Homology, Homotopy and Applications, vol. 24(1), 2022, pp.205–215

REALISABILITY OF THE GROUP OF SELF-HOMOTOPY EQUIVALENCES AND LOCAL HOMOTOPY THEORY

MAHMOUD BENKHALIFA

(*communicated by Jelena Grbić*)

Abstract

We prove that any group G occurs as the quotient $\mathcal{E}(X)/\mathcal{E}_*(X)$, where $\mathcal{E}(X)$ denotes the group of self-homotopy equivalence of a certain CW-complex X and $\mathcal{E}_*(X)$ denotes its subgroup of the elements inducing the identity on the homology groups.

1. Introduction

For a simply connected space X, we are interested in the group of homotopy classes of self-homotopy equivalences, $\mathcal{E}(X)$, and the so-called Kahn's realisability problem of groups [**15**]. Namely, if a given group G can occur as $\mathcal{E}(X)$ for some space X.

For finite groups, in [**11**] Costoya and Viruel solved completely this problem by constructing a rational elliptic space X having formal dimension $n = 208 + 80|\mathcal{G}|$, where $\mathcal{G}$ is a certain finite graph associated with G and $|\mathcal{G}|$ denotes its order. The space X satisfies $\pi_k(X) = 0$ for all $k \geqslant 120$. Later on in [**4**], it is proven that we can realise the finite group G by a non-elliptic space having formal dimension $n = 120$, independently of the order of $\mathcal{G}$. Let's also mention that, in [**10**], the authors showed that rational points of every Lie type are also realised by a rational space such that $\mathcal{E}_*(X) = 1$. Here $\mathcal{E}_*(X)$ denotes the normal subgroup of $\mathcal{E}(X)$ of the elements inducing the identity on the homology groups.

It is worth noting that Kahn's realisability problem has been solved for generic spaces in [**7**], but it is still open in the realm of CW-complexes for infinite groups. As a way to approach this problem, in [**9**, Problem 19] the question of whether an arbitrary group can appear as the distinguished quotient $\mathcal{E}(X)/\mathcal{E}_*(X)$ is raised by A. Viruel. Recall that the group $\mathcal{E}(X)/\mathcal{E}_*(X)$ was already considered in [**6**] and [**8**] in case where X is a CW-complex with connectivity and dimension constrains

Inspired by the ideas developed in [**9**], this paper aims to solve completely Viruel's problem for any arbitrary group. More precisely, we establish

Theorem 1.1. *Any group G occurs as $\mathcal{E}(X)/\mathcal{E}_*(X)$, where the CW-complex X can be chosen as follows*

1. *X is a 19-connected and 388-dimensional $\mathbb{Z}_{(p)}$-localised space, where $p > 185$ is a prime number, which is of finite type if G is finite;*

Received April 1, 2021, revised June 6, 2021, June 12, 2021; published on April 6, 2022.
2010 Mathematics Subject Classification: 55P10.
Key words and phrases: Kahn's realisability problem of groups, group of homotopy self-equivalences, Anick's R-local homotopy theory.
Article available at `http://dx.doi.org/10.4310/HHA.2022.v24.n1.a11`

2. *$H_{115}(X,\mathbb{Z}_{(p)})$ and $H_{388}(X,\mathbb{Z}_{(p)})$ are free $\mathbb{Z}_{(p)}$-modules of infinite basis iff G is infinite;*
3. *$H_{20}(X,\mathbb{Z}_{(p)}) \cong H_{26}(X,\mathbb{Z}_{(p)}) \cong H_{34}(X,\mathbb{Z}_{(p)}) \cong H_{52}(X,\mathbb{Z}_{(p)}) \cong H_{68}(X,\mathbb{Z}_{(p)}) \cong \mathbb{Z}_{(p)}$;*
4. *$\mathcal{E}_*(X) \cong \mathrm{Hom}\big(H_{388}(X,\mathbb{Z}_{(p)}), \pi_{388}(X^{115})\big)$, where X^{115} is the 115-skeleton of X;*
5. *$\mathcal{E}_*(X)$ is infinite.*

Notice that as $\mathcal{E}_*(X)$ is infinite, it follows that our construction does not provide a solution to the original Kahn's problem (see Remark 5.3).

Acknowledgments

The author is deeply grateful to the referee for careful reading of the article and for the valuable suggestions which greatly improved the manuscript.

2. Preliminaries

2.1. Anick's $\mathbb{Z}_{(p)}$-local homotopy theory

In this paragraph we review briefly Anick's differential graded Lie algebra framework for $\mathbb{Z}_{(p)}$-local homotopy theory [**3**]. Let $\mathbf{CW}_m^{k+1}$ denote the category of m-connected, finite CW-complexes of dimension no greater than $k+1$ with m-skeleton reduced to a point. Let $\mathbf{CW}_m^{k+1}(\mathbb{Z}_{(p)})$ denote the category obtained by $\mathbb{Z}_{(p)}$-localising the spaces in $\mathbf{CW}_m^{k+1}$. When $k < \min(m+2p-3, mp-1)$ the homotopy category of $\mathbf{CW}_m^{k+1}(\mathbb{Z}_{(p)})$ is equivalent to the homotopy category $\mathbf{DGL}_m^k(\mathbb{Z}_{(p)})$ of the free differential graded Lie algebras $(\mathbb{L}(W),\partial)$ in which W is a free graded $\mathbb{Z}_{(p)}$-module satisfying $W_n = 0$ for $n < m$ and $n > k$.

Given a space X in $\mathbf{CW}_m^{k+1}(\mathbb{Z}_{(p)})$, Anick's model recovers homotopy data via

$$\pi_*(X) \cong H_{*-1}((\mathbb{L}(W)),\partial) \quad \text{and} \quad H_*(X,\mathbb{Z}_{(p)}) \cong H_{*-1}(W,d),$$

where d is the linear part of ∂ (see [**1**, Theorem 8.5]). Moreover, Anick's theory directly implies an identification of the form

$$\mathcal{E}(X) \cong \mathrm{aut}((\mathbb{L}(W),\partial))/\simeq, \tag{1}$$

where the latter is the group of differential graded Lie homotopy self-equivalences of $(\mathbb{L}(W),\partial)$ modulo the relation of homotopy in $\mathbf{DGL}_m^k(\mathbb{Z}_{(p)})$ (see [**1**, pp. 425–6]). We write $\mathcal{E}(\mathbb{L}(W)) \cong \mathrm{aut}((\mathbb{L}(W),\partial))/\simeq$ for this group. Similarly, we have

$$\mathcal{E}_*(X) \cong \mathrm{aut}_*((\mathbb{L}(W),\partial))/\simeq, \tag{2}$$

we denote the latter group by $\mathcal{E}_*(\mathbb{L}(W))$, where $\mathrm{aut}_*((\mathbb{L}(W),\partial))$ is the group of differential graded Lie automorphisms inducing the identity automorphism on $H_*(W,d)$.

2.2. Strongly connected digraphs and theorem of de Groot

A digraph (i.e. a directed graph) $\mathcal{G} = (V(\mathcal{G}), E(\mathcal{G}))$, where $V(\mathcal{G})$ denotes the set of the vertices of $\mathcal{G}$ and $E(\mathcal{G})$ the set of its edges, is strongly connected if for every $v,u \in V(\mathcal{G})$, there exists an integer $m \in \mathbb{N}$ and vertices $v = v_0, v_1, \dots, v_m = u$ such that $(v_i, v_{i+1}) \in E(G), i = 0,1,\dots,m$.

The following theorem, due to J. de Groot [**12**, p. 96], plays a central role in this paper.

Theorem 2.1. *Any group G is isomorphic to the automorphism group of a some strongly connected digraph $\mathcal{G}$.*

Remark 2.2. It should be noted that, in [**12**], de Groot is not considering strongly connected digraphs, but just graphs or digraphs. But any simple graph can be seen as a symmetric digraph, [**14**, §1.1], if it is connected, then the associated digraph is strongly connected.

2.3. Two short exact sequences

Let p be a prime and let $(\mathbb{L}(W_q \oplus W_{\leqslant n}), \partial)$, with $q > n+1$, be an object in $\mathbf{DGL}^k_m(\mathbb{Z}_{(p)})$, where $k < \min(m+2p-3, mp-1)$. Assume that ∂ is decomposable (i.e. the linear part of ∂ is zero).

We define the map $b_q \colon W_q \to H_{q-1}(\mathbb{L}(W_{\leqslant n}))$ by setting $b_q(w) = \{\partial(w)\}$. Here $\{\partial(w)\}$ denotes the homology class of $\partial(w) \in \mathbb{L}_{q-1}(W_{\leqslant n})$.

Let $\mathcal{C}^q$ be the subgroup of $\operatorname{aut}(W_q) \times \mathcal{E}(\mathbb{L}(W_{\leqslant n}))$ consisting of the couples $(\xi, [\alpha])$ making the following diagram commutes

$$\begin{array}{ccc} W_q & \xrightarrow{\quad\xi\quad} & W_q \\ \Big\downarrow{\scriptstyle b_q} & & \Big\downarrow{\scriptstyle b_q} \\ H_{q-1}(\mathbb{L}(W_{\leqslant n})) & \xrightarrow{H_{q-1}(\alpha)} & H_{q-1}(\mathbb{L}(W_{\leqslant n})) \end{array} \tag{3}$$

and let

$$\mathcal{C}^q_* = \big\{[\beta] \in \mathcal{E}_*(\mathbb{L}(W_{\leqslant n})) \quad \text{such that} \quad (id, [\beta]) \in \mathcal{C}^q\big\}. \tag{4}$$

Clearly $\mathcal{C}^q_*$ is a subgroup of $\mathcal{E}_*(\mathbb{L}(W_{\leqslant n}))$. In [**5**, Theorems 2.6 and 2.9] it is shown that

Theorem 2.3. *There exist two short exact sequences of groups*

$$\operatorname{Hom}\big(W_q, H_q(\mathbb{L}(W_{\leqslant n}))\big) \rightarrowtail \mathcal{E}(\mathbb{L}(W_q \oplus W_{\leqslant n})) \twoheadrightarrow \mathcal{C}^q, \tag{5}$$

$$\operatorname{Hom}\big(W_q, H_q(\mathbb{L}(W_{\leqslant n}))\big) \rightarrowtail \mathcal{E}_*(\mathbb{L}(W_q \oplus W_{\leqslant n})) \twoheadrightarrow \mathcal{C}^q_*. \tag{6}$$

Remark 2.4. For the sequence (6), we need to use the fact that ∂ is decomposable, the fact that $q > n+1$ and Remark 2.8 in [**5**].

3. Graded Lie $\mathbb{Z}_{(p)}$-algebras associated with strong connected digraphs

Definition 3.1. Let $\mathcal{G}$ be a strongly connected digraph with more than one vertex and $p > 185$. We define the following free differential graded Lie $\mathbb{Z}_{(p)}$-algebra

$$\mathcal{L}(\mathcal{G}) = \Big(\mathbb{L}(w_1, w_2, w_3, w_4, w_5, x_v, z_{(v,u)} \mid v \in V(\mathcal{G}), (v,u) \in E(\mathcal{G})), \partial\Big).$$

The degrees of the generators are as follows:

$$|w_1| = 19, \quad |w_2| = 25, \quad |w_3| = 51, \quad |w_4| = 33, \quad |w_5| = 67,$$

$$|x_v| = 114, \quad \forall v \in V(\mathcal{G}), \qquad |z_{(v,u)}| = 387, \quad \forall (v,u) \in E(\mathcal{G}).$$

The differential is given by

$$\partial(w_1) = \partial(w_2) = \partial(w_4) = \partial(x_v) = 0, \quad \partial(w_3) = [w_2, w_2], \quad \partial(w_5) = [w_4, w_4], \tag{7}$$
$$\partial(z_{(v,u)}) = (\operatorname{ad} x_v)^3([w_1, w_2]) + (\operatorname{ad} w_1)^7([w_2, [x_v, x_u]]) + (\operatorname{ad} w_1)^{19}(w_2) + X_v + Y + Z,$$

where

$$X_v = \big[[x_v, [w_2, w_3]], [x_v, [w_1, [w_1, [w_1, w_2]]]]\big], \qquad Y = (\operatorname{ad} w_1)^{12}([[w_2, w_4], [w_4, w_5]]),$$

$$Z = \big[(\operatorname{ad} w_1)^5(w_4), [(\operatorname{ad} w_2)^3(w_4), (\operatorname{ad} w_2)^2([w_4, w_5])]\big]. \tag{8}$$

Recall that the iterated Lie bracket of length $k+1$ is defined by

$$(\operatorname{ad} x)^k(y) = [x, [x, [\dots, [x, y] \dots]]], \qquad \text{where } x \text{ is involved } k \text{ times.}$$

For the sake of simplicity, let's denote

$$\mathcal{L}(\mathcal{G}_{\leqslant 114}) = \Big(\mathbb{L}(w_1, w_2, w_3, w_4, w_5, x_v \mid v \in V(\mathcal{G})), \partial\Big). \tag{9}$$

Lemma 3.2. *For every $v \in V(\mathcal{G})$, the following cycles are not boundaries in $\mathcal{L}(\mathcal{G}_{\leqslant 114})$,*

$$[x_v, [x_u, [x_t, [w_2, w_1]]]], \quad (\operatorname{ad} x_v)^3([w_1, w_2]), \quad (\operatorname{ad} w_1)^7([w_2, [x_v, x_u]]), \quad (\operatorname{ad} w_1)^{19}(w_2).$$

Proof. Since the only non zero differentials are $\partial(w_3) = [w_2, w_2]$, $\partial(w_5) = [w_4, w_4]$ and none of the given cycles contain w_2 or w_4 twice, it follows that they cannot be boundaries. □

Lemma 3.3. *The element $C = [w_3, [w_1, [w_1, w_2]]] - [w_2, [w_1, [w_1, w_3]]] \in \mathcal{L}(\mathcal{G}_{\leqslant 114})$ is a 114-cycle. Moreover, for every $v \in V(\mathcal{G})$, the 338-cycle $\mathcal{A}_v = [x_v, [x_v, [C, [w_1, w_2]]]]$ is not a boundary in $\mathcal{L}(\mathcal{G}_{\leqslant 114})$.*

Proof. First, the condition $p > 187$ ensures that $\mathcal{L}(\mathcal{G})$ is an object of $\mathbf{DGL}_{19}^{387}(\mathbb{Z}_{(p)})$ allowing us to use the framework in [**2**]. Next, by [**2**, Lemma 1.9] the following map is injective,

$$H_i(\mathcal{L}(\mathcal{G}_{\leqslant 114})) \to H_i(\mathcal{T}(\mathcal{G}_{\leqslant 114})), \qquad \forall i \leqslant 386. \tag{10}$$

Here $\mathcal{T}(\mathcal{G}_{\leqslant 114}) = \mathbb{T}(w_1, w_2, w_3, w_4, w_5, \{x_v\}_{v \in V(\mathcal{G})})$ denotes the universal algebra of $\mathcal{L}(\mathcal{G}_{\leqslant 114})$. Recall that $\mathcal{T}(\mathcal{G}_{\leqslant 114})$ can be considered as a graded Lie algebra by defining

$$[a, b] = ab - (-1)^{|a||b|} ba. \tag{11}$$

First, as $\partial(w_3) = [w_2, w_2]$, we deduce that

$$\partial(C) = [[w_2, w_2], [w_1, [w_1, w_2]]] - [w_2, [w_1, [w_1, [w_2, w_2]]]]. \tag{12}$$

Expanding $[[w_2, w_2], [w_1, [w_1, w_2]]]$ and $[w_2, [w_1, [w_1, [w_2, w_2]]]]$ in $\mathcal{T}(\mathcal{G}_{\leqslant 114})$ and using equation (11) we get

$$[[w_2, w_2], [w_1, [w_1, w_2]]] = 2w_2 w_1^2 w_2^2 - 2w_2^3 w_1^2 - 2w_1^2 w_2^3 + 2w_2^2 w_1^2 w_2,$$

$$[w_2, [w_1, [w_1, [w_2, w_2]]]] = 2w_2 w_1^2 w_2^2 - 2w_2^3 w_1^2 - 2w_1^2 w_2^3 + 2w_2^2 w_1^2 w_2. \tag{13}$$

Consequently, $\partial(C) = 0$ as desired.

Next, recall that the differential ∂ in $\mathcal{T}(\mathcal{G}_{\leqslant 114})$ is given by

$$\partial(w_1) = \partial(w_2) = \partial(x_v) = 0, \quad \partial(w_3) = [w_2, w_2] = 2w_2^2, \;\; \partial(w_5) = [w_4, w_4] = 2w_4^2. \tag{14}$$

On the one hand, expanding the cycle C in $\mathcal{T}(\mathcal{G}_{\leqslant 114})$ by using (11), we get the monomial $w_3w_1^2w_2 - w_2w_1^2w_3$ which cannot be reached by ∂ according to (14).

On the other hand, expanding the cycle $\mathcal{A}_v$ in $\mathcal{T}(\mathcal{G}_{\leqslant 114})$, we get the monomial $x_v^2w_3w_1^2w_2w_1w_2$ which cannot be reached by ∂. Hence, C and $\mathcal{A}_v$ are not boundaries in $\mathcal{T}(\mathcal{G}_{\leqslant 114})$. By the injective map (10), we derive that C and $\mathcal{A}_v$ are also not boundaries in $\mathcal{L}(\mathcal{G}_{\leqslant 114})$. □

Lemma 3.4. *For every $s, s' \in V(\mathcal{G})$ such that $s \neq s'$, the following brackets*

$$[x_s, [x_s, [x_{s'}, [w_1, w_2]]]], \quad [x_s, [x_{s'}, [x_s, [w_1, w_2]]]], \quad [x_{s'}, [x_s, [x_s, [w_1, w_2]]]]$$

are linearly independent. Moreover, any linear combination of those brackets is not a boundary in $\mathcal{L}(\mathcal{G}_{\leqslant 114})$.

Proof. Indeed, set $A = [w_1, w_2]$, if

$$a[x_s, [x_s, [x_{s'}, A]]] + b[x_s, [x_{s'}, [x_s, A]]] + c[x_{s'}, [x_s, [x_s, A]]] = 0,$$

then, expanding the bracket $a[x_s, [x_s, [x_{s'}, A]]]$ in $\mathcal{T}(\mathcal{G}_{\leqslant 114})$, we get the monomial $ax_s^2x_{s'}A$. Now expanding $[x_s, [x_{s'}, [x_s, A]]]$ and $[x_{s'}, [x_s, [x_s, A]]]$ we obtain

$$\begin{aligned}[x_s, [x_{s'}, [x_s, A]]] &= x_sx_{s'}x_sA - x_sx_{s'}Ax_s - x_s^2Ax_{s'} + x_sAx_sx_{s'} \\ &\quad - x_{s'}x_sAx_s + x_{s'}x_sAx_s + x_sAx_{s'}x_s - Ax_sx_{s'}x_s,\end{aligned} \tag{15}$$

$$[x_{s'}, [x_s, [x_s, A]]] = x_{s'}x_s^2A - 2x_{s'}x_sAx_s - x_{s'}Ax_s^2 - x_s^2Ax_{s'} - 2x_sAx_sx_{s'} - Ax_s^2x_{s'}. \tag{16}$$

Therefore $ax_s^2x_{s'}A$ does not appear in (15) and (16). As a result $a = 0$.

Likewise, as the expression $bx_sx_{s'}x_sw_1w_2$ does not appear in (16), it follows that $b = 0$. Consequently, $c = 0$.

Finally, since the monomials $ax_s^2x_{s'}w_2w_1$, $bx_sx_{s'}x_sw_1w_2$ and $cx_{s'}x_sx_sw_1w_2$ cannot be reached by the differential ∂ according to (14), it follows that any linear combination of the given brackets is not a boundary in $\mathcal{L}(\mathcal{G}_{\leqslant 114})$. □

Lemma 3.5. *For every $v \in V(\mathcal{G})$, the brackets X_v, Y and Z given in* (8) *are linearly independent. Moreover, any linear combination of those brackets is not a boundary in $\mathcal{L}(\mathcal{G}_{\leqslant 114})$.*

Proof. Indeed, recall that

$$X_v = \big[[x_v, [w_2, w_3]], [x_v, [w_1, [w_1, [w_1, w_2]]]]\big], \qquad Y = (\operatorname{ad} w_1)^{12}([[w_2, w_4], [w_4, w_5]]),$$

$$\big[(\operatorname{ad} w_1)^5(w_4), [(\operatorname{ad} w_2)^3(w_4), (\operatorname{ad} w_2)^2([w_4, w_5])]\big].$$

As Y contains exactly 2 generators w_4, Z has 3 and X_v does not contain w_4, we derive that X_v, Y and Z are linearly independent.

Next, we claim that any linear combination $aX_v + bY + cZ$ cannot be a boundary. Indeed, using the same argument given in the previous lemmas, expanding the three brackets in the universal algebra $\mathcal{T}(\mathcal{G}_{\leqslant 114})$, we get the following monomials

$$ax_v w_2 w_3 x_v w_1^3 w_2, \qquad bw_1^{12} w_2 w_4^2 w_5, \qquad cw_1^5 w_4 w_2^3 w_4 w_2^2 w_4 w_5,$$

and none of them can be reached by the differential ∂ according to (14). □

Corollary 3.6. *Let $p > 185$. If $\mathcal{L}(\mathcal{G})$ and $\mathcal{L}(\mathcal{G}_{\leqslant 114})$ are the differential graded Lie algebra given in Definition* 3.1 *and* (9) *respectively, then we have two short exact sequences*

$$\mathrm{Hom}\big(W_{387}, H_{387}(\mathcal{L}(\mathcal{G}_{\leqslant 114}))\big) \rightarrowtail \mathcal{E}(\mathcal{L}(\mathcal{G})) \twoheadrightarrow \mathcal{C}^{387}, \tag{17}$$

$$\mathrm{Hom}\big(W_{387}, H_{387}(\mathcal{L}(\mathcal{G}_{\leqslant 114}))\big) \rightarrowtail \mathcal{E}_*(\mathcal{L}(\mathcal{G})) \twoheadrightarrow \mathcal{C}_*^{387}. \tag{18}$$

Proof. First, the condition $p > 185$ ensures that $\mathcal{L}(\mathcal{G})$ is an object of the category $\mathbf{DGL}_{19}^{387}(\mathbb{Z}_{(p)})$. Then, the two sequences follow by applying Theorem 2.3 to $\mathcal{L}(\mathcal{G})$. □

Remark 3.7. The commutativity of the diagram (3) means that for every $w \in W_q$, there exists $t \in \mathbb{L}_q(W_{\leqslant n})$ such that $\alpha \circ \partial(w) - \partial \circ \xi(w) = \partial(t)$. Hence, in the terms of Corollary 3.6, there exists $\phi_{(v,u)} \in \mathcal{L}(\mathcal{G}_{\leqslant 114})$ such that

$$\alpha \circ \partial(z_{(v,u)}) - \partial \circ \xi(z_{(v,u)}) = \partial(\phi_{(v,u)}). \tag{19}$$

4. Construction of the isomorphism $\Psi\colon \mathrm{aut}(\mathcal{G}) \to C^{387}$

Since the Hall basis of $\mathcal{L}_{114}(\mathcal{G}_{\leqslant 114})$ is formed by

$$\{x_s\}_{s\in V(\mathcal{G})}, \qquad [w_3,[w_1,[w_1,w_2]]], \qquad [[w_1,w_2],[w_1,w_3]], \qquad [w_2,[w_1,[w_1,w_3]]],$$

it follows that, for every $(\xi,[\alpha]) \in \mathcal{C}^{387}$, we can write

$$\begin{aligned}
&\alpha(w_1) = \beta w_1, \quad \alpha(w_2) = \lambda w_2, \quad \alpha(w_3) = q w_3, \quad \alpha(w_5) = r w_5,\\
&\alpha(x_v) = \sum_{s\in V(\mathcal{G})} a(v,s)x_s + a_{v,1}[w_3,[w_1,[w_1,w_2]]] + a_{v,2}[[w_1,w_2],[w_1,w_3]]\\
&\qquad\quad + a_{v,3}[w_2,[w_1,[w_1,w_3]]],\\
&\xi(z_{(v,u)}) = \rho_{(v,u)} z_{(v,u)} + \sum_{(r,s)\in E(\mathcal{G})} \rho_{(v,u),(r,s)} z_{(r,s)},
\end{aligned} \tag{20}$$

where almost all of the coefficients $\rho_{(v,u),(r,s)}, \rho_{(v,u)}$ and $a(v,s)$ are zero and where $\beta, \lambda, q, \gamma, r \neq 0$. It is worth noting that as α is an equivalence of homotopy, then its induces an isomorphism on the indecomposables, therefore at least one of the coefficients $a(v,s)$ is not zero. Likewise, as ξ is an isomorphism at least one of the coefficients $\rho_{(v,u),(r,s)}, \rho_{(v,u)}$ is not zero.

Proposition 4.1. *If $(\xi,[\alpha]) \in \mathcal{C}^{387}$, then the coefficients in* (20) *satisfy the following*

1. $a_{v,1} = -a_{v,3}$ *and* $a_{v,2} = 0$ *for all* $v \in V(\mathcal{G})$,
2. $q = \lambda^2$ *and* $r = \gamma^2$.

Proof. First, since $\partial(\alpha(x_v)) = \alpha(\partial(x_v)) = 0$ and $\partial(w_3) = [w_2, w_2]$, it follows that

$$a_{v,1}[[w_2, w_2], [w_1, [w_1, w_2]]] + a_{v,2}[[w_1, w_2], [w_1, [w_2, w_2]]] \\ + a_{v,3}[w_2, [w_1, [w_1, [w_2, w_2]]]] = 0.$$

But from (12) we know that $[[w_2, w_2], [w_1, [w_1, w_2]]] = [w_2, [w_1, [w_1, [w_2, w_2]]]]$ and in $\mathcal{T}_{114}(\mathcal{G}_{\leqslant 114})$ we have

$$\begin{aligned}[][[w_1, w_2], [w_1, [w_2, w_2]]] &= 2w_1w_2w_1w_2^2 - 2w_1w_2^3w_1 + 2w_2w_1^2w_2^2 - 2w_2w_1w_2^2w_1 \\ &\quad - 2w_1w_2^2w_1w_2 + 2w_2^2w_1^2w_2 - 2w_1w_2^3w_1 + 2w_2^2w_1w_2w_1, \\ [[w_2, w_2], [w_1, [w_1, w_2]]] &= 2w_2w_1^2w_2^2 - 2w_2^3w_1^2 - 2w_1^2w_2^3 + 2w_2^2w_1^2w_2.\end{aligned}$$

As a result, $a_{v,1} = -a_{v,3}$ and $a_{v,2} = 0$.

Secondly, since $\partial(\alpha(w_5)) = \alpha(\partial(w_5))$ and $\partial(\alpha(w_3)) = \alpha(\partial(w_3))$, we derive

$$\partial(\alpha(w_5)) = r[w_4, w_4], \qquad \alpha(\partial(w_5)) = [\alpha(w_4), \alpha(w_4)] = \gamma^2[w_4, w_4],$$

$$\partial(\alpha(w_3)) = q[w_2, w_2], \qquad \alpha(\partial(w_3)) = [\alpha(w_2), \alpha(w_2)] = \lambda^2[w_4, w_4].$$

Consequently, $q = \lambda^2$ and $r = \gamma^2$. □

Proposition 4.2. *Let $(\xi, [\alpha]) \in \mathcal{C}^{387}$. There exists unique $\phi \in \operatorname{aut}(\mathcal{G})$ such that*

$$\begin{aligned} \xi(z_{(v,u)}) &= z_{(\phi(v),\phi(u))}, && \forall (v, u) \in E(\mathcal{G}), \\ \alpha(x_v) &= x_{\phi(v)}, && \forall v \in V(\mathcal{G}), \\ \alpha(w_i) &= w_i, && i = 1, 2, 3, 4, 5. \end{aligned} \tag{21}$$

Proof. Notice that the strong connectivity of the graph implies that for every $v \in V(\mathcal{G})$, v is the starting vertex of an edge $(v, w) \in E(\mathcal{G})$. Therefore the coefficients $a(v, s), a_{v,1}, a_{v,2}, a_{v,3}, a_{v,4}$ and $a_{v,5}$ involved in $\alpha(x_v)$ (see (20)) can be entirely determined by using (19). Indeed, we have

$$\begin{aligned} \alpha(\partial(z_{(v,u)}))) &= \big(\operatorname{ad}(\alpha(x_v))\big)^3([\alpha(w_1), \alpha(w_2)]) + \big(\operatorname{ad}(\alpha(w_1))\big)^7([\alpha(w_2), [\alpha(x_v), \alpha(x_u)]]) \\ &\quad + \big(\operatorname{ad}(\alpha(w_1))\big)^{19}(\alpha(w_2)) + \alpha(X_v) + \alpha(Y) + \alpha(Z), \\ \partial(\xi(z_{(v,u)}))) &= \rho_{(v,u)}\partial(z_{(v,u)}) + \sum_{(r,s)\in E(\mathcal{G})} \rho_{(v,u),(r,s)}\partial(z_{(r,s)}) \\ &= \rho_{(v,u)}(\operatorname{ad} x_v)^3([w_1, w_2]) + \rho_{(v,u)}(\operatorname{ad} x_1)^7([w_2, [x_v, x_u]]) \\ &\quad + \rho_{(v,u)}(\operatorname{ad} w_1)^{19}(w_2) + \rho_{(v,u)}(X_v + Y + Z) \\ &\quad + \sum_{(r,s)\in E(\mathcal{G})} \rho_{(v,u),(r,s)}\Big((\operatorname{ad} x_r)^3([w_1, w_2]) \\ &\quad + (\operatorname{ad} w_1)^7([w_2, [x_r, x_s]]) + (\operatorname{ad} w_1)^{19}(w_2) + X_r + Y + Z\Big), \end{aligned} \tag{22}$$

where X_v, X_r, Y, Z are given in (8). Next Proposition 4.1 implies that

$$\alpha(x_v) = \sum_{s\in V(\mathcal{G})} a(v, s)x_s + a_{v,1}\Big([w_3, [w_1, [w_1, w_2]]] - [w_2, [w_1, [w_1, w_3]]]\Big), \tag{23}$$

where almost all of the coefficients $a(v,s)$ are zero and at least one of them is not nil. Expanding the expression

$$(\operatorname{ad}\alpha(x_v))^3([\alpha(w_1),\alpha(w_2)]) = \beta\lambda\operatorname{ad}(\alpha(x_v))^3([w_1,w_2]), \tag{24}$$

leads to the following brackets

$$\begin{aligned}
&\beta\lambda a^2(v,s)a(v,s')[x_s,[x_s,[x_{s'},[w_2,w_1]]]], && v,s,s'\in V(\mathcal{G}),\\
&\beta\lambda a^2(v,s)a(v,s')[x_s,[x_{s'},[x_s,[w_2,w_1]]]], && v,s,s'\in V(\mathcal{G}),\\
&\beta\lambda a^2(v,s)a(v,s')[x_{s'},[x_s,[x_s,[w_2,w_1]]]], && v,s,s'\in V(\mathcal{G}).
\end{aligned}$$

But none of the brackets in the expression (22), giving $\partial(\xi(z_{(v,u)}))$, is formed using three generators $x_s, x_s, x_{s'}$ with $s\neq s'$. Moreover, by Lemma 3.4, the following expression

$$\beta\lambda a^2(v,s)a(v,s')\Big([x_s,[x_s,[x_{s'},[w_1,w_2]]]] + [x_s,[x_{s'},[x_s,[w_1,w_2]]]] + [x_{s'},[x_s,[x_s,[w_1,w_2]]]]\Big)$$

is neither trivial nor a boundary. Now based on the formula (19), we deduce that all of the coefficients $\beta\lambda a^2(v,s)a(v,s')$ are nil. Since $\beta,\lambda\neq 0$, it follows that only one coefficient among $a(v,s), s\in V(\mathcal{G})$ is not zero. Let us denote it by $a(v,t_v)$. As a result, the formula (23) becomes

$$\alpha(x_v) = a(v,t_v)x_t + a_{v,1}\Big([w_3,[w_1,[w_1,w_2]]] - [w_2,[w_1,[w_1,w_3]]]\Big). \tag{25}$$

Using (25) and expanding again (24), we get the following (only one) bracket

$$\beta\lambda a^2(v,t_v)a_{v,1}[x_t,[x_t,[C,[w_2,w_1]]]],$$

where

$$C = [w_3,[w_1,[w_1,w_2]]] - [w_2,[w_1,[w_1,w_3]]].$$

This bracket does not appear in the expression (22) and it cannot be a boundary by Lemma 3.3. Therefore, using the formula (19) and taking into account that $a(v,t_v),\beta,\lambda$ are not nil, we deduce that $a_{v,1}=0$. Thus, there is a unique vertex $t\in V(\mathcal{G})$ such that $\alpha(x_v) = a(v,t_v)x_t$.

Consequently, on the one hand, we derive the following

$$\alpha(Y) = \beta^{12}\lambda\gamma^4 Y, \qquad \alpha(Z) = \beta^5\lambda^5\gamma^5 Z, \qquad \alpha(X_v) = \beta^3\lambda^4 a^2_{(v,t)}X_t, \quad \forall v\in V(\mathcal{G}).$$

On the other hand, the formulas (22) become

$$\begin{aligned}
\alpha(\partial(z_{(v,u)})) &= \beta\lambda a^3(v,t_v)\big(\operatorname{ad}(x_t)\big)^3([w_1,w_2])\\
&\quad + \beta^7\lambda a(v,t_v)a(u,t_u)\big(\operatorname{ad}(w_1)\big)^7([w_2,[x_t,x_{t'}]])\\
&\quad + \beta^{19}\lambda\big(\operatorname{ad}(w_1)\big)^{19}(w_2) + \beta^3\lambda^4 a^2(v,t_v)X_t + \beta^{12}\lambda\gamma^4 Y + \beta^5\lambda^5\gamma^5 Z,\\
\partial(\xi(z_{(v,u)})) &= \rho_{(v,u)}(\operatorname{ad}x_v)^3([w_1,w_2]) + \rho_{(v,u)}(\operatorname{ad}x_1)^7([w_2,[x_v,x_u]])\\
&\quad + \rho_{(v,u)}(\operatorname{ad}w_1)^{19}(w_2) + \rho_{(v,u)}(X_v+Y+Z)\\
&\quad + \sum_{(r,s)\in E(\mathcal{G})}\rho_{(v,u),(r,s)}\Big((\operatorname{ad}x_r)^3([w_1,w_2]) + (\operatorname{ad}w_1)^7([w_2,[x_r,x_s]])
\end{aligned}$$

$$+ (\operatorname{ad} w_1)^{19}(w_2) + X_r + Y + Z\Big). \tag{26}$$

Likewise, due to Lemmas 3.2, 3.3 and 3.5, all the brackets in the expression

$$\partial(\xi(z_{(v,u)})) - \alpha(\partial(z_{(v,u)})),$$

are not boundaries and comparing the coefficients in (19), we get $\rho_{(v,u)} = 0$ and all the coefficients $\rho_{(v,u),(r,s)} = 0$ except $\rho_{(v,u),(t_v,t_u)} \neq 0$ which satisfies the following equations

$$\begin{aligned}\rho_{(v,u),(t_v,t_u)} &= \beta\lambda a^3(v,t_v) = \beta^7\lambda a(v,t_v)a(u,t_u) = \beta^3\lambda^4 a^2(v,t_v) = \beta^{19}\lambda = \beta^{12}\lambda\gamma^4 \\ &= \beta^5\lambda^5\gamma^5. \end{aligned}\tag{27}$$

From $\beta^{19}\lambda = \beta^{12}\lambda\gamma^4 = \beta^5\lambda^5\gamma^5$, we deduce that

$$\beta^7 = \gamma^4, \quad \beta^{12} = \lambda^4\gamma^5, \quad \beta^7 = \lambda^4\gamma \Longrightarrow \gamma^3 = \lambda^4,$$

therefore,

$$(\beta^7)^{12} = \gamma^{48} = (\beta^{12})^7 = \lambda^{28}\gamma^{35} \Longrightarrow \gamma^{13} = \lambda^{28} = (\gamma^3)^7 = \gamma^{21}.$$

It follows that $\gamma^8 = 1$. As $\gamma^3 = \lambda^4$ and $\beta^7 = \gamma^4$, we deduce that $\beta = \gamma = 1$. Now (27) becomes

$$\rho_{(v,u),(t_v,t_u)} = \lambda a^3(v,t_v) = \lambda a(v,t_v)a(u,t_u) = a^2(v,t_v) = \lambda = \lambda^5.$$

Next, from $\lambda a^3(v,t_v) = \lambda$, we derive that $a(v,t_v) = 1$. As a result we get

$$\rho_{(v,u),(t_v,t_u)} = \beta = \lambda = a(v,t_v) = a(u,t_u) = \gamma = 1,$$

and from Proposition 4.1, it follows that $q = r = 1$. Thus, by going back the formulas (20), we have proved that for every $v \in V(\mathcal{G})$, there is a unique vertex $t_v \in V(\mathcal{G})$ and for every $(v,u) \in E(\mathcal{G})$, there is a unique vertex $(t_v,t_u) \in E(\mathcal{G})$ such that

$$\xi(z_{(v,u)}) = z_{(t_v,t_u)}, \qquad \alpha(x_v) = x_{t_v}, \qquad \alpha(w_i) = w_i, \qquad i = 1,2,3,4,5. \tag{28}$$

Thus, if we define $\phi\colon \mathcal{G} \to \mathcal{G}$, $\phi(v) = t_v$, $\phi((v,u)) = (t_v,t_u)$, we obtain (21). □

As a consequence of Proposition 4.2, we define a map $\Psi\colon \mathcal{C}^{387} \to \operatorname{aut}(\mathcal{G})$ by setting

$$\Psi\big((\xi,[\alpha])\big) = \phi.$$

Theorem 4.3. *The map Ψ is an isomorphism of groups.*

Proof. For every $\sigma \in \operatorname{aut}(\mathcal{G})$, we define

$$\begin{aligned} \xi_\sigma(z_{(v,u)}) &= z_{(\sigma(v),\sigma(u))}, && \forall (v,u) \in E(\mathcal{G}),\\ \alpha_\sigma(x_v) &= x_{\sigma(v)}, && \forall v \in V(\mathcal{G}),\\ \alpha_\sigma(w_i) &= w_i, && i = 1,2,3,4,5. \end{aligned}$$

Clearly, we have $\partial \circ \xi_\sigma(z_{(v,u)}) = \partial(z_{(\sigma(v),\sigma(u))}) = \alpha_\sigma \circ \partial(z_{(v,u)})$ which implies that $(\xi_\sigma,[\alpha_\sigma]) \in C^{387}$. Hence, we get a map $\Phi\colon \operatorname{aut}(\mathcal{G}) \to C^{387}$, $\Phi(\sigma) = (\xi_\sigma,[\alpha_\sigma])$ and it is easy to check that it is the inverse of Ψ. Finally, Φ is a homomorphism of groups because we have:

$$\Phi(\sigma_1 \circ \sigma_2) = (\xi_{\sigma_1\circ\sigma_2},[\alpha_{\sigma_1} \circ \alpha_{\sigma_2}]) = (\xi_{\sigma_1},[\alpha_{\sigma_1}]).(\xi_{\sigma_2},[\alpha_{\sigma_2}]) = \Phi(\sigma_1)\Phi(\sigma_2),$$

for all $\sigma_1, \sigma_2 \in \operatorname{aut}(\mathcal{G})$. □

5. The main result

Now, we are in a position to prove the main theorem of this paper.

Corollary 5.1. *The group* $\mathcal{C}_*^{387}$*, defined in* (4)*, is trivial.*

Proof. The proof is a straightforward consequence of Proposition 4.2, since $\xi = id$ forces $\phi = id$ and therefore $\alpha = id$. □

From the short exact sequence (18) and Corollary 5.1, we derive the following result

Corollary 5.2. $\mathrm{Hom}\big(W_{387}, H_{387}(\mathcal{L}(\mathcal{G}_{\leqslant 114}))\big) \cong \mathcal{E}_*(\mathcal{L}(\mathcal{G}))$.

Remark 5.3. It worth mentioning that the group $\mathcal{E}_*(\mathcal{L}(\mathcal{G}))$ is not trivial. Indeed, it is easy to see that, for $v \in \mathcal{G}$, the bracket $[[w_2, w_4], (\mathrm{ad}\, w_1)^{10}[w_2, x_v]]$ is a cycle defining a non trivial homology class in $H_{387}(\mathcal{L}(\mathcal{G}_{\leqslant 114}))$. Here we use the exact sequence (18).

Theorem 5.4. *For any group G, there is an object $\mathcal{L}(\mathcal{G})$ in $\mathbf{DGL}_{19}^{387}(\mathbb{Z}_{(p)})$ such that*

$$G \cong \mathcal{E}(\mathcal{L}(\mathcal{G}))/\mathcal{E}_*(\mathcal{L}(\mathcal{G})).$$

Proof. First, by Theorem 2.1, to the group G corresponds a strongly connected digraph $\mathcal{G}$ such that $G \cong \mathrm{aut}(\mathcal{G})$. Next to the graph $\mathcal{G}$, we can assign the object $\mathcal{L}(\mathcal{G})$ in $\mathbf{DGL}_{19}^{387}(\mathbb{Z}_{(p)})$. Then from Corollary 5.1, the short exact sequence (17) and Theorem 4.3, we derive

$$\mathcal{E}(\mathcal{L}(\mathcal{G}))/\mathcal{E}_*(\mathcal{L}(\mathcal{G})) \cong \mathcal{E}(\mathcal{L}(\mathcal{G}))/\mathrm{Hom}\big(W_{387}, H_{387}(\mathcal{L}(\mathcal{G}_{\leqslant 114}))\big) \cong \mathcal{C}^{387} \cong \mathrm{aut}(\mathcal{G}) \cong G,$$

as we wanted. □

Using the Anick's $\mathbb{Z}_{(p)}$-local homotopy theory framework and the identifications (1) and (2), we obtain the following

Theorem 5.5. *Any group G occurs as $\mathcal{E}(X)/\mathcal{E}_*(X)$. More precisely, the CW-complex X can be chosen as*

1. *The space X is an object of $\mathbf{CW}_{19}^{388}(\mathbb{Z}_{(p)})$, where $p > 185$, which is of finite type if G is finite;*
2. *$H_{115}(X, \mathbb{Z}_{(p)})$ and $H_{388}(X, \mathbb{Z}_{(p)})$ are free $\mathbb{Z}(p)$-modules of infinite basis iff G is infinite;*
3. *$H_{20}(X, \mathbb{Z}_{(p)}) \cong H_{26}(X, \mathbb{Z}_{(p)}) \cong H_{34}(X, \mathbb{Z}_{(p)}) \cong H_{52}(X, \mathbb{Z}_{(p)}) \cong H_{68}(X, \mathbb{Z}_{(p)}) \cong \mathbb{Z}_{(p)}$;*
4. *$\mathcal{E}_*(X) \cong \mathrm{Hom}\big(H_{388}(X, \mathbb{Z}_{(p)}), \pi_{388}(X^{115})\big)$, where X^{115} is the 115-skeleton of X,*
5. *$\mathcal{E}_*(X)$ is infinite.*

Remark 5.6. It would be of great interest if we can modify the construction of $\mathcal{L}(\mathcal{G})$ to have $\mathcal{E}_*(\mathcal{L}(\mathcal{G})) = 1$. If it is possible, then Kahn's realisability problem of groups will be completely resolved for CW-complexes.

References

- **[1]** D. J. Anick, *Hopf algebras up to homotopy*, J. Amer. Math. Soc. 2 (1989), 417–453.
- **[2]** D. J. Anick, *An R-local Milnor-Moore theorem*, Adv. Math. 77 (1989), 116–136.
- **[3]** D. J. Anick, *R-local homotopy theory*, Lect. Notes Math. 1418 (1990), 78–85.
- **[4]** M. Benkhalifa, *On the group of self-homotopy equivalences of an elliptic space*, Proc. Am. Math. Soc. 148(6) (2020), 2695–2706.
- **[5]** M. Benkhalifa and S. B. Smith, *The effect of cell attachment on the group of self-equivalences of an R-local space*, J. Homotopy Relat. Struct. 4(1) (2015), 135–144.
- **[6]** M. Benkhalifa, *The group of self-homotopy equivalences of a simply connected and 4-dimensional CW-complex*, Int. Electron. J. Algebra 19 (2016), 19–34.
- **[7]** P. J. Chocano, M. A. Manuel and F. Ruiz del Portal, *Topological realizations of groups in Alexandroff spaces*, Rev. R. Acad. Cienc. Exactas Fís. Nat., Ser. A Mat. 115 (2021), Article No. 25.
- **[8]** C. Costoya, D. Mendez and A. Viruel, *The group of self-homotopy equivalences of A_2^n-polyhedra*, J. Group Theory 23(4) (2020), 575–591.
- **[9]** C. Costoya, D. Mendez and A. Viruel, *Realisability problem in arrow categories*, Collect. Math. 71 (2020), 1–23.
- **[10]** C. Costoya and A. Viruel, *On the realizability of group actions*, Adv. Math. 336 (2018), 299–315.
- **[11]** C. Costoya and A. Viruel, *Every finite group is the group of self homotopy equivalences of an elliptic space*, Acta Math. 213 (2019), 49–62.
- **[12]** J. de Groot, *Groups represented by homeomorphism groups*, Math. Ann. 138 (1959), 80–102.
- **[13]** Y. Félix, *Problems on mapping spaces and related subjects, Homotopy theory of function spaces and related topics*, Contemp. Math., Vol. 519, Amer. Math. Soc. (2010), 217–230.
- **[14]** P. Hell and J. Nešetřil, *Graphs and homomorphisms*, Oxford Lect. Ser. Math. Appl., Vol. 28, Oxford University Press, Oxford (2004).
- **[15]** D. W. Kahn, *Realization problem for the group of homotopy classes of self-homotopy equivalences*, Math. Ann. 220 (1976), 37–46.

Mahmoud Benkhalifa mbenkhalifa@sharjah.ac.ae

Department of Mathematics, College of Sciences, University of Sharjah, United Arab Emirates