)!,\frac{1}{2}(m-1)!(n-m+2)!),\\ \beta&=(0,(m-1)!(n-m+2)!).\end{aligned}$$

We recall $p_1=(n+1)n(n-1)\cdots(n-m+3)$ and $p_2=(n+2)(n+1)n(n-1)\cdots(n-m+2)$, we have the following proposition.

**Proposition 3.5.** *Let $n$ be odd and $n\geqslant 3$. Then the following hold:*
*(a) if $m$ is odd and $m\geqslant 3$, then there is a isomorphism*

$$H^1_{m,n}\cong\mathbb{Z}_{\frac{1}{2(m-1)!}p_1}\oplus\mathbb{Z}_{\frac{2}{(m-1)!}p_2},$$

*(b) if $m$ is even and $m\geqslant 4$, then there is a isomorphism*

$$H^1_{m,n}\cong\mathbb{Z}_{\frac{2}{(m-1)!}p_1}\oplus\mathbb{Z}_{\frac{1}{2(m-1)!}p_2}.$$

*Proof.* For part $(a)$, since $\begin{vmatrix}2(n-m+2) & m\\ 0 & \frac{1}{2(n-m+2)}\end{vmatrix}=1$, the subgroup $H^2_{m,n}$ is also generated by $2(n-m+2)\alpha+m\beta=(m-1)!(n-m+2)!u_1$ and $\frac{1}{2(n-m+2)}\beta=\frac{1}{2}(m-1)!(n-m+1)!u_2$. Thus

$$\begin{aligned}H^1_{m,n}&\cong\frac{\langle (m-1)!(n-m+2)!u_1,\frac{1}{2}(m-1)!(n-m+1)!u_2\rangle}{\langle\frac{1}{2}(n+1)!u_1,(n+2)!u_2\rangle}\\ &\cong\mathbb{Z}_{\frac{1}{2(m-1)!}p_1}\oplus\mathbb{Z}_{\frac{2}{(m-1)!}p_2}.\end{aligned}$$

For part $(b)$, since $\begin{vmatrix}\frac{1}{2}(n-m+2) & \frac{1}{4}m\\ 0 & \frac{2}{(n-m+2)}\end{vmatrix}=1$, the subgroup $H^2_{m,n}$ is also generated by $\frac{1}{2}(n-m+2)\alpha+\frac{1}{4}m\beta=\frac{1}{4}(m-1)!(n-m+2)!u_1$ and $\frac{2}{(n-m+2)}\beta=2(m-1)!(n-m+1)!u_2$. Thus

$$H^1_{m,n}\cong\mathbb{Z}_{\frac{2}{(m-1)!}p_1}\oplus\mathbb{Z}_{\frac{1}{2(m-1)!}p_2}.$$ □

**Case two:** If $n$ is an even integer and $n\geqslant 4$.
Similarly we define a homomorphism

$$\lambda'\colon[X,\Omega W_{n+1}]\to H^{2n+2}(X)\oplus H^{2n+4}(X),$$

by $\lambda'(\alpha)=(\alpha^*(a_{2n+2}),\alpha^*(a_{2n+4}))$. As in [**6**, Lemma 2.1], we have the following lemma.

**Lemma 3.6.** *The map of $\lambda'\colon[X,\Omega W_{n+1}]\to H^{2n+2}(X)\oplus H^{2n+4}(X)$ is monic and* $\operatorname{Im}\lambda'=\{(a,b)\mid a\equiv b \bmod 2\}$.

The following lemma was proved in [**6**, Lemma 2.2].

**Lemma 3.7.** $\operatorname{Im}\lambda'\circ(\Omega\pi)_*$ *is generated by* $((n+1)!,\frac{1}{2}(n+2)!)$ *and* $(0,(n+2)!)$.

We have the following theorem that was proved in [**6**, Theorem 2.4].

**Theorem 3.8.** *The following holds.*

*(a) if* $n+1 \equiv 1 \bmod 4$, *then* $[X, U(n+1)] \cong \mathbb{Z}/\frac{1}{2}(n+2)! \oplus \mathbb{Z}/(n+1)!$,

*(b) if* $n+1 \equiv 3 \bmod 4$, *then* $[X, U(n+1)] \cong \mathbb{Z}/(n+2)! \oplus \mathbb{Z}/\frac{1}{2}(n+1)!$.

In what follows we study the group $[X, SU(n)]$.

If $n+1 \equiv 1 \,(\mathrm{mod}\ 4)$, we put $u_1 = (0,2)$ and $u_2 = (1, \frac{1}{2}(n+2))$. Then $u_1, u_2 \in \operatorname{Im} \lambda'$ and are generators of $\operatorname{Im} \lambda'$. Similarly let ${H'}^1_{m,n}$ be the subgroup of $[X, U(n+1)]$ generated by $i \circ \langle \ell_1, \varepsilon_{m,n} \rangle$ and $i \circ \langle \ell_2, \varepsilon_{m,n} \rangle$. Let ${H'}^2_{m,n}$ be the subgroup generated by $\tilde{\gamma}(i \circ \ell_1 \wedge i \circ \varepsilon_{m,n})$ and $\tilde{\gamma}(i \circ \ell_2 \wedge i \circ \varepsilon_{m,n})$. Arguing as for Proposition 3.5, by definition of $\lambda'$

$$\lambda'(\tilde{\gamma}(i \circ \ell_1 \wedge i \circ \varepsilon_{m,n})) = ((m-1)!(n-m+1)!, \tfrac{1}{2}(m-1)!(n-m+2)!),$$

$$\lambda'(\tilde{\gamma}(i \circ \ell_2 \wedge i \circ \varepsilon_{m,n})) = (0, (m-1)!(n-m+2)!).$$

Therefore ${H'}^2_{m,n}$ generated by

$$\alpha = ((m-1)!(n-m+1)!, \frac{1}{2}(m-1)!(n-m+2)!),$$
$$\beta = (0, (m-1)!(n-m+2)!).$$

We have the following proposition.

**Proposition 3.9.** *Let $n$ be even and $n \geqslant 4$. Then the following hold:*
*$(a)$ if $m$ is odd and $m \geqslant 3$, then there is a isomorphism*

$$ {H'}^1_{m,n} \cong \mathbb{Z}_{\frac{2}{(m-1)!}p_2} \oplus \mathbb{Z}_{\frac{1}{2(m-1)!}p_1},$$

*$(b)$ if $m$ is even and $m \geqslant 4$, then there is a isomorphism*

$$ {H'}^1_{m,n} \cong \mathbb{Z}_{\frac{1}{2(m-1)!}p_2} \oplus \mathbb{Z}_{\frac{2}{(m-1)!}p_1}.$$

*Proof.* For part $(a)$, since $\begin{vmatrix} 2(n-m+2) & m \\ 0 & \frac{1}{2(n-m+2)} \end{vmatrix} = 1$, the subgroup ${H'}^2_{m,n}$ is also generated by $2(n-m+2)\alpha + m\beta = 2(m-1)!(n-m+2)!u_2$ and $\frac{1}{2(n-m+2)}\beta = \frac{1}{4}(m-1)!(n-m+1)!u_1$. Thus

$$\begin{aligned} {H'}^1_{m,n} &\cong \frac{\langle \frac{1}{4}(m-1)!(n-m+1)!u_1, 2(m-1)!(n-m+2)!u_2 \rangle}{\langle \frac{1}{2}(n+2)!u_1, (n+1)!u_2 \rangle} \\ &\cong \mathbb{Z}_{\frac{2}{(m-1)!}p_2} \oplus \mathbb{Z}_{\frac{1}{2(m-1)!}p_1}. \end{aligned}$$

For part $(b)$, since $\begin{vmatrix} \frac{1}{2}(n-m+2) & \frac{1}{4}m \\ 0 & \frac{2}{(n-m+2)} \end{vmatrix} = 1$, the subgroup ${H'}^2_{m,n}$ is also generated by $\frac{1}{2}(n-m+2)\alpha + \frac{1}{4}m\beta = \frac{1}{2}(m-1)!(n-m+2)!u_2$ and $\frac{2}{(n-m+2)}\beta = (m-1)!(n-m+1)!u_1$. Thus

$$ {H'}^1_{m,n} \cong \mathbb{Z}_{\frac{1}{2(m-1)!}p_2} \oplus \mathbb{Z}_{\frac{2}{(m-1)!}p_1}. \qquad \square$$

Similarly when $n+1 \equiv 3 \pmod 4$, we can show that ${H'}^2_{m,n}$ is generated by $\alpha = ((m-1)!(n-m+1)!, \frac{1}{2}(m-1)!(n-m+2)!)$ and $\beta = (0, (m-1)!(n-m+2)!)$, therefore if $m$ is odd and $m \geqslant 3$, then ${H'}^1_{m,n}$ is also isomorphic to

$$\mathbb{Z}_{\frac{2}{(m-1)!}p_2} \oplus \mathbb{Z}_{\frac{1}{2(m-1)!}p_1},$$

and if $m$ is even and $m \geqslant 4$, then ${H'}^1_{m,n}$ is also isomorphic to

$$\mathbb{Z}_{\frac{1}{2(m-1)!}p_2} \oplus \mathbb{Z}_{\frac{2}{(m-1)!}p_1}.$$

The following lemma was proved in [**14**, Lemma 3.10].

**Lemma 3.10.** *The map* $i_*\colon [X, SU(n)] \to [X, U(n+1)]$ *is a monomorphism.*

Let $J_{m,n}$ be the subgroup of $[X, SU(n)]$ generated by $\langle \ell_1, \varepsilon_{m,n}\rangle$ and $\langle \ell_2, \varepsilon_{m,n}\rangle$, then we have the following theorem.

**Theorem 3.11.** *There is an isomorphism*

$$J_{m,n} \cong \begin{cases} \mathbb{Z}_{\frac{1}{2(m-1)!}p_1} \oplus \mathbb{Z}_{\frac{2}{(m-1)!}p_2} & \text{if } m \text{ is an odd and } m \geqslant 3,\\ \mathbb{Z}_{\frac{2}{(m-1)!}p_1} \oplus \mathbb{Z}_{\frac{1}{2(m-1)!}p_2} & \text{if } m \text{ is an even and } m \geqslant 4,\end{cases}$$

*where* $\mathbb{Z}_{\frac{1}{2(m-1)!}p_1}$ *is generated by* $\langle 2(n-m+2)\ell_1 + m\ell_2, \varepsilon_{m,n}\rangle$ *and* $\mathbb{Z}_{\frac{2}{(m-1)!}p_2}$ *is generated by* $\langle \frac{1}{2(n-m+2)}\ell_2, \varepsilon_{m,n}\rangle$*. Also* $\mathbb{Z}_{\frac{2}{(m-1)!}p_1}$ *is generated by* $\langle \frac{1}{2}(n-m+2)\ell_1 + \frac{1}{4}m\ell_2, \varepsilon_{m,n}\rangle$ *and* $\mathbb{Z}_{\frac{1}{2(m-1)!}p_2}$ *is generated by* $\langle \frac{2}{(n-m+2)}\ell_2, \varepsilon_{m,n}\rangle$.

*Proof.* By definition of $J_{m,n}$ and $H^1_{m,n}$ we have $i_*(J_{m,n}) = H^1_{m,n}$. By Lemma 3.10, the map $i_*$ is a monomorphism so $i_*$ send $J_{m,n}$ isomorphically onto $H^1_{m,n}$. When $n$ is odd the statement follows from Proposition 3.5 and when $n$ is even the statement follows from Proposition 3.9. □

## 4. Proof of Theorem 1.1

Consider the homotopy fibration sequence

$$\mathcal{G}_{k,m}(SU(n)) \to SU(n) \xrightarrow{\alpha_k} \Omega_0^{2m-1}SU(n) \to B\mathcal{G}_{k,m}(SU(n)) \xrightarrow{ev} BSU(n).$$

Applying the functor $[\Sigma^{2n-2m+1}\mathbb{C}P^2, \quad]$, we get the following exact sequence:

$$\begin{aligned}[\Sigma^{2n-2m+1}\mathbb{C}P^2, \mathcal{G}_{k,m}(SU(n))] &\overset{(\Omega ev)_*}{\longrightarrow} [\Sigma^{2n-2m+1}\mathbb{C}P^2, SU(n)]\\ &\overset{(\alpha_k)_*}{\longrightarrow} [X, SU(n)]\\ &\longrightarrow [\Sigma^{2n-2m+1}\mathbb{C}P^2, B\mathcal{G}_{k,m}(SU(n))]\\ &\longrightarrow [\Sigma^{2n-2m+1}\mathbb{C}P^2, BSU(n)].\end{aligned}$$

Since $[\Sigma^{2n-2m+1}\mathbb{C}P^2,\, BSU(n)] \cong \tilde{K}^0(\Sigma^{2n-2m+1}\mathbb{C}P^2) = 0$, exactness implies that

$$[\Sigma^{2n-2m+1}\mathbb{C}P^2, B\mathcal{G}_{k,m}(SU(n))] \cong \operatorname{Coker}(\alpha_k)_*.$$

We know that $[\Sigma^{2n-2m+1}\mathbb{C}P^2, SU(n)]$ is generated by $\ell_1$ and $\ell_2$. Equivalently, if $m$ is an odd and $m \geqslant 3$ then $[\Sigma^{2n-2m+1}\mathbb{C}P^2, SU(n)]$ is generated by $2(n-m+2)\ell_1 + m\ell_2$ and $\frac{1}{2(n-m+2)}\ell_2$ and if $m$ is an even and $m \geqslant 4$, is generated by $\frac{1}{2}(n-m+2)\ell_1 + \frac{1}{4}m\ell_2$ and $\frac{2}{(n-m+2)}\ell_2$. By definitions of $\alpha_k$ and $J_{m,n}$, the image of $(\alpha_k)_*$ is $J_{m,n}$. Write

$|G|$ for the order of a group $G$. If $A$ is the order of $[X, SU(n)]$, then by exactness of the sequence we have

$$\big|[\Sigma^{2n-7}\mathbb{C}P^2, B\mathcal{G}_{k,m}(SU(n))]\big| = \big|\operatorname{Coker}(\alpha_k)_*\big|$$
$$= \begin{cases} \dfrac{A}{(\frac{1}{2(m-1)!}p_1, k)(\frac{2}{(m-1)!}p_2, k)} & \text{if } m \text{ is an odd and } m \geqslant 3, \\ \dfrac{A}{(\frac{2}{(m-1)!}p_1, k)(\frac{1}{2(m-1)!}p_2, k)} & \text{if } m \text{ is an even and } m \geqslant 4. \end{cases}$$

Now suppose that $\mathcal{G}_{k,m}(SU(n)) \simeq \mathcal{G}_{k',m}(SU(n))$. Then there is an isomorphism of groups

$$[\Sigma^{2n-2m+1}\mathbb{C}P^2, B\mathcal{G}_{k,m}(SU(n))] \cong [\Sigma^{2n-2m+1}\mathbb{C}P^2, B\mathcal{G}_{k',m}(SU(n))],$$

thus $\big|[\Sigma^{2n-2m+1}\mathbb{C}P^2, B\mathcal{G}_{k,m}]\big| = \big|[\Sigma^{2n-2m+1}\mathbb{C}P^2, B\mathcal{G}_{k',m}]\big|$. That is, if $m$ is an odd and $m \geqslant 3$ then

$$\frac{A}{(\frac{1}{2(m-1)!}p_1, k)(\frac{2}{(m-1)!}p_2, k)} = \frac{A}{(\frac{1}{2(m-1)!}p_1, k')(\frac{2}{(m-1)!}p_2, k')},$$

and if $m$ is an even and $m \geqslant 4$ then

$$\frac{A}{(\frac{2}{(m-1)!}p_1, k)(\frac{1}{2(m-1)!}p_2, k)} = \frac{A}{(\frac{2}{(m-1)!}p_1, k')(\frac{1}{2(m-1)!}p_2, k')}.$$

Therefore if $\mathcal{G}_{k,m}(SU(n)) \simeq \mathcal{G}_{k',m}(SU(n))$ in case $m$ is an odd and $m \geqslant 3$, we get the equation

$$(\frac{1}{2(m-1)!}p_1, k)(\frac{2}{(m-1)!}p_2, k) = (\frac{1}{2(m-1)!}p_1, k')(\frac{2}{(m-1)!}p_2, k'),$$

note that $p_2 = (n+2)p_1(n-m+2)$. We need to show that $(\frac{2}{(m-1)!}p_2, k) = (\frac{2}{(m-1)!}p_2, k')$. It suffices to prove it after $p$-localization for all primes $p$. For any number $m$ and prime $p$, the $p$-component of $m$ is the power $p^r$ such that $p^r$ divides $m$ but $p^{r+1}$ does not. So we may assume $k = p^r$ and $k' = p^s$ for some non-negative numbers $r$ and $s$. Denote the $p$-components of $p_1/2(m-1)!$ and $2p_2/(m-1)!$ by $p^a$ and $p^b$ respectively. Then the equation (5) says the following relation

$$\min\{p^a, p^r\} \times \min\{p^b, p^r\} = \min\{p^a, p^s\} \times \min\{p^b, p^s\}. \qquad (*)$$

We need to show that $\min\{p^b, p^r\} = \min\{p^b, p^s\}$ $(**)$. Relabeling if necessary, we may assume $a \leqslant b$ and $r \leqslant s$. Then there are 6 cases:

1. $a \leqslant b \leqslant r \leqslant s$,
2. $a \leqslant r \leqslant b \leqslant s$,
3. $a \leqslant r \leqslant s \leqslant b$,
4. $r \leqslant a \leqslant b \leqslant s$,
5. $r \leqslant a \leqslant s \leqslant b$,
6. $r \leqslant s \leqslant a \leqslant b$.

Except for cases 4 and 5, the relation $(*)$ implies $(**)$ directly. For case 4, the relation $(*)$ implies $2r = a + b$. It follows that $r = a = b$, otherwise $2r < a + b = 2r$ leading to contradiction. So $\min\{p^b, p^r\} = p^b = \min\{p^b, p^s\}$. For case 5 we can use a similar argument to show $(**)$. Similarly in case $m$ is an even and $m \geqslant 4$, we can conclude that $(\frac{1}{2(m-1)!}p_2, k) = (\frac{1}{2(m-1)!}p_2, k')$.

## 5. Calculation of $[\Sigma^{2n}\mathbb{C}P^{n-1}, SU(n)]$

In this section, we will study the group $[\Sigma^{2n}\mathbb{C}P^{n-1}, SU(n)]$. Consider the fibre sequence

$$\Omega U(\infty) \xrightarrow{\Omega\pi} \Omega W_{n+1} \xrightarrow{\delta} U(n+1) \xrightarrow{j} U(\infty) \xrightarrow{\pi} W_{n+1}.$$

Applying the functor $[X = \Sigma^{2n}\mathbb{C}P^{n-1}, ]$ to (6), there is an exact sequence of groups:

$$[X, \Omega U(\infty)] \xrightarrow{(\Omega\pi)_*} [X, \Omega W_{n+1}] \xrightarrow{\delta_*} U_{n+1}(X) \xrightarrow{j_*} [X, U(\infty)] \xrightarrow{\pi_*} [X, W_{n+1}].$$

Since $\Sigma^{2n+1}\mathbb{C}P^{n-1}$ is a CW-complex consisting only of odd dimensional cells, therefore we have

$$[\Sigma^{2n}\mathbb{C}P^{n-1}, U(\infty)] \cong [\Sigma^{2n+1}\mathbb{C}P^{n-1}, BU(\infty)] \cong \tilde{K}^0(\Sigma^{2n+1}\mathbb{C}P^{n-1}) \cong 0,$$

therefore $[X, U(\infty)] = 0$ and we get the following exact sequence

$$\tilde{K}^0(X) \xrightarrow{(\Omega\pi)_*} [X, \Omega W_{n+1}] \xrightarrow{\delta_*} U_{n+1}(X) \to 0.$$

Therefore we have the following lemma.

**Lemma 5.1.** *$U_{n+1}(X) \cong \mathrm{Coker}(\Omega\pi)_*$.*

Define a homomorphism

$$\lambda\colon [X, \Omega W_{n+1}] \to H^{2n+2}(X) \oplus H^{2n+4}(X) \oplus \cdots \oplus H^{4n-2}(X)$$

by $\lambda(\alpha) = (\alpha^*(a_{2n+2}), \alpha^*(a_{2n+4}), \dots, \alpha^*(a_{4n-2}))$, where $\alpha \in [X, \Omega W_{n+1}]$ and $a_{2n+2}$, $a_{2n+4}, \dots, a_{4n-2}$ are generators of $H^{2n+2}(\Omega W_{n+1}) \cong H^{2n+4}(\Omega W_{n+1}) \cong \cdots \cong H^{4n-2}(\Omega W_{n+1}) \cong \mathbb{Z}$, respectively.

Recall that $H^*(\mathbb{C}P^{n-1}) = \mathbb{Z}[t]/(t^n)$, where $|t| = n-1$ and $K(\mathbb{C}P^{n-1}) = \mathbb{Z}[x]/(x^n)$. Note that $\tilde{K}^0(X = \Sigma^{2n}\mathbb{C}P^{n-1}) \cong \tilde{K}^0(\mathbb{C}P^{n-1})$ is a free abelian group generated by $\zeta_n\hat{\otimes}x, \zeta_n\hat{\otimes}x^2, \dots, \zeta_n\hat{\otimes}x^{n-1}$, where $\zeta_n$ a generator of $\tilde{K}^0(S^{2n})$. We have

$$\begin{aligned}
&\mathrm{ch}\, x = t + \frac{1}{2!}t^2 + \frac{1}{3!}t^3 + \cdots + \frac{1}{(n-1)!}t^{n-1},\\
&\mathrm{ch}\, x^2 = t^2 + \frac{1}{2!}t^3 + \cdots + A,\\
&\vdots\\
&\mathrm{ch}\, x^{n-1} = B,
\end{aligned}$$

where

$$A = \mathrm{ch}_{n-1}(x^2) = \sum_{\substack{i+j=n-1,\\ 1\leqslant i\leqslant[\frac{n-1}{2}]}} \mathrm{ch}_i\, x\, \mathrm{ch}_j\, x = \sum_{k=1}^{[\frac{n-1}{2}]} \frac{1}{k!(n-k-1)!}t^{n-1} = A't^{n-1},$$

$$
\begin{aligned}
\mathrm{ch}_n(x^n) = {} & \mathrm{ch}_1 x \sum_{\substack{i_1+\cdots+i_{n-1}=n-1,\\ 0\leqslant i_1\leqslant i_2\leqslant\cdots\leqslant i_{n-1}}} \mathrm{ch}_{i_1} x^{i_1}\cdots \mathrm{ch}_{i_{n-1}} x^{i_{n-1}} \\
& + \mathrm{ch}_2 x^2 \sum_{\substack{i_1+\cdots+i_k=n-2, k=[\frac{n-2}{2}],\\ 2\leqslant i_1\leqslant i_2\leqslant\cdots\leqslant i_k}} \mathrm{ch}_{i_1} x^{i_1}\cdots \mathrm{ch}_{i_k} x^{i_k} \\
& + \mathrm{ch}_3 x^3 \sum_{\substack{i_1+\cdots+i_k=n-3, k=[\frac{n-3}{3}],\\ 3\leqslant i_1\leqslant i_2\leqslant\cdots\leqslant i_k}} \mathrm{ch}_{i_1} x^{i_1}\cdots \mathrm{ch}_{i_k} x^{i_k} + \cdots \\
& + \mathrm{ch}_k x^k \sum_{i_1=n-k, k=[\frac{n}{2}]} \mathrm{ch}_{i_1} x^{i_1},
\end{aligned}
$$

and $B = \mathrm{ch}_{n-1}(x^{n-1}) = B' t^{n-1}$. We have the following lemma.

**Lemma 5.2.** $\mathrm{Im}\,\lambda\circ(\Omega\pi)_*$ *is generated by* $\alpha_1, \alpha_2, \ldots, \alpha_{n-1}$ *where*

$$
\begin{aligned}
\alpha_1 &= ((n+1)!, \frac{1}{2}(n+2)!, \ldots, \frac{1}{(n-1)!}(2n-1)!), \\
\alpha_2 &= (0, (n+2)!, \frac{1}{2}(n+3)!, \ldots, (2n-1)!A'), \\
&\vdots \\
\alpha_{n-1} &= (0, 0, 0, \ldots, (2n-1)!B').
\end{aligned}
$$

*Proof.* According to the definition of the map $\lambda$, we have

$$
\begin{aligned}
\lambda\circ(\Omega\pi)_*(\zeta_n\hat{\otimes}x) = ((\Omega\pi\circ(\zeta_n\hat{\otimes}x))^*(a_{2n+2}), (\Omega\pi\circ(\zeta_n\hat{\otimes}x))^*(a_{2n+4}), \ldots,& \\
(\Omega\pi\circ(\zeta_n\hat{\otimes}x))^*(a_{4n-2})).&
\end{aligned}
$$

The calculation of the first component is as follows

$$
\begin{aligned}
(\Omega\pi\circ(\zeta_n\hat{\otimes}x))^*(a_{2n+2}) &= a_{2n+2}\circ\Omega\pi(\zeta_n\hat{\otimes}x) \\
&= (n+1)!\,\mathrm{ch}_{n+1}(\zeta_n\hat{\otimes}x) \\
&= (n+1)!\sigma^{2n}t,
\end{aligned}
$$

the calculation the second component is as follows

$$
\begin{aligned}
(\Omega\pi\circ(\zeta_n\hat{\otimes}x))^*(a_{2n+4}) &= a_{2n+4}\circ\Omega\pi(\zeta_n\hat{\otimes}x) \\
&= (n+2)!\,\mathrm{ch}_{n+2}(\zeta_n\hat{\otimes}x) \\
&= \frac{1}{2}(n+2)!\sigma^{2n}t^2
\end{aligned}
$$

and the calculation the last component is as follows

$$
\begin{aligned}
(\Omega\pi\circ(\zeta_n\hat{\otimes}x))^*(a_{4n-2}) &= a_{4n-2}\circ\Omega\pi(\zeta_n\hat{\otimes}x) \\
&= (2n-1)!\,\mathrm{ch}_{2n-1}(\zeta_n\hat{\otimes}x) \\
&= \frac{1}{(n-1)!}(2n-1)!\sigma^{2n}t^{n-1}.
\end{aligned}
$$

Therefore

$$\lambda \circ (\Omega\pi)_*(\zeta_n \hat{\otimes} x) = ((n+1)!, \frac{1}{2}(n+2)!, \dots, \frac{1}{(n-1)!}(2n-1)!).$$

Similarly we can show that

$$\begin{aligned} &\lambda \circ (\Omega\pi)_*(\zeta_n \hat{\otimes} x^2) = (0, (n+2)!, \frac{1}{2}(n+3)!, \dots, (2n-1)!A'), \\ &\vdots \\ &\lambda \circ (\Omega\pi)_*(\zeta_n \hat{\otimes} x^{n-1}) = (0, 0, \dots, (2n-1)!B'). \end{aligned}$$

□

Let $\gamma$ be the commutator map $U(n+1) \wedge U(n+1) \to U(n+1)$. Since $U(\infty)$ is an infinite loop space it is homotopy commutative. Therefore the Samelson product $\langle j, j \rangle$ is null homotopic, implying that there is a lift

$$\begin{array}{ccc} & & \Omega W_{n+1} \\ & \overset{\tilde{\gamma}}{\nearrow} & \downarrow \delta \\ U(n+1) \wedge U(n+1) & \xrightarrow{\gamma} & U(n+1) \\ & & \downarrow j \\ & & U(\infty) \end{array}$$

for some map $\tilde{\gamma}$ such that $\delta \circ \tilde{\gamma} \simeq \gamma$. By using the method in [**5**, Proposition 5.2], similarly we have $\tilde{\gamma}$ so that

$$\begin{aligned} \tilde{\gamma}^*(a_{2n+2}) &= \sum_{i+j=n} x_{2i+1} \otimes x_{2j+1}, \\ \tilde{\gamma}^*(a_{2n+4}) &= \sum_{i+j=n+1} x_{2i+1} \otimes x_{2j+1}. \end{aligned}$$

In the following, we bring an application that also by Hamanaka and Kono has been studied in [**7**].

- $n = 3$.

As in [**7**, Lemma 2.1], we have the following lemma.

**Lemma 5.3.** *In case* $n = 3$, *the map* $\lambda\colon [X, \Omega W_{n+1}] \to H^{2n+2}(X) \oplus H^{2n+4}(X) \oplus \cdots \oplus H^{4n-2}(X)$ *is monic.*

By Lemma 5.2, $\operatorname{Im} \lambda \circ (\Omega\pi)_*$ is generated by $\alpha_1 = (4!, \frac{1}{2}5!) = \frac{1}{2}4!\upsilon_1$ and $\alpha_2 = (0, 5!) = 5!\upsilon_2$, where $\upsilon_1 = (2, 5), \upsilon_2 = (0, 1) \in \operatorname{Im} \lambda$ and $\upsilon_1$ and $\upsilon_2$ generate $\operatorname{Im} \lambda$. Therefore by Lemma 5.1 we get the following theorem.

**Theorem 5.4.** *There is an isomorphism* $[X, U(4)] \cong \mathbb{Z}_{120} \oplus \mathbb{Z}_{12}$.

Let $\ell_1\colon \Sigma\mathbb{C}P^2 \hookrightarrow SU(3)$ the inclusion map and $\varepsilon_1\colon S^5 \longrightarrow SU(3)$ is a generator of $\pi_5(SU(3)) \cong \mathbb{Z}$. Also let $\ell_2$ be the composition

$$\ell_2\colon \Sigma\mathbb{C}P^2 \xrightarrow{q_1} S^5 \xrightarrow{\varepsilon_1} SU(3),$$

where $q_1$ is projection map. Let $N$ be the subgroup of $[X, U(4)]$ generated by $i \circ \langle \ell_1, \varepsilon_1 \rangle$ and $i \circ \langle \ell_2, \varepsilon_1 \rangle$, where $i\colon SU(3) \to U(4)$ is the inclusion map. Note that

$$\begin{aligned} {\varepsilon_1}^*(x_3) &= 0, \quad {\varepsilon_1}^*(x_5) = 2s_1, \\ {\ell_1}^*(x_3) &= \sigma(t), \quad {\ell_1}^*(x_5) = \sigma(t^2), \\ {\ell_2}^*(x_3) &= 0, \quad {\ell_2}^*(x_5) = 2\sigma(t^2), \end{aligned}$$

where $s_1$ and $t$ be the generators of $H^5(S^5)$ and $H^2(\mathbb{C}P^2)$ respectively. Let $M$ be the subgroup generated by $\tilde{\gamma} \circ (i \circ \ell_1 \wedge i \circ \varepsilon_1)$ and $\tilde{\gamma} \circ (i \circ \ell_2 \wedge i \circ \varepsilon_1)$. So according to the Lemma 5.1, $N$ is isomorphic to $M/\operatorname{Im}(\Omega\pi_*)$. We show that

$$\begin{aligned} \lambda(\tilde{\gamma} \circ (i \circ \ell_1 \wedge i \circ \varepsilon_1)) &= (2, 2), \\ \lambda(\tilde{\gamma} \circ (i \circ \ell_2 \wedge i \circ \varepsilon_1)) &= (0, 4). \end{aligned}$$

According to the definition of the map $\lambda$, we have

$$\lambda(\tilde{\gamma} \circ (i \circ \ell_1 \wedge i \circ \varepsilon_1)) = ((\tilde{\gamma} \circ (i \circ \ell_1 \wedge i \circ \varepsilon_1))^*(a_8), (\tilde{\gamma} \circ (i \circ \ell_1 \wedge i \circ \varepsilon_1))^*(a_{10}).$$

The calculation of the first component is as follows:

$$\begin{aligned} (\tilde{\gamma} \circ (i \circ \ell_1 \wedge i \circ \varepsilon_1))^*(a_8) &= (i \circ \ell_1 \wedge i \circ \varepsilon_1)^* \circ \tilde{\gamma}^*(a_8) \\ &= (i \circ \ell_1 \wedge i \circ \varepsilon_1)^*(x_3 \otimes x_5) \\ &= {\ell_1}^*(x_3) \otimes {\varepsilon_1}^*(x_5) = 2! s_1 \sigma(t), \end{aligned}$$

the calculation of the second component is as follows:

$$\begin{aligned} (\tilde{\gamma} \circ (i \circ \ell_1 \wedge i \circ \varepsilon_1))^*(a_{10}) &= (i \circ \ell_1 \wedge i \circ \varepsilon_1)^* \circ \tilde{\gamma}^*(a_{10}) \\ &= (i \circ \ell_1 \wedge i \circ \varepsilon_1)^*(x_5 \otimes x_5) \\ &= {\ell_1}^*(x_5) \otimes {\varepsilon_1}^*(x_5) = 2! s_1 \sigma(t^2). \end{aligned}$$

Therefore $\lambda(\tilde{\gamma} \circ (i \circ \ell_1 \wedge i \circ \varepsilon_1)) = (2, 2)$. Similarly we can show that $\lambda(\tilde{\gamma} \circ (i \circ \ell_2 \wedge i \circ \varepsilon_1)) = (0, 4)$. Therefore the subgroup $M$ is generated by $\rho_1 = (2, 2)$ and $\rho_2 = (0, 4)$, since $\begin{vmatrix} 4 & 3 \\ 0 & \frac{1}{4} \end{vmatrix} = 1$, the subgroup $M$ is also generated by $4\rho_1 + 3\rho_2 = 4(2, 5)$ and $\frac{1}{4}\rho_2 = (0, 1)$, therefore we get the following lemma.

**Lemma 5.5.** *$N$ isomorphic to $\mathbb{Z}_{120} \oplus \mathbb{Z}_3$.*

Let $P_1$ be the subgroup of $[\Sigma^6\mathbb{C}P^2, SU(3)]$ generated by $\langle \ell_1, \varepsilon_1 \rangle$ and $\langle \ell_2, \varepsilon_1 \rangle$, then we get the following theorem.

**Theorem 5.6.** *There is an isomorphism $P_1 \cong \mathbb{Z}_{120} \oplus \mathbb{Z}_3$, where $\mathbb{Z}_3$ generated by $\langle 4\ell_1 + 3\ell_2, \varepsilon_1 \rangle$ and $\mathbb{Z}_{120}$ generated by $\langle \frac{1}{4}\ell_2, \varepsilon_1 \rangle$.*

## 6. The Samelson product $\langle \varepsilon_{m,n}, k.j_n \rangle$ when $m < n$

Let $j_n\colon \Sigma\mathbb{C}P^{n-1} \to SU(n)$ be the canonical map. In this section, we will study the order of the Samelson product $\langle \varepsilon_{m,n}, k.j_n \rangle\colon S^{2m-1} \wedge \Sigma\mathbb{C}P^{n-1} \to SU(n)$ when $m < n$. This gives a lower bound on the order of the boundary map $SU(n) \xrightarrow{\alpha_k} \Omega_0^{2m-1}SU(n)$ which is important for determining the homotopy types of gauge groups. In the general case, the calculation of the order of the Samelson product $S^{2m-1} \wedge \Sigma\mathbb{C}P^{n-1} \to SU(n)$ by use of unstable K-theory is not possible and is out of reach.

Consider the fibre sequence

$$\Omega U(\infty) \xrightarrow{\Omega\pi'} \Omega W_n \xrightarrow{\delta'} U(n) \xrightarrow{j'} U(\infty) \xrightarrow{\pi'} W_n.$$

Applying the functor $[X = \Sigma^{2m}\mathbb{C}P^{n-1},]$ to (8), there is an exact sequence of groups:

$$[X, \Omega U(\infty)] \xrightarrow{(\Omega\pi')_*} [X, \Omega W_n] \xrightarrow{\delta'_*} U_n(X) \xrightarrow{j'_*} [X, U(\infty)] \xrightarrow{\pi'_*} [X, W_n].$$

Since $[X, U(\infty)] = 0$ we get the following exact sequence

$$\tilde{K}^0(X) \xrightarrow{(\Omega\pi')_*} [X, \Omega W_n] \xrightarrow{\delta'_*} U_n(X) \to 0.$$

Therefore we have the following lemma.

**Lemma 6.1.** $U_n(X) \cong \mathrm{Coker}(\Omega\pi')_*$.

Define a homomorphism

$$\lambda' \colon [X, \Omega W_n] \to H^{2n}(X) \oplus H^{2n+2}(X) \oplus \cdots \oplus H^{2n+2m-2}(X)$$

by $\lambda(\alpha) = (\alpha^*(a_{2n}), \alpha^*(a_{2n+2}), \ldots, \alpha^*(a_{2n+2m-2}))$, where $\alpha \in [X, \Omega W_n]$ and $a_{2n}$, $a_{2n+2}, \ldots, a_{2n+2m-2}$ are generators of $H^{2n}(\Omega W_n) \cong H^{2n+2}(\Omega W_n) \cong \cdots \cong H^{2n+2m-2}(\Omega W_n) \cong \mathbb{Z}$, respectively. We know $\tilde{K}^0(X = \Sigma^{2m}\mathbb{C}P^{n-1}) \cong \tilde{K}^0(\mathbb{C}P^{n-1})$ is a free abelian group generated by $\zeta_m \hat{\otimes} x, \zeta_m \hat{\otimes} x^2, \ldots, \zeta_m \hat{\otimes} x^{n-1}$, where $\zeta_m$ a generator of $\tilde{K}^0(S^{2m})$. We have the following lemma.

**Lemma 6.2.** *$\mathrm{Im}\,\lambda' \circ (\Omega\pi')_*$ is generated by $\beta_1, \beta_2, \ldots, \beta_{n-1}$ where*

$$\beta_1 = \left(\frac{n!}{(n-m)!}, \frac{(n+1)!}{(n-m+1)!}, \ldots, \frac{(n+m-1)!}{(n-1)!}\right),$$
$$\beta_2 = (n!{A_1}', (n+1)!{A_2}', \ldots, (n+m-1)!{A_3}'),$$
$$\vdots$$
$$\beta_{n-1} = (n!{B_1}', (n+1)!{B_2}', \ldots, (n+m-1)!{B_3}'),$$

*where*

$$A_1 = \mathrm{ch}_{n-m}(x^2) = {A_1}'t^{n-m}, \qquad B_1 = \mathrm{ch}_{n-m}(x^{n-1}) = {B_1}'t^{n-m},$$
$$A_2 = \mathrm{ch}_{n-m+1}(x^2) = {A_2}'t^{n-m+1}, \qquad B_2 = \mathrm{ch}_{n-m+1}(x^{n-1}) = {B_2}'t^{n-m+1},$$
$$A_3 = \mathrm{ch}_{n-1}(x^2) = {A_3}'t^{n-1}, \qquad B_3 = \mathrm{ch}_{n-1}(x^{n-1}) = {B_3}'t^{n-1}.$$

*Proof.* The proof is similar to the proof of Lemma 5.2. □

In the following we give some applications. As in [**6**, Lemma 2.1], we have the following lemma.

**Lemma 6.3.** *If $m = 2$, then the map $\lambda' \colon [X, \Omega W_n] \to H^{2n}(X) \oplus H^{2n+2}(X) \oplus \cdots \oplus H^{2n+2m-2}(X)$ is monic.*

- $n = 3, m = 2$.

$\mathrm{Im}\,\lambda' \circ (\Omega\pi')_*$ is generated by

$$\beta_1 = (3!, \frac{1}{2}4!), \qquad \beta_2 = (0, 4!),$$

also $\lambda' \circ (\Omega\pi')_*$ is generated by $4\beta_1 - \beta_2 = 24(1,1)$, therefore we can deduce the following result:

**Proposition 6.4.** *The order of the Samelson product $S^3 \wedge \Sigma\mathbb{C}P^2 \to SU(3)$ is equal to* 24.

- $n = 4, m = 2$.

Im $\lambda' \circ (\Omega\pi')_*$ is generated by

$$\beta_1 = (\frac{1}{2}4!, \frac{1}{3!}5!), \quad \beta_2 = (4!, \frac{1}{2}5!), \quad \beta_3 = (0, 2.5!)$$

also $\lambda' \circ (\Omega\pi')_*$ is generated by $15\beta_1 - 5\beta_2 + \frac{1}{4}\beta_3 = \frac{1}{2}5!(1,1)$, therefore we can deduce the following result:

**Proposition 6.5.** *The order of the Samelson product $S^3 \wedge \Sigma\mathbb{C}P^3 \to SU(4)$ is equal to* $\frac{1}{2}5!$.

The previous two cases have also been studied in [**6**] and [**4**], respectively.

- $n = 5, m = 2$.

Im $\lambda' \circ (\Omega\pi')_*$ is generated by

$$\beta_1 = (\frac{1}{3!}5!, \frac{1}{4!}6!), \quad \beta_2 = (\frac{1}{2}5!, \frac{5}{12}6!), \quad \beta_3 = (2.5!, \frac{3}{2}6!), \quad \beta_4 = (0, 5.6!),$$

also $\lambda' \circ (\Omega\pi')_*$ is generated by $(2\beta_2 - \frac{1}{6}\beta_4) + (\frac{1}{6}\beta_3 - 2\beta_1) = 5!(1,1)$, therefore we can deduce the following result:

**Proposition 6.6.** *The order of the Samelson product $S^3 \wedge \Sigma\mathbb{C}P^4 \to SU(5)$ is equal to* 5!.

- $n = 6, m = 2$.

Im $\lambda' \circ (\Omega\pi')_*$ is generated by

$$\beta_1 = (\frac{1}{4!}6!, \frac{1}{5!}7!), \quad \beta_2 = (\frac{5}{12}6!, \frac{3}{4!}7!), \quad \beta_3 = (\frac{3}{2}6!, \frac{15}{12}7!),$$
$$\beta_4 = (5.6!, \frac{9}{2}7!), \quad \beta_5 = (0, 12.7!),$$

also $\lambda' \circ (\Omega\pi')_*$ is generated by $(\frac{7}{6}\beta_3 - 35\beta_1 - \frac{7}{3.4!}\beta_5) + (\frac{1}{3.4!}\beta_4 - \frac{1}{6}\beta_2) = \frac{1}{4!}7!(1,1)$, therefore we can deduce the following result:

**Proposition 6.7.** *The order of the Samelson product $S^3 \wedge \Sigma\mathbb{C}P^5 \to SU(6)$ is equal to* $\frac{1}{4!}7!$.

*Proof of Theorem 1.2.* By Propositions 6.6 and 6.7 we conclude Theorem 1.2. □

## Acknowledgments

The author is greatly grateful to Professor Stephen Theriault for his immense encouragement and much valuable advice. Also, the author would like to thank the referee for a careful reading of the paper and for his helpful comments.

## References


[1] M. F. Atiyah and R. Bott, The Yang–Mills equations over Riemann Surfaces, *Philos. Trans. Roy. Soc. London Ser.* A **308** (1983), 523–615.

[2] T. Cutler, The homotopy type of $Sp(3)$-gauge groups, *Topol. Appl.* **236** (2018), 44–58.

[3] M. C. Crabb and W. A. Sutherland, Counting homotopy types of gauge groups, *Proc. London Math. Soc.* **83** (2000), 747–768.

[4] T. Cutler and S. Theriault, The homotopy types of $SU(4)$-gauge groups, *arXiv:1909.04643.*

[5] H. Hamanaka and A. Kono, On $[X, U(n)]$ when dim $X = 2n$, *J. Math. Kyoto Univ.* **43** (2) (2003), 333–348.

[6] H. Hamanaka and A. Kono, Unstable $K^1$-group and homotopy type of certain gauge groups, *Proc. Roy. Soc. Edinburgh Sect. A* **136** (2006), 149–155.

[7] H. Hamanaka and A. Kono, Homotopy type of gauge groups of $SU(3)$-bundles over $S^6$, *Topology and its Applications.* **154** (2007), 1377–1380.

[8] H. Hamanaka, S. Kaji and A. Kono, Samelson products in $Sp(2)$, *Topol. Appl.* **155** (2008), 1207–1212.

[9] A. Kono, A note on the homotopy type of certain gauge groups, *Proc. Roy. Soc. Edinburgh Sect. A* **117** (1991), 295–297.

[10] Y. Kamiyama, D. Kishimoto, A. Kono, and S. Tsukuda, Samelson products of $SO(3)$ and applications, *Glasg. Math. J.* **49** (2007), 405–409.

[11] D. Kishimoto, S. Theriault, and M. Tsutaya, Homotopy types of $G_2$-gauge groups, *Topol. Appl.* **228** (2017), 92–107.

[12] G. E. Lang, The evaluation map and EHP sequences, *Pacific J. Math.* **44** (1973), 201–210.

[13] S. Mohammadi and M. A. Asadi–Golmankhaneh, The homotopy types of $SU(4)$-gauge groups over $S^8$, *Topol. Appl.* **266** (2019), 106845.

[14] S. Mohammadi and M. A. Asadi–Golmankhaneh, The homotopy types of $SU(n)$-gauge groups over $S^6$, *Topol. Appl.* **270** (2020), 106952.

[15] S. D. Theriault, The homotopy types of $SU(5)$-gauge groups, *Osaka J. Math.* **52** (2015), 15–31.

[16] S. D. Theriault, The homotopy types of $Sp(2)$-gauge groups, *Kyoto J. Math.* **50** (2010), 591–605.



Sajjad Mohammadi `sj.mohammadi@urmia.ac.ir`

Department of Mathematics, Faculty of Sciences, Urmia University,
P.O. Box 5756151818, Urmia, Iran