A WELLS TYPE EXACT SEQUENCE FOR NON-DEGENERATE UNITARY SOLUTIONS OF THE YANG–BAXTER EQUATION

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Abstract

Cycle sets are known to give non-degenerate unitary solutions of the Yang–Baxter equation and linear cycle sets are enriched versions of these algebraic systems. The paper explores the recently developed cohomology and extension theory for linear cycle sets. We derive a four term exact sequence relating 1-cocycles, second cohomology and certain groups of automorphisms arising from central extensions of linear cycle sets. This is an analogue of a similar exact sequence for group extensions known due to Wells. We also relate the exact sequence for linear cycle sets with that for their underlying abelian groups via the forgetful functor and also discuss generalities on dynamical 2-cocycles.

1. Introduction

The quantum Yang–Baxter equation is a fundamental equation arising in theoretical physics and has deep connections with mathematics specially braid groups and knot theory. A solution of the quantum Yang–Baxter equation is a linear map $R\colon V \otimes V \to V \otimes V$ satisfying

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12},$$

where V is a vector space over a field and $R_{ij}\colon V \otimes V \otimes V \to V \otimes V \otimes V$ acts as R on the (i,j) tensor factor and as the identity on the remaining factor. If $F\colon V \otimes V \to V \otimes V$ is the flip operator $F(v \otimes w) = w \otimes v$, then $R\colon V \otimes V \to V \otimes V$ is a solution of the quantum Yang–Baxter equation if and only if $\overline{R} = F \circ R$ satisfies the braid relation

$$\overline{R}_{12}\overline{R}_{23}\overline{R}_{12} = \overline{R}_{23}\overline{R}_{12}\overline{R}_{23},$$

in which case one says that $\overline{R}$ is a solution of the Yang–Baxter equation. Topologically, the braid relation is simply the third Reidemeister move of planar diagrams of links as shown in Figure 1.
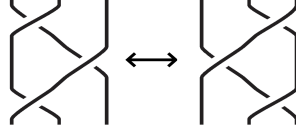

Figure 1: The braid relation

If X is a basis of the vector space V, then a map $r\colon X \times X \to X \times X$ satisfying $r_{12}r_{23}r_{12} = r_{23}r_{12}r_{23}$ induces a solution of the Yang–Baxter equation. In this case, we say that (X, r) is a set-theoretic solution of the Yang–Baxter equation. Writing $r(x, y) = (\sigma_x(y), \tau_y(x))$ for $x, y \in X$, we say that the solution r is non-degenerate if σ_x and τ_x are invertible for all $x \in X$. The problem of finding these set-theoretic solutions was posed by Drinfeld [9] and has attracted a lot of attention.

A *(left) cycle set*, as defined by Rump [24], is a non-empty set X with a binary operation $\cdot$ having bijective left translations $X \to X$, $x \mapsto y \cdot x$, and satisfying the relation

$$(x \cdot y) \cdot (x \cdot z) = (y \cdot x) \cdot (y \cdot z) \tag{1}$$

for all $x, y, z \in X$. A cycle set is *non-degenerate* if the squaring map $a \mapsto a \cdot a$ is invertible. It is known that every finite cycle set is non-degenerate [24, Theorem 2]. Rump showed that cycle sets are in bijection with non-degenerate unitary set-theoretic solutions of the Yang–Baxter equation. These solutions give a rich class of structures and are connected with semigroups of special type, Bieberbach groups [14], biquandles [12, 16], colourings of plane curves [11], Hopf algebras [10] and Garside groups [7], to name a few. Special solutions, particularly, the ones possessing self-distributivity are intimately connected to invariants of knots and links in the 3-space and thickened surfaces [12, 16, 17]. Cycle sets have been proved to be very useful in understanding the structure of solutions of the Yang–Baxter equation and for obtaining general classification results. Cycle sets as braces give only unitary solutions whereas skew braces, racks, bi-quandles etc. give general solutions. The structure of cycle sets is still far from being completely understood, and many important questions on the topic are yet not answered. The reader is referred to [4, 5, 7, 8, 11, 13, 19, 20, 21, 26, 27, 28, 29] for some recent works.

A *(left) brace* is an abelian group $(A, +)$ with an additional group operation $\circ$ such that

$$a \circ (b + c) + a = a \circ b + a \circ c \tag{2}$$

holds for all $a, b, c \in A$. Braces were introduced by Rump [26] in a slightly different but equivalent form where he showed that these algebraic systems give set-theoretic solutions of the Yang–Baxter equation. The preceding definition is due to Cedó, Jespers and Okniński [5]. Each abelian group is trivially a brace with $a + b = a \circ b$. In addition, regular rings give a large supply of braces. Relations between the additive and the multiplicative groups of a brace have been explored in many recent works, for example, [6, 15, 22].

A *(left) linear cycle set* is a cycle set $(X, \cdot)$ with an abelian group operation $+$

satisfying the conditions

$$x \cdot (y + z) = x \cdot y + x \cdot z \tag{3}$$

and

$$(x + y) \cdot z = (x \cdot y) \cdot (x \cdot z) \tag{4}$$

for all $x, y, z \in X$. This notion goes back to Rump [26], who showed it to be equivalent to the brace structure via the relation

$$x \cdot y = x^{-1} \circ (x + y),$$

where x^{-1} is inverse with respect to $\circ$. An abelian group can be viewed as a linear cycle set by taking $x \cdot y = y$ for all $x, y \in X$, and referred as a *trivial* linear cycle set. Rump [26] showed that linear cycle sets are closely related to radical rings.

It was pointed out in [2] that an extension theory for cycle sets (equivalently braces) would be crucial for a classification of these objects. This led to development of an extension theory by Lebed and Vendramin [19, 20]. A homology and cohomology theory for linear cycle sets (and hence for braces) was developed recently in [19]. As in case of groups, Lie algebras, quandles or any nice algebraic system, the second cohomology groups were shown to classify central cycle set extensions. A cohomology theory for general cycle sets was developed in [20]. We will follow the linear cycle set language of [19] since it gives a neat construction of cohomology and extension theory. It is worth noting that the right and the two-sided analogues of braces and cycle sets can be defined analogously and have been considered in the literature.

In this paper, we derive an exact sequence relating 1-cocycles, certain group of automorphisms and second cohomology groups of linear cycle sets. This can be thought of as a linear cycle set analogue of a fundamental exact sequence for groups due to Wells [32]. For notational convenience, sometimes, we will denote the value of a map ϕ at a point x by ϕ_x. We use the notation Aut, Z^1 and H^2 to denote group of automorphisms, group of 1-cocycles and second cohomology group of a linear cycle set, respectively. To distinguish linear cycle sets from groups, we use the bold notation $\mathbf{Aut}$, $\mathbf{Z}^1$ and $\mathbf{H}^2$ to denote group of automorphisms, group of 1-cocycles and second cohomology of a group, respectively.

Section 2 contains preliminaries and some basic results. We prove that there is a natural group homomorphism from the second linear cycle set cohomology to the second symmetric cohomology of the underlying abelian group (Proposition 2.6) and also examine this homomorphism for trivial cycle sets (Proposition 2.7). Section 3 prepares the foundation for the main result. Given a linear cycle set $(X, \cdot, +)$ and an abelian group A, we define an action of $\mathrm{Aut}(X) \times \mathbf{Aut}(A)$ on $H^2_N(X; A)$. As a consequence, we obtain a lower bound on the size of $H^2_N(X; A)$ (Corollary 3.2). In Section 4, we prove our main theorem (Theorem 4.5) that associates to each central extension of linear cycle sets a four term exact sequence relating group of 1-cocycles, certain group of automorphisms and second cohomology groups. Section 5 explores properties of the important connecting map in this exact sequence (Theorem 5.3). In Section 6, we relate the exact sequence with the corresponding Wells exact sequence for the underlying extension of abelian groups via the forgetful functor (Theorem 6.1). Finally, in Section 7, we discuss some generalities on bi-groupoids and dynamical extensions of (linear) cycle sets.

Acknowledgments

2.   Preliminaries and some basic results

Recall that a bi-groupoid is a non-empty set with two binary algebraic operations. We begin with the following immediate observation.

Lemma 2.1. *Let $(X, \cdot, *)$ be a bi-groupoid and define $S\colon X \times X \to X \times X$ given by $S(x,y) = (x \cdot y, y * x)$ for $x, y \in X$. Then the following hold:*

1. *The pair (X, S) is a set theoretic solution of the Yang–Baxter equation if and only if the equalities*

$$(x \cdot (y \cdot z)) = (x \cdot y) \cdot ((y * x) \cdot z),$$
$$((y \cdot z) * x) \cdot (z * y) = ((y * x) \cdot z) * (x \cdot y) \ and$$
$$(z * y) * ((y \cdot z) * x) = z * (y * x)$$

   hold for all $x, y, z \in X$.

2. *If $x \cdot y = y$ for all $x, y \in X$, then the pair (X, S) is a set theoretic solution of the Yang–Baxter equation if and only if the operation $*$ is left distributive, i.e.*

$$z * (y * x) = (z * y) * (z * x) \tag{5}$$

   for all $x, y, z \in X$.

Recall that $(X, *)$ is a *(left) rack* if the maps $x \mapsto y * x$ are bijections and (5) holds for all $x, y, z \in X$. Thus, assertion (2) of the preceding lemma gives a non-degenerate solution of the Yang–Baxter equation if and only if $(X, *)$ is a rack. Racks are useful in defining invariants of framed links in the 3-space. We look for conditions under which a rack is a cycle set. Following [18], we say that a rack X is *abelian* if it satisfies the equality

$$x * (y * z) = y * (x * z)$$

for all $x, y, z \in X$. Note that this condition is equivalent to the group of inner automorphisms of X

$$\mathrm{Inn}(X) = \langle S_x, \ x \in X \mid S_x(y) := x * y, \ x, y \in X \rangle$$

being abelian.

Proposition 2.2. *If $(X, *)$ is a rack such that $\mathrm{Inn}(X)$ is abelian, then it is a cycle set.*

Proof. It follows from the rack axiom that

$$x * (y * z) = (x * y) * (x * z)$$

and

$$y * (x * z) = (y * x) * (y * z)$$

for all $x, y, z \in X$. Since $\mathrm{Inn}(X)$ is abelian, we get

$$(x * y) * (x * z) = (y * x) * (y * z),$$

which is desired. $\square$

2.1.   Cohomology and extensions of linear cycle sets

A *morphism* between linear cycle sets X and Y is a map $\varphi \colon X \to Y$ satisfying $\varphi(x + x') = \varphi(x) + \varphi(x')$ and $\varphi(x \cdot x') = \varphi(x) \cdot \varphi(x')$ for all $x, x' \in X$. The *kernel* of φ is defined by $\mathrm{Ker}(\varphi) = \varphi^{-1}(0)$. The notion of *image*, of a *short exact sequence* of linear cycle sets, and of *linear cycle subsets* are defined in the usual manner.

Two linear cycle set extensions $A \overset{i}{\rightarrowtail} E \overset{\pi}{\twoheadrightarrow} X$ and $A \overset{i'}{\rightarrowtail} E' \overset{\pi'}{\twoheadrightarrow} X$ are called *equivalent* if there exists a linear cycle set isomorphism $\varphi \colon E \to E'$ such that the diagram

$$
\begin{array}{ccccc}
A & \overset{i}{\longrightarrow} & E & \overset{\pi}{\longrightarrow} & X \\
\downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle \mathrm{id}} \\
A & \overset{i'}{\longrightarrow} & E' & \overset{\pi'}{\longrightarrow} & X
\end{array}
$$

commutes. A cohomology theory for linear cycle sets is developed in the recent works of Lebed and Vendramin [**19, 20**]. Following [**19**], a *2-cocycle* for a linear cycle set $(X, \cdot, +)$ with coefficients in the (additively written) abelian group A consists of two maps $f, g \colon X \times X \to A$ satisfying the conditions

$$g(x, y) = g(y, x), \tag{6}$$

$$g(x, y) + g(x + y, z) = g(y, z) + g(x, y + z), \tag{7}$$

$$f(x + y, z) = f(x \cdot y, x \cdot z) + f(x, z) \text{ and} \tag{8}$$

$$f(x, y + z) - f(x, y) - f(x, z) = g(x \cdot y, x \cdot z) - g(y, z) \tag{9}$$

for all $x, y, z \in X$. Note that, if (f, g) is a 2-cocycle of a linear cycle set $(X, \cdot, +)$ with coefficients in an abelian group A, then conditions (6)–(7)–(8)–(9) imply that

$$f(0, x) = f(x, 0) = 0 \text{ and}$$
$$g(0, x) = g(x, 0) = g(0, 0)$$

for all $x \in X$. A pair of maps $f, g \colon X \times X \to A$ is called a *2-coboundary* if there exists a map $\lambda \colon X \to A$ such that

$$f(x, y) = \lambda(x \cdot y) - \lambda(y) \text{ and} \tag{10}$$

$$g(x, y) = \lambda(x + y) - \lambda(x) - \lambda(y) \tag{11}$$

for all $x, y \in X$.

A 2-cocycle (f, g) is called normalised if $g(0, 0) = 0$, whereas a 2-coboundary (f, g) is called normalised if the map $\lambda \colon X \to A$ satisfies $\lambda(0) = 0$. We denote the group

of normalised 2-cocycles by $Z_N^2(X; A)$, and the group of normalised 2-coboundaries by $B_N^2(X; A)$. The quotient $Z_N^2(X; A)/B_N^2(X; A)$ is the *normalised cohomology* group $H_N^2(X; A)$ of X with coefficients in A. We shall also need the group of normalised 1-cocycles defined as

$$Z_N^1(X; A) = \big\{ \lambda \colon X \to A \mid \lambda(x + y) = \lambda(x) + \lambda(y) \text{ and } \lambda(x \cdot y) = \lambda(y) \text{ for all } x, y \in X \big\}.$$

The following is an analogue of a similar classical result for groups [**19**, Lemma 5.2].

Lemma 2.3. *Let $(X, \cdot, +)$ be a linear cycle set, A an abelian group and two choices $f, g \colon X \times X \to A$ of maps. Then the set $X \times A$ with the operations*

$$(x, a) + (y, b) = \big(x + y, a + b + g(x, y) \big) \text{ and}$$
$$(x, a) \cdot (y, b) = \big(x \cdot y, b + f(x, y) \big)$$

for $a, b \in A$, $x, y \in X$, is a linear cycle set if and only if (f, g) is a 2-cocycle.

The linear cycle set of Lemma 2.3 is denoted by $X \oplus_{f,g} A$. A reformulation of Lemma 2.3 for braces is as follows [**19**, Lemma 5.4].

Lemma 2.4. *Let $(X, \circ, +)$ be a brace, A be an abelian group, and $f, g \colon X \times X \to A$ be two maps. Then the set $X \times A$ with the operations*

$$(x, a) + (y, b) = \big(x + y, a + b + g(x, y) \big) \text{ and}$$
$$(x, a) \circ (y, b) = \big(x \circ y, a + b + f(x, y) \big)$$

for $a, b \in A$, $x, y \in X$, is a brace if and only if for the corresponding linear cycle set $(X, \cdot, +)$, the pair $(\overline{f}, g)$ is a 2-cocycle, where

$$\overline{f}(x, y) = -f(x, x \cdot y) + g(x, y) \tag{12}$$

for all $x, y \in X$.

A linear cycle subset X' of a linear cycle set X is called *central* if $x \cdot x' = x'$ and $x' \cdot x = x$ for all $x \in X$ and $x' \in X'$. A *central extension* of a linear cycle set $(X, \cdot, +)$ by an abelian group A is the datum of a short exact sequence of linear cycle sets

$$0 \to A \xrightarrow{i} E \xrightarrow{\pi} X \to 0, \tag{13}$$

where A is endowed with the trivial cycle set structure, and its image $i(A)$ is central in E. Notice that an extension

$$0 \to A \to E \to X \to 0 \tag{14}$$

of abelian groups can be thought of as a central extension of linear cycle sets by viewing each group as a trivial linear cycle set. Let $\mathrm{Ext}(X, A)$ denote the set of equivalence classes of central extensions of X by A. The linear cycle set $X \oplus_{f,g} A$ from Lemma 2.3 is a central extension of X by A in the obvious way. More precisely, we have a central extension of linear cycle sets

$$0 \to A \xrightarrow{i} X \oplus_{f,g} A \xrightarrow{\pi} X \to 0,$$

where $i(a) = (0, a)$ and $\pi(x, a) = x$ for all $a \in A$ and $x \in X$. Trivially, the underlying extension

$$0 \to A \xrightarrow{i} (X \oplus_{f,g} A, +) \xrightarrow{\pi} (X, +) \to 0$$

of abelian groups is also central. As in case of groups, all central linear cycle set extensions of X by A arise in this manner [**19**, Lemma 5.6].

Lemma 2.5. *Let* $0 \to A \xrightarrow{i} E \xrightarrow{\pi} X \to 0$ *be a central linear cycle set extension and* $s \colon X \to E$ *be a set-theoretic section of* π.

1. *The maps* $f, g \colon X \times X \to E$ *defined by*

$$f(x, y) = s(x) \cdot s(y) - s(x \cdot y) \;\; and$$
$$g(x, y) = s(x) + s(y) - s(x + y)$$

   take values in $i(A)$ *and* (f, g) *is a 2-cocycle.*

2. *The cocycle above is normalised if and only if* $s(0) = 0$.

3. *Extensions* E *and* $X \oplus_{f,g} A$ *are equivalent.*

4. *A cocycle* (f', g') *obtained from another section* s' *of* π *is cohomologous to* (f, g). *If both cocycles are normalised, then they are cohomologous in the normalised sense.*

Lemma 2.5 yields a bijective correspondence

$$\text{Ext}(X, A) \longleftrightarrow \text{H}^2_{\text{N}}(X; A). \tag{15}$$

Thus, central extensions of linear cycle sets (and hence of braces) are completely determined by their second normalised cohomology groups.

The rest of the paper centres around the extension $X \oplus_{f,g} A$ of a linear cycle set $(X, \cdot, +)$ by an abelian group A with respect to a given 2-cocycle (f, g). Thus, in view of lemmas 2.3 and 2.5, we would only use the 2-cocycle conditions (6)–(9) rather than the defining axioms of a linear cycle set.

2.2. Homomorphism from linear cycle set cohomology to group cohomology

Let G be a group, A a G-module and $\mathbf{H}^2(G; A)$ the second group cohomology of G with coefficients in A. It is well-known that $\mathbf{H}^2(G; A)$ classifies group extensions of G by A inducing the given action of G on A [**3**]. Recall that a choice of group theoretical 2-cocycle $g \colon G \times G \to A$ satisfying (6) is called a *symmetric 2-cocycle*. Let $\mathbf{H}^2_{sym}(G; A)$ be the subgroup of $\mathbf{H}^2(G; A)$ consisting of cohomology classes of group theoretical symmetric 2-cocycles (that is, maps satisfying conditions (6) and (7)). An easy check shows that if both G and A are abelian groups, then $\mathbf{H}^2_{sym}(G; A)$ classifies extensions of the form

$$0 \to A \to E \to X \to 0,$$

where E is an abelian group. It follows that A is necessarily a trivial G-module in this case. There is a natural group homomorphism from linear cycle set cohomology to symmetric group cohomology.

Proposition 2.6. *Let $(X, \cdot, +)$ be a linear cycle set and A an abelian group viewed as a trivial $(X, +)$-module. Then there is a group homomorphism*

$$\Lambda \colon \mathrm{H}^2_N(X; A) \to \boldsymbol{H}^2_{sym}((X, +); A)$$

given by $\Lambda[(f, g)] = [g]$.

Proof. Given a normalised 2-cocycle $(f, g) \in \mathrm{Z}^2_N(X; A)$ for the linear cycle set $(X, \cdot, +)$ with coefficients in the abelian group A, conditions (6) and (7) imply that g is a group theoretical symmetric 2-cocycle of the abelian group $(X, +)$ with coefficients in the abelian group A. Further, if $(f, g) \in \mathrm{B}^2_N(X; A)$, then condition (11) imply that g is a group theoretical 2-coboundary. Thus, there is a well-defined map

$$\Lambda \colon \mathrm{H}^2_N(X; A) \to \boldsymbol{H}^2_{sym}((X, +); A) \text{ given by } \Lambda[(f, g)] = [g].$$

That Λ is a group homomorphism follows from the fact that addition of 2-cocycles is point-wise for both linear cycle sets and groups. □

Given two abelian groups G and A, let $\mathrm{Bilin}(G \times G, A)$ denote the group of bilinear maps from $G \times G$ to A. For trivial cycle sets, the map Λ is surjective and its kernel can be determined precisely.

Proposition 2.7. *Let $(X, \cdot, +)$ be a trivial linear cycle set and A an abelian group viewed as a trivial $(X, +)$-module. Then*

$$\mathrm{H}^2_N(X; A) \cong \mathrm{Bilin}((X, +) \times (X, +), A) \times \boldsymbol{H}^2_{sym}((X, +); A).$$

Proof. We begin by noting that if $f, g \colon X \times X \to A$ is a 2-cocycle of the trivial linear cycle set $(X, \cdot, +)$, then conditions (6)–(7) imply that g is a group theoretical symmetric 2-cocycle, whereas conditions (8)–(9) imply that f is a bilinear map, that is, $f \in \mathrm{Bilin}((X, +) \times (X, +), A)$. Further, by conditions (10)–(11), (f, g) is a 2-coboundary if there exists a map $\lambda \colon X \to A$ such that

$$f(x, y) = 0$$

and

$$g(x, y) = \lambda(x + y) - \lambda(x) - \lambda(y)$$

for all $x, y \in X$. Thus, it follows that

$$\mathrm{H}^2_N(X; A) \cong \mathrm{Bilin}((X, +) \times (X, +), A) \times \mathbf{H}^2_{sym}((X, +); A),$$

and the map Λ is simply projection onto the second factor. □

Given two abelian groups $(X, +)$ and A, where the second group A is viewed as a trivial $(X, +)$-module, it follows from Proposition 2.7 that the product group $\mathrm{Bilin}((X, +) \times (X, +), A) \times \mathbf{H}^2_{sym}((X, +); A)$ classifies meta-trivial linear cycle sets, that is, extensions of a trivial linear cycle set by a trivial linear cycle set.

Question 2.8. *What can we say about the homomorphism Λ for non-trivial linear cycle sets?*

3.   Action of automorphisms on cohomology of linear cycle sets

Let $(X, \cdot, +)$ be a linear cycle set and A an abelian group. Let $\mathrm{Aut}(X)$ denote the group of all linear cycle set automorphisms of X and $\mathbf{Aut}(A)$ the usual automorphism group of A. For $(\phi, \theta) \in \mathrm{Aut}(X) \times \mathbf{Aut}(A)$ and $(f, g) \in \mathrm{Z}_\mathrm{N}^2(X; A)$, we define

$$^{(\phi,\theta)}(f, g) = \left(^{(\phi,\theta)}f, \ ^{(\phi,\theta)}g\right),$$

where

$$^{(\phi,\theta)}f(x, y) := \theta\big(f\big(\phi^{-1}(x), \phi^{-1}(y)\big)\big)$$

and

$$^{(\phi,\theta)}g(x, y) := \theta\big(g\big(\phi^{-1}(x), \phi^{-1}(y)\big)\big)$$

for all $x, y \in X$.

Proposition 3.1. *The group* $\mathrm{Aut}(X) \times \mathbf{Aut}(A)$ *acts by automorphisms on the group* $\mathrm{H}_\mathrm{N}^2(X; A)$ *as*

$$^{(\phi,\theta)}[f, g] = [^{(\phi,\theta)}(f, g)]$$

for $(\phi, \theta) \in \mathrm{Aut}(X) \times \mathbf{Aut}(A)$ *and* $(f, g) \in \mathrm{Z}_\mathrm{N}^2(X; A)$.

Proof. For $(\phi, \theta) \in \mathrm{Aut}(X) \times \mathbf{Aut}(A)$ and $(f, g) \in \mathrm{Z}_\mathrm{N}^2(X; A)$, we first show that $^{(\phi,\theta)}(f, g)$ is a normalised 2-cocycle of the cycle set X. For $x, y \in X$, we have

$$^{(\phi,\theta)}f(x + y, z) = \theta\big(f\big(\phi^{-1}(x + y), \phi^{-1}(z)\big)\big)$$
$$= \theta\big(f\big(\phi^{-1}(x) + \phi^{-1}(y), \phi^{-1}(z)\big)\big)$$
$$= \theta\big(f\big(\phi^{-1}(x) \cdot \phi^{-1}(y), \phi^{-1}(x) \cdot \phi^{-1}(z)\big)\big) + \theta\big(f\big(\phi^{-1}(x), \phi^{-1}(z)\big)\big) \text{ due to } (8)$$
$$= \theta\big(f\big(\phi^{-1}(x \cdot y), \phi^{-1}(x \cdot z)\big)\big) + \theta\big(f\big(\phi^{-1}(x), \phi^{-1}(z)\big)\big)$$
$$= {}^{(\phi,\theta)}f(x \cdot y, x \cdot z) + {}^{(\phi,\theta)}f(x, z),$$

$$^{(\phi,\theta)}f(x, y + z) - {}^{(\phi,\theta)}f(x, y) - {}^{(\phi,\theta)}f(x, z)$$
$$= \theta\big(f\big(\phi^{-1}(x), \phi^{-1}(y) + \phi^{-1}(z)\big) - f\big(\phi^{-1}(x), \phi^{-1}(y)\big) - f\big(\phi^{-1}(x), \phi^{-1}(z)\big)\big)$$
$$= \theta\big(g\big(\phi^{-1}(x) \cdot \phi^{-1}(y), \phi^{-1}(x) \cdot \phi^{-1}(z)\big) - g\big(\phi^{-1}(y), \phi^{-1}(z)\big)\big) \text{ due to } (9)$$
$$= \theta\big(g\big(\phi^{-1}(x \cdot y), \phi^{-1}(x \cdot z)\big)\big) - \theta\big(g\big(\phi^{-1}(y), \phi^{-1}(z)\big)\big)$$
$$= {}^{(\phi,\theta)}g(x \cdot y, x \cdot z) - {}^{(\phi,\theta)}g(y, z)$$

and

$$^{(\phi,\theta)}g(x, y) + {}^{(\phi,\theta)}g(x + y, z) = \theta\big(g\big(\phi^{-1}(x), \phi^{-1}(y)\big) + g\big(\phi^{-1}(x) + \phi^{-1}(y), \phi^{-1}(z)\big)\big)$$
$$= \theta\big(g\big(\phi^{-1}(y), \phi^{-1}(z)\big) + g\big(\phi^{-1}(x), \phi^{-1}(y) + \phi^{-1}(z)\big)\big) \text{ due to } (7)$$
$$= \theta\big(g\big(\phi^{-1}(y), \phi^{-1}(z)\big)\big) + \theta\big(g\big(\phi^{-1}(x), \phi^{-1}(y + z)\big)\big)$$
$$= {}^{(\phi,\theta)}g(y, z) + {}^{(\phi,\theta)}g(x, y + z).$$

Further, $^{(\phi,\theta)}g(x, y) = {}^{(\phi,\theta)}g(y, x)$ and $^{(\phi,\theta)}g(0, 0) = 0$. Hence, $^{(\phi,\theta)}(f, g) \in \mathrm{Z}_\mathrm{N}^2(X; A)$.

Now, given $(\phi_i, \theta_i) \in \mathrm{Aut}(X) \times \mathbf{Aut}(A)$ for $i = 1, 2$, we see that

$$
\begin{aligned}
{}^{(\phi_1,\theta_1)(\phi_2,\theta_2)} f(x,y) &= {}^{(\phi_1\phi_2,\theta_1\theta_2)} f(x,y) \\
&= \theta_1\theta_2\big(f\big(\phi_2^{-1}\phi_1^{-1}(x), \phi_2^{-1}\phi_1^{-1}(y)\big)\big) \\
&= \theta_1\big({}^{(\phi_2,\theta_2)} f\big(\phi_1^{-1}(x), \phi_1^{-1}(y)\big)\big) \\
&= {}^{(\phi_1,\theta_1)}\big({}^{(\phi_2,\theta_2)} f\big)(x,y)
\end{aligned}
$$

for all $x, y \in X$. Similarly, one can show that

$$
{}^{(\phi_1,\theta_1)(\phi_2,\theta_2)} g(x,y) = {}^{(\phi_1,\theta_1)}\big({}^{(\phi_2,\theta_2)} g\big)(x,y).
$$

It can now easily be deduced that the group $\mathrm{Aut}(X) \times \mathbf{Aut}(A)$ acts on $\mathrm{Z}_{\mathrm{N}}^2(X; A)$ by automorphisms. It only remains to be shown that the action preserves $\mathrm{B}_{\mathrm{N}}^2(X; A)$. Let $(f, g) \in \mathrm{B}_{\mathrm{N}}^2(X; A)$. Then there exists $\lambda \colon X \to A$ such that conditions (10) and (11) holds. For $x, y \in X$, we have

$$
\begin{aligned}
{}^{(\phi,\theta)} f(x,y) &= \theta\big(f\big(\phi^{-1}(x), \phi^{-1}(y)\big)\big) \\
&= \theta\big(\lambda\big(\phi^{-1}(x) \cdot \phi^{-1}(y)\big) - \lambda\big(\phi^{-1}(y)\big)\big) \\
&= \theta\big(\lambda\big(\phi^{-1}(x \cdot y)\big)\big) - \theta\big(\lambda\big(\phi^{-1}(y)\big)\big) \\
&= \lambda'(x \cdot y) - \lambda'(y)
\end{aligned}
$$

and

$$
\begin{aligned}
{}^{(\phi,\theta)} g(x,y) &= \theta\big(\lambda\big(\phi^{-1}(x) + \phi^{-1}(y)\big) - \lambda\big(\phi^{-1}(x)\big) - \lambda\big(\phi^{-1}(y)\big)\big) \\
&= \lambda'(x + y) - \lambda'(x) - \lambda'(y),
\end{aligned}
$$

where $\lambda' = \theta\lambda\phi^{-1} \colon X \to A$. Hence, ${}^{(\phi,\theta)}(f, g) \in \mathrm{B}_{\mathrm{N}}^2(X; A)$ and we are done. $\qquad \square$

Applying the orbit-stabiliser theorem to the action of $\mathrm{Aut}(X) \times \mathbf{Aut}(A)$ on $\mathrm{H}_{\mathrm{N}}^2(X; A)$ yields

Corollary 3.2. *If X is a finite linear cycle set, A a finite abelian group and a pair $(f, g) \in \mathrm{Z}_{\mathrm{N}}^2(X; A)$, then*

$$
|\,\mathrm{H}_{\mathrm{N}}^2(X; A)| \geqslant \frac{|\,\mathrm{Aut}(X) \times \mathbf{Aut}(A)|}{|(\mathrm{Aut}(X) \times \mathbf{Aut}(A))_{[f,g]}|},
$$

where $(\mathrm{Aut}(X) \times \mathbf{Aut}(A))_{[f,g]}$ is the stabiliser subgroup of $\mathrm{Aut}(X) \times \mathbf{Aut}(A)$ at $[f, g]$.

In general, we have $\mathrm{Aut}(X) \leqslant \mathbf{Aut}((X, +))$ for any linear cycle set $(X, \cdot, +)$ and the equality holds for trivial linear cycle sets. Corollary 3.2 and Proposition 2.7 then yields the following.

Corollary 3.3. *If G and A are finite abelian groups with A viewed as a trivial G-module and $(f, g) \in \mathrm{Z}_{\mathrm{N}}^2(G; A)$, then*

$$
|\,\boldsymbol{H}_{sym}^2(G; A)| \geqslant \frac{|\,\boldsymbol{Aut}(G) \times \boldsymbol{Aut}(A)|}{|\,\mathrm{Bilin}(G \times G, A)|\,|(\boldsymbol{Aut}(G) \times \boldsymbol{Aut}(A))_{[f,g]}|},
$$

where $(\boldsymbol{Aut}(G) \times \boldsymbol{Aut}(A))_{[f,g]}$ is the stabiliser subgroup of $\boldsymbol{Aut}(G) \times \boldsymbol{Aut}(A)$ at $[f, g]$.

4. An exact sequence relating automorphisms and cohomology

Let $(X, \cdot, +)$ be a linear cycle set, A an abelian group and

$$\mathcal{E} \colon 0 \to A \xrightarrow{i} E \xrightarrow{\pi} X \to 0 \tag{16}$$

a central extension of X by A. In view of Lemma 2.5, there exists a normalised 2-cocycle (f, g) such that

$$E \cong X \oplus_{f,g} A.$$

Further, the extension $\mathcal{E}$ determines a unique cohomology class $[f, g] \in \mathrm{H}^2_{\mathrm{N}}(X; A)$.

Fix a central extension (16) of a linear cycle set $(X, \cdot, +)$ by an abelian group A and its corresponding cohomology class $[f, g] \in \mathrm{H}^2_{\mathrm{N}}(X; A)$ as in Lemma 2.5. For each $(\phi, \theta) \in \mathrm{Aut}(X) \times \mathbf{Aut}(A)$, we have $^{(\phi, \theta)}[f, g] \in \mathrm{H}^2_{\mathrm{N}}(X; A)$. Since the group $\mathrm{H}^2_{\mathrm{N}}(X; A)$ acts freely and transitively on itself by (left) translation, there exists a unique element $\Theta_{[f,g]}(\phi, \theta) \in \mathrm{H}^2_{\mathrm{N}}(X; A)$ such that

$$\Theta_{[f,g]}(\phi, \theta) + \ ^{(\phi, \theta)}[f, g] = [f, g].$$

This gives a map

$$\Theta_{[f,g]} \colon \ \mathrm{Aut}(X) \times \mathbf{Aut}(A) \to \mathrm{H}^2_{\mathrm{N}}(X; A). \tag{17}$$

We denote $\Theta_{[f,g]}$ by Θ for convenience of notation. Our aim is to relate certain group of automorphisms of E to groups $\mathrm{Aut}(X)$, $\mathbf{Aut}(A)$, $\mathrm{H}^2_{\mathrm{N}}(X; A)$ and $\mathrm{Z}^1_{\mathrm{N}}(X; A)$. For this purpose, we define

$$\mathrm{Aut}_A(E) = \big\{ \psi \in \mathrm{Aut}(E) \mid \psi(x, a) = (\phi(x), \lambda(x) + \theta(a)) \text{ for some}$$
$$(\phi, \theta) \in \mathrm{Aut}(X) \times \mathrm{Aut}(A) \text{ and map } \lambda \colon X \to A \big\}.$$

Proposition 4.1. $\mathrm{Aut}_A(E)$ *is a subgroup of* $\mathrm{Aut}(E)$.

Proof. Let $\psi_i(x, a) = (\phi_i(x), \lambda_i(x) + \theta_i(a))$ for $i = 1, 2$. Then

$$
\begin{aligned}
\psi_1 \psi_2(x, a) &= \psi_1\big(\phi_2(x), \ \lambda_2(x) + \theta_2(a)\big) \\
&= \big(\phi_1 \phi_2(x), \ \lambda_1\big(\phi_2(x)\big) + \theta_1\big(\lambda_2(x)\big) + \theta_1 \theta_2(a)\big) \\
&= \big(\phi_1 \phi_2(x), \ \lambda(x) + \theta_1 \theta_2(a)\big),
\end{aligned} \tag{18}
$$

where $\lambda \colon X \to A$ is given by

$$\lambda(x) = \lambda_1\big(\phi_2(x)\big) + \theta_1\big(\lambda_2(x)\big) \tag{19}$$

for $x \in X$ and $a \in A$. Consequently, $\mathrm{Aut}_A(E)$ is closed under composition. It remains to show that if $\psi \in \mathrm{Aut}_A(E)$, then $\psi^{-1} \in \mathrm{Aut}_A(E)$. But a direct computation shows that if $\psi(x, a) = (\phi(x), \lambda(x) + \theta(a))$, then

$$\psi^{-1}(x, a) = \big(\phi^{-1}(x), \ \theta^{-1}\big(-\lambda(\phi^{-1}(x))\big) + \theta^{-1}(a)\big),$$

and hence $\mathrm{Aut}_A(E)$ is a subgroup of $\mathrm{Aut}(E)$. $\qquad\square$

Proposition 4.2. *Let* $(\phi, \theta) \in \mathrm{Aut}(X) \times \mathbf{Aut}(A)$ *and* $\lambda \colon X \to A$ *a map. Then the map* $\psi \colon E \to E$ *given by* $\psi(x, a) = (\phi(x), \lambda(x) + \theta(a))$ *is an automorphism of* E *if*

and only if

$$\lambda(x + y) + \theta\big(g(x, y)\big) = \lambda(x) + \lambda(y) + g\big(\phi(x), \phi(y)\big) \tag{20}$$

and

$$\lambda(x \cdot y) + \theta\big(f(x, y)\big) = \lambda(y) + f\big(\phi(x), \phi(y)\big) \tag{21}$$

for all $x, y \in X$.

Proof. A direct computation shows that

$$\psi\big((x, a) + (y, b)\big) = \psi\big(x + y, \ a + b + g(x, y)\big)$$
$$= \big(\phi(x + y), \ \lambda(x + y) + \theta\big(a + b + g(x, y)\big)\big),$$

$$\psi(x, a) + \psi(y, b) = \big(\phi(x), \ \lambda(x) + \theta(a)\big) + \big(\phi(y), \lambda(y) + \theta(b)\big)$$
$$= \big(\phi(x) + \phi(y), \ \lambda(x) + \lambda(y) + \theta(a) + \theta(b) + g\big(\phi(x), \phi(y)\big)\big),$$

$$\psi\big((x, a) \cdot (y, b)\big) = \psi\big(x \cdot y, \ b + f(x, y)\big)$$
$$= \big(\phi(x \cdot y), \ \lambda(x \cdot y) + \theta\big(b + f(x, y)\big)\big) \text{ and}$$

$$\psi(x, a) \cdot \psi(y, b) = \big(\phi(x), \ \lambda(x) + \theta(a)\big) \cdot \big(\phi(y), \lambda(y) + \theta(b)\big)$$
$$= \big(\phi(x) \cdot \phi(y), \ \lambda(y) + \theta(b) + f\big(\phi(x), \phi(y)\big)\big).$$

The result now follows immediately from the preceding equalities. □

In view of (18), the map

$$\Psi \colon \mathrm{Aut}_A(E) \to \mathrm{Aut}(X) \times \mathbf{Aut}(A)$$

given by $\Psi(\psi) = (\phi, \theta)$ is a group homomorphism.

Proposition 4.3. $\mathrm{Im}(\Psi) = \Theta^{-1}\{0\}$.

Proof. First note that

$$\Theta^{-1}\{0\} = \big\{(\phi, \theta) \in \mathrm{Aut}(X) \times \mathbf{Aut}(A) \mid {}^{(\phi, \theta)}[f, g] = [f, g]\big\},$$

the stabiliser subgroup of $\mathrm{Aut}(X) \times \mathbf{Aut}(A)$ at $[f, g]$. Suppose that $(\phi, \theta) \in \Theta^{-1}\{0\}$. Then, by definition of cohomologous 2-cocycles, there exists a map $\lambda \colon X \to A$ such that

$$\begin{aligned}{}^{(\phi, \theta)}g(x, y) - g(x, y) &= \lambda(x + y) - \lambda(x) - \lambda(y) \text{ and} \\ {}^{(\phi, \theta)}f(x, y) - f(x, y) &= \lambda(x \cdot y) - \lambda(y)\end{aligned}$$

for all $x, y \in X$. The preceding equations can be written as

$$\theta\big(g(x, y)\big) - g\big(\phi(x), \phi(y)\big) = \lambda\big(\phi(x) + \phi(y)\big) - \lambda\big(\phi(x)\big) - \lambda\big(\phi(y)\big) \text{ and} \tag{22}$$
$$\theta\big(f(x, y)\big) - f\big(\phi(x), \phi(y)\big) = \lambda\big(\phi(x) \cdot \phi(y)\big) - \lambda\big(\phi(y)\big) \tag{23}$$

for all $x, y \in X$. We define $\psi \colon E \to E$ by setting

$$\psi(x, a) = \big(\phi(x), \ -\lambda\big(\phi(x)\big) + \theta(a)\big)$$

for $x \in X$ and $a \in A$. Notice that equations (22) and (23) are precisely equations (20)

and (21), respectively. Hence, it follows from Proposition 4.2 that $\psi \in \mathrm{Aut}_A(E)$. Since $\Psi(\psi) = (\phi, \theta)$, we get $\Theta^{-1}\{0\} \subseteq \mathrm{Im}(\Psi)$. Conversely, if $(\phi, \theta) \in \mathrm{Im}(\Psi)$, then there exists a map $\lambda \colon X \to A$ such that $\psi \colon E \to E$ given by $\psi(x, a) := (\phi(x),\ \lambda(x) + \theta(a))$ lies in $\mathrm{Aut}_A(E)$. Again, by Proposition 4.2, the map ψ being a morphism of linear cycle sets gives equations (22) and (23). As seen above, this implies that $^{(\phi,\theta)}[f, g] = [f, g]$, which completes the proof. $\qquad \square$

Proposition 4.4. $\mathrm{Z}_{\mathrm{N}}^1(X; A) \cong \mathrm{Ker}(\Psi)$.

Proof. Observe that $\psi \in \mathrm{Ker}(\Psi)$ if and only if there exists a map $\lambda \colon X \to A$ such that $\psi(x, a) = (x,\ \lambda(x) + a)$ for all $x \in X$ and $a \in A$. Now, ψ is a morphism of linear cycle sets if and only if conditions (23) and (22) hold with $\phi = \mathrm{id}_X$ and $\theta = \mathrm{id}_A$. These conditions take the form

$$\lambda(x \cdot y) = \lambda(y) \text{ and} \tag{24}$$
$$\lambda(x + y) = \lambda(x) + \lambda(y) \tag{25}$$

for $x, y \in X$, and hence $\lambda \in \mathrm{Z}_{\mathrm{N}}^1(X; A)$. Conversely, given $\lambda \in \mathrm{Z}_{\mathrm{N}}^1(X; A)$, we see that $\psi \colon E \to E$ defined as $\psi(x, a) = (x,\ \lambda(x) + a)$ is an element of $\mathrm{Ker}(\Psi)$. In view of (19), it follows that the map

$$\iota \colon \mathrm{Z}_{\mathrm{N}}^1(X; A) \to \mathrm{Ker}(\Psi)$$

given by $\iota(\lambda) = \psi$ is an isomorphism of groups. $\qquad \square$

Combining (17), Proposition 4.3 and Proposition 4.4 gives the following exact sequence

Theorem 4.5. *Let X be a linear cycle set, A an abelian group and $E = X \oplus_{f,g} A$ the central extension of X by A corresponding to the 2-cocycle $(f, g) \in \mathrm{Z}_{\mathrm{N}}^2(X; A)$. Then there exists an exact sequence of groups*

$$1 \longrightarrow \mathrm{Z}^1(X; A) \overset{\iota}{\longrightarrow} \mathrm{Aut}_A(E) \overset{\Psi}{\longrightarrow} \mathrm{Aut}(X) \times \boldsymbol{Aut}(A) \overset{\Theta}{\longrightarrow} \mathrm{H}_{\mathrm{N}}^2(X; A), \tag{26}$$

where exactness at $\mathrm{Aut}(X) \times \boldsymbol{Aut}(A)$ means that $\mathrm{Im}(\Psi) = \Theta^{-1}\{0\}$.

Corollary 4.6. *Let X be a linear cycle set and A an abelian group such that $\mathrm{H}_{\mathrm{N}}^2(X; A)$ is trivial. Then every automorphism in $\mathrm{Aut}(X) \times \boldsymbol{Aut}(A)$ extends to an automorphism in $\mathrm{Aut}_A(E)$.*

Restricting the action of $\mathrm{Aut}(X) \times \boldsymbol{Aut}(A)$ on $\mathrm{H}_{\mathrm{N}}^2(X; A)$ to that of its subgroups $\mathrm{Aut}(X)$ and $\boldsymbol{Aut}(A)$ gives the following result.

Corollary 4.7. *Every automorphism in $\mathrm{Aut}(X)_{[f]}$ and $\boldsymbol{Aut}(A)_{[g]}$ can be extended to an automorphism in $\mathrm{Aut}_A(E)$.*

5. Properties of map Θ

Let $(X, \cdot, +)$ be a linear cycle set and A an abelian group. Since we know the group $\mathrm{Aut}(X) \times \boldsymbol{Aut}(A)$ acts on the group $\mathrm{H}_{\mathrm{N}}^2(X; A)$, we have their semi-direct product $\mathrm{H}_{\mathrm{N}}^2(X; A) \rtimes \big(\mathrm{Aut}(X) \times \boldsymbol{Aut}(A) \big)$. Further, the group $\mathrm{H}_{\mathrm{N}}^2(X; A)$ acts on itself by (left) translation.

Proposition 5.1. $H_N^2(X; A) \rtimes \big(\mathrm{Aut}(X) \times \boldsymbol{Aut}(A) \big)$ *acts on* $H_N^2(X; A)$ *by setting*

$$^{[\alpha](\phi,\theta)}[\beta] = {}^{[\alpha]}\big({}^{(\phi,\theta)}[\beta] \big)$$

for $(\phi, \theta) \in \mathrm{Aut}(X) \times \boldsymbol{Aut}(A)$ *and* $[\alpha], [\beta] \in H_N^2(X; A)$.

Proof. For $(\phi_1, \theta_1), (\phi_2, \theta_2) \in \mathrm{Aut}(X) \times \mathbf{Aut}(A)$ and $[\alpha_1], [\alpha_2], [\beta] \in H_N^2(X; A)$, we compute

$$\big([\alpha_1](\phi_1,\theta_1) \big)\big([\alpha_2](\phi_2,\theta_2) \big) [\beta] = \big([\alpha_1] \, {}^{(\phi_1,\theta_1)}[\alpha_2] \big)\big((\phi_1,\theta_1)(\phi_2,\theta_2) \big) \, [\beta]$$

$$= \big([\alpha_1] \, {}^{(\phi_1,\theta_1)}[\alpha_2] \big) \Big({}^{\big((\phi_1,\theta_1)(\phi_2,\theta_2) \big)}[\beta] \Big)$$

$$= {}^{[\alpha_1]} \Big({}^{(\phi_1,\theta_1)[\alpha_2]} \big({}^{(\phi_1,\theta_1)} \big({}^{(\phi_2,\theta_2)}[\beta] \big) \big) \Big)$$

$$= {}^{[\alpha_1]} \Big({}^{(\phi_1,\theta_1)}[\alpha_2] + {}^{(\phi_1,\theta_1)} \big({}^{(\phi_2,\theta_2)}[\beta] \big) \Big),$$

since $^{(\phi_1,\theta_1)}[\alpha_2] \in H_N^2(X; A)$, which acts on itself by translation

$$= {}^{[\alpha_1]} \Big({}^{(\phi_1,\theta_1)} \big([\alpha_2] + {}^{(\phi_2,\theta_2)}[\beta] \big) \Big),$$

since $\mathrm{Aut}(X) \times \mathbf{Aut}(A)$ acts by automorphisms on $H_N^2(X; A)$

$$= \big([\alpha_1](\phi_1,\theta_1) \big) \Big({}^{\big([\alpha_2](\phi_2,\theta_2) \big)}[\beta] \Big),$$

since $H_N^2(X; A)$ acts on itself by translation.

Hence, $H_N^2(X; A) \rtimes \big(\mathrm{Aut}(X) \times \mathbf{Aut}(A) \big)$ acts on $H_N^2(X; A)$. $\square$

Let G be a group and A an abelian group equipped with an action of G. Then

$$\mathbf{Z}^1(G; A) = \big\{ f \colon G \to A \mid f(xy) = f(x) + {}^x f(y) \text{ for all } x, y \in G \big\}$$

is called the group of *1-cocycles* and

$$\mathbf{B}^1(G; A) = \big\{ f \colon G \to A \mid \text{there exists } a \in A \text{ such that } f(x) = {}^x a - a \text{ for all } x \in G \big\}$$

the group of *1-coboundaries* [**3**, Chapter 4]. Further, a complement of a subgroup H in a group G is another subgroup K of G such that $G = HK$ and $H \cap K = 1$. The following result relating 1-cocycles and complements is well-known [**31**, 11.1.2].

Lemma 5.2. *Let H be an abelian group and G a group acting on H by automorphisms. Then the map $f \mapsto \{f(g)g \mid g \in G\}$ gives a bijection from the set $\mathbf{Z}^1(G; H)$ of 1-cocycles to the set $\{K \mid G = HK \text{ and } H \cap K = 1\}$ of complements of H in G.*

Theorem 5.3. *Let $(X, \cdot, +)$ be a linear cycle set, A an abelian group along with $\Theta_{[\alpha]} \colon \mathrm{Aut}(X) \times \boldsymbol{Aut}(A) \longrightarrow H_N^2(X; A)$ the map corresponding to a cohomology class $[\alpha] \in H_N^2(X; A)$. Then the following hold:*

1. *$\Theta_{[\alpha]}$ is a group theoretical 1-cocycle.*
2. *Any two such maps corresponding to distinct linear cycle set cohomology classes are cohomologous as group theoretical 1-cocycles.*

Proof. Suppose that

$$\Theta = \Theta_{[\alpha]} \text{ for } [\alpha] \in H_N^2(X; A) \text{ and } \mathbf{g} \in H_N^2(X; A) \rtimes \big(\mathrm{Aut}(X) \times \mathbf{Aut}(A) \big).$$

Then for elements $[\alpha], {}^{\mathbf{g}}[\alpha] \in H_N^2(X; A)$, there exists a unique $[\beta] \in H_N^2(X; A)$ such that $^{[\beta]}[\alpha] = {}^{\mathbf{g}}[\alpha]$. Viewing $[\beta]$ as an element of $H_N^2(X; A) \rtimes \big(\mathrm{Aut}(X) \times \mathbf{Aut}(A) \big)$, it

follows that $[\beta]^{-1}\mathbf{g} \in \mathbb{S}_{[\alpha]}$, the stabiliser subgroup of $\mathrm{H}^2_{\mathrm{N}}(X; A) \rtimes \big(\mathrm{Aut}(X) \times \mathbf{Aut}(A)\big)$ at $[\alpha]$, and hence

$$\mathrm{H}^2_{\mathrm{N}}(X; A) \rtimes \big(\mathrm{Aut}(X) \times \mathbf{Aut}(A)\big) = \mathrm{H}^2_{\mathrm{N}}(X; A)\mathbb{S}_{[\alpha]}.$$

Further, since $\mathrm{H}^2_{\mathrm{N}}(X; A)$ acts freely on itself, it follows that $\mathbb{S}_{[\alpha]}$ is a complement of $\mathrm{H}^2_{\mathrm{N}}(X; A)$ in $\mathrm{H}^2_{\mathrm{N}}(X; A) \rtimes \big(\mathrm{Aut}(X) \times \mathbf{Aut}(A)\big)$. By the bijection of Lemma 5.2, let $f \colon \mathrm{Aut}(X) \times \mathbf{Aut}(A) \to \mathrm{H}^2_{\mathrm{N}}(X; A)$ be the unique 1-cocycle corresponding to the complement $\mathbb{S}_{[\alpha]}$ of $\mathrm{H}^2_{\mathrm{N}}(X; A)$ in $\mathrm{H}^2_{\mathrm{N}}(X; A) \rtimes \big(\mathrm{Aut}(X) \times \mathbf{Aut}(A)\big)$. Then

$$\mathbb{S}_{[\alpha]} = \big\{ f(\phi, \theta)(\phi, \theta) \mid (\phi, \theta) \in \mathrm{Aut}(X) \times \mathbf{Aut}(A) \big\},$$

that is,

$$[\alpha] = {}^{f(\phi,\theta)(\phi,\theta)}[\alpha] = {}^{f(\phi,\theta)}\big({}^{(\phi,\theta)}[\alpha]\big).$$

Now, by definition of Θ as in (17), we obtain $f(\phi, \theta) = \Theta(\phi, \theta)$, and hence Θ is a 1-cocycle.

For the second assertion, let $\Theta = \Theta_{[\alpha]}$ and $\Theta' = \Theta'_{[\alpha']}$ for $[\alpha], [\alpha'] \in \mathrm{H}^2_{\mathrm{N}}(X; A)$. Then for any $(\phi, \theta) \in \mathrm{Aut}(X) \times \mathbf{Aut}(A)$, we have

$$\Theta(\phi,\theta)\big({}^{(\phi,\theta)}[\alpha]\big) = [\alpha] \text{ and } \Theta'(\phi,\theta)\big({}^{(\phi,\theta)}[\alpha']\big) = [\alpha'].$$

Since $\mathrm{H}^2_{\mathrm{N}}(X; A)$ acts transitively on itself by (left) translation, there exists a unique $[\beta] \in \mathrm{H}^2_{\mathrm{N}}(X; A)$ such that ${}^{[\beta]}[\alpha'] = [\alpha]$. This gives

$$\Theta(\phi,\theta)\big({}^{(\phi,\theta)[\beta]}\big({}^{(\phi,\theta)}[\alpha']\big)\big) = {}^{[\beta]}\big(\Theta'(\phi,\theta)\big({}^{(\phi,\theta)}[\alpha']\big)\big).$$

Since $[\beta]$, $\Theta(\phi,\theta)$, ${}^{(\phi,\theta)}[\beta]$ and $\Theta'(\phi,\theta)$ all lie in $\mathrm{H}^2_{\mathrm{N}}(X; A)$, which acts freely on itself, we must have

$$\Theta(\phi,\theta) + {}^{(\phi,\theta)}[\beta] = \Theta'(\phi,\theta) + [\beta].$$

Thus, Θ and Θ' differ by a 1-coboundary, which completes the proof. □

6. Comparison with Wells exact sequence for groups

In [**32**], Wells derived an exact sequence relating 1-cocycles, automorphisms and second cohomology of groups corresponding to a given extension of groups. The sequence has found applications in some long standing problems on automorphisms of finite groups. We refer the reader to [**23**, Chapter 2] for a detailed account of the same, and recall the construction of this exact sequence for central extension of groups. Consider a central extension

$$\mathcal{E}' \colon 0 \to N \to G \to H \to 0 \tag{27}$$

of (additively written) groups. In this case N is a trivial H-module, and hence the group $\mathbf{Z}^1(H; N)$ of 1-cocycles is simply the group of all homomorphisms from H to N. Let $\mathbf{H}^2(H; N)$ be the second group cohomology of H with coefficients in N, and $g \colon H \times H \to N$ be a group theoretical normalised 2-cocycle corresponding to the

extension (27). It follows from classical extension theory of groups that

$$G \cong H \times_g N,$$

where $H \times_g N$ has underlying set $H \times N$ and group operation

$$(x, a) + (y, b) = (x + y, a + b + g(x, y))$$

for $x, y \in H$ and $a, b \in N$.

Let $\mathbf{Aut}_N(G)$ be the group of automorphisms of G keeping N invariant as a set. In view of the identification $G \cong H \times_g N$, we have

$$\mathbf{Aut}_N(G) = \big\{ \psi \in \mathrm{Aut}(G) \mid \psi(x, a) = (\phi(x), \lambda(x) + \theta(a)) \text{ for some}$$
$$(\phi, \theta) \in \mathrm{Aut}(H) \times \mathrm{Aut}(N) \text{ and map } \lambda \colon H \to N \big\}.$$

There is a monomorphism of groups

$$\jmath \colon \mathbf{Z}^1(H, N) \to \mathbf{Aut}_N(G)$$

given by $\jmath(\lambda) = \psi$, where $\psi(x, a) = (x, \lambda(x) + a)$ [**23**, Proposition 2.45]. Also, there is a natural homomorphism

$$\Phi \colon \mathbf{Aut}_N(G) \to \mathbf{Aut}(H) \times \mathbf{Aut}(N)$$

given by

$$\Phi(\psi) = (\phi, \theta).$$

As in Section 3, there is an action of $\mathbf{Aut}(H) \times \mathbf{Aut}(N)$ on $\mathbf{H}^2(H; N)$. In fact, given any $(\phi, \theta) \in \mathbf{Aut}(H) \times \mathbf{Aut}(N)$ and $[h] \in \mathbf{H}^2(H; N)$, setting

$$^{(\phi,\theta)}[h] = [^{(\phi,\theta)}h],$$

where $^{(\phi,\theta)}h(x, y) = \theta\big(h\big(\phi^{-1}(x), \phi^{-1}(y)\big)\big)$ for $x, y \in X$, defines this action. Further, the action restricts to an action on the subgroup $\mathbf{H}^2_{sym}(H; N)$ of $\mathbf{H}^2(H; N)$ consisting of symmetric cohomology classes. Notice that the group $\mathbf{H}^2(H; N)$ acts freely and transitively on itself by (left) translation. Now, for each $(\phi, \theta) \in \mathbf{Aut}(H) \times \mathbf{Aut}(N)$, we have cohomology classes $^{(\phi,\theta)}[g], [g] \in \mathbf{H}^2(H; N)$. Thus, there exists a unique element $\Omega(\phi, \theta) \in \mathbf{H}^2(H; N)$ such that

$$\Omega(\phi, \theta) + \,^{(\phi,\theta)}[g] = [g].$$

This gives a map

$$\Omega \colon \mathbf{Aut}(H) \times \mathbf{Aut}(N) \to \mathbf{H}^2(H; N), \tag{28}$$

which depends on the equivalence class of the extension $\mathcal{E}'$ or equivalently on its corresponding cohomology class. Further, Ω is a 1-cocycle with respect to the action of $\mathbf{Aut}(H) \times \mathbf{Aut}(N)$ on $\mathbf{H}^2(H; N)$ [**23**, Corollary 2.41]. With the preceding set-up, Wells derived the following exact sequence of groups

$$1 \longrightarrow \mathbf{Z}^1(H, N) \overset{\jmath}{\longrightarrow} \mathbf{Aut}_N(G) \overset{\Phi}{\longrightarrow} \mathbf{Aut}(H) \times \mathbf{Aut}(N) \overset{\Omega}{\longrightarrow} \mathbf{H}^2(H; N). \tag{29}$$

Let $\mathcal{L}$ and $\mathcal{A}$ denote the categories of linear cycle sets and abelian groups, respectively. Then there is a forgetful functor

$$\mathcal{F} \colon \mathcal{L} \to \mathcal{A}$$

that maps a linear cycle set to its underlying abelian group. The preceding discussion

shows that the functor $\mathcal{F}$ induces a map from the exact sequence (26) to the exact sequence (29).

Theorem 6.1. *Let $(X, \cdot, +)$ be a linear cycle set, A an abelian group viewed as a trivial $(X, +)$-module and $E = X \oplus_{f,g} A$ the central extension corresponding to the 2-cocycle $(f, g) \in \mathrm{Z}^2_{\mathrm{N}}(X; A)$. Then the following diagram of groups commutes*

$$
\begin{array}{ccccccc}
\mathrm{Z}^1(X; A) & \xrightarrow{\ \iota\ } & \mathrm{Aut}_A(E) & \xrightarrow{\ \Psi\ } & \mathrm{Aut}(X) \times \boldsymbol{Aut}(A) & \xrightarrow{\ \Theta\ } & \mathrm{H}^2_{\mathrm{N}}(X; A) \\
\Big\uparrow{\scriptstyle inclusion} & & \Big\uparrow{\scriptstyle inclusion} & & \Big\uparrow{\scriptstyle inclusion} & & \Big\downarrow{\scriptstyle \Lambda} \\
\boldsymbol{Z}^1((X, +); A) & \xrightarrow{\ \mathrm{J}\ } & \boldsymbol{Aut}_A((E, +)) & \xrightarrow{\ \Phi\ } & \boldsymbol{Aut}((X, +)) \times \boldsymbol{Aut}(A) & \xrightarrow{\ \Omega\ } & \boldsymbol{H}^2_{sym}((X, +); A),
\end{array}
$$

where Λ is as in Proposition 2.6.

7. Extensions of bi-groupoids and dynamical cocycles

During the last decade many new examples of bi-groupoids, namely, bi-racks, bi-quandles, braces, skew braces, linear cycle sets, etc, have been introduced in connection to virtual knot theory and solutions of the Yang–Baxter equation. Let X and S be two non-empty sets, with choices $\alpha, \alpha' \colon X \times X \to \mathrm{Map}(S \times S, S)$ and $\beta, \beta' \colon S \times S \to \mathrm{Map}(X \times X, X)$ of maps. Then $X \times S$ with the binary operations

$$(x, s) \cdot (y, t) = \big(\beta_{s,t}(x, y),\ \alpha_{x,y}(s, t)\big) \tag{30}$$

and

$$(x, s) * (y, t) = \big(\beta'_{s,t}(x, y),\ \alpha'_{x,y}(s, t)\big) \tag{31}$$

forms a bi-groupoid. Naturally, to obtain a bi-groupoid of special type it is essential to have functions $\alpha, \alpha', \beta, \beta'$ with nice properties. Using defining axioms of a cycle set, we can deduce a generalisation of [**30**, Lemma 2.1], which itself is a linear cycle set analogue of a similar result for quandles [**1**, Lemma 2.1].

Proposition 7.1. *Let X and S be two sets, with $\alpha \colon X \times X \to \mathrm{Map}(S \times S, S)$ and $\beta \colon S \times S \to \mathrm{Map}(X \times X, X)$ two maps. Then the set $X \times S$ with the binary operation*

$$(x, s) \cdot (y, t) = \big(\beta_{s,t}(x, y),\ \alpha_{x,y}(s, t)\big) \tag{32}$$

forms a cycle set if and only if the following conditions hold:

1. *the map $(y, t) \mapsto \big(\beta_{s,t}(x, y),\ \alpha_{x,y}(s, t)\big)$ is a bijection for all $(x, s) \in X \times S$,*

2. *$\beta_{\alpha_{x,y}(s,t), \alpha_{x,z}(s,q)}\big(\beta_{s,t}(x, y), \beta_{s,q}(x, z)\big) = \beta_{\alpha_{y,x}(t,s), \alpha_{y,z}(t,q)}\big(\beta_{t,s}(y, x), \beta_{t,q}(y, z)\big)$*

   and

   $\alpha_{\beta_{s,t}(x,y), \beta_{s,q}(x,z)}\big(\alpha_{x,y}(s, t), \alpha_{x,z}(s, q)\big) = \alpha_{\beta_{t,s}(y,x), \beta_{t,q}(y,z)}\big(\alpha_{y,x}(t, s), \alpha_{y,z}(t, q)\big)$
   for all $x, y, z \in X$ and $s, t, q \in S$.

If one of the sets is a cycle set, then we obtain

Corollary 7.2. *Let $(X, \cdot)$ be a cycle set, S a set and $\alpha \colon X \times X \to \mathrm{Map}(S \times S, S)$ a map. Then the set $X \times S$ with the binary operation*

$$(x, s) \cdot (y, t) = \big(x \cdot y,\ \alpha_{x,y}(s, t)\big) \tag{33}$$

forms a cycle set if and only if the following conditions hold:

1. the map $(y, t) \mapsto \big(x \cdot y,\ \alpha_{x,y}(s, t)\big)$ is a bijection for each $(x, s) \in X \times S$,

2. $\alpha_{x \cdot y, x \cdot z}\Big(\alpha_{x,y}(s, t),\ \alpha_{x,z}(s, q)\Big) = \alpha_{y \cdot x, y \cdot z}\Big(\alpha_{y,x}(t, s),\ \alpha_{y,z}(t, q)\Big)$ for all choices $x, y, z \in X$ and $s, t, q \in S$.

A map α satisfying condition (2) of the preceding corollary is referred as a *dynamical cocycle* of X with values in S and the cycle set structure on $X \times S$ is called a *dynamical extension* of X by α. One can prove a similar result for linear cycle sets. Before stating the result, note that condition 1 in the definition of a linear cycle set is redundant [**25**, Section 3].

Proposition 7.3. *Let X and S be two sets, $\alpha, \alpha' \colon X \times X \to \mathrm{Map}(S \times S, S)$ and $\beta, \beta' \colon S \times S \to \mathrm{Map}(X \times X, X)$ maps. Then the set $X \times S$ with the binary operations*

$$(x, s) \cdot (y, t) = \big(\beta_{s,t}(x, y),\ \alpha_{x,y}(s, t)\big) \tag{34}$$

and

$$(x, s) + (y, t) = \big(\beta'_{s,t}(x, y),\ \alpha'_{x,y}(s, t)\big) \tag{35}$$

forms a linear cycle set if and only if the following conditions hold:

1. the map $(y, t) \mapsto \big(\beta_{s,t}(x, y),\ \alpha_{x,y}(s, t)\big)$ is a bijection for each $(x, s) \in X \times S$,

2. $\beta_{s, \alpha'_{y,z}(t, q)}\Big(x,\ \beta'_{t,q}(y, z)\Big) = \beta'_{\alpha_{x,y}(s, t), \alpha_{x,z}(s, q)}\Big(\beta_{s,t}(x, y),\ \beta_{s,q}(x, z)\Big)$

   and

   $\alpha_{x, \beta'_{t,q}(y, z)}\Big(s,\ \alpha'_{y,z}(t, q)\Big) = \alpha'_{\beta_{s,t}(x,y), \beta_{s,q}(x,z)}\Big(\alpha_{x,y}(s, t),\ \alpha_{x,z}(s, q)\Big)$,

3. $\beta_{\alpha'_{x,y}(s,t), q}\Big(\beta'_{s,t}(x, y),\ z\Big) = \beta_{\alpha_{x,y}(s,t), \alpha_{x,z}(s,q)}\Big(\beta_{s,t}(x, y),\ \beta_{s,q}(x, z)\Big)$

   and

   $\alpha_{\beta'_{s,t}(x,y), z}\Big(\alpha'_{x,y}(s, t),\ q\Big) = \alpha_{\beta_{s,t}(x,y), \beta_{s,q}(x,z)}\Big(\alpha_{x,y}(s, t),\ \alpha_{x,z}(s, q)\Big)$

for all $x, y, z \in X$ and $s, t, q \in S$.

As before, if one of the sets is already a linear cycle set, then we have

Corollary 7.4. *Let $(X, \cdot, +)$ be a linear cycle set, S a set and two choices of maps $\alpha, \alpha' \colon X \times X \to \mathrm{Map}(S \times S, S)$. Then the set $X \times S$ with the binary operations*

$$(x, s) \cdot (y, t) = \big(x \cdot y,\ \alpha_{x,y}(s, t)\big) \tag{36}$$

and

$$(x, s) + (y, t) = \big(x + y,\ \alpha'_{x,y}(s, t)\big) \tag{37}$$

forms a linear cycle set if and only if the following conditions hold:

1. the map $(y, t) \mapsto \big(x \cdot y,\ \alpha_{x,y}(s, t)\big)$ is a bijection for each $(x, s) \in X \times S$,

2. $\alpha_{x, y+z}\Big(s,\ \alpha'_{y,z}(t, q)\Big) = \alpha'_{x \cdot y, x \cdot z}\Big(\alpha_{x,y}(s, t),\ \alpha_{x,z}(s, q)\Big)$,

3. $\alpha_{x+y, z}\Big(\alpha'_{x,y}(s, t),\ q\Big) = \alpha_{x \cdot y, x \cdot z}\Big(\alpha_{x,y}(s, t),\ \alpha_{x,z}(s, q)\Big)$

for all $x, y, z \in X$ and $s, t, q \in S$.

The pair of maps $\alpha, \alpha' \colon X \times X \to \mathrm{Map}(S \times S, S)$ satisfying conditions (2)–(4) of the preceding corollary is called a *dynamical cocycle* of the linear cycle set X with values in S and the linear cycle set structure on $X \times S$ is called the *dynamical extension* of X by S. Taking

$$\alpha_{x,y}(s,t) = t + f(x,y)$$

and

$$\alpha'_{x,y}(s,t) = s + t + g(x,y)$$

for some 2-cocycle $(f,g) \in \mathrm{Z}^2_{\mathrm{N}}(X;A)$, we see that the dynamical extension generalises the extension obtained in Lemma 2.3.

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