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OPTIMAL COLORED TVERBERG THEOREMS FOR PRIME POWERS

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Abstract

The colored Tverberg theorem of Blagojević, Matschke, and Ziegler (Theorem 1.4) provides optimal bounds for the colored Tverberg problem, under the condition that the number of intersecting rainbow simplices $r = p$ is a prime number.

Our Theorem 1.6 extends this result to an optimal colored Tverberg theorem for multisets of colored points, which is valid for each prime power $r = p^k$, and includes Theorem 1.4 as a special case for $k = 1$. One of the principal new ideas is to replace the ambient simplex Δ^N, used in the original Tverberg theorem, by an "abridged simplex" of smaller dimension, and to compensate for this reduction by allowing vertices to repeatedly appear a controlled number of times in different rainbow simplices. Configuration spaces, used in the proof, are combinatorial pseudomanifolds which can be represented as multiple chessboard complexes. Our main topological tool is the Eilenberg-Krasnoselskii theory of degrees of equivariant maps for non-free actions.

A quite different generalization arises if we consider colored classes that are (approximately) two times smaller than in the classical colored Tverberg theorem. Theorem 1.8, which unifies and extends some earlier results of this type, is based on the constraint method and uses the high connectivity of the configuration space.

1. Introduction

"Tverberg problems" (or Tverberg theorems) is a common name for a class of theorems, problems and conjectures about finite sets of points in Euclidean spaces, more precisely about intersection patterns of the associated convex sets.

The original Tverberg theorem [**Tve66**] claims that every set $X \subset \mathbb{R}^d$, with at least $(r-1)(d+1)+1$ elements, can be partitioned $X = X_1 \sqcup \cdots \sqcup X_r$ into r nonempty,

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pairwise disjoint subsets $X_1, \dots, X_r$ such that the corresponding convex hulls have a nonempty intersection

$$\bigcap_{i=1}^{r} \mathrm{conv}(X_i) \neq \emptyset\,. \tag{1}$$

(Such a partition is usually referred to as a *Tverberg partition*, while a point in (1) is often called a *Tverberg point*.)

For example if $r = 3$ and $d = 2$, the Tverberg theorem reduces to an elementary statement saying that for each collection X of 7 points in the plane, either one of the points in X is covered by two vertex disjoint triangles, formed by the remaining 6 points, or alternatively, there exist two line segments, with distinct end-points in X, which intersect in a point covered by the triangle with vertices in the remaining three points.

At the time when it was proved (1966), Tverberg theorem was an important achievement in combinatorial geometry, both deep and beautiful. It was however difficult to predict that it will give birth to a new branch of topological combinatorics and to the present day inspire its development, see [**BBZ, BS, Sk18, Ž17**] for more detailed historical overview and a guide to the literature.

1.1. Tverberg theorem from a topological viewpoint

Sir Christopher Zeeman, in his lecture "A brief history of topology" (Berkeley, 1993), isolated three major problems (the manifold problem, the embedding problem, and the knotting problem) that shaped early history of topology.

Embedding problem: Given a manifold M, what is the least dimension d such that M is embeddable in $\mathbb{R}^d$.

In the same vein a map $f\colon M \to \mathbb{R}^d$ is an *r-embedding* ($r \in \mathbb{N}$) if for each $y \in \mathbb{R}^d$ there are at most $(r-1)$ points in the pre-image $f^{-1}(y)$. If such a map does not exist then (by definition) M is *not r-embeddable*. Otherwise we say that M is r-embeddable, where a 2-embedding clearly corresponds to the embedding in the usual sense.

If we replace a manifold by a (geometric realisation of a finite) simplicial complex K, and "slightly" modify the definitions, the scenery changes and we are in the realm of combinatorial topology (topological combinatorics). A continuous map $f\colon K \to \mathbb{R}^d$ is called an *almost r-embedding* if $f(\Delta_1) \cap \dots \cap f(\Delta_r) = \emptyset$ for each collection $\{\Delta_i\}_{i=1}^r$ of pairwise disjoint faces of K. If an almost r-embedding of K in $\mathbb{R}^d$ does not exist we say that K is *not almost r-embeddable* in $\mathbb{R}^d$. The obvious implication

$$K \text{ is not almost } r\text{-embeddable in } \mathbb{R}^d \quad \Rightarrow \quad K \text{ is not } r\text{-embeddable in } \mathbb{R}^d$$

leads to combinatorial proofs and interesting refinements of important topological statements. For example the minimal, 6-vertex triangulation of the projective plane $\mathbb{R}P^2$ does not admit an almost embedding (almost 2-embedding) in $\mathbb{R}^3$, see [**M03**, Example 5.8.5].

By similar arguments [**M03**, Section 5.6], one can prove the classical *Van Kampen–Flores theorem* which says that the d-skeleton of the $(2d+2)$-dimensional simplex is not almost embeddable in $\mathbb{R}^{2d}$. Recall that in a very special case $d = 1$ this reduces to the non-planarity of K_5 (the complete graph on five vertices).

A new impetus for the study of these concepts came from the area of discrete and computational geometry, after it was observed that some important, intrinsically affine results, admit a topological reformulation involving (almost) r-embeddability.

Classical *Radon's theorem* (1921), one of the earliest results of combinatorial convexity, claims that every set of $d+2$ points in $\mathbb{R}^d$ can be divided into two disjoint subsets whose convex hulls intersect. Almost sixty years later (1979), Bajmóczy and Bárány [**M03**, Section 5.1.3] proved a topological analogue and a generalisation of Radon's theorem by showing that the $(d+1)$-dimensional simplex Δ^{d+1} is not almost 2-embeddable in $\mathbb{R}^d$.

Tverberg theorem includes Radon's theorem as a special case. It also admits a "linear mapping" reformulation saying that for each linear (affine) map $a\colon \Delta^N \longrightarrow \mathbb{R}^d$ $(N = (r-1)(d+1))$ there exist r nonempty disjoint faces $\Delta_1, \dots, \Delta_r$ of the simplex Δ^N such that $a(\Delta_1) \cap \dots \cap a(\Delta_r) \neq \emptyset$.

It was quite natural to conjecture a non-linear version of this result and, eventually, the *Topological Tverberg theorem* was born.

Theorem 1.1 ([**BSS, Öz87, V96**])**.** *Let $r = p^k$ be a prime power, $d \geqslant 1$, and $N = (r-1)(d+1)$. Then for every continuous map $f\colon \Delta^N \to \mathbb{R}^d$, defined on an N-dimensional simplex, there exist disjoint faces $\Delta_1, \dots, \Delta_r$ of Δ^N such that*

$$f(\Delta_1) \cap \dots \cap f(\Delta_r) \neq \emptyset .$$

In other words the simplex Δ^N is not almost r-embeddable in $\mathbb{R}^d$.

The history of this result is also "non-linear". Bárány, Shlosman, and Szűcs (1981) obtained the result when $r = p$ is a prime number [**BSS**], Özaydin proved the generalization to the prime power case $r = p^k$ (his preprint [**Öz87**] was not published in a journal but it was known to some experts and appeared in reviews such as [**Živ96**]), and finally Volovikov [**V96**] independently discovered and published the result in its present form.

It is known that the condition on r is essential. Indeed, as demonstrated in [**BFZ2**], if r is not a prime power the topological Tverberg theorem fails if d is sufficiently large.

1.2. Colored Tverberg theorems

From the topological viewpoint, a natural step after the appearance of Theorem 1.1 was to find other non-trivial examples of simplicial complexes K which are not almost r-embeddable in $\mathbb{R}^d$. An interesting class of complexes comes from higher dimensional analogues and generalizations of the Van Kampen–Flores theorem, see [**Ž17**, Section 21.4.3] and [**JPZ-1**] for examples and references. "Colored Tverberg theorems", which originated in discrete and computational geometry, provide another, quite different class of examples.

A coloring of vertices of a simplex Δ^N by $k+1$ colors is a partition of vertices $V = Vert(\Delta^N) = C_0 \sqcup C_1 \sqcup \dots \sqcup C_k$ into "monochromatic" subsets C_i. A subset $\Delta \subseteq V$ is called a *rainbow simplex or a rainbow face* if $|\Delta \cap C_i| \leqslant 1$ for each $i = 0, \dots, k$. If the cardinality of C_i is t_i then the join $K_{t_0,t_1,\dots,t_k} := [t_0] * [t_1] * \dots * [t_k]$ is isomorphic to the subcomplex of all rainbow simplices in Δ^N. A "colored Tverberg theorem" is any statement of the form

$$K_{t_0,t_1,\dots,t_k} \text{ is not almost } r\text{-embeddable in } \mathbb{R}^d \tag{2}$$

and the general "colored Tverberg problem" is to clarify for which values of parameters r, d, k and t_i the statement (2) is true. Here are some examples.

$$K_{3,3} \text{ is not almost 2-embeddable in } \mathbb{R}^2, \tag{3}$$

$$K_{3,3,3} \text{ is not affinely almost 3-embeddable in } \mathbb{R}^2, \tag{4}$$

$$K_{5,5,5} \text{ is not almost 3-embeddable in } \mathbb{R}^3, \tag{5}$$

$$K_{4,4,4,4} \text{ is not almost 4-embeddable in } \mathbb{R}^3. \tag{6}$$

Let $T(r,k,d)$ be defined as the smallest number t such that (2) is true if $t_i \geqslant t$ for each $i = 0,\dots,k$. Originally even the finiteness of the function $T(r,k,d)$ was under question (and had important consequences, see Section 1.4).

1.3. A-B-C classification of colored Tverberg theorems

Following [**Ž17**, Section 21.4], variants of the colored Tverberg problem are classified as *type A, B, or C* depending on whether $k = d, k < d$ or $k > d$, where $k+1$ is the number of colors and d is the dimension of the target space.

The main difference between the types A and C, on one side, and B on the other, is that in the type B case the number r must satisfy the inequality $\frac{r-1}{r}d \leqslant k$, while in the types A and C there are no a priori constraints.

In agreement with this classification, (4) and (6) are illustrative examples of *topological type A colored Tverberg theorems* while (3) and (5) are classified as topological type B colored Tverberg theorems.

The following general results are the main representatives of the first two classes of colored Tverberg theorems. In particular (3), (5) and (6) are their easy consequences.

Theorem 1.2 (Type A [**BMZ**]). *Suppose that $r+1$ is a prime number and $d \geqslant 1$. Then the complex $K_{r,r,\dots,r} := [r]^{*(d+1)}$, obtained as a join of $d+1$ copies of the 0-dimensional complex $[r]$, is not almost r-embeddable in $\mathbb{R}^d$.*

Theorem 1.3 (Type B [**VŽ94, ŽV92**]). *Assume that $r = p^\nu$ is a prime power, $d \geqslant 1$, and let k be an integer such that $\frac{r-1}{r}d \leqslant k < d$. Then the complex*

$$K_{2r-1,2r-1,\dots,2r-1} := [2r-1]^{*(k+1)}$$

obtained as a join of $k+1$ copies of the 0-dimensional complex $[2r-1]$, is not almost r-embeddable in $\mathbb{R}^d$.

Both Theorems 1.2 and 1.3 are optimal in their own kind. For example Theorem 1.3 in the case $r = 2$ reduces to (3), which is optimal since $K_{2,t}$ is a planar graph for each t. Similarly [**VŽ94**], under the conditions of Theorem 1.3 the complex $K_{2r-2,t,\dots,t} = [2r-2] * [t]^{*k}$ is always r-embeddable in $\mathbb{R}^d$ for each integer t.

For this reason these theorems may be referred to as the *Optimal type A colored Tverberg theorem*, respectively the *Optimal type B colored Tverberg theorem*.

1.4. Historical interlude about the origin of colored Tverberg theorems

The search for "colorful" analogues and relatives of the Tverberg theorem was originally dictated by some important questions about the asymptotic behavior of finite collections of points (lines, hyperplanes) in Euclidean spaces, see reviews [**Z11**] and [**Ž17**] for introduction.

Bárány, Füredy, and Lovász [**BFL90**] were the first (already in 1988) to observe the importance of the function $T(r,d) := T(r,d,d)$ (defined at the end of Section 1.2), for the "Halving hyperplanes problem" and related k-set problem [**Ž17**, Section 21.4.2].

Making a conjecture $T(r,d) < +\infty$ for all r and d, they proved that if $h_d(n)$ is the number of essentially different halving hyperplanes of a set of size n in $\mathbb{R}^d$ then

$$h_d(n) = O(n^{d-\epsilon_d}), \quad \text{where} \quad \epsilon_d = T(d+1,d)^{-(d+1)}\,.$$

This was not an isolated example. Soon after that [**ABFK**] (see also [**AK92**] for a subsequent application) Alon, Bárány, Füredy, and Kleitman proved that some of the, at the time, central problems of discrete and computational geometry *(Point selection problem, Weak ϵ-net problem, Hitting set problem, etc.)* can all be reduced to the *Weak colored Tverberg problem*, which claims that $T(d+1,d) < +\infty$.

The inequality $T(d+1,d) < 4d+1$, obtained by Živaljević and Vrećica (1992) as a consequence of their *Topological colored Tverberg theorem* [**ŽV92**], claiming that $T(r,d) \leqslant 2r-1$ if r is a prime number, eventually supplied the missing (topological) link for all these results.

This influx of topological methods and ideas into discrete and computational geometry was met with enthusiasm in both mathematical and theoretical computer science communities. The *Handbook of Discrete and Computational Geometry* [**DCG**] obtained a topological chapter (see [**Ž17**] for the 3$^{\text{rd}}$-edition version) and computationally oriented mathematicians were persuaded to learn more topology [**M03**].

A new wave of excitement came in the Fall of 2009 when Blagojević, Matschke, and Ziegler proved the *Optimal type A colored Tverberg theorem* (Theorem 1.2) and its equally important type C relatives (Theorems 1.4 and 1.5), see [**BMZ**] and [**Z11**].

Already in (1992) Bárány and Larman [**BL92**] raised the question of finding the exact value of $T(r,d)$ and conjectured that $T(r,d) = r$. They established this conjecture for $r = 2$ and for $d = 2$, in particular they proved (4). The *Optimal type A colored Tverberg theorem* provides an affirmative answer to the Bárány and Larman conjecture if $r+1$ is a prime number. In particular (6) follows as a special case for $r = 4$ and $d = 3$.

1.5. Optimal type C colored Tverberg theorems

The following relative of Theorem 1.1 is referred to as the *Optimal type C colored Tverberg theorem* [**BMZ**]. (It is not difficult to see that Theorem 1.2 is its easy consequence.)

Theorem 1.4 ([**BMZ**]). *Let $r \geqslant 2$ be a prime, $d \geqslant 1$, and $N := (r-1)(d+1)$. Let Δ^N be an N-dimensional simplex with a partition (coloring) of its vertex set into $d+2$ parts,*

$$V = [N+1] = C_0 \sqcup \cdots \sqcup C_d \sqcup C_{d+1}\,,$$

with $|C_i| = r-1$ *for* $i \leqslant d$ *and* $|C_{d+1}| = 1$*. Then for every continuous map of a simplex* $f\colon \Delta^N \to \mathbb{R}^d$*, there are* r *disjoint "rainbow simplices"* $\Delta_1, \dots, \Delta_r$ *of* Δ^N *satisfying*

$$f(\Delta_1) \cap \dots \cap f(\Delta_r) \neq \emptyset\,,$$

where by definition a face Δ *of* Δ^N *is a rainbow simplex if and only if* $|\Delta \cap C_j| \leqslant 1$ *for each* $j = 0, \dots, d+1$*.*

Note that Theorem 1.4 does not include Theorem 1.1 as a special case. Indeed, we need a stronger condition in the colored Tverberg theorem, where r is a prime rather than a prime power. It remains an interesting question if this condition on r can be relaxed.

In the following section we argue that a proper extension of Theorem 1.4 (to the case of prime powers $r = p^k$) may require multisets of colored points. This may not be an accident, as it appears to be dictated by the topology of a naturally associated configuration space. For a motivating example, comparing the old and new results, see Section 1.7.

The following extension of Theorem 1.4 is the main result of [**BMZ**].

Theorem 1.5 ([**BMZ**])**.** *Let* $r \geqslant 2$ *be a prime,* $d \geqslant 1$*, and* $N := (r-1)(d+1)$*. Let* Δ^N *be an* N*-dimensional simplex with a partition of its vertex set into* $m+1$ *parts,*

$$V = [N+1] = C_0 \sqcup \dots \sqcup C_m\,,$$

with $|C_i| \leqslant r-1$ *for* $i = 0, \dots, m$*. Then for every continuous map* $f\colon \Delta^N \to \mathbb{R}^d$*, there are* r *disjoint rainbow simplices* $\Delta_1, \dots, \Delta_r$ *in* Δ^N *satisfying*

$$f(\Delta_1) \cap \dots \cap f(\Delta_r) \neq \emptyset\,.$$

1.6. Optimal colored Tverberg theorem for multisets of points

Our (first) main new result (Theorem 1.6) is valid for each prime power $r = p^k$, and includes Theorem 1.4 as a special case for $k = 1$. One of the guiding ideas is to replace the simplex Δ^N (used in both Theorems 1.1 and 1.4) by a simplex of smaller dimension, and to compensate for this by allowing its vertices to appear a controlled number of times in different faces of Δ^N.

Theorem 1.6. *Let* $r = p^k$ *be a prime power,* $d \geqslant 1$*, and* $N := k(p-1)(d+1)$*. Let* Δ^N *be an* N*-dimensional simplex whose vertices are colored by* $d+2$ *colors, meaning that there is a partition* $V = C_0 \sqcup C_1 \sqcup \dots \sqcup C_d \sqcup C_{d+1}$ *into* $d+2$ *monochromatic subsets. We also assume that:*

(1) *Each of the sets* $C_0, \dots, C_d$ *has* $(p-1)k$ *vertices. The vertices in each* C_i *are assigned multiplicities, as prescribed by the vector* $\mathbb{L} = (1, p, \dots, p^{k-1})^{\times(p-1)} \in \mathbb{N}^{k(p-1)}$*.*

(2) *The (exceptional) color class* C_{d+1} *contains a single vertex with multiplicity one.*

We claim that under these conditions for any continuous map $f\colon \Delta^N \to \mathbb{R}^d$ *there exist* r *(not necessarily disjoint or even different) faces* $\Delta_1, \dots, \Delta_r$ *of* Δ^N *such that:*

(A) $f(\Delta_1) \cap \dots \cap f(\Delta_r) \neq \emptyset$*.*

(B) *The number of occurrences of each vertex of* Δ^N *in all faces* Δ_i *does not exceed the prescribed multiplicity of that vertex.*

(C) *All faces* Δ_i *are* rainbow simplices*, in the sense that their vertices have different colors,* $(\forall i)(\forall j)\,|Vert(\Delta_i)\cap C_j|\leqslant 1$.

Theorem 1.5 is deduced from Theorem 1.4 by a direct combinatorial argument. By a similar reduction procedure we obtain a result (Theorem 7.7 in Section 7) which extends Theorem 1.6, and unifies Theorems 1.4, 1.6, and 1.5 in a single statement.

1.7. Motivating example and a comparison of new and old results

The following example illustrates the optimality of Theorem 1.6 and illuminates its relationship with other statements of Tverberg type.

Remark 1.7. (On multisets) The multiset terminology and notation used in the paper is fairly standard and self-explanatory. If $X = \{x_1^{\alpha_1}, \dots, x_k^{\alpha_k}\} = x^\alpha = x_1^{\alpha_1}\dots x_k^{\alpha_k}$ is a multiset (in monomial notation) then the associated *multiplicity vector* is $\alpha = (\alpha_1, \dots, \alpha_k)$. (From this point of view the ordinary sets are square-free monomials.)

Most of the time it is sufficient to describe the multiplicity vector α, which itself can be treated as a multiset! For example if $X = \{a, a, b, b, b, c, c, c\} = a^2b^3c^3$ then the corresponding multiplicity vector is $\alpha = (2, 3, 3)$ which, as a multiset, can be recorded as $\alpha = \{2, 3^2\}$ or even as a monomial $\alpha = 23^2$. Another example is the *multiplicity vector* $\mathbb{L} = (1, p, \dots, p^{k-1})^{\times(p-1)} \in \mathbb{N}^{k(p-1)}$, used in the formulation of Theorem 1.6, which (treated as a multiset) can be recorded as $1^{p-1}p^{p-1}\dots(p^{k-1})^{p-1}$.

Let $\mathcal{S} = \{A_i^{\alpha_i}\}_{i=0}^{d+1}$ be a *multiset* of points in $\mathbb{R}^d$ where $\{A_i\}_{i=0}^d$ are the vertices of a non-degenerate simplex and A_{d+1} its barycenter. Each of the points $A_i\,(i = 0, \dots, d+1)$ is assigned a different color and a multiplicity $\alpha_i \in \{1, r-1\}$, where $\alpha_i = 1$ only for $i = d+1$.

If we allow a perturbation of $\mathcal{S}$ or informally a "scattering of points", where each $\{A_i^{\alpha_i}\}$ is replaced by a collection $\{A_i^j\}_{j=1}^{\alpha_i}$ of (possibly distinct) points in $\mathbb{R}^d$ then:

(1) The classical (affine) Tverberg theorem (Theorem 1.1) guarantees the existence of a partition $\mathcal{S} = S_1 \sqcup \dots \sqcup S_r$ such that $\mathrm{Conv}(S_1) \cap \dots \cap \mathrm{Conv}(S_r) \neq \emptyset$;

(2) Theorem 1.4 says that, under assumption that r is a prime, such a partition exists with an additional property that each set S_k has at most one point in each color;

(3) Theorem 1.6 claims that the conclusion of Theorem 1.4 is true under assumption that r is a prime power, if we allow only a partial scattering where in each color the points create clusters (partition) of the type $1^{p-1}p^{p-1}\dots(p^{k-1})^{p-1}$. (The clusters themselves can be scattered in an arbitrary fashion.)

Figure 1 symbolically illustrates the similarities and differences between different theorems of Tverberg type (Theorems 1.1, 1.4 and 1.6). Each theorem applies to a specific configuration of points, obtained by a perturbation (scattering) of the canonical initial configuration $\mathcal{S}$. This perturbation is unconditional (in the case of Theorems 1.1 and 1.4) or conditional (Theorem 1.6).

Summarizing, the multisets and the corresponding multiplicity vectors in Theorem 1.6 provide information what kind of perturbations of the initial configuration are allowed, if we want to preserve the conclusion of the Type C colored Tverberg theorem in the prime power case.

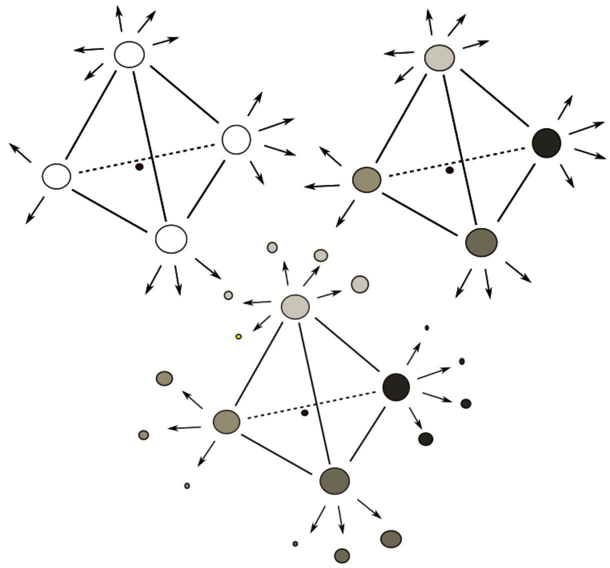

Figure 1

1.8. Colored Tverberg theorem for small color classes

For comparison we include a quite different extension of the colored Tverberg theorem to prime powers. We are able to avoid multiple occurrences of points in different rainbow simplices in Theorem 1.6 (the condition (B)), at a price of increasing the number of color classes (and decreasing the number of vertices in each color class).

The following theorem can be interpreted as a prime power extension (relative) of the results from [**BFZ1**, Section 9].

Theorem 1.8. *Let $r = p^\alpha$ be a prime power, $d \geqslant 1$, and $N := (r-1)(d+2)$. Let Δ^N be an N-dimensional simplex with a partition (coloring) of its vertex set into t color classes,*

$$V = [N+1] = C_1 \sqcup \cdots \sqcup C_t \,,$$

with $|C_i| \leqslant q = \frac{r+1}{2}$ for all i. Then for every continuous map $f\colon \Delta^N \to \mathbb{R}^d$, there are r disjoint "rainbow simplices" $\Delta_1, \ldots, \Delta_r$ of Δ^N satisfying

(1)
$$f(\Delta_1) \cap \cdots \cap f(\Delta_r) \neq \emptyset \,,$$
where (as before) a face Δ of Δ^N is a rainbow simplex if and only if $|\Delta \cap C_j| \leqslant 1$ for each $j = 1, \ldots, t$.

(2) *The dimension of each of the simplices Δ_i is at most k, where k is defined from $rk + s = (r-1)d$, $k > 0$, and $0 \leqslant s < r$.*

(3) *There are at most s simplices whose dimensions is k.*

The organizations of the paper is the following. The proof of Theorem 1.6 is given in Section 4. The role of chessboard complexes and their generalizations, as configuration spaces for theorems of Tverberg type, is briefly reviewed in Section 2. In Section 2 we also formulate our main topological result of Borsuk-Ulam type (Theorem 2.1), used in the proof of Theorem 1.6, and its companion (Theorem 2.2), about degrees of maps from multiple chessboard complexes. The proof of Theorems 2.2 and 2.1 is postponed until Section 6.

In Section 3 we develop the theory of multiple chessboard complexes in the generality needed for applications in the proofs of Theorems 1.6 and 2.2. The focus is on multiple chessboard complexes which turn out to be pseudomanifolds (Sections 3.1 and 3.3).

In Section 7 we formulate and prove extensions of Theorem 1.6 (Theorems 7.7 and 7.8). In Section 7.1 we show how the method of "unavoidable complexes" can be adapted for applications to problems about multisets of points.

In Section 5 we outline the proof of [**KB**, Theorem 2.1] (slightly extended to the case of pseudomanifolds), as one of the central result illustrating the Eilenberg-Krasnoselskii comparison principle for degrees of equivariant maps, in the case of non-free group actions.

Finally, in Section 8 we give a proof of Theorem 1.8.

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2. Chessboard complexes and equivariant maps

The central role of *chessboard complexes*, as proper configuration spaces for *colored Tverberg problem and its relatives*, was recognized in [**ŽV92**] almost thirty years ago. To the present day these complexes remain, together with their generalizations (the multiple chessboard complexes) in the focus of research in this area of geometric combinatorics.

Recall that the (standard) chessboard complex $\Delta_{p,q}$ is the complex of all non-attacking placements of rooks in a $(p \times q)$-chessboard (a placement is non-attacking if it is not allowed to have more than one rook in the same row or in the same column). More generally, the multiple chessboard complex $\Delta_{p,q}^{\mathbb{A},\mathbb{B}}$ (see Section 3), where $\mathbb{A} \in \mathbb{N}^q$ and $\mathbb{B} \in \mathbb{N}^p$, arises if we allow more than one rook in each row (each column), where their precise number is determined by vectors $\mathbb{A}$ and $\mathbb{B}$.

Among the central results in this area are the *topological type C colored Tverberg theorem* (Theorem 2.2 in [**BMZ**], reproduced here as Theorem 1.4) and the *topological type B colored Tverberg theorem* [**ŽV92, VŽ94**]. Both of these results are obtained by applications of the *Configuration Space/Test Map scheme* involving chessboard complexes (see [**Ž17**]).

The associated *test maps* are respectively (7) (in the type C case) and (8) (for the type B result),

$$f\colon (\Delta_{r-1,r})^{*(d+1)} * [r] \xrightarrow{\mathbb{Z}/p} W_r^{\oplus(d+1)}, \tag{7}$$

$$f\colon (\Delta_{2r-1,r})^{*(k+1)} \xrightarrow{(\mathbb{Z}/p)^k} W_r^{\oplus d}, \tag{8}$$

where $W_r \subset \mathbb{R}^r$ is the standard $(r-1)$-dimensional representation of the group S_r.

Both theorems are consequences of the corresponding Borsuk–Ulam-type statements claiming that in either case the G-equivariant map f must have a zero, where in (7) $r = p$ is a prime number and $G = \mathbb{Z}/p$, and in (8) $r = p^k$ is a power of a prime and $G = (\mathbb{Z}/p)^k$.

The following theorem extends (7) and serves as a basis for a new type C topological Tverberg theorem, which extends (in a natural way) the result of Blagojević, Matschke and Ziegler to the prime power case.

Theorem 2.1. *Let $G = (\mathbb{Z}_p)^k$ be a p-toral group of order $r = p^k$. Let $\Delta^{\mathbb{1};\mathbb{L}}_{k(p-1),p^k}$ be the multiple chessboard complex (based on a $k(p-1) \times p^k$ chessboard), where $\mathbb{1} = (1,\dots,1) \in \mathbb{R}^{p^k}$ and $\mathbb{L} = (1,p,\dots,p^{k-1})^{\times(p-1)} \in \mathbb{R}^{k(p-1)}$. Let $\partial\Delta_{[p^k]} \cong S^{p^k-2}$ be the boundary of a simplex with p^k vertices. Then there does not exist a G-equivariant map*

$$f\colon (\Delta^{\mathbb{1};\mathbb{L}}_{k(p-1),p^k})^{*(d+1)} * [p^k] \longrightarrow (\partial\Delta_{[p^k]})^{*(d+1)} \cong (S^{p^k-2})^{*(d+1)} \cong S^{(p^k-1)(d+1)-1}\,.$$

Theorem 2.1 is a consequence of the following theorem about degrees of equivariant maps.

Theorem 2.2. *Let $G = (\mathbb{Z}_p)^k$ be a p-toral group of order $r = p^k$. Let $\Delta^{\mathbb{1};\mathbb{L}}_{k(p-1),p^k}$ be the multiple chessboard complex (based on a $k(p-1) \times p^k$ chessboard), where $\mathbb{1} = (1,\dots,1) \in \mathbb{R}^{p^k}$ and $\mathbb{L} = (1,p,\dots,p^{k-1})^{\times(p-1)} \in \mathbb{R}^{k(p-1)}$. Let $\partial\Delta_{[p^k]} \cong S^{p^k-2}$ be the boundary of a simplex with p^k vertices. Then $\deg(f) \neq 0 \pmod p$ for any G-equivariant map*

$$f\colon (\Delta^{\mathbb{1};\mathbb{L}}_{k(p-1),p^k})^{*(d+1)} \longrightarrow (\partial\Delta_{[p^k]})^{*(d+1)} \cong (S^{p^k-2})^{*(d+1)} \cong S^{(p^k-1)(d+1)-1}\,.$$

3. Chessboard pseudomanifolds

Following [**JVZ-1, JVZ-2**], a multiple chessboard complex

$$\Delta^{\mathbb{K};\mathbb{L}}_{m,n} = \Delta^{k_1,\dots,k_n;l_1,\dots,l_m}_{m,n}$$

is an abstract simplicial complex with vertices in $[m] \times [n]$, where the simplices have at most k_i elements in the row $[m] \times \{i\}$ and at most l_j elements in each column $\{j\} \times [n]$.

We shall be mainly interested in complexes $\Delta^{\mathbb{1};\mathbb{L}}_{m,n} = \Delta^{1,\dots,1;l_1,\dots,l_m}_{m,n}$ where at most one rook is permitted in each of the rows of the chessboard $[m] \times [n]$.

Proposition 3.1. *The multiple chessboard complex $\Delta^{\mathbb{1};\mathbb{L}}_{m,n}$ is a pseudomanifold if*

$$n = l_1 + l_2 + \cdots + l_m + 1\,. \tag{9}$$

More precisely, the links of simplices of codimension 1 and 2 are spheres of dimensions 0 and 1, while in codimension 3 both 2-spheres and 2-dimensional tori T^2 may appear.

Proof. Let $S \in \Delta^{\mathbb{1};\mathbb{L}}_{m,n}$ and let $s_i := |S \cap (\{i\} \times [n])|$. The link $\mathrm{Link}(S)$ is clearly isomorphic to the multiple chessboard complex $\Delta^{\mathbb{1};\mathbb{T}}_{m,n}$ where $\mathbb{T} = (t_1, \dots, t_m)$ and $t_i := l_i - s_i$. (Here we allow that $t_j = 0$ for some $j \in [m]$.) The proof is completed by an explicit description of all multiple chessboard complexes that arise as links of simplices in codimension $\leqslant 3$.

If $\mathrm{codim}(S) = 1$ then there exists j_0 such that $l_{j_0} = s_{j_0} + 1$ and $l_j = s_j$ for each $j \neq j_0$. The condition (9) guarantees that $t_{j_0} = 1$, which together with $t_j = 0$ for $j \neq j_0$ implies $\mathrm{Link}(S) \cong \Delta_{1,2} \cong S^0$.

If $\mathrm{codim}(S) = 2$ then there are two possibilities. Either (I) there exists j_0 such that $l_{j_0} = s_{j_0} + 2$ and $l_j = s_j$ for each $j \neq j_0$, or (II) there exists $j_0 \neq j_1$ such that both $l_{j_0} = s_{j_0} + 1, l_{j_1} = s_{j_1} + 1$ and $l_j = s_j$ for each $j \neq j_0, j_1$. In the first case $\mathrm{Link}(S) \cong \partial\Delta_{[3]} \cong S^1$, while in the second $\mathrm{Link}(S) \cong \Delta_{2,3} \cong S^1$.

If $\mathrm{codim}(S) = 3$ then the number of non-zero entries in the vector $\mathbb{T} = (t_1, \dots, t_m)$ is $1, 2$ or 3. In the first case $\mathrm{Link}(S) \cong \partial\Delta_{[4]} \cong S^2$. In the second case $\mathrm{Link}(S) \cong \Delta^{\mathbb{1};\mathbb{T}}_{2,4}$, where $\mathbb{T} = (2, 1)$, hence $\Delta^{\mathbb{1};\mathbb{T}}_{2,4} \cong S^2$.

Finally, in the third case $\mathrm{Link}(S) \cong \Delta_{3,4} \cong T^2$, since $\Delta_{3,4}$ is an orientable surface which has zero Euler characteristic. □

3.1. Hierarchy of pseudomanifolds $\Delta^{\mathbb{1};\mathbb{L}}_{m,n}$

From here on we tacitly assume that the chessboard complex $\Delta^{\mathbb{1};\mathbb{L}}_{m,n}$ satisfies the condition $n = l_1 + \dots + l_m + 1$. If $\mathbb{L} = \mathbb{1} \in \mathbb{N}^m$ then $\Delta^{\mathbb{1};\mathbb{L}}_{m,n} = \Delta_{n-1,n}$ is a standard chessboard complex [**BLVZ**], while in the case $m = 1$ the complex $\Delta^{\mathbb{1};\mathbb{L}}_{m,n} \cong \partial\Delta_{[n]}$ is the boundary sphere $\partial\Delta_{[n]} \cong S^{n-2}$ of the simplex $\Delta_{[n]} := 2^{[n]}$.

The pseudomanifolds $\Delta^{\mathbb{1};\mathbb{L}}_{m,n}$ form a poset category where the complexes $\Delta_{n-1,n}$ and $\partial\Delta_{[n]}$ play the role of the initial and terminal object. The morphisms in this category are the *θ-collapse maps* Ω_θ, described in the following definition.

Definition 3.2. Assuming $m' \geqslant m$, choose an epimorphism $\theta\colon [m'] \to [m]$, then let $\widehat{\theta}\colon [m'] \times [n] \to [m] \times [n]$ be the associated map of chessboards where $\widehat{\theta}(i, j) = (\theta(i), j)$. We say that a sequence $\mathbb{B} = (b_1, \dots, b_m)$ is obtained by a θ-collapse from a sequence $\mathbb{A} = (a_1, \dots, a_{m'})$ if $b_i = \sum_{\theta(j)=i} a_j$. Define $\Omega_\theta\colon \Delta^{\mathbb{1};\mathbb{A}}_{m',n} \to \Delta^{\mathbb{1};\mathbb{B}}_{m,n}$ as the induced map of multiple chessboard complexes where $\Omega_\theta(S) := \widehat{\theta}(S)$, for each simplex $S \in \Delta^{\mathbb{1};\mathbb{A}}_{m,n}$. (Informally, the map Ω_θ merges together some columns of $\Delta^{\mathbb{1};\mathbb{A}}_{m',n}$, as dictated by θ.)

The special cases $\Omega_\theta\colon \Delta_{n-1,n} \to \Delta^{\mathbb{1};\mathbb{L}}_{m,n}$ and $\Omega_\theta\colon \Delta^{\mathbb{1};\mathbb{L}}_{m,n} \to \partial\Delta_{[n]}$ are of particular importance. These maps are completely determined by the vector $\mathbb{L}$ and in this case both the corresponding Ω_θ and the associated map $\theta\colon [n-1] \to [m]$ are referred to as $\mathbb{L}$-collapse maps.

The group S_n, permuting the rows of the chessboard $[m] \times [n]$, acts on the multiple chessboard complex $\Delta^{\mathbb{1};\mathbb{L}}_{m,n}$. The simplicial map $\Omega_\theta\colon \Delta^{\mathbb{1};\mathbb{L}'}_{m',n} \to \Delta^{\mathbb{1};\mathbb{L}}_{m,n}$, associated to a collapse map $\theta\colon [m'] \to [m]$, is clearly S_n-equivariant.

3.2. Orientability of pseudomanifolds $\Delta^{\mathbb{1};\mathbb{L}}_{m,n}$

Before the calculation of the degree of the collapse map, we check the orientability of the pseudomanifold $\Delta^{\mathbb{1};\mathbb{L}}_{m,n}$ and calculate the associated orientation character. Recall that the *concordance* of two actions is a standard necessary condition in the study of degrees of equivariant maps, see [**KB**, Remark 1.1].

Proposition 3.3. *The pseudomanifold $\Delta^{\mathbb{1};\mathbb{L}}_{m,n}$ is always orientable. It has a fundamental class $\tau \in H_d(\Delta^{\mathbb{1};\mathbb{L}}_{m,n};\mathbb{Z}) \cong \mathbb{Z}$ where $d = \dim(\Delta^{\mathbb{1};\mathbb{L}}_{m,n}) = n-2$. A permutation $g \in S_n$ reverses the orientation (changes the sign of τ) if and only if g is odd.*

Proof. Let $\Omega_\theta \colon \Delta^{\mathbb{1};\mathbb{L}}_{m,n} \to \partial\Delta_{[n]}$ be the collapse map associate to the constant map $\theta\colon [m] \to [1]$ (Definition 3.2). In other words Ω_θ is the map induced by the projection $[m] \times [n] \to [1] \times [n]$ of chessboards, where a simplex $S \in \Delta^{\mathbb{1};\mathbb{L}}_{m,n}$ is mapped to a simplex $S' \in \partial\Delta_{[n]}$ if and only if

$$(\forall i \in [n])\,[(\{i\} \times [m]) \cap S \neq \emptyset \;\Leftrightarrow\; i \in S']\,.$$

Let $\widehat{S}$ be the simplex $S \in \Delta^{\mathbb{1};\mathbb{L}}_{m,n}$ oriented by listing its vertices in the increasing order of rows. Note that if $\Omega_\theta(S) = S' \in \partial\Delta_{[n]}$ then $\Omega_\theta(\widehat{S}) = \widehat{S'}$.

Choose an orientation $\mathcal{O}'$ on the sphere $\partial\Delta_{[n]}$ and use this orientation to define, via the collapse map Ω_θ, an orientation $\mathcal{O}$ on $\Delta^{\mathbb{1};\mathbb{L}}_{m,n}$. More explicitly, an ordered simplex $\widehat{S}$ is positively oriented with respect to $\mathcal{O}$ if and only if $\widehat{S'}$ is positively oriented with respect to the orientation $\mathcal{O}'$. It is not difficult to check that $\mathcal{O}$ is indeed an orientation on the pseudomanifold $\Delta^{\mathbb{1};\mathbb{L}}_{m,n}$ which has all the properties listed in Proposition 3.3. □

Corollary 3.4. *As a consequence of Proposition 3.3 the S_n-pseudomanifolds $\Delta^{\mathbb{1};\mathbb{L}}_{m,n}$ and $\Delta^{\mathbb{1};\mathbb{L}'}_{m',n}$ are* concordant *in the sense that each $g \in S_n$ either changes the orientation of both of the complexes if none of them.*

3.3. Degree of the collapse map Ω_θ

In the following proposition we calculate the degree of the map Ω_θ. This calculation will play a central role in the proof of Theorem 2.2.

Proposition 3.5. *The degree of the map $\Omega_\theta \colon \Delta^{\mathbb{1};\mathbb{A}}_{m',n} \to \Delta^{\mathbb{1};\mathbb{B}}_{m,n}$ is,*

$$\deg(\Omega_\theta) = \binom{\mathbb{B}}{\mathbb{A}} = \frac{b_1!\, b_2! \dots b_m!}{a_1!\, a_2! \dots a_{m'}!}. \tag{10}$$

In the special case when $m = 1$ we obtain that the degree of the map Ω_θ is the multinomial coefficient,

$$\deg(\Omega_\theta) = \frac{(a_1 + a_2 + \dots + a_{m'})!}{a_1!\, a_2! \dots a_{m'}!} \tag{11}$$

and in the special case $a_1 = a_2 = \ldots = a_{m'} = 1$ (10) reduces to the formula,

$$\deg(\Omega_\theta) = b_1!\, b_2! \dots b_m!. \tag{12}$$

Proof. Each simplicial map $\Omega_\theta \colon \Delta^{\mathbb{1};\mathbb{A}}_{m',n} \to \Delta^{\mathbb{1};\mathbb{B}}_{m,n}$ is non-degenerate in the sense that it maps bijectively the top dimensional simplices of $\Delta^{\mathbb{1};\mathbb{A}}_{m',n}$ to top dimensional simplices of $\Delta^{\mathbb{1};\mathbb{B}}_{m,n}$. Moreover, it is an orientation preserving map so in order to calculate the degree of Ω_θ it is sufficient to calculate the cardinality of the preimage $\Omega_\theta^{-1}(c_0)$ of the barycenter c_0 of a chosen top dimensional simplex of $\Delta^{\mathbb{1};\mathbb{B}}_{m,n}$.

Since the degree is multiplicative it is sufficient to establish formula (12). A simple calculation shows that the cardinality of the set $\Omega_\theta^{-1}(c_0)$ is, in the case of a map $\Omega_\theta \colon \Delta_{n-1,n} \to \Delta^{\mathbb{1};\mathbb{B}}_{m,n}$, indeed given by the formula (12). □

4. Proof of Theorem 1.6

By convention $\Delta = \Delta_C$ is a simplex spanned by a set C, in particular $\Delta^N \cong \Delta_C$ where $C = Vert(\Delta^N) = C_0 \sqcup C_1 \sqcup \cdots \sqcup C_d \sqcup C_{d+1}$.

Recall that a set $S \subset C$ (and the corresponding face $\Delta_S \subseteq \Delta_C$) is called a *rainbow set* (rainbow face) if $|S \cap C_i| \leqslant 1$ for all $i = 0, 1, \ldots, d+1$. It follows that the set of all rainbow simplices is a subcomplex of Δ_C which has a representation as a join of 0-dimensional simplicial complexes:

$$\mathfrak{R} = \mathfrak{Rainbow} := C_0 * C_1 * \cdots * C_d * C_{d+1} \subset \Delta_C\,. \tag{13}$$

By assumption $|C_i| = m := k(p-1)$ for $i = 0, 1, \ldots, d$ and $|C_{d+1}| = 1$, or more explicitly $C_i = \{c^i_{\alpha,\beta}\}\,(0 \leqslant \alpha \leqslant k-1; 1 \leqslant \beta \leqslant p-1)$ for all $0 \leqslant i \leqslant d$, and $C_{d+1} = \{c_0\}$. Theorem 1.6 claims that for each continuous map $f\colon \Delta_C \to \mathbb{R}^d$ there exist rainbow faces $\Delta_1, \ldots, \Delta_r \in \mathfrak{R}$ such that:

(1) Vertex c_0 appears in at most one of the faces Δ_i;
(2) For all i, α, β the vertex $c^i_{\alpha,\beta}$ may appear in not more than p^α faces $\Delta_1, \ldots, \Delta_r$;
(3) $f(\Delta_1) \cap \cdots \cap f(\Delta_r) \neq \emptyset$.

An r-tuple $(\Delta_1, \ldots, \Delta_r)$ of rainbow simplices is naturally associated to the join of simplices $\Delta_1 * \cdots * \Delta_r \in \mathfrak{R}^{*r}$. Our immediate objective is to identify the subcomplex $\mathfrak{R}^{*r}_{\mathbb{L}} \subset \mathfrak{R}^{*r}$ which collects all r-tuples $(\Delta_1, \ldots, \Delta_r)$ satisfying conditions (1) and (2).

By assumption $\Delta_{i,\nu} := \Delta_i \cap C_\nu$ is either empty or a singleton, for each rainbow simplex Δ_i. A moment's reflection reveals that the union $\cup\{\Delta_{i,\nu}\}_{i=1}^r$ is a simplex in $\Delta^{\mathbb{1};\mathbb{L}}_{k(p-1),p^k}$, for $0 \leqslant \nu \leqslant d$ and a simplex in $[r] = [p^k]$ if $\nu = d+1$. It immediately follows that

$$\mathfrak{R}^{*r}_{\mathbb{L}} \cong (\Delta^{\mathbb{1};\mathbb{L}}_{k(p-1),p^k})^{*(d+1)} * [r]\,.$$

Let $\hat{f}\colon \mathfrak{R} \to \mathbb{R}^d$ be the restriction of the map $f\colon \Delta_C \to \mathbb{R}^d$. The corresponding map defined on the r-tuples of rainbow simplices, satisfying conditions (1) and (2) is the map

$$\hat{F}\colon (\Delta^{\mathbb{1};\mathbb{L}}_{k(p-1),p^k})^{*(d+1)} * [r] \longrightarrow (\mathbb{R}^d)^{*r}\,.$$

By composing with the projection $(\mathbb{R}^d)^{*r} \to (\mathbb{R}^d)^{*r}/D$ (where $D \cong \mathbb{R}^d$ is the diagonal) and the embedding $(\mathbb{R}^d)^{*r}/D \hookrightarrow (W_r)^{\oplus(d+1)}$, where $W_r \cong \mathbb{R}^r/\mathbb{R}$ denotes the standard $(r-1)$-dimensional representation of S_r, we obtain a map

$$\check{F}\colon (\Delta^{\mathbb{1};\mathbb{L}}_{k(p-1),p^k})^{*(d+1)} * [r] \longrightarrow (W_r)^{\oplus(d+1)},$$

which has a zero in a simplex $(\Delta_1, \ldots, \Delta_r)$ if and only if $f(\Delta_1) \cap \cdots \cap f(\Delta_r) \neq \emptyset$. Since the sphere $S((W_r)^{\oplus(d+1)}) \cong (S(W_r))^{*(d+1)}$ is equivariantly homeomorphic to $(\partial\Delta_{[r]})^{*(d+1)}$ a zero exists by Theorem 2.1, which concludes the proof of Theorem 1.6. □

5. Comparison principle for equivariant maps

The following theorem is proved in [**KB**, Theorem 2.1 in Section 2]. Note the condition that the H_i-fixed point sets S^{H_i} are *locally k-connected* for $k \leqslant \dim(M^{H_i}) - 1$

is automatically satisfied if S is a representation sphere. So in this case it is sufficient to show that the sphere S^{H_i} is (globally) $(\dim(M^{H_i}) - 1)$-connected which is equivalent to the condition

$$\dim(M^{H_i}) \leqslant \dim(S^{H_i})\, (i = 1, \ldots, m)\,. \tag{14}$$

Theorem 5.1. *Let G be a finite group acting on a compact topological manifold $M = M^n$ and on a sphere $S \cong S^n$ of the same dimension. Let $N \subset M$ be a closed invariant subset and let $(H_1), (H_2), \ldots, (H_m)$ be the orbit types in $M \setminus N$. Assume that the set S^{H_i} is both globally and locally k-connected for all $k = 0, 1, \ldots, \dim(M^{H_i}) - 1$, where $i = 1, \ldots, m$. Then for every pair of G-equivariant maps $\Phi, \Psi\colon M \longrightarrow S$, which are equivariantly homotopic on N, there is the following relation*

$$\deg(\Psi) \equiv \deg(\Phi) \pmod{GCD\{|G/H_1|, \ldots, |G/H_k|\}}\,. \tag{15}$$

The proof of the following extension of Theorem 5.1 to manifolds with singularities doesn't require new ideas. By a *singular topological manifold* we mean a topological manifold with a codimension 2 singular set. In particular Theorem 5.2 applies to pseudomanifolds $\Delta^{\mathbb{1};\mathbb{L}}_{m,n}$, introduced in Section 3.

Theorem 5.2. *Let G be a finite group acting on a compact "singular topological manifold" $M = M^n$ and on a sphere $S \cong S^n$ of the same dimension. Let $N \subset M$ be a closed invariant subset and let $(H_1), (H_2), \ldots, (H_m)$ be the orbit types in the complement $M \setminus N$. Assume that the set S^{H_i} is both globally and locally k-connected for all $k = 0, 1, \ldots, \dim(M^{H_i}) - 1$, where $i = 1, \ldots, m$. Then for every pair of G-equivariant maps $\Phi, \Psi\colon M \longrightarrow S$ which are equivariantly homotopic on N, there is the following relation*

$$\deg(\Psi) \equiv \deg(\Phi) \pmod{GCD\{|G/H_1|, \ldots, |G/H_k|\}}\,. \tag{16}$$

Proof. Following into footsteps of the proof of Theorem 5.1 (see [**KB**, Theorem 2.1]) we define a G-equivariant map

$$f_0\colon (M \times \{0, 1\}) \cup (N \times [0, 1]) \longrightarrow B \setminus \{O\}, \tag{17}$$

where $B = \mathrm{Cone}(S)$ is a cone over the sphere S (with the apex O), Ψ and Φ are restrictions of f_0 on $M \times \{0\}$ (respectively $M \times \{1\}$) and the restriction of f_0 on $N \times [0, 1]$ is a homotopy between $\Psi|_N$ and $\Phi|_N$.

If $f\colon M \times [0, 1] \to B$ is a G-equivariant extension of f_0 then ([**KB**, Lemma 2.1]) $\deg(f) = \pm(\deg(\Psi) - \deg(\Phi))$ and the relation (16) will follow if

$$\deg(f) = \sum_{i=1}^{m} a_i \cdot |G/H_i| \tag{18}$$

for some integers $a_i \in \mathbb{Z}$.

The proof of the following lemma ([**KB**, Lemma 2.2]) is quite general, in particular it holds for "singular topological manifolds".

Lemma 5.3. *There exists a G-equivariant extension $f\colon M \times [0, 1] \to B$ of the map f_0 satisfying the following conditions:*

(α) $K = f^{-1}(O) = \bigcup_{j=1}^{m} T_j$ *where $T_u \cap T_v = \emptyset$ for $u \neq v$;*

(β) $T_j = G(K_j)$ *for a compact set* K_j*;*

(γ) $K_j = H_j(K_j)$ *is* H_j*-invariant;*

(δ) $g(K_j) \cap h(K_j) = \emptyset$ *if* $gh^{-1} \notin H_j$ $(j = 1, \dots, m)$.

The proof of Theorem 5.2 is completed as in [**KB**, Section 2.1.3] by observing that "singular topological manifolds" also have absolute and relative fundamental classes.

More explicitly, if F_j is the restriction f to a sufficiently small neighborhood of K_j then

$$\deg(f) = \sum_{j=1}^{m} \deg(F_j)\,.$$

By the same argument as in [**KB**] we deduce from Lemma 5.3 that $\deg(F_j) = a_j \cdot |G/H_j|$ for some $a_j \in \mathbb{Z}$, and the relation (16) is an immediate consequence. ☐

6. Proof of Theorem 2.2

We are supposed to show that the degree $\deg(f)$ of each G-equivariant map

$$f\colon (\Delta^{\mathbb{1};\mathbb{L}}_{k(p-1),p^k})^{*(d+1)} \longrightarrow (\partial\Delta_{[p^k]})^{*(d+1)} \cong (S^{p^k-2})^{*(d+1)} \cong S^{(p^k-1)(d+1)-1}, \tag{19}$$

where $G = (\mathbb{Z}_p)^k$ is a p-toral group, is non-zero modulo p. Following the *Comparison principle for equivariant maps* (Section 5) we should:

(A) Exhibit a particular map (19) such that $\deg(f) \neq 0$ modulo p;

(B) Check if the conditions of Theorem 5.2 are satisfied.

The following proposition provides the needed example for the first part of the proof.

Proposition 6.1. *The* θ*-collapse map*

$$\Omega_\theta\colon \Delta^{\mathbb{1};\mathbb{L}}_{k(p-1),p^k} \longrightarrow \partial\Delta_{[p^k]}\,, \tag{20}$$

where $\theta\colon [k(p-1)] \to [1]$ *is a constant map, has a non-zero degree modulo* p.

Proof. We calculate the degree of the map (20) by applying the formula (11). Recall that $\mathbb{L} = (1, p, \dots, p^{k-1})^{\times(p-1)} \in \mathbb{R}^{k(p-1)}$ so in this case

$$\deg(\Omega_\theta) = \frac{(p^k-1)!}{[(p^{k-1})!\,(p^{k-2})!\dots p!\,1!]^{p-1}}\,. \tag{21}$$

The well-known formula for the highest power of p dividing $m!$ is an infinite sum with a finite number of non-zero terms

$$\mathrm{ord}_p(m!) = \left\lfloor \frac{m}{p} \right\rfloor + \left\lfloor \frac{m}{p^2} \right\rfloor + \dots + \left\lfloor \frac{m}{p^s} \right\rfloor + \dots\,.$$

By applying this formula we obtain

$$\mathrm{ord}_p((p^k-1)!) = \mathrm{ord}_p((p^k)!) - k = p^{k-1} + p^{k-2} + \dots + 1 - k$$

and by applying the same formula to the denominator of (21) we obtain exactly the same quantity. ☐

In light of Proposition 6.1 the map

$$(\Omega_\theta)^{*(d+1)}\colon (\Delta^{\mathbb{1};\mathbb{L}}_{k(p-1),p^k})^{*(d+1)} \longrightarrow (\partial\Delta_{[p^k]})^{*(d+1)} \cong (S^{p^k-2})^{*(d+1)} \cong S^{(p^k-1)(d+1)-1} \tag{22}$$

has a non-zero degree $\deg((\Omega_\theta)^{*(d+1)}) = (\deg(\Omega_\theta))^{d+1}$ modulo p, which completes part (A) of the proof.

For the part (B) of the proof of Theorem 2.2 note (Section 5) that it is sufficient to check the inequality (14). Let us begin with the observation that the boundary $\partial\Delta_{[r]}$ $(r = p^k)$ is S_r-equivariantly homeomorphic to the unit sphere $S(W_r)$ in the standard S_r-representation $W_r := \{x \in \mathbb{R}^r \mid x_1 + \cdots + x_r = 0\}$. As a consequence, for each subgroup $H \subseteq S_r$ the corresponding fixed point set $\partial\Delta^H_{[r]} \cong S(W_r)^H = S(W_r^H)$ is also a sphere.

The action of H decomposes $[r]$ into orbits $[r] = O_1 \sqcup \cdots \sqcup O_t$. From here easily follows a combinatorial description of the fixed point set $\partial\Delta^H_{[r]}$. A point $x \in \partial\Delta_{[r]}$, with barycentric coordinates $\{\lambda_i\}_{i=1}^r$, is fixed by H if and only if the barycentric coordinates are constant in each of the orbits. Summarising, $\partial\Delta^H_{[r]}$ is precisely the boundary of the simplex with vertices $\{o_i\}_{i=1}^t$, where o_i is the barycenter of the face $\Delta_{O_i} \subset \Delta_{[r]}$.

Let $\Delta^{\mathbb{1};\mathbb{L}}_{m,r}$ be a multiple chessboard complex, where $\mathbb{L} = (l_1, \ldots, l_m)$ assuming $m = k(p-1)$. It is not difficult to see that the barycenter $b_{i,j}$ of (geometric realization of) the simplex $\{i\} \times O_j$ is in the fixed point set $(\Delta^{\mathbb{1};\mathbb{L}}_{m,r})^H$ if and only if $|O_j| \leqslant l_i$.

More generally, a point x is in $(\Delta^{\mathbb{1};\mathbb{L}}_{m,r})^H$ if and only if it can be expressed as a convex combination

$$x = \sum_{(i,j)\in S} \lambda_{i,j} b_{i,j},$$

where S is a subset of $[m] \times [t]$ satisfying

(1) If $(i,j), (i',j) \in S$ then $i = i'$;

(2) $(\forall i \in [m])\ \sum\{|O_j| \mid (i,j) \in S\} \leqslant l_i$.

The θ-collapse map Ω_θ, where $\theta\colon [m] \to [1]$ is the constant map, maps $(\Delta^{\mathbb{1};\mathbb{L}}_{m,r})^H$ to $\partial\Delta^H_{[r]}$. Moreover $\Omega_\theta(b_{i,j}) = o_j$ and, in light of (1) and (2), the simplex with vertices $\{b_{i,j}\}_{(i,j)\in S}$ is mapped bijectively to a face of $\partial\Delta^H_{[r]}$. The following inequality is an immediate consequence,

$$\dim((\Delta^{\mathbb{1};\mathbb{L}}_{m,r})^H) \leqslant \dim(\partial\Delta^H_{[r]})\,.$$

From here and (22) we obtain the inequality

$$\dim[(\Delta^{\mathbb{1};\mathbb{L}}_{k(p-1),p^k})^{*(d+1)}]^H \leqslant \dim[(\partial\Delta_{[p^k]})^{*(d+1)}]^H,$$

which finishes the proof of part (B) and concludes the proof of the theorem. □

Remark 6.2. It follows from the part (B) of the proof of Theorem 2.2 that $\Delta = (\Delta^{\mathbb{1};\mathbb{L}}_{m,r})^H$ is also a "chessboard complex". Indeed, $S \subseteq [m] \times [t]$ is a simplex in Δ if and only if S has at most one rook in each row $[m] \times \{j\}$ and the total weight of the set $(\{i\} \times [t]) \cap S$ is at most l_i, where the weight of each element (i,j) is $|O_j|$.

It follows that Δ can be classified as a complex of the type $\Delta^{\mathbb{1},\mathcal{L}}_{m,t}$ (cf. [**JVZ-1**, Definition 2.3]), where $\mathcal{L}$ is family of threshold (simplicial) complexes.

6.1. Proof of Theorem 2.1

If such a map f exists, then there is a commutative diagram

$$\begin{array}{ccc}
(\Delta^{\mathbb{1};\mathbb{L}}_{k(p-1),p^k})^{*(d+1)} * [p^k] & \xrightarrow{f} & (\partial\Delta_{[p^k]})^{*(d+1)} \\
\pi\Big\uparrow & & \cong\Big\uparrow \\
(\Delta^{\mathbb{1};\mathbb{L}}_{k(p-1),p^k})^{*(d+1)} & \xrightarrow{\widehat{f}} & (\partial\Delta_{[p^k]})^{*(d+1)},
\end{array} \tag{23}$$

where π is the inclusion map and $\widehat{f} = f \circ \pi$. The map π is homotopic to a constant map (since Image(π) is contained in Star(v) for any vertex $v \in [p^k]$). It follows that deg(f) $= 0$ which contradicts Theorem 2.2. □

7. Generalizations by the method of constraints

In this section we prove Theorems 7.7 and 7.8 as extensions and relatives of Theorem 1.6. Theorem 7.7 unifies the optimal colored Tverberg theorem (Theorem 1.5), and the (primary) colored Tverberg theorem for multisets of points (Theorem 1.6).

All these results (in agreement with [**Ž17**]) can be classified as Type C colored Tverberg theorems (characterized by the condition that the number of colors is at least $d + 2$, where d is the dimension of the ambient euclidean space).

Theorems 7.7 and 7.8 are deduced from Theorem 1.6 by a combinatorial reduction procedure closely related to the method of "constraining functions" and "unavoidable complexes" [**BFZ1**, Sections 3 and 4], see also [**JJTVZ**], [**JMVZ**] and the review paper [**Ž17**] for more information. Here we show in Section 7.1 how this method can be modified and extended to yield results about multisets of points.

7.1. Unavoidable complexes

A multiset with vertices in V can be also described as a pair $\mathbb{V} = (V, m)$ where $m\colon V \to \mathbb{N}$ is a function assigning non-negative multiplicities to elements of V. If $V = \{v_1, \dots, v_s\}$ and $m(v_i) = m_i$ we usually write $\mathbb{V} = \{v_1^{m_1}, \dots, v_s^{m_s}\}$. We have introduced $\mathbb{L}$ (in Theorem 1.6) as the vector $(1, p, \dots, p^{k-1})^{\times(p-1)} \in \mathbb{N}^{k(p-1)}$. With a mild abuse of language (Remark 1.7) we use the same notation for the corresponding multiset. More precisely a multiset is of the type $\mathbb{L}$ if $(1, p, \dots, p^{k-1})^{\times(p-1)}$ is the associated multiplicity vector.

Definition 7.1 (Unavoidable complexes)**.** Let $\mathbb{V} = (V, m)$ be a multiset and r a positive integer. A simplicial complex $K \subseteq 2^V$ is $(r, \mathbb{V})$-unavoidable if for each $\mathbb{V}$-proper collection $\{\Delta_i\}_{i=1}^r$ of (not necessarily distinct) subsets of V, at least one of the subsets Δ_i is in K. By definition a collection $\{\Delta_i\}_{i=1}^r$ is $\mathbb{V}$-proper if for each $v \in V$ the cardinality of the set $\{i \mid v \in \Delta_i\}$ is at most $m(v)$.

Definition 7.2 (Tverberg complexes)**.** Let $\mathbb{V} = (V, m)$ be a multiset. Assume that r and d are positive integers. A simplicial complex $K \subseteq 2^V$ is a Tverberg complex of the type $(r, d, \mathbb{V})$ if for each continuous map $f\colon K \to \mathbb{R}^d$ there exists a $\mathbb{V}$-proper collection $\{\Delta_i\}_{i=1}^r$ of simplices in K such that

$$f(\Delta_1) \cap \cdots \cap f(\Delta_r) \neq \emptyset\,. \tag{24}$$

Example 7.3. In the notation of Theorem 1.6,

$$K_d := C_0 * C_1 * \cdots * C_{d+1} \cong [k(p-1)]^{*(d+1)} * [1]$$

is a Tverberg complex of the type $(r, d, \mathbb{V})$ where $r = p^k$ is a prime power and $\mathbb{V}$ is a disjoint union of $d+1$ copies of the multiset (of the type) $\mathbb{L}$ and a singleton $[1]$.

Unavoidable complexes, originally introduced in [**BFZ1**, Definition 4.1] under the name "Tverberg unavoidable subcomplexes", play the fundamental role in the "constraint method" [**BFZ1**, Sections 3 and 4]. Here we show how the constraint method can be extended to the case of multisets.

Proposition 7.4. *Let $\mathbb{V} = (V, m)$ be a multiset with vertices in V. Assume $K \subseteq 2^V$ is a Tverberg complex of the type $(r, d+1, \mathbb{V})$, where r and d are positive integers. Let L be a $(r, \mathbb{V})$-unavoidable complex. Then $K \cap L$ is a Tverberg complex of the type $(r, d, \mathbb{V})$.*

Proof. We are supposed to prove, following Definition 7.2, that for each continuous map $f\colon K \cap L \to \mathbb{R}^d$ there exists a $\mathbb{V}$-proper collection $\{\Delta_i\}_{i=1}^r$ of simplices in $K \cap L$ which satisfies the condition (24).

The first step is to include $f\colon K \cap L \to \mathbb{R}^d$ into a commutative diagram (25) where e and i are the inclusion maps.

$$\begin{array}{ccc} K & \xrightarrow{F} & \mathbb{R}^{d+1} \\ {\scriptstyle e}\uparrow & & {\scriptstyle i}\uparrow \\ K \cap L & \xrightarrow{f} & \mathbb{R}^d . \end{array} \tag{25}$$

Let $\bar{f}$ be an extension ($\bar{f} \circ e = f$) of the map f to K. Suppose that $\rho\colon K \to \mathbb{R}$ is the function $\rho(x) := \mathrm{dist}(x, K \cap L)$, measuring the distance of the point $x \in K$ from $K \cap L$. Let $F = (\bar{f}, \rho)\colon K \to \mathbb{R}^{d+1}$.

By assumption $K \subseteq 2^V$ is a Tverberg complex of the type $(r, d+1, \mathbb{V})$ so there exists a $\mathbb{V}$-proper family $\{\Delta_1, \dots, \Delta_r\}$ of faces of K such that

$$F(\Delta_1) \cap \cdots \cap F(\Delta_r) \neq \emptyset . \tag{26}$$

More explicitly, there exist $x_i \in \Delta_i$ such that $F(x_i) = F(x_j)$ for each $i, j = 1, \dots, r$.

By assumption the complex L is $(r, \mathbb{V})$-unavoidable, hence $\Delta_i \in L$ for some $i \in [r]$. This implies $\rho(x_i) = 0$ and in turn $\rho(x_j) = 0$ for each $j = 1, \dots, r$.

If Δ_i' is the minimal face of K containing x_i then $\Delta_i' \in L$ for each $i = 1, \dots, r$ and $f(\Delta_1') \cap \ldots \cap f(\Delta_r') \neq \emptyset$. □

The proof of Proposition 7.4 is modeled on the proof of [**BFZ1**, Lemma 4.3]. The following result is an immediate corollary, see [**BFZ1**, Theorem 4.4].

Corollary 7.5. *Let $\mathbb{V} = (V, m)$ be a multiset with vertices in V. Assume $K \subseteq 2^V$ is a Tverberg complex of the type $(r, d+c, \mathbb{V})$, where r, d and c are positive integers. Let $L_1, \dots, L_c$ be a family of $(r, \mathbb{V})$-unavoidable complexes. Then $K \cap L_1 \cap \cdots \cap L_c$ is a Tverberg complex of the type $(r, d, \mathbb{V})$.*

Proof. By induction, relying on Proposition 7.4, we prove that $K \cap L_1 \cap \cdots \cap L_j$ is a Tverberg complex of the type $(r, d+c-j, \mathbb{V})$ for each $j = 1, \dots, c$. □

In the following proposition we exhibit a class of $(r,\mathbb{V})$-unavoidable subcomplexes of Δ_V, suitable for application of Corollary 7.5. (It should be compared to the example (i) in [**BFZ1**, Lemma 4.2].)

Proposition 7.6. *Let $\mathbb{V}=(V,m)$ be a multiset with vertices in V. Let $S\subset V$ be a subset such that*

$$m(S):=\sum_{v\in S} m(v)\leqslant r-1\,. \tag{27}$$

Then the complex $\Delta_{V\setminus S}=\{A\in 2^V \mid A\cap S=\emptyset\}$ is $(r,\mathbb{V})$-unavoidable.

In words, the face $\Delta_{V\setminus S}$ of Δ_V is a $(r,\mathbb{V})$-unavoidable subcomplex of Δ_V if the total m-weight of S does not exceed $r-1$.

7.2. Extensions and relatives of Theorem 1.6

Our primary example of a Tverberg complex of the type $(r,d+c,\mathbb{V})$ is the complex

$$K_{d+c}:=C_0'*C_1'*\cdots*C_{d+c+1}'\cong[k(p-1)]^{*(d+c+1)}*[1] \tag{28}$$

described in Example 7.3 (with a slight change of notation). In this case the ambient simplex is $\Delta_V\cong\Delta^M$ where $M=k(p-1)(d+c+1)$, $r=p^k$ is a prime power, and $\mathbb{V}$ is a disjoint union of $d+c+1$ copies of the multiset $\mathbb{L}$ with multiplicities $(1,p,\dots,p^{k-1})^{p-1}$ and a singleton $[1]$,

$$\mathbb{V}=\mathbb{L}^{\oplus(d+c+1)}\oplus[1]\,. \tag{29}$$

Choose pairwise disjoint subsets $S_1,\dots,S_c$ of V such that the m-weight (27) of each S_i does not exceed $r-1$. Let $S:=S_1\cup\cdots\cup S_c$ and let $\Delta^N:=\Delta_{V\setminus S}$.

By Corollary 7.5 the complex

$$K_{d+c}\cap\Delta_{V\setminus S_1}\cap\cdots\cap\Delta_{V\setminus S_c}=K_{d+c}\cap\Delta_{V\setminus S}$$

is a Tverberg complex of the type $(r,d,\mathbb{V})$. This result, formally translated in the language of Theorem 1.6, may be reformulated as follows.

Theorem 7.7. *Let $r=p^k$ be a prime power, $c\geqslant 1, d\geqslant 1$ and $N:=k(p-1)(d+1)$. Let Δ^N be an N-dimensional simplex whose vertices are colored by $d+c+2$ colors. More explicitly there is a partition $U=C_0\sqcup C_1\sqcup\cdots\sqcup C_{d+c+1}$ of vertices of Δ^N into $d+c+2$ monochromatic sets (some of the sets C_i are allowed to be empty). Let us assume that $\hat{m}\colon U\to\mathbb{N}$ is a function assigning positive multiplicities to the vertices of Δ^N, which turns U into a multiset $\mathbb{U}=(U,\hat{m})$. Assume that $\mathbb{U}=(U,\hat{m})$ can be enlarged to the multiset $\mathbb{V}=(V,m)=\mathbb{L}^{\oplus(d+c+1)}\oplus[1]$ where $U\subseteq V$ and $m\colon V\to\mathbb{N}$ is an extension of $\hat{m}$. The enlargement is performed by adding of c (pairwise disjoint) sets $S_1,\dots,S_c$ such that the m-weight (27) of each S_i does not exceed $r-1$.*

Then for any continuous map $f\colon\Delta^N\to\mathbb{R}^d$ there exist r faces $\Delta_1,\dots,\Delta_r$ of Δ^N such that:

- (A) $f(\Delta_1)\cap\cdots\cap f(\Delta_r)\neq\emptyset$.
- (B) *The number of occurrences of each vertex of Δ^N in all faces Δ_i, does not exceed the prescribed multiplicity of that vertex.*
- (C) *All faces Δ_i are* rainbow simplices, *in the sense that* $(\forall i)(\forall j)\,|Vert(\Delta_i)\cap C_j|\leqslant 1$. □

Theorem 7.7 reduces to Theorem 1.5 in the case $k = 1$.

If we assume in Theorem 7.7 that $C_j = \emptyset$ for $j > d + 1$ we obtain the following extension of Theorem 1.6. More directly it can be obtained from the case $c = 1$ of Corollary 7.5. Note that the number of colors and the total number of vertices (counted with multiplicities) is the same as in Theorem 1.6. The main difference is that the color classes are treated equally (there are no "exceptional" colors).

Theorem 7.8. *Let $r = p^k$ be a prime power, $d \geqslant 1$, and $N = k(p-1)(d+1)$. Let $\Delta = \Delta^N$ be a simplex whose vertices are colored by $d+2$ colors, meaning that there is a partition $V = C_0 \sqcup C_1 \sqcup \cdots \sqcup C_d \sqcup C_{d+1}$ into $d+2$ monochromatic subsets. Assume that:*

(1) *The vertices of C_i are assigned multiplicities from the set $\{1, p, \dots, p^{k-1}\}$ so that each multiplicity is assigned to not more than $p-1$ elements of C_i;*

(2) *The total sum of all multiplicities over all the vertices of Δ is $(r-1)(d+1)+1$.*

Under these conditions for any continuous map $f\colon \Delta \to \mathbb{R}^d$ there exist r faces $\Delta_1, \dots, \Delta_r$ of Δ such that:

(A) *$f(\Delta_1) \cap \cdots \cap f(\Delta_r) \neq \emptyset$.*

(B) *The number of occurrences of each vertex of Δ^N in all faces Δ_i does not exceed the prescribed multiplicity of that vertex.*

(C) *All faces Δ_i are rainbow simplices.*

8. Proof of Theorem 1.8

The following Borsuk–Ulam type theorem for fixed-point free actions of the group $G = (\mathbb{Z}_p)^\alpha$ is a useful tool for proving topological relatives of Tverberg's theorem, see [**V96**], [**M03**, Section 6.2], or [**Ž17**, Theorem 21.5.2].

Theorem 8.1. *Let p be a prime number and $G = (\mathbb{Z}_p)^\alpha$ an elementary abelian p-group. Suppose that X and Y are fixed-point free G-spaces such that $\widetilde{H}^i(X, \mathbb{Z}_p) \cong 0$ for all $i \leqslant n$ and Y is an n-dimensional cohomology sphere over $\mathbb{Z}_p$. Then there does not exist a G-equivariant map $f\colon X \to Y$.*

The following theorem has already been formulated in [**JPVZ**], in a less general form. Note that even if r is a prime, it does not immediately follow from Theorem 1.4.

Theorem 8.2. *Let $r = p^\alpha$ be a prime power, $d \geqslant 1$, and $N := (r-1)(d+1)$. Let Δ^N be an N-dimensional simplex with a partition (coloring) of its vertex set into t color classes,*

$$V = [N+1] = C_1 \sqcup \cdots \sqcup C_t\,,$$

where $|C_i| \leqslant q = \frac{r+1}{2}$ for all i. Then for every continuous map $f\colon \Delta^N \to \mathbb{R}^d$, there are r disjoint rainbow simplices $\Delta_1, \dots, \Delta_r$ of Δ^N satisfying

$$f(\Delta_1) \cap \cdots \cap f(\Delta_r) \neq \emptyset\,.$$

Proof. If $m_i := |C_i|$ then (by assumption) $\sum_{i=1}^t m_i = N+1$. The configuration space of r-tuples of rainbow simplices is the join $\mathcal{C} = \Delta_{r,m_1} * \cdots * \Delta_{r,m_t}$. It can be visualized as the chessboard complex associated with t "small" chessboards of height r

positioned side by side. The condition is that in each of these small chessboard the rooks placement is non-taking.

By assumption $r \geqslant 2m_i - 1$ so by [**ŽV92**, Proposition 1] it follows that Δ_{r,m_i} is $m_i - 2$-connected. Since $\sum_{i=1}^{t}(m_i - 2) + 2(t-1) = (r-1)(d+1) - 1$, it follows that the complex $\mathcal{C}$ is $(N-1)$-connected.

The reduction based on the standard *configuration space/test map scheme* [**M03, Ž17**] shows that if f violates the statement of the theorem, then there exists at least one $(\mathbb{Z}_p)^\alpha$-equivariant map

$$\mathcal{C} \longrightarrow S^{(d+1)(r-1)-1}\,.$$

This, however, is in contradiction with Theorem 8.1. □

One of the difficulties in the proof of Theorem 1.8 is that the configuration space (naturally associated to the problem) is a "symmetrized deleted join" of complexes and, as a consequence, it may not be so easy to check all conditions, needed for application of Theorem 8.1.

Recall that the *deleted join* [**M03**, Section 6] of a family

$$\mathcal{K} = \langle K_i \rangle_{i=1}^{r} = \langle K_1, \dots, K_r \rangle$$

of subcomplexes of $2^{[m]}$ is the complex $\mathcal{K}^*_\Delta = K_1 *_\Delta \cdots *_\Delta K_r \subseteq (2^{[m]})^{*r}$ where $A = A_1 \sqcup \cdots \sqcup A_r \in \mathcal{K}^*_\Delta$ if and only if A_j are pairwise disjoint and $A_i \in K_i$ for each $i = 1, \dots, r$. In the case $K_1 = \cdots = K_r = K$ this reduces to the usual definition of r-fold deleted join K^{*r}_Δ, see [**M03**].

The *symmetrized deleted join* of $\mathcal{K}$ is, following [**JVZ-2**, Section 2.2] and [**JPZ-1**, Definition 2.2], defined as

$$SymmDelJoin(\mathcal{K}) := \bigcup_{\pi \in S_r} K_{\pi(1)} *_\Delta \cdots *_\Delta K_{\pi(r)} \subseteq \left(2^{[m]}\right)^{*r}_{\Delta}\,,$$

where the union is over the set of all permutations of r elements and $(2^{[m]})^{*r}_\Delta \cong [r]^{*m}$ is the r-fold deleted join of a simplex with m vertices.

Proof of Theorem 1.8. As in the proof of Theorem 8.2 let $\mathcal{C} = \Delta_{r,m_1} * \cdots * \Delta_{r,m_t}$ be the configuration space of all r-tuples $(\Delta_1, \dots, \Delta_r)$ of (pairwise vertex disjoint) rainbow simplices, where $m_i := |C_i|$ for each $i = 1, \dots, t$.

As before $\sum_{i=1}^{t} m_i = N + 1$ and the only difference with the setting of Theorem 8.2 is that the dimension of the simplex Δ^N is now $N = (r-1)(d+2)$ (rather than $N = (r-1)(d+1)$).

Let $\Delta_N^{(\nu)} = (\Delta^N)^{(\nu)}$ be the ν-dimensional skeleton of the simplex Δ^N. It is not difficult to see that the symmetrized deleted join

$$\Sigma = SymmDelJoin\left(\Delta_N^{(k)}, \dots, \Delta_N^{(k)}, \Delta_N^{(k-1)}, \dots, \Delta_N^{(k-1)}\right)$$

is precisely the configuration space of all (pairwise vertex disjoint) r-tuples satisfying the conditions (2) and (3) in Theorem 1.8.

Suppose (for contradiction) that f is a counterexample, violating the statement of the theorem. By the standard reduction, f induces an equivariant map

$$F\colon \mathcal{C} \to W_r^{\oplus(d+1)},$$

which does not have a zero in Σ. By [**F17**, Lemma 2.10] there exists an equivariant map,

$$\Phi\colon (\Delta^N)^{*r}_{\Delta} \to W_r,$$

which has the remarkable property that $\Phi^{-1}(0) = \Sigma$. It immediately follows that the map

$$(F, \Phi)\colon \mathcal{C} \to W_r^{\oplus(d+2)}$$

has no zeros, however this is in contradiction with Theorem 8.2. $\square$

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