Homology, Homotopy and Applications, vol. 24(1), 2022, pp.129–141

RELATIVE $\mathbb{A}^1$-HOMOLOGY AND ITS APPLICATIONS

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(*communicated by Charles A. Weibel*)

Abstract

In this paper, we prove an $\mathbb{A}^1$-homology version of the Whitehead theorem with dimension bound. We also prove an excision theorem for $\mathbb{A}^1$-homology, Suslin homology and $\mathbb{A}^1$-homotopy sheaves. In order to prove these results, we develop a general theory of relative $\mathbb{A}^1$-homology and $\mathbb{A}^1$-homotopy sheaves. As an application, we compute the relative $\mathbb{A}^1$-homology of a hyperplane embedding $\mathbb{P}^{n-1} \hookrightarrow \mathbb{P}^n$.

1. Introduction

In this paper, we study a relative version of the $\mathbb{A}^1$-homology sheaves of smooth schemes over a field and give applications to motives and $\mathbb{A}^1$-homotopy theory. Let k be a field. In [**Vo**], Voevodsky constructed a triangulated category of motives $\mathbf{DM}_-^{eff}(k)$ over k as a triangulated subcategory of the derived category of Nisnevich sheaves with transfers, with a functor M from the category of smooth k-schemes $\mathcal{S}m_k$ to $\mathbf{DM}_-^{eff}(k)$. For $X \in \mathcal{S}m_k$, $M(X)$ is called the motive of X. Its homology sheaves with transfers $\mathbf{H}_*^S(X) = H_*(M(X))$ are called the Suslin homology sheaves (*cf.* [**As**, Section 2]), whose sections over $\operatorname{Spec} k$ give the Suslin homology group introduced by Suslin–Voevodsky [**SV**] when k is perfect.

In [**MV**], Morel–Voevodsky established the $\mathbb{A}^1$-homotopy theory and defined an $\mathbb{A}^1$-version of homotopy groups, called $\mathbb{A}^1$-homotopy sheaves, as Nisnevich sheaves on $\mathcal{S}m_k$. Morel [**Mo2**] introduced an $\mathbb{A}^1$-version of the singular homology, called $\mathbb{A}^1$-homology sheaves, as an analogue of Suslin homology by instead using Nisnevich sheaves without transfers. As with motives, there is a functor $C^{\mathbb{A}^1}$ from $\mathcal{S}m_k$ to a triangulated subcategory of the derived category of Nisnevich sheaves on $\mathcal{S}m_k$. The $\mathbb{A}^1$-homology sheaves $\mathbf{H}_*^{\mathbb{A}^1}(X)$ of $X \in \mathcal{S}m_k$ are defined as the homology sheaves $H_*(C^{\mathbb{A}^1}(X))$.

The purpose of this paper is threefold. Firstly, we prove an $\mathbb{A}^1$-homological Whitehead theorem with dimension bound and the excision theorem. Secondly, as a tool for proving them, we develop a general theory of *relative* $\mathbb{A}^1$-homology, namely $\mathbb{A}^1$-homology sheaves for morphisms $f\colon X \to Y$. Thirdly, as an example, we compute the relative $\mathbb{A}^1$-homology of a hyperplane embedding $\mathbb{P}^{n-1} \hookrightarrow \mathbb{P}^n$.

This work was supported by JSPS KAKENHI Grant Number JP19J21433.
Received May 3, 2020, revised October 27, 2020, January 22, 2021; published on April 6, 2022.
2010 Mathematics Subject Classification: 14F42, 19E15.
Key words and phrases: $\mathbb{A}^1$-homology, $\mathbb{A}^1$-homotopy, algebraic geometry.
Article available at `http://dx.doi.org/10.4310/HHA.2022.v24.n1.a7`

Our $\mathbb{A}^1$-Whitehead theorem detects whether a morphism $X \to Y$ is an $\mathbb{A}^1$-weak equivalence in terms of the $\mathbb{A}^1$-fundamental groups and the $\mathbb{A}^1$-homology up to degree $\max\{\dim X + 1, \dim Y\}$. See [**Ar**, Thm. 6.4.15] for the classical homological Whitehead theorem in topology.

Theorem 1.1 (See Theorem 3.2)**.** *Assume k perfect. Let $f\colon (X,*) \to (Y,*)$ be a morphism of $\mathbb{A}^1$-simply connected pointed smooth k-schemes and let $d = \max\{\dim X + 1, \dim Y\}$. If f induces an isomorphism $\mathbf{H}_i^{\mathbb{A}^1}(X) \xrightarrow{\cong} \mathbf{H}_i^{\mathbb{A}^1}(Y)$ for all $2 \leqslant i < d$ and an epimorphism $\mathbf{H}_d^{\mathbb{A}^1}(X) \twoheadrightarrow \mathbf{H}_d^{\mathbb{A}^1}(Y)$, then f is an $\mathbb{A}^1$-weak equivalence.*

The Whitehead theorem for $\mathbb{A}^1$-*homotopy* sheaves is established by Morel–Voevodsky [**MV**], and the novelty here is the detection by $\mathbb{A}^1$-*homology* sheaves and the degree bound $d = \max\{\dim X + 1, \dim Y\}$. Next, our excision theorem for $\mathbb{A}^1$-homology sheaves is stated as follows.

Theorem 1.2 (See Theorem 3.5)**.** *Let X be a smooth k-scheme and U a Zariski open set of X whose complement has codimension r. Then the morphisms*

$$\mathbf{H}_i^{\mathbb{A}^1}(U) \to \mathbf{H}_i^{\mathbb{A}^1}(X) \quad \textit{and} \quad \mathbf{H}_i^{S}(U) \to \mathbf{H}_i^{S}(X)$$

are isomorphisms for every $i < r-1$ and epimorphisms for $i = r-1$.

Asok [**As**, Prop. 3.8] proved a similar result in degree 0. We also obtain the $\mathbb{A}^1$-homotopy version of the excision theorem.

Corollary 1.3 (See Corollary 3.6)**.** *Assume k perfect. Let $(X,*)$ be a pointed smooth k-scheme and $(U,*)$ a pointed Zariski open set of $(X,*)$ whose complement has codimension r. If $(X,*)$ and $(U,*)$ are $\mathbb{A}^1$-simply connected, then the morphism*

$$\pi_i^{\mathbb{A}^1}(U,*) \to \pi_i^{\mathbb{A}^1}(X,*)$$

is an isomorphism for every $i < r-1$ and an epimorphism for $i = r-1$.

This is a variation of the $\mathbb{A}^1$-excision theorem of Asok–Doran [**AD**] which assumes that $\pi_i^{\mathbb{A}^1}(X,*) = 0$ for all $i < r-1$ and k infinite. In order to prove these results, we develop a general theory of relative $\mathbb{A}^1$-homotopy and $\mathbb{A}^1$-homology sheaves. If $f\colon X \to Y$ is a morphism of smooth k-schemes, we define its $\mathbb{A}^1$-homotopy $\pi_i^{\mathbb{A}^1}(f)$, $\mathbb{A}^1$-homology $\mathbf{H}_i^{\mathbb{A}^1}(f)$ and Suslin homology $\mathbf{H}_i^S(f)$.

Finally, as an application of the results above, we compute the relative $\mathbb{A}^1$-homology sheaves of the pair $(\mathbb{P}^n, \mathbb{P}^{n-1})$. Let $\underline{\mathbf{K}}_n^{MW}$ be the unramified Milnor–Witt K-theory defined by Morel [**Mo2**].

Theorem 1.4 (See Theorem 4.1)**.** *Assume k perfect. For $0 \leqslant i \leqslant n$, $n > 0$, we have*

$$\mathbf{H}_i^{\mathbb{A}^1}(\mathbb{P}^n, \mathbb{P}^{n-1}) \cong \begin{cases} \underline{\mathbf{K}}_n^{MW} & (i = n), \\ 0 & (i < n). \end{cases}$$

In particular, when $i < n$, we have $\mathbf{H}_i^{\mathbb{A}^1}(\mathbb{P}^n) \cong \mathbf{H}_i^{\mathbb{A}^1}(\mathbb{P}^{i+1})$.

Similar stabilization $\mathbf{H}_i^S(\mathbb{P}^n) \cong \mathbf{H}_i^S(\mathbb{P}^{i+1})$ in $i < n$ also holds for the Suslin homology (Corollary 4.3). The $\mathbb{A}^1$-homology $\mathbf{H}_i^{\mathbb{A}^1}(\mathbb{P}^{i+1})$ can be described in terms of $\underline{\mathbf{K}}_i^{MW}$ (Corollary 4.2).

This paper is organized as follows. In Section 2, we prove a weak version of the relative $\mathbb{A}^1$-Hurewicz theorem. In Section 3, we prove Theorems 1.1–1.2. In Section 4, we prove Theorem 1.4.

Notation. In this paper, we fix a field k. We denote by $\mathcal{S}m_k$ the category of smooth k-schemes. Every sheaf is considered on the Nisnevich topology on $\mathcal{S}m_k$. Objects of an abelian category are regarded as complexes concentrated in degree zero.

Acknowledgments

I would like to thank my adviser Shohei Ma for many useful advices. I would also like to thank Tom Bachmann for pointing out a mistake in the previous version. I would like to thank the referee for many detailed comments which helped us to improve the presentation.

2. Relative $\mathbb{A}^1$-homotopy and $\mathbb{A}^1$-homology

In this section, we give basic definitions of relative $\mathbb{A}^1$-homotopy and $\mathbb{A}^1$-homology, and establish a weak relative $\mathbb{A}^1$-Hurewicz theorem. We also compare $\mathbb{A}^1$-homology and Suslin homology. We refer to [**MV**], [**MVW**], [**SV**], [**Mo2**] and [**As**] for the basic theory of $\mathbb{A}^1$-homology and $\mathbb{A}^1$-homotopy.

2.1. Basic definitions

Let $\mathcal{S}pc_k$ be the category of simplicial Nisnevich sheaves on $\mathcal{S}m_k$ (called k-spaces) equipped with the $\mathbb{A}^1$-model structure of [**MV**]. We denote by $\pi_0^{\mathbb{A}^1}(\mathcal{X})$ the sheaf of $\mathbb{A}^1$-connected components of a k-space $\mathcal{X}$ and denote by $\pi_n^{\mathbb{A}^1}(\mathcal{X}, *)$ the n-th $\mathbb{A}^1$-homotopy sheaf of a pointed k-space $(\mathcal{X}, *)$ for $n \geqslant 0$. A k-space $\mathcal{X}$ is called $\mathbb{A}^1$-connected if $\pi_0^{\mathbb{A}^1}(\mathcal{X}) \cong \operatorname{Spec} k$, and a pointed k-space $(\mathcal{X}, *)$ is called $\mathbb{A}^1$-n-connected if $\mathcal{X}$ is $\mathbb{A}^1$-connected and if $\pi_i^{\mathbb{A}^1}(\mathcal{X}, *) = 0$ for all $1 \leqslant i \leqslant n$. Especially, $(\mathcal{X}, *)$ is called $\mathbb{A}^1$-simply connected if it is $\mathbb{A}^1$-1-connected. We consider a relative version of these definitions.

Definition 2.1. For a morphism of pointed k-spaces $f\colon (\mathcal{X}, *) \to (\mathcal{Y}, *)$, the i-th $\mathbb{A}^1$-homotopy sheaf of f is defined by

$$\pi_i^{\mathbb{A}^1}(f) = \begin{cases} \pi_{i-1}^{\mathbb{A}^1}(F_{\mathbb{A}^1}(f)) & (i > 0), \\ \operatorname{Spec} k & (i = 0), \end{cases}$$

where $F_{\mathbb{A}^1}(f)$ is the homotopy fiber with respect to the $\mathbb{A}^1$-model structure of $\mathcal{S}pc_k$. When f is an inclusion, we write $\pi_i^{\mathbb{A}^1}(\mathcal{Y}, \mathcal{X}) = \pi_i^{\mathbb{A}^1}(f)$. A morphism (or a pair) is called $\mathbb{A}^1$-n-connected if its $\mathbb{A}^1$-homotopy sheaves in degree $\leqslant n$ are isomorphic to $\operatorname{Spec} k$.

Since $F_{\mathbb{A}^1}(f) \to \mathcal{X} \xrightarrow{f} \mathcal{Y}$ is a homotopy fiber sequence under the $\mathbb{A}^1$-model structure, we obtain by [**AE**, Prop. 4.21] the long exact sequence

$$\cdots \to \pi_i^{\mathbb{A}^1}(\mathcal{X}, *) \to \pi_i^{\mathbb{A}^1}(\mathcal{Y}, *) \to \pi_i^{\mathbb{A}^1}(f) \to \pi_{i-1}^{\mathbb{A}^1}(\mathcal{X}, *) \to \cdots$$
$$\cdots \to \pi_0^{\mathbb{A}^1}(\mathcal{X}, *) \to \pi_0^{\mathbb{A}^1}(\mathcal{Y}, *) \to 0.$$

We fix a commutative unital ring R. Let $\mathcal{M}od_k(R)$ be the category of Nisnevich sheaves of R-modules on $\mathcal{S}m_k$ and $\mathbf{D}(k,R)$ be its unbounded derived category. We denote by $\mathbf{D}_{\mathbb{A}^1}(k,R)$ the full subcategory of $\mathbf{D}(k,R)$ consisting of $\mathbb{A}^1$-local complexes and $\mathcal{M}od_k^{\mathbb{A}^1}(R)$ for the full subcategory of $\mathcal{M}od_k(R)$ consisting of strictly $\mathbb{A}^1$-invariant sheaves. We write $\mathcal{A}b_k = \mathcal{M}od_k(\mathbb{Z})$ and $\mathcal{A}b_k^{\mathbb{A}^1} = \mathcal{M}od_k^{\mathbb{A}^1}(\mathbb{Z})$. For $\mathcal{X} \in \mathcal{S}pc_k$, we denote by $R(\mathcal{X})$ the simplicial Nisnevich sheaf of R-modules freely generated by $\mathcal{X}$ and $C(\mathcal{X};R)$ for its normalized chain complex. Let $L_{\mathbb{A}^1}$ be a left adjoint of the inclusion $\mathbf{D}_{\mathbb{A}^1}(k,R) \hookrightarrow \mathbf{D}(k,R)$ (see [**CD**, Thm. 2.5]). We write $C^{\mathbb{A}^1}(\mathcal{X};R) = L_{\mathbb{A}^1}(C(\mathcal{X};R))$ and $\mathbf{H}_*^{\mathbb{A}^1}(\mathcal{X};R) = H_*(C(\mathcal{X};R))$. We consider a relative version of these definitions.

Definition 2.2. Let $f\colon \mathcal{X} \to \mathcal{Y}$ be a morphism of k-spaces. We denote by $C(f;R)$ the mapping cone of $C(\mathcal{X};R) \to C(\mathcal{Y};R)$. We write $C^{\mathbb{A}^1}(f;R) = L_{\mathbb{A}^1}(C(f;R))$. We define the i-th $\mathbb{A}^1$-homology sheaf $\mathbf{H}_i^{\mathbb{A}^1}(f;R)$ as the homology of $C^{\mathbb{A}^1}(\mathcal{X};R)$ in degree i. When f is an inclusion, we write $C^{\mathbb{A}^1}(\mathcal{Y},\mathcal{X};R) = C^{\mathbb{A}^1}(f;R)$ and $\mathbf{H}_i^{\mathbb{A}^1}(\mathcal{Y},\mathcal{X};R) = \mathbf{H}_i^{\mathbb{A}^1}(f;R)$.

2.2. Relative $\mathbb{A}^1$-Hurewicz theorem

We denote by $\mathcal{G}r_k^{\mathbb{A}^1}$ the category of strongly $\mathbb{A}^1$-invariant sheaves of groups on $\mathcal{S}m_k$ (see [**Mo2**, Def. 1.7]). For a functor $F\colon \mathscr{C} \to \mathscr{D}$ between categories, a morphism $f\colon A \to A'$ in $\mathscr{D}$ is called *universal with respect to F* if it induces a bijection $\mathrm{Hom}_{\mathscr{D}}(A', F(B)) \cong \mathrm{Hom}_{\mathscr{D}}(A, F(B))$ for every $B \in \mathscr{C}$. Morel [**Mo2**] proved the following $\mathbb{A}^1$-Hurewicz theorem which relates $\mathbb{A}^1$-homotopy and $\mathbb{A}^1$-homology.

Theorem 2.3 (See [**Mo2**, Thm. 6.35 and 6.37])**.** *Let $(\mathcal{X},*)$ be a pointed k-space. Then there exists a natural morphism*

$$h\colon \pi_n^{\mathbb{A}^1}(\mathcal{X},*) \to \mathbf{H}_n^{\mathbb{A}^1}(\mathcal{X};\mathbb{Z})$$

such that if $(\mathcal{X},*)$ is $\mathbb{A}^1$-$(n-1)$-connected for $n \geqslant 1$, then h is universal with respect to the inclusion $\mathcal{A}b_k^{\mathbb{A}^1} \hookrightarrow \mathcal{G}r_k^{\mathbb{A}^1}$.

Morel proves that h is an isomorphism assuming k perfect; for the above assertion his argument works for general k. The following is a relative version of Theorem 2.3.

Proposition 2.4. *Let $f\colon (\mathcal{X},*) \to (\mathcal{Y},*)$ be a morphism from an $\mathbb{A}^1$-simply connected pointed k-space to an $\mathbb{A}^1$-connected k-space. Suppose that f is $\mathbb{A}^1$-$(n-1)$-connected for $n \geqslant 2$. Then there exists a universal morphism*

$$h\colon \pi_n^{\mathbb{A}^1}(f) \to \mathbf{H}_n^{\mathbb{A}^1}(f;\mathbb{Z})$$

with respect to the inclusion $\mathcal{A}b_k^{\mathbb{A}^1} \hookrightarrow \mathcal{G}r_k^{\mathbb{A}^1}$.

Proof. We write $\mathbf{H}_n(-) = H_n(C(-;\mathbb{Z}))$. Let $\mathrm{Ex}_{\mathbb{A}^1}$ be the resolution functor as in [**MV**, §3.2]. By applying the relative Hurewicz theorem of simplicial sets [**GJ**, Thm. 3.11] to all stalks, we have a natural isomorphism $\pi_i^{\mathbb{A}^1}(f) \cong \mathbf{H}_i(\mathrm{Ex}_{\mathbb{A}^1}(f))$ for all $1 \leqslant i \leqslant n$ and $\mathbf{H}_0(\mathrm{Ex}_{\mathbb{A}^1}(f)) = 0$. Thus we obtain $\mathbf{H}_i(\mathrm{Ex}_{\mathbb{A}^1}(f)) = 0$ for every $i \leqslant n-1$. By the isomorphism $\pi_n^{\mathbb{A}^1}(f) \cong \mathbf{H}_n(\mathrm{Ex}_{\mathbb{A}^1}(f))$, there exists an isomorphism

$$\mathrm{Hom}_{\mathcal{A}b_k}(\pi_n^{\mathbb{A}^1}(f), A) \cong \mathrm{Hom}_{\mathcal{A}b_k}(\mathbf{H}_n(\mathrm{Ex}_{\mathbb{A}^1}(f)), A) \tag{1}$$

for every $A \in \mathcal{A}b_k^{\mathbb{A}^1}$. Since $\mathbf{H}_i(\mathrm{Ex}_{\mathbb{A}^1}(f)) = 0$ for all $i \leqslant n-1$, the adjunction on $L_{\mathbb{A}^1}$

shows that

$$\begin{aligned}\mathrm{Hom}_{\mathcal{A}b_k}(\mathbf{H}_n(\mathrm{Ex}_{\mathbb{A}^1}(f)), A) &\cong \mathrm{Hom}_{\mathbf{D}(k,\mathbb{Z})}(C(\mathrm{Ex}_{\mathbb{A}^1}(f);\mathbb{Z}), A[n]) \\ &\cong \mathrm{Hom}_{\mathbf{D}(k,\mathbb{Z})}(C^{\mathbb{A}^1}(\mathrm{Ex}_{\mathbb{A}^1}(f);\mathbb{Z}), A[n]).\end{aligned}$$

Then $H_i(C^{\mathbb{A}^1}(\mathrm{Ex}_{\mathbb{A}^1}(f);\mathbb{Z})) = 0$ for all $i \leqslant n-1$ by [**Mo2**, Thm. 6.22]. Thus

$$\mathrm{Hom}_{\mathbf{D}(k,\mathbb{Z})}(C^{\mathbb{A}^1}(\mathrm{Ex}_{\mathbb{A}^1}(f);\mathbb{Z}), A[n]) \cong \mathrm{Hom}_{\mathcal{A}b_k}(\mathbf{H}_n^{\mathbb{A}^1}(\mathrm{Ex}_{\mathbb{A}^1}(f);\mathbb{Z}), A).$$

The morphism of distinguished triangles

$$\begin{array}{ccccccc} C^{\mathbb{A}^1}(\mathcal{X};\mathbb{Z}) & \longrightarrow & C^{\mathbb{A}^1}(\mathcal{Y};\mathbb{Z}) & \longrightarrow & C^{\mathbb{A}^1}(f;\mathbb{Z}) & \longrightarrow & \\ \cong\downarrow & & \cong\downarrow & & \downarrow & & \\ C^{\mathbb{A}^1}(\mathrm{Ex}_{\mathbb{A}^1}(\mathcal{X});\mathbb{Z}) & \longrightarrow & C^{\mathbb{A}^1}(\mathrm{Ex}_{\mathbb{A}^1}(\mathcal{Y});\mathbb{Z}) & \longrightarrow & C^{\mathbb{A}^1}(\mathrm{Ex}_{\mathbb{A}^1}(f);\mathbb{Z}) & \longrightarrow & \end{array}$$

in $\mathbf{D}(k,\mathbb{Z})$ induced by the natural transformation $\mathrm{Id} \to \mathrm{Ex}_{\mathbb{A}^1}$ gives a quasi-isomorphism $C^{\mathbb{A}^1}(f;\mathbb{Z}) \to C^{\mathbb{A}^1}(\mathrm{Ex}_{\mathbb{A}^1}(f);\mathbb{Z})$. Therefore, we obtain

$$\mathrm{Hom}_{\mathcal{A}b_k}(\mathbf{H}_n^{\mathbb{A}^1}(\mathrm{Ex}_{\mathbb{A}^1}(f);\mathbb{Z}), A) \cong \mathrm{Hom}_{\mathcal{A}b_k}(\mathbf{H}_n^{\mathbb{A}^1}(f;\mathbb{Z}), A). \tag{2}$$

By the isomorphisms (1)–(2), we have

$$\mathrm{Hom}_{\mathcal{A}b_k}(\pi_n^{\mathbb{A}^1}(f), A) \cong \mathrm{Hom}_{\mathcal{A}b_k}(\mathbf{H}_n^{\mathbb{A}^1}(f;\mathbb{Z}), A) = \mathrm{Hom}_{\mathcal{A}b_k^{\mathbb{A}^1}}(\mathbf{H}_n^{\mathbb{A}^1}(f;\mathbb{Z}), A) \tag{3}$$

for all $A \in \mathcal{A}b_k^{\mathbb{A}^1}$. On the other hand, [**Mo2**, Thm. 6.22] leads to the adjunction

$$\mathcal{M}od_k(R) \underset{\longleftarrow}{\xrightarrow{H_0\circ L_{\mathbb{A}^1}}} \mathcal{M}od_k^{\mathbb{A}^1}(R)\ , \tag{4}$$

and this induces a universal morphism

$$h' \colon \pi_n^{\mathbb{A}^1}(f) \to H_0(L_{\mathbb{A}^1}(\pi_n^{\mathbb{A}^1}(f)))$$

with respect to $\mathcal{A}b_k^{\mathbb{A}^1} \hookrightarrow \mathcal{A}b_k$. By the isomorphism (3), we have

$$\mathrm{Hom}_{\mathcal{A}b_k^{\mathbb{A}^1}}(\mathbf{H}_n^{\mathbb{A}^1}(f;\mathbb{Z}), A) \cong \mathrm{Hom}_{\mathcal{A}b_k}(\pi_n^{\mathbb{A}^1}(f), A) \cong \mathrm{Hom}_{\mathcal{A}b_k^{\mathbb{A}^1}}(H_0(L_{\mathbb{A}^1}(\pi_n^{\mathbb{A}^1}(f))), A).$$

Thus Yoneda's lemma in $\mathcal{A}b_k^{\mathbb{A}^1}$ shows that

$$H_0(L_{\mathbb{A}^1}(\pi_n^{\mathbb{A}^1}(f))) \cong \mathbf{H}_n^{\mathbb{A}^1}(f;\mathbb{Z}).$$

Therefore, the composite morphism

$$h \colon \pi_n^{\mathbb{A}^1}(f) \xrightarrow{h'} H_0(L_{\mathbb{A}^1}(\pi_n^{\mathbb{A}^1}(f))) \cong \mathbf{H}_n^{\mathbb{A}^1}(f;\mathbb{Z})$$

is universal with respect to $\mathcal{A}b_k^{\mathbb{A}^1} \hookrightarrow \mathcal{A}b_k$. Since $\pi_n^{\mathbb{A}^1}(f)$ and $\mathbf{H}_n^{\mathbb{A}^1}(f;\mathbb{Z})$ are strongly $\mathbb{A}^1$-invariant, the morphism h is universal with respect to the inclusion $\mathcal{A}b_k^{\mathbb{A}^1} \hookrightarrow \mathcal{G}r_k^{\mathbb{A}^1}$. □

When k is perfect, Proposition 2.4 gives an isomorphism between the relative $\mathbb{A}^1$-homotopy and the $\mathbb{A}^1$-homology sheaves.

Corollary 2.5. *Let f be as in Proposition 2.4. If k is perfect, then there exists a natural isomorphism $\pi_n^{\mathbb{A}^1}(f) \cong \mathbf{H}_n^{\mathbb{A}^1}(f;\mathbb{Z})$.*

Proof. Since $\pi_n^{\mathbb{A}^1}(f) \in \mathcal{A}b_k^{\mathbb{A}^1}$ by [**Mo2**, Cor. 6.2], Yoneda's lemma in $\mathcal{A}b_k^{\mathbb{A}^1}$ gives a natural isomorphism $\pi_n^{\mathbb{A}^1}(f) \cong \mathbf{H}_n^{\mathbb{A}^1}(f;\mathbb{Z})$. □

By Corollary 2.5, we obtain the following.

Corollary 2.6. *Assume k perfect. Let* $f\colon (\mathcal{X}, *) \to (\mathcal{Y}, *)$ *be a morphism of* $\mathbb{A}^1$*-simply connected pointed* k*-spaces and* $n \geqslant 2$ *an integer. If* $\mathbf{H}_i^{\mathbb{A}^1}(f;\mathbb{Z}) = 0$ *for all* $2 \leqslant i \leqslant n$*, then* f *is* $\mathbb{A}^1$*-*n*-connected.*

Proof. We use induction on i. The assertion is clear for $i = 0$. We next consider the case $i = 1$. There exists an exact sequence

$$\pi_1^{\mathbb{A}^1}(\mathcal{Y}, *) \to \pi_1^{\mathbb{A}^1}(f) \to \pi_0^{\mathbb{A}^1}(\mathcal{X}, *).$$

Since $\pi_1^{\mathbb{A}^1}(\mathcal{Y}, *) = \pi_0^{\mathbb{A}^1}(\mathcal{X}, *) = 0$, we have $\pi_1^{\mathbb{A}^1}(f) = 0$. Finally, let $i \geqslant 2$ and $\pi_{i-1}^{\mathbb{A}^1}(f) = 0$. Then Corollary 2.5 shows that $\pi_i^{\mathbb{A}^1}(f) \cong \mathbf{H}_i^{\mathbb{A}^1}(f;\mathbb{Z}) = 0$. □

2.3. $\mathbb{A}^1$-homology and Suslin homology

Next, we compare $\mathbb{A}^1$-homology and Suslin homology. Let $\mathbf{NST}_k(R)$ be the category of Nisnevich sheaves with transfers with coefficients in R, $\mathbf{D}_{tr}(k, R)$ be the unbounded derived category of $\mathbf{NST}_k(R)$, and $R_{tr}\colon \mathcal{S}m_k \to \mathbf{NST}_k(R)$ be the functor as in [**MVW**, Def. 2.8] (with R-coefficients). Following [**Vo**], we denote by $\mathbf{DM}^{eff}(k, R)$ the full subcategory of $\mathbf{D}_{tr}(k, R)$ consisting of $\mathbb{A}^1$-local complexes. Let $L^{tr}_{\mathbb{A}^1}$ be a left adjoint of the inclusion $\mathbf{DM}^{eff}(k, R) \hookrightarrow \mathbf{D}_{tr}(k, R)$ (see [**CD**, Thm. 2.5]). We write $M(X; R) = L^{tr}_{\mathbb{A}^1}(R_{tr}(X))$ for each $X \in \mathcal{S}m_k$. The homology sheaves $\mathbf{H}^S_*(X; R) = H_*(M(X; R))$ are called the *Suslin homology sheaves* of X (cf. [**SV**]). We introduce a relative version.

Definition 2.7. Let $f\colon X \to Y$ be a morphism in $\mathcal{S}m_k$. Then we denote by $R_{tr}(f)$ the mapping cone of the morphism $R_{tr}(X) \to R_{tr}(Y)$. We write $M(f;\, R) = L^{tr}_{\mathbb{A}^1}(R_{tr}(f))$. We define the i-th Suslin homology sheaf $\mathbf{H}^S_i(f; R)$ as the homology of $M(f; R)$ in degree i. When f is an embedding, we write $R_{tr}(Y, X) = R_{tr}(f)$ and $\mathbf{H}^S_i(Y, X; R) = \mathbf{H}^S_i(f; R)$.

Let $\mathbf{NST}_k^{\mathbb{A}^1}(R)$ be the full subcategory of $\mathbf{NST}_k(R)$ consisting of strictly $\mathbb{A}^1$-invariant sheaves. If f is a morphism in $\mathcal{S}m_k$, we have a morphism $\mathbf{H}^{\mathbb{A}^1}_*(f; R) \to \mathbf{H}^S_*(f; R)$ in $\mathcal{M}od_k^{\mathbb{A}^1}(R)$. The following is an analogue of the result of Asok [**As**, Cor. 3.4] in higher degree.

Proposition 2.8. *Let* $f\colon X \to Y$ *be a morphism in* $\mathcal{S}m_k$ *and let* $n \geqslant 0$*. If* $\mathbf{H}_i^{\mathbb{A}^1}(f; R) = 0$ *for all* $i < n$*, then the natural morphism* $\mathbf{H}_n^{\mathbb{A}^1}(f; R) \to \mathbf{H}^S_n(f; R)$ *is universal with respect to the canonical functor* $\mathbf{NST}_k^{\mathbb{A}^1}(R) \to \mathcal{M}od_k^{\mathbb{A}^1}(R)$*.*

Proof. By induction on n and Yoneda's lemma in $\mathbf{NST}_k^{\mathbb{A}^1}(R)$, we may assume that $\mathbf{H}^S_i(f; R) = 0$ for all $i < n$. For $A \in \mathbf{NST}_k^{\mathbb{A}^1}(R)$, we have

$$\mathrm{Hom}_{\mathcal{M}od_k^{\mathbb{A}^1}(R)}(\mathbf{H}_n^{\mathbb{A}^1}(f; R), A) \cong \mathrm{Hom}_{\mathbf{D}(k,R)}(C(f; R), A[n]), \tag{5}$$

$$\mathrm{Hom}_{\mathbf{NST}_k^{\mathbb{A}^1}(R)}(\mathbf{H}^S_i(f; R), A) \cong \mathrm{Hom}_{\mathbf{D}_{tr}(k,R)}(R_{tr}(f), A[n]). \tag{6}$$

On the other hand, the adjunction $\mathbf{D}(k, R) \rightleftarrows \mathbf{D}_{tr}(k, R)$ gives

$$\mathrm{Hom}_{\mathbf{D}_{tr}(k,R)}(R_{tr}(f), A[n]) \xrightarrow{\cong} \mathrm{Hom}_{\mathbf{D}(k,R)}(R(f), A[n]) \tag{7}$$

such that the diagram

$$\begin{array}{ccc}
\mathrm{Hom}_{\mathbf{NST}_k^{\mathbb{A}^1}(R)}(\mathbf{H}_n^S(f;R),A) & \xrightarrow{\cong} & \mathrm{Hom}_{\mathbf{D}_{tr}(k,R)}(R_{tr}(f),A[n]) \\
\downarrow & & \cong\downarrow \\
\mathrm{Hom}_{\mathcal{M}od_k^{\mathbb{A}^1}(R)}(\mathbf{H}_n^{\mathbb{A}^1}(f;R),A) & \xrightarrow{\cong} & \mathrm{Hom}_{\mathbf{D}(k,R)}(R(f),A[n])
\end{array}$$

commutes. □

By Proposition 2.8 and Yoneda's lemma in $\mathbf{NST}_k^{\mathbb{A}^1}(R)$, if $\mathbf{H}_i^{\mathbb{A}^1}(f;R)=0$ for all $i<n$, then $\mathbf{H}_i^S(f;R)=0$ for all $i<n$. By using the $\mathbb{A}^1$-Hurewicz theorem, we obtain an analogue of the Hurewicz theorem relating $\mathbb{A}^1$-homotopy and Suslin homology.

Corollary 2.9. *(1) Let $(X,*)$ be a pointed smooth k-scheme which is $\mathbb{A}^1$-$(n-1)$-connected for $n\geqslant 1$. Then there exists a universal morphism $\pi_n^{\mathbb{A}^1}(X,*)\to\mathbf{H}_n^S(X;\mathbb{Z})$ with respect to the functor $\mathbf{NST}_k^{\mathbb{A}^1}(\mathbb{Z})\to\mathcal{G}r_k^{\mathbb{A}^1}$.*

(2) Let f be an $\mathbb{A}^1$-$(n-1)$-connected morphism of $\mathbb{A}^1$-simply connected pointed smooth k-schemes for $n\geqslant 2$. Then there exists a universal morphism $\pi_n^{\mathbb{A}^1}(f)\to\mathbf{H}_n^S(f;\mathbb{Z})$ with respect to the functor $\mathbf{NST}_k^{\mathbb{A}^1}(\mathbb{Z})\to\mathcal{G}r_k^{\mathbb{A}^1}$.

Proof. (1) follows from Proposition 2.8 and Theorem 2.3, and (2) follows from Propositions 2.8 and 2.4. □

3. Proof of the main theorems

In this section, we prove Theorems 1.1 and 1.2.

3.1. $\mathbb{A}^1$-Whitehead theorem with dimension bound

For the proof of Theorems 1.1 and 1.2, we consider the Nisnevich cohomology of morphisms. For a morphism $f\colon X\to Y$ in $\mathcal{S}m_k$, $A\in\mathcal{A}b_k$ and $n\geqslant 0$, we define

$$H^n_{Nis}(f;A)=\mathrm{Hom}_{\mathbf{D}(k,\mathbb{Z})}(C(f;\mathbb{Z}),A[n]).$$

We write $H^n_{Nis}(X,U;A)=H^n_{Nis}(i;A)$ for an embedding $i\colon U\hookrightarrow X$.

Proposition 3.1. *Let $f\colon X\to Y$ be a morphism in $\mathcal{S}m_k$. We write $d=\max\{\dim X+1,\dim Y\}$.*

(1) If $\mathbf{H}_i^{\mathbb{A}^1}(f;R)=0$ for all $i\leqslant d$, then f induces an isomorphism $C^{\mathbb{A}^1}(X;R)\cong C^{\mathbb{A}^1}(Y;R)$ in $\mathbf{D}(k,R)$.

(2) If $\mathbf{H}_i^S(f;R)=0$ for all $i\leqslant d$, then f induces an isomorphism of motives $M(X;R)\cong M(Y;R)$ in $\mathbf{DM}^{eff}(k,R)$.

Proof. (1) For each $m>d$, we only need to show that if $\mathbf{H}_i^{\mathbb{A}^1}(f;R)=0$ for all $i\leqslant m$, then $\mathbf{H}_{m+1}^{\mathbb{A}^1}(f;R)=0$. By (5), there exists a natural isomorphism

$$H^{m+1}_{Nis}(f;A)\cong\mathrm{Hom}_{\mathcal{M}od_k^{\mathbb{A}^1}(R)}(\mathbf{H}_{m+1}^{\mathbb{A}^1}(f;R),A)$$

for every $A\in\mathcal{M}od_k^{\mathbb{A}^1}(R)$. The left hand side vanishes by the exact sequence

$$\cdots\to H^m_{Nis}(X;A)\to H^{m+1}_{Nis}(f;A)\to H^{m+1}_{Nis}(Y;A)\to\cdots$$

and [**Ni**, Thm. 1.32]. Therefore, Yoneda's lemma in $\mathcal{M}od_k^{\mathbb{A}^1}(R)$ shows that $\mathbf{H}_{m+1}^{\mathbb{A}^1}(f;R)=0$.

(2) For each $m > d$, we only need to show that if $\mathbf{H}_i^S(f;R) = 0$ for all $i \leqslant m$, then $\mathbf{H}_{m+1}^S(f;R) = 0$. By (6), there exists a natural isomorphism

$$\mathrm{Hom}_{\mathbf{D}_{tr}(k,R)}(R_{tr}(f), A[m+1]) \cong \mathrm{Hom}_{\mathbf{NST}_k^{\mathbb{A}^1}(R)}(\mathbf{H}_{m+1}^S(f;R), A)$$

for every $A \in \mathbf{NST}_k^{\mathbb{A}^1}(R)$. By (7), the left hand side coincides with $H_{Nis}^{m+1}(f;A)$, and thus vanishes by the proof of (1). Since Yoneda's lemma in $\mathbf{NST}_k^{\mathbb{A}^1}(R)$ shows that $\mathbf{H}_{m+1}^{\mathbb{A}^1}(f;R) = 0$, we have $M(f;R) = 0$. □

We can now prove the $\mathbb{A}^1$-Whitehead theorem with dimension bound.

Theorem 3.2. *Assume k perfect. Let $f\colon (X,*) \to (Y,*)$ be a morphism of $\mathbb{A}^1$-simply connected pointed smooth k-schemes. If $\mathbf{H}_i^{\mathbb{A}^1}(f;R) = 0$ for all $2 \leqslant i \leqslant \max\{\dim X + 1, \dim Y\}$, then f is an $\mathbb{A}^1$-weak equivalence.*

Proof. The morphism $\mathbf{H}_0^{\mathbb{A}^1}(X;\mathbb{Z}) \to \mathbf{H}_0^{\mathbb{A}^1}(Y;\mathbb{Z})$ is an isomorphism by [**As**, Prop. 3.5]. Moreover, $\mathbf{H}_1^{\mathbb{A}^1}(X;\mathbb{Z}) = \mathbf{H}_1^{\mathbb{A}^1}(Y;\mathbb{Z}) = 0$ by [**Mo2**, Thm. 6.35]. Thus we have $\mathbf{H}_0^{\mathbb{A}^1}(f;\mathbb{Z}) = \mathbf{H}_1^{\mathbb{A}^1}(f;\mathbb{Z}) = 0$. Therefore, our assumption and Proposition 3.1 show that $\mathbf{H}_i^{\mathbb{A}^1}(f;\mathbb{Z}) = 0$ for all $i \in \mathbb{Z}$. Thus we have $\pi_i^{\mathbb{A}^1}(f) = 0$ for every $i \geqslant 0$ by Corollary 2.6. Since f induces $\pi_i^{\mathbb{A}^1}(X,*) \cong \pi_i^{\mathbb{A}^1}(Y,*)$ for all $i \geqslant 0$, the morphism f is an $\mathbb{A}^1$-weak equivalence by [**MV**, Prop. 2.14]. □

Theorem 3.2 implies the following.

Corollary 3.3. *Assume k perfect. Let $f\colon (X,*) \to (Y,*)$ be a morphism of pointed smooth k-schemes and let $d = \max\{\dim X + 1, \dim Y\}$. Suppose that $(X,*)$ is $\mathbb{A}^1$-simply connected and $(Y,*)$ is $\mathbb{A}^1$-connected. If f is $\mathbb{A}^1$-d-connected, then f is an $\mathbb{A}^1$-weak equivalence.*

Proof. By $d \geqslant 1$, the exact sequence

$$0 = \pi_1^{\mathbb{A}^1}(X,*) \to \pi_1^{\mathbb{A}^1}(Y,*) \to \pi_1^{\mathbb{A}^1}(f) = 0$$

shows that $(Y,*)$ is $\mathbb{A}^1$-simply connected. On the other hand, by Proposition 2.4, $\mathbf{H}_i^{\mathbb{A}^1}(f;R) = 0$ for all $i \leqslant d$. Thus f is an $\mathbb{A}^1$-weak equivalence by Theorem 3.2. □

Proposition 3.1 also has the following application.

Corollary 3.4. *Assume k perfect. Let $f\colon (X,*) \to (Y,*)$ be a morphism of pointed smooth k-schemes. We assume that $\mathbf{H}_i^{\mathbb{A}^1}(f;R) = 0$ for all $i \leqslant \max\{\dim X + 1, \dim Y\}$. Then the morphism $S^2 \wedge f\colon S^2 \wedge X \to S^2 \wedge Y$ is an $\mathbb{A}^1$-weak equivalence. Moreover, if X and Y are $\mathbb{A}^1$-connected, then the morphism $S^1 \wedge f\colon S^1 \wedge X \to S^1 \wedge Y$ is an $\mathbb{A}^1$-weak equivalence.*

Proof. For a k-space $\mathcal{X}$, the suspension $S^1 \wedge \mathcal{X}$ is $\mathbb{A}^1$-connected by [**Mo2**, Thm. 6.38]. Similarly, if $\mathcal{X}$ is $\mathbb{A}^1$-connected, then $S^1 \wedge \mathcal{X}$ is $\mathbb{A}^1$-simply connected. Since f induces isomorphisms for all $\mathbb{A}^1$-homology sheaves by Proposition 3.1, so does $S^1 \wedge f$. Therefore, Corollary 2.6 shows that $S^2 \wedge f$ is an $\mathbb{A}^1$-weak equivalence. Similarly, $S^1 \wedge f$ is an $\mathbb{A}^1$-weak equivalence when X and Y are $\mathbb{A}^1$-connected. □

3.2. $\mathbb{A}^1$-excision theorem

We next prove an excision theorem for $\mathbb{A}^1$- and Suslin homology. This is an analogue of [**As**, Prop. 3.8] in higher degree.

Theorem 3.5. *Let X be a smooth k-scheme and U a Zariski open set of X whose complement has codimension r. Then the morphisms $\mathbf{H}_i^{\mathbb{A}^1}(U;R) \to \mathbf{H}_i^{\mathbb{A}^1}(X;R)$ and $\mathbf{H}_i^S(U;R) \to \mathbf{H}_i^S(X;R)$ are isomorphisms for every $i < r-1$ and epimorphisms for $i = r-1$.*

Proof. It suffices to prove that $\mathbf{H}_i^{\mathbb{A}^1}(X,U;R) = \mathbf{H}_i^S(X,U;R) = 0$ for all $i < r$. By Proposition 2.8, we only need to prove this for the $\mathbb{A}^1$-homology sheaves. We use induction on i. The case $i < 0$ follows from [**Mo2**, Thm. 6.22]. We assume $i \geqslant 0$ and $\mathbf{H}_j^{\mathbb{A}^1}(X,U;R) = 0$ for all $j < i$. By (5), we have

$$H^i_{Nis}(X,U;A) \cong \mathrm{Hom}_{\mathcal{M}od_k^{\mathbb{A}^1}(R)}(\mathbf{H}_i^{\mathbb{A}^1}(X,U;R),A)$$

for every $A \in \mathcal{M}od_k^{\mathbb{A}^1}(R)$. Then the left hand side vanishes by [**Mo1**, Lem. 6.4.4]. Therefore, we have $\mathbf{H}_i^{\mathbb{A}^1}(X,U;R) = 0$ by Yoneda's lemma in $\mathcal{M}od_k^{\mathbb{A}^1}(R)$. □

Theorem 3.5 gives the excision theorem for $\mathbb{A}^1$-homotopy.

Corollary 3.6. *Assume k perfect. Let $(X,*)$ be a pointed smooth k-scheme and $(U,*)$ a pointed Zariski open set of $(X,*)$ whose complement has codimension r. If $(X,*)$ and $(U,*)$ are $\mathbb{A}^1$-simply connected, then the morphism $\pi_i^{\mathbb{A}^1}(U) \to \pi_i^{\mathbb{A}^1}(X)$ is an isomorphism for every $i < r-1$ and an epimorphism for $i = r-1$. In other words, the pair (X,U) is $\mathbb{A}^1$-$(r-1)$-connected.*

Proof. Since $\mathbf{H}_i^{\mathbb{A}^1}(X,U;\mathbb{Z}) = 0$ for all $i < r$ by Theorem 3.5, the pair (X,U) is $\mathbb{A}^1$-$(r-1)$-connected by Corollary 2.6. □

4. $\mathbb{A}^1$-homology of a hyperplane embedding

In this section, as an example of relative $\mathbb{A}^1$-homology, we compute the $\mathbb{A}^1$-homology of a hyperplane embedding $\mathbb{P}^{n-1} \hookrightarrow \mathbb{P}^n$ in degree $\leqslant n$. By a suitable linear change of coordinates, we may regard $\mathbb{P}^{n-1}$ as the hyperplane in $\mathbb{P}^n$ defined by $x_n = 0$, where $x_0, \dots, x_n$ denote homogeneous coordinates on $\mathbb{P}^n$. Let $\underline{\mathbf{K}}_n^{MW}$ be the unramified Milnor–Witt K-theory defined by Morel [**Mo2**]. For $A \in \mathcal{A}b_k$, we write $A \otimes^{\mathbb{A}^1} R = H_0(L_{\mathbb{A}^1}(A \otimes R))$ which is called $\mathbb{A}^1$-*tensor product* by Morel [**Mo1**]. Our main result is the following.

Theorem 4.1. *For $0 \leqslant i \leqslant n$, $n > 0$, we have*

$$\mathbf{H}_i^{\mathbb{A}^1}(\mathbb{P}^n,\mathbb{P}^{n-1};R) \cong \begin{cases} \underline{\mathbf{K}}_n^{MW} \otimes^{\mathbb{A}^1} R & (i = n), \\ 0 & (i < n). \end{cases}$$

In particular, when $i < n$, we have $\mathbf{H}_i^{\mathbb{A}^1}(\mathbb{P}^n;R) \cong \mathbf{H}_i^{\mathbb{A}^1}(\mathbb{P}^{i+1};R)$.

When $i = n$, we have the following description.

Corollary 4.2. *There exists a morphism* $\underline{\mathbf{K}}^{MW}_{n+1} \otimes^{\mathbb{A}^1} R \to \mathbf{H}^{\mathbb{A}^1}_n(\mathbb{P}^n; R)$ *such that*

$$\mathbf{H}^{\mathbb{A}^1}_n(\mathbb{P}^{n+1}; R) \cong \mathrm{Coker}(\underline{\mathbf{K}}^{MW}_{n+1} \otimes^{\mathbb{A}^1} R \to \mathbf{H}^{\mathbb{A}^1}_n(\mathbb{P}^n; R)).$$

Proof. By Theorem 4.1, the homology exact sequence

$$\mathbf{H}^{\mathbb{A}^1}_{n+1}(\mathbb{P}^{n+1}, \mathbb{P}^n; R) \to \mathbf{H}^{\mathbb{A}^1}_n(\mathbb{P}^n; R) \to \mathbf{H}^{\mathbb{A}^1}_n(\mathbb{P}^{n+1}; R) \to \mathbf{H}^{\mathbb{A}^1}_n(\mathbb{P}^{n+1}, \mathbb{P}^n; R) = 0$$

gives an isomorphism

$$\begin{aligned}\mathbf{H}^{\mathbb{A}^1}_n(\mathbb{P}^{n+1}; R) &\cong \mathrm{Coker}(\mathbf{H}^{\mathbb{A}^1}_{n+1}(\mathbb{P}^{n+1}, \mathbb{P}^n; R) \to \mathbf{H}^{\mathbb{A}^1}_n(\mathbb{P}^n; R)) \\ &\cong \mathrm{Coker}(\underline{\mathbf{K}}^{MW}_{n+1} \otimes^{\mathbb{A}^1} R \to \mathbf{H}^{\mathbb{A}^1}_n(\mathbb{P}^n; R)).\end{aligned}$$

□

Theorem 4.1 and Proposition 2.8 imply the following.

Corollary 4.3. *For all* $i < n$, *we have* $\mathbf{H}^S_i(\mathbb{P}^n, \mathbb{P}^{n-1}; R) = 0$. *Moreover, there exists a universal morphism* $\underline{\mathbf{K}}^{MW}_n \otimes^{\mathbb{A}^1} R \to \mathbf{H}^S_n(\mathbb{P}^n, \mathbb{P}^{n-1}; R)$ *with respect to the canonical functor* $\mathbf{NST}^{\mathbb{A}^1}_k(R) \to \mathcal{M}od^{\mathbb{A}^1}_k(R)$. *In particular, when* $i < n$ *we have*

$$\mathbf{H}^S_i(\mathbb{P}^n; R) \cong \mathbf{H}^S_i(\mathbb{P}^{i+1}; R).$$

Remark 4.4. The vanishing $\mathbf{H}^{\mathbb{A}^1}_i(\mathbb{P}^n, \mathbb{P}^{n-1}; R) = 0$ for $i < n$ is an example where the Lefschetz hyperplane theorem holds. However, it is not true in general that $\mathbf{H}^{\mathbb{A}^1}_i(X, H; R) = 0$ for $i < \dim X$ and $H \subseteq X$ a very ample divisor. Indeed, let $C \subseteq \mathbb{P}^2$ be a smooth plane curve of degree $\geqslant 3$. Since C is not rational, it is not $\mathbb{A}^1$-connected by [**AM**, Prop. 2.1.12]. Therefore, the canonical morphism $\mathbf{H}^{\mathbb{A}^1}_0(C; R) \to R$ is not an isomorphism by [**As**, Thm. 4.14]. Thus the morphism $\mathbf{H}^{\mathbb{A}^1}_0(C; R) \to \mathbf{H}^{\mathbb{A}^1}_0(\mathbb{P}^2; R) \cong R$ is not an isomorphism.

4.1. A basic distinguished triangle

For the proof of Theorem 4.1, we compute the mapping cone of $C^{\mathbb{A}^1}(\mathbb{A}^n - \{0\}; R) \to C^{\mathbb{A}^1}(\mathbb{P}^{n-1}; R)$. We first give a Zariski excision result for $\mathbb{A}^1$-homology.

Lemma 4.5. *Let* $\{U, V\}$ *be a Zariski covering of a smooth* k*-scheme* X. *Then the morphism* $(U, U \cap V) \to (X, V)$ *induces a quasi-isomorphism*

$$C^{\mathbb{A}^1}(U, U \cap V; R) \cong C^{\mathbb{A}^1}(X, V; R).$$

Proof. Since the functor $L_{\mathbb{A}^1}$ is exact (see, *e.g.*, [**CD**, Thm. 2.5]), we only need to show that the morphism $(U, U \cap V) \to (X, V)$ induces a quasi-isomorphism

$$C(U, U \cap V; R) \cong C(X, V; R). \tag{8}$$

For each open set $W \subseteq X$, we regard $R(W)$ as a subsheaf of $R(X)$. Then by the short exact sequence in [**AD**, proof of Prop. 3.32], we see that $R(U \cap V) = R(U) \cap R(V)$ and $R(X) = R(U) + R(V)$. Thus we have an isomorphism

$$R(U)/R(U \cap V) = R(U)/(R(U) \cap R(V)) \cong (R(U) + R(V))/R(V) = R(X)/R(V).$$

Finally, $R(U)/R(U \cap V)$ and $R(X)/R(V)$ are canonically quasi-isomorphic to $C(U, U \cap V; R)$ and $C(X, V; R)$, respectively. Therefore, we obtain (8). □

We obtain the following distinguished triangle in $\mathbf{D}(k, R)$.

Proposition 4.6. *For $n \geqslant 1$ and $* \in \mathbb{P}^n(k)$, we have a distinguished triangle*

$$C^{\mathbb{A}^1}(\mathbb{A}^n - \{0\}; R) \to C^{\mathbb{A}^1}(\mathbb{P}^{n-1}; R) \to C^{\mathbb{A}^1}(\mathbb{P}^n, *; R) \to$$

in $\mathbf{D}(k, R)$, where the first morphism is induced by the $\mathbb{G}_m$-quotient and the second morphism is induced by $(\mathbb{P}^{n-1}, \emptyset) \to (\mathbb{P}^n, *)$.

Proof. We write $U = \mathbb{P}^n - \{(0 : \ldots : 0 : 1)\}$. Since the projection $\rho\colon U \to \mathbb{P}^{n-1}$ is a vector bundle, it is an $\mathbb{A}^1$-weak equivalence by [**MV**, Example 2.2]. We denote by V the Zariski open set of $\mathbb{P}^n$ defined by $x_n \neq 0$. Since the diagram

$$\begin{array}{ccc} \mathbb{A}^n - \{0\} & \xrightarrow{/\mathbb{G}_m} & \mathbb{P}^{n-1} \\ {\scriptstyle\cong}\uparrow & & \uparrow{\scriptstyle\rho} \\ U \cap V & \xrightarrow{\subseteq} & U \end{array}$$

commutes, we obtain the commutative diagram

$$\begin{array}{ccc} C^{\mathbb{A}^1}(\mathbb{A}^n - \{0\}; R) & \longrightarrow & C^{\mathbb{A}^1}(\mathbb{P}^{n-1}; R) \\ {\scriptstyle\cong}\downarrow & & \downarrow{\scriptstyle\cong} \\ C^{\mathbb{A}^1}(U \cap V; R) & \longrightarrow & C^{\mathbb{A}^1}(U; R). \end{array} \tag{9}$$

On the other hand, Lemma 4.5 gives the commutative diagram

$$\begin{array}{ccc} C^{\mathbb{A}^1}(\mathbb{P}^{n-1}; R) & \longrightarrow & C^{\mathbb{A}^1}(\mathbb{P}^n, V; R) \\ {\scriptstyle\cong}\downarrow & & \downarrow{\scriptstyle\cong} \\ C^{\mathbb{A}^1}(U; R) & \longrightarrow & C^{\mathbb{A}^1}(U, U \cap V; R). \end{array} \tag{10}$$

By the diagrams (9) and (10), we obtain an isomorphism of triangles

$$\begin{array}{ccccccc} C^{\mathbb{A}^1}(\mathbb{A}^n - \{0\}; R) & \longrightarrow & C^{\mathbb{A}^1}(\mathbb{P}^{n-1}; R) & \longrightarrow & C^{\mathbb{A}^1}(\mathbb{P}^{n-1}, V; R) & \longrightarrow & \\ {\scriptstyle\cong}\downarrow & & {\scriptstyle\cong}\downarrow & & {\scriptstyle\cong}\downarrow & & \\ C^{\mathbb{A}^1}(U \cap V; R) & \longrightarrow & C^{\mathbb{A}^1}(U; R) & \longrightarrow & C^{\mathbb{A}^1}(U, U \cap V; R) & \longrightarrow & . \end{array}$$

Since the lower triangle is distinguished, so is the upper triangle. Therefore, we only need to show that for a k-rational point $* \in V(k)$, the morphism $(\mathbb{P}^{n-1}, *) \to (\mathbb{P}^{n-1}, V)$ induces a quasi-isomorphism $C^{\mathbb{A}^1}(\mathbb{P}^n, *; R) \cong C^{\mathbb{A}^1}(\mathbb{P}^{n-1}, V; R)$. Since $V \cong \mathbb{A}^n$, the exact sequence

$$0 = \mathbf{H}_i^{\mathbb{A}^1}(V; R) \to \mathbf{H}_i^{\mathbb{A}^1}(\mathbb{P}^n; R) \to \mathbf{H}_i^{\mathbb{A}^1}(\mathbb{P}^n, V; R) \to \mathbf{H}_{i-1}^{\mathbb{A}^1}(V; R) = 0$$

shows that $\mathbf{H}_i^{\mathbb{A}^1}(\mathbb{P}^n; R) \cong \mathbf{H}_i^{\mathbb{A}^1}(\mathbb{P}^n, V; R)$ for all $i \geqslant 2$. For $i = 0, 1$, there exists an exact sequence

$$\begin{aligned} 0 = \mathbf{H}_1^{\mathbb{A}^1}(V; R) \to \mathbf{H}_1^{\mathbb{A}^1}(\mathbb{P}^n; R) \to \mathbf{H}_1^{\mathbb{A}^1}(\mathbb{P}^n, V; R) & \\ \to \mathbf{H}_0^{\mathbb{A}^1}(V; R) \to \mathbf{H}_0^{\mathbb{A}^1}(\mathbb{P}^n; R) \to \mathbf{H}_0^{\mathbb{A}^1}(\mathbb{P}^n, V; R) \to 0. & \end{aligned}$$

Then the morphism $\mathbf{H}_0^{\mathbb{A}^1}(V; R) \to \mathbf{H}_0^{\mathbb{A}^1}(\mathbb{P}^n; R)$ is an isomorphism by [**As**, Prop. 3.5]. Therefore, we have $\mathbf{H}_0^{\mathbb{A}^1}(\mathbb{P}^n, *; R) = \mathbf{H}_0^{\mathbb{A}^1}(\mathbb{P}^n, V; R) = 0$ and $\mathbf{H}_1^{\mathbb{A}^1}(\mathbb{P}^n; R) \cong \mathbf{H}_1^{\mathbb{A}^1}(\mathbb{P}^n, V; R)$. Thus $C^{\mathbb{A}^1}(\mathbb{P}^n, *; R) \to C^{\mathbb{A}^1}(\mathbb{P}^{n-1}, V; R)$ is a quasi-isomorphism. $\square$

4.2. Proof of Theorem 4.1

We can now prove Theorem 4.1.

Proof of Theorem 4.1. We first prove $\mathbf{H}_i^{\mathbb{A}^1}(\mathbb{P}^n, \mathbb{P}^{n-1}; R) = 0$ for all $i < n$. The $\mathbb{A}^1$-weak equivalence ρ as in the proof of Proposition 4.6 gives an isomorphism $\mathbf{H}_i^{\mathbb{A}^1}(\mathbb{P}^{n-1}; R) \xrightarrow{\cong} \mathbf{H}_i^{\mathbb{A}^1}(\mathbb{P}^n - \{0\}; R)$. On the other hand, $\mathbf{H}_i^{\mathbb{A}^1}(\mathbb{P}^n - \{0\}; R) \to \mathbf{H}_i^{\mathbb{A}^1}(\mathbb{P}^n; R)$ is an isomorphism for all $i < n-1$ and an epimorphism for $i = n-1$ by Theorem 3.5. Thus we have $\mathbf{H}_i^{\mathbb{A}^1}(\mathbb{P}^n, \mathbb{P}^{n-1}; R) = 0$.

Next, we prove $\mathbf{H}_n^{\mathbb{A}^1}(\mathbb{P}^n, \mathbb{P}^{n-1}; R) \cong \underline{\mathbf{K}}_n^{MW} \otimes^{\mathbb{A}^1} R$ for all $n \geqslant 2$. By Proposition 4.6, there exists a morphism of distinguished triangles

$$\begin{array}{ccccccc}
C^{\mathbb{A}^1}(\mathbb{P}^{n-1}; R) & \longrightarrow & C^{\mathbb{A}^1}(\mathbb{P}^n; R) & \longrightarrow & C^{\mathbb{A}^1}(\mathbb{P}^n, \mathbb{P}^{n-1}; R) & \longrightarrow & \\
\Vert & & {\scriptstyle\alpha}\downarrow & & {\scriptstyle\beta}\downarrow & & \\
C^{\mathbb{A}^1}(\mathbb{P}^{n-1}; R) & \longrightarrow & C^{\mathbb{A}^1}(\mathbb{P}^n, *; R) & \longrightarrow & C^{\mathbb{A}^1}(\mathbb{A}^n - \{0\}; R)[1] & \longrightarrow & ,
\end{array}$$

where α is induced by $(\mathbb{P}^n, \emptyset) \to (\mathbb{P}^n, *)$. Taking the homology exact sequence, we obtain $\mathbf{H}_n^{\mathbb{A}^1}(\mathbb{P}^n, \mathbb{P}^{n-1}; R) \cong \mathbf{H}_{n-1}^{\mathbb{A}^1}(\mathbb{A}^n - \{0\}; R)$ for all $n \geqslant 2$ by the five lemma. Note that the adjunctions $\mathcal{A}b_k \rightleftarrows \mathcal{M}od_k(R)$ and (4) show that the functor $- \otimes^{\mathbb{A}^1} R\colon \mathcal{A}b_k \to \mathcal{M}od_k^{\mathbb{A}^1}(R)$ is left adjoint to the canonical functor $\mathcal{M}od_k^{\mathbb{A}^1}(R) \to \mathcal{A}b_k$. Moreover, for every $X \in \mathcal{S}m_k$ which is $\mathbb{A}^1$-$(n-1)$-connected, the adjunction $\mathcal{A}b_k \rightleftarrows \mathcal{M}od_k^{\mathbb{A}^1}(R)$ leads to a natural isomorphism

$$\mathbf{H}_n^{\mathbb{A}^1}(X; R) \cong \mathbf{H}_n^{\mathbb{A}^1}(X; \mathbb{Z}) \otimes^{\mathbb{A}^1} R. \tag{11}$$

Hence, we have

$$\begin{aligned}
\mathrm{Hom}_{\mathcal{M}od_k(R)}(\mathbf{H}_{n-1}^{\mathbb{A}^1}(\mathbb{A}^n - \{0\}; R), A) &\cong \mathrm{Hom}_{\mathcal{A}b_k}(\mathbf{H}_{n-1}^{\mathbb{A}^1}(\mathbb{A}^n - \{0\}; \mathbb{Z}), A) && (12)\\
&\cong \mathrm{Hom}_{\mathcal{A}b_k}(\pi_{n-1}^{\mathbb{A}^1}(\mathbb{A}^n - \{0\}), A) && (13)\\
&\cong \mathrm{Hom}_{\mathcal{A}b_k}(\underline{\mathbf{K}}_n^{MW}, A) && (14)\\
&\cong \mathrm{Hom}_{\mathcal{M}od_k^{\mathbb{A}^1}(R)}(\underline{\mathbf{K}}_n^{MW} \otimes^{\mathbb{A}^1} R, A), && (15)
\end{aligned}$$

where (12) and (15) follow from the adjunction $\mathcal{A}b_k \rightleftarrows \mathcal{M}od_k^{\mathbb{A}^1}$, (13) from Theorem 2.3 and (14) from [**Mo2**, Thm. 6.40]. Thus Yoneda's lemma in $\mathcal{M}od_k^{\mathbb{A}^1}(R)$ shows that

$$\mathbf{H}_n^{\mathbb{A}^1}(\mathbb{P}^n, \mathbb{P}^{n-1}; R) \cong \mathbf{H}_{n-1}^{\mathbb{A}^1}(\mathbb{A}^n - \{0\}; R) \cong \underline{\mathbf{K}}_n^{MW} \otimes^{\mathbb{A}^1} R.$$

Finally, we prove $\mathbf{H}_1^{\mathbb{A}^1}(\mathbb{P}^1; R) \cong \underline{\mathbf{K}}_1^{MW} \otimes^{\mathbb{A}^1} R$. For $R = \mathbb{Z}$, this is a direct consequence of [**MV**, Lem. 2.15 and Cor. 2.18] and [**Mo2**, Thm. 3.37]. Since $\mathbb{P}^1$ is $\mathbb{A}^1$-connected, the general case follows from (11). □

References

[**AE**] B. Antieau and E. Elmanto, *A primer for unstable motivic homotopy theory*, Survey on recent developments in algebraic geometry, 305–370, Proc. Sympos. Pure Math., **95**, Amer. Math. Soc, Providence, RI (2017).

[**As**] A. Asok, *Birational invariants and $\mathbb{A}^1$-connectedness*, J. Reine Angew. Math., **681** (2012) 39–64.

[**Ar**] M. Arkowitz, *Introduction to homotopy theory*, Universitext, Springer, New York (2011).

[**AD**] A. Asok and B. Doran, *$\mathbb{A}^1$-homotopy groups, excision, and solvable quotients*, Adv. Math. **221**(4) (2009) 1144–1190.

[**AM**] A. Asok and F. Morel, *Smooth varieties up to $\mathbb{A}^1$-homotopy and algebraic h-cobordisms*, Adv. Math. **227**(5) (2011) 1990–2058.

[**CD**] D.-C. Cisinski and F. Déglise, *Local and stable homological algebra in Grothendieck abelian categories*, Homology, Homotopy Appl. **11**(1) (2009) 219–260.

[**GJ**] P. Goerss and R. Jardine, *Simplicial homotopy theory*, Progress in Mathematics, **174**, Birkhäuser Verlag, Basel (1999).

[**Mo1**] F. Morel, *The stable $\mathbb{A}^1$-connectivity theorems*, K-theory, **35**(1–2) (2005) 1–68.

[**Mo2**] F. Morel, *$\mathbb{A}^1$-algebraic topology over a field*, Lecture Notes in Math., **2052**, Springer, Heidelberg (2012).

[**MV**] F. Morel and V. Voevodsky, *$\mathbb{A}^1$-homotopy theory of schemes*, Publ. Math. IHES, **90** (1999) 45–143.

[**MVW**] V. Mazza, V. Voevodsky, and C. Weibel, *Lecture notes on motivic cohomology*, Volume 2 of Clay Mathematics Monographs, America Mathematical Society, Providence, RI (2006).

[**Ni**] Y. Nisnevich, *The completely decomposed topology on schemes and associated descent spectral sequence in algebraic K-theory*, Algebraic K-theory: connections with geometry and topology (Lake Louise, AB, 1987), NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., **279**, Kluwer Acad. Publ., Dordrecht (1989) 241–342.

[**SV**] A. Suslin and V. Voevodsky, *Singular homology of abstract varieties*, Invent. Math., **123**(1) (1996) 61–94.

[**Vo**] V. Voevodsky, *Triangulated categories of motives over a field*, Cycles, transfers, and motivic homology theories, Ann. of Math. Stud., **143**, Princeton Univ. Press, Princeton, NJ (2000) 188–238.

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