NONDEGENERATE HOMOTOPY AND GEOMETRIC FLOWS

JIŘÍ MINARČÍK AND MICHAL BENEŠ

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Abstract

Formulating geometric flows of space curves using quantities derived from the Frenet frame restricts the motion to one connected component of the space of locally convex curves. A new invariant quantity called tangent turning sign is proposed to determine the nondegenerate homotopy type of the initial curve and identify its possible shapes during the geometric flow.

1. Introduction

Geometric flows of space curves are useful for understanding and modeling important natural phenomena such as turbulence [**19**] or dynamics of dislocation lines [**15**], and can be used to optimize the shape of filaments under topological constraints [**23**]. It is often natural and useful to formulate these problems in terms of the local geometric quantities derived from the Frenet frame. However, as the construction of the Frenet frame requires positive curvature, this formulation effectively restricts the resulting motion within the space of locally convex space curves. We aim to further develop and utilize ideas from regular homotopy theory to better understand the consequences posed by these limitations.

Besides their practical applications, geometric flows of various types have shown to be a powerful tool for answering and elucidating topological questions. Along with the obvious example of the important role of the Ricci flow in Perelman's proof of the Poincaré Conjecture [**17, 18**], one can also point out the recent efforts in using minimization of geometric functionals for finding the optimal embedding of objects with prescribed topology. This includes the gradient flows of O'Hara energies [**5, 16**] or the optimization of the Willmore energy for finding optimal surfaces, like the Clifford torus [**10**]. This work explores the opposite direction by employing Feldman's results [**6, 9**] to gain new insights in the long-term behavior of space curve flows. For other recent use of topology in the study of geometric flows, see e.g. [**11**].

The contribution of this work is twofold. First, the connection between a family of geometric evolution equations and results from nondegenerate homotopy theory is established. Then, a new invariant quantity called tangent turning sign is introduced

to help us distinguish between different nondegenerate homotopy classes of space curves. In this way, the scope of possible trajectories of evolving curves can be inferred a priori, from the initial geometric configuration.

2. Background

This paper deals with an abstract family of geometric flows of filaments in three dimensional Euclidean space. Specific examples of such flows model the motion of scroll waves in excitable media [12, 8], dynamics of elastic rods [4, 3], relaxation of magnetic field lines in the solar corona [20], movement of dislocation and disclination loops in crystalline materials [15] or motion of vortex filaments via the localized induction approximation [19].

This section defines necessary notation and introduces the notion of Frenet frame dependent geometric flows. Consider a family of regular, closed curves $\{\Gamma_t\}_{t \in [0, t_{max})}$, where $t_{max} > 0$ is the terminal time and for each $t \in [0, t_{max})$ the curve Γ_t is given by a map $X(\cdot, t) \colon S^1 \to \mathbb{R}^3$. The tangent vector T is given by $T = \partial_s X = \|\partial_u X\|^{-1} \partial_u X$, where ∂_s denotes the arclength derivative. Assuming that $\partial_s^2 X$ is non-zero, we can define the normal and the binormal vector as $N = \|\partial_s T\|^{-1} \partial_s T$ and $B = T \times N$, respectively. Curves that meet this assumption at each point are called locally convex, nondegenerate (see [6, 9]) or curves without inflectional points (see [20]). Consider the following formulation of general geometric flow of space curves:

$$\partial_t X = \alpha T + \beta N + \gamma B \qquad \text{on } S^1 \times [0, t_{max}), \qquad (1)$$

$$X|_{t=0} = X_0 \qquad \text{in } S^1, \qquad (2)$$

where $X_0 \colon S^1 \to \mathbb{R}^3$ is parametrization of the initial curve and α, β and γ are functions which may depend on local geometric quantities such as the curvature κ and the torsion τ or on global geometric quantities such as the length.

The problem with formulation (1) lies in the fact that the vectors N and B and the torsion τ are undefined when $\kappa = 0$. In some cases, the right-hand side of (1) can be modified in a way that avoids this issue.

Remark 2.1. Consider a geometric flow given by α, β and γ such that for all curves and all points $u_0 \in S^1$, where the curvature κ vanishes, we have

$$\lim_{u \to u_0} (\beta^2 + \gamma^2) = 0. \qquad (3)$$

Then equation (1) can be modified to

$$\partial_t X = \begin{cases} \alpha T + \beta N + \gamma B, & \kappa > 0, \\ \alpha T, & \kappa = 0. \end{cases} \qquad (4)$$

This modification can be applied for example to the curve shortening flow [1, 2] or the binormal flow [19] given by $\partial_t X = \kappa N$ and $\partial_t X = \kappa B$, respectively.

Definition 2.2 (Frenet frame dependent geometric flows). Geometric flow, given by α, β and γ and the equation (1), which does not satisfy the condition (3) is called Frenet frame dependent.
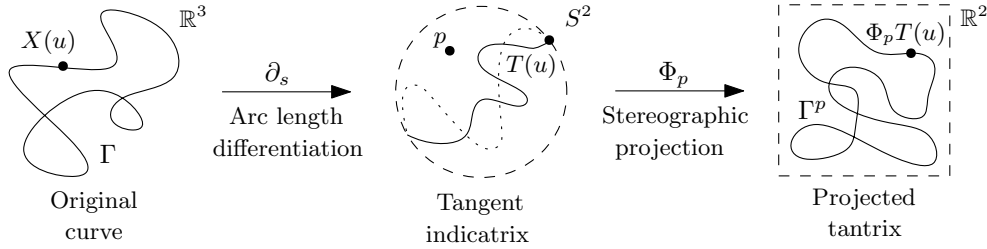
Figure 1: Construction of Γ^p from Definition 3.2.

3. Tangent turning sign

In Section 2, we introduced the notion of Frenet frame dependent geometric flows of space curves and provided examples that show their usefulness. The existence of these flows is, however, limited by the assumption of non vanishing curvature at each time and point along the curve. This property restricts such motion to one simply connected component of the space of locally convex space curve. In this section, we investigate the properties of this space in order to gain new insights into the long term behavior of this family of geometric motion laws. We introduce a new geometric quantity called tangent turning sign and show that it remains constant during any Frenet frame dependent motion. This allows us to infer possible geometric shapes of space curves which can be obtained by evolving a given initial curve. The notions from this paragraph are formalized in the following definition.

Definition 3.1 (Nondegenerate homotopy)**.** Let $\mathcal{M}$ denote the space of all locally convex, closed space curves with a $\mathcal{C}^2$-class parametrization. A regular homotopy between two curves from $\mathcal{M}$ is called nondegenerate homotopy provided each intermediate curve generated by the homotopy belongs to the space $\mathcal{M}$. The equivalence between two curves from $\mathcal{M}$, induced by the nondegenerate homotopy is denoted by $\sim$ and the associated quotient space is denoted by $\mathcal{M}/\sim$.

The notion of nondegenerate homotopy is motivated by the fact that a curve $\Gamma \in \mathcal{M}$ may be obtained by a Frenet frame dependent geometric flow from an initial condition Γ_0 only if Γ and Γ_0 belong to the same equivalence class in $\mathcal{M}/\sim$. This reasoning naturally leads to the following question: *What is the cardinality of $\mathcal{M}/\sim$ and how to classify a given curve from $\mathcal{M}$?*

The first part of the question has been answered by Feldman in [**6**]. By considering the Frenet frame of a locally convex curve as a mapping $\mathcal{F} \colon S^1 \to SO(3)$ and by using the properties of the fundamental group of $SO(3)$, namely that $\pi_1(SO(3)) \cong \mathbb{Z}_2$, Feldman proved that $\mathcal{M}/\sim$ has two equivalence classes.

However, this result does not address the second part of the question. Namely, how to determine the equivalence class for a given curve from $\mathcal{M}$. This section provides a solution by defining new topological invariant that allows us to easily classify any locally convex space curve and that also provides a simple intuition for the structure of $\mathcal{M}/\sim$. The invariant is introduced in the following definition.

Definition 3.2 (Tangent turning sign). Let $\Gamma \in \mathcal{M}$ be a space curve with parametrization X. Choose any $p \in S^2 \setminus \operatorname{Ran} T$ and denote by

$$\Phi_p \colon S^2 \setminus \{p\} \to \mathbb{R}^2$$

the stereographic projection from the point p. This gives us a new planar curve Γ^p (see Figure 1) parametrized by $\Phi_p \circ T$. The *tangent turning sign* $\mathcal{T}_\Gamma$ of the original space curve Γ is then defined as

$$\mathcal{T}_\Gamma = (-1)^{d(\Gamma^p)}, \tag{5}$$

where $d(\Gamma^p)$ is the degree of the Gauss map for the curve Γ^p, also referred to as the turning number.

Note that the construction of the tangent turning sign in Definition 3.2 is based on an arbitrary choice of the point p. This ambiguity is addressed in the following proposition.

Proposition 3.3. *The tangent turning sign is well-defined.*

Proof. Let Γ be an arbitrary curve from $\mathcal{M}$. We have to show that the value of $\mathcal{T}_\Gamma$ is independent of the choice of p. Consider two different points $p_1, p_2 \in S^2 \setminus \operatorname{Ran} T$. To begin, assume that the points p_1 and p_2 lie in the same connected component $C \subset S^2 \setminus \operatorname{Ran} T$. Then there is a regular path $p \colon [0, 1] \to C$ such that $p(0) = p_1$ and $p(1) = p_2$. This defines a regular homotopy

$$h_t := \Phi_{p(t)} \circ T$$

between the projected curves Γ^{p_1} and Γ^{p_2}. Thus, by the Whitney–Graustein Theorem [22], we have $d(\Gamma^{p_1}) = d(\Gamma^{p_2})$.

It remains to show that the definition of $\mathcal{T}_\Gamma$ is consistent for points $p_1 \in C_1$ and $p_2 \in C_2$, when C_1 and C_2 are different connected components of $S^2 \setminus \operatorname{Ran} T$. Without loss of generality, assume that C_1 and C_2 share a common border, i.e. $\partial C_1 \cap \partial C_2 \neq \varnothing$. This allows us to find a neighborhood $\mathcal{N} \subset S^2$ such that $p_1, p_2 \in \mathcal{N}$ and

$$\mathcal{N} \setminus \operatorname{Ran} T \subset C_1 \cup C_2. \tag{6}$$

Applying the stereographic projections Φ_{p_1} and Φ_{p_2} leads to configurations depicted in Figure 3a. The common border of C_1 and C_2 is projected in two different ways, which leads to a change of the turning number. However, by adding two circles of the same orientation to Γ^{p_1} as in Figure 3b, the curve can be morphed by a regular homotopy into Γ^{p_2}. Since the addition of the circular parts to Γ^{p_1} has changed its total signed curvature by $\pm 4\pi$, we have

$$|d(\Gamma^{p_1}) - d(\Gamma^{p_2})| = 2.$$

Thus the sign in (5) does not change and the tangent turning sign is defined properly. $\square$

The following example shows the construction of Γ^p and the corresponding value of $\mathcal{T}_\Gamma$ for several specific curves from $\mathcal{M}$.

Example 3.4. Consider the following parametric functions:

$$X_1(u) = \frac{1}{2} \begin{pmatrix} 1 + \cos(2u) \\ \sin(2u) \\ 2\sin u \end{pmatrix}, \qquad X_2(u) = \frac{1}{6} \begin{pmatrix} \cos(2u)(5 + \cos(3u)) \\ \sin(2u)(5 + \cos(3u)) \\ \sin(3u) \end{pmatrix},$$

$$X_3(u) = \begin{pmatrix} \cos(4u)\cos u \\ \sin(4u)\cos u \\ \sin u \end{pmatrix}, \qquad X_4(u) = \frac{1}{6} \begin{pmatrix} \cos u(5 + \cos(10u)) \\ \sin u(5 + \cos(10u)) \\ \sin(10u) \end{pmatrix},$$

for $u \in 2S^1$. The curves defined by these parametric functions are shown in Figure 2 along with their tangent indicatrices and their stereographical projections from the point $p = (0, 0, 1)^T$.

Since the turning numbers read $d(\Gamma_1^p) = d(\Gamma_2^p) = 2$, $d(\Gamma_3^p) = 5$ and $d(\Gamma_4^p) = 11$, the tangent turning sign is positive for the first two curves Γ_1 and Γ_2, and negative for the remaining curves Γ_3 and Γ_4.

As eluded to in the beginning of this section, we wish to show that $\mathcal{T}_\Gamma$ does not change during any Frenet frame dependent flow. The proof is based on the following statements.

Lemma 3.5. *Let h_t be a nondegenerate homotopy and let $T(\cdot, t)$ denote the tangent vector map of the curve given by h_t. Then for each $t \in [0, 1]$, there is a neighborhood H of t and $p \in S^2$ such that*

$$p \notin \mathrm{Ran}\left(T|_{S^1 \times (H \cap [0,1])}\right). \tag{7}$$

Proof. Let t be a fixed element of $[0, 1]$. The tangent indicatrix cannot fill the sphere, i.e.

$$\mathrm{Ran}\, T(\cdot, t) \neq S^2,$$

because $T(\cdot, t)$ is assumed to be differentiable. This follows the fact that there are no differentiable, space-filling functions (see [**14**]). Furthermore, since $\mathrm{Ran}\, T(\cdot, t)$ is closed, there is $p \in S^2$ such that

$$\mathrm{Ran}\, T(\cdot, t) \cap B_p^\varepsilon = \varnothing,$$

for some $\varepsilon > 0$, where

$$B_p^\varepsilon = \{x \in S^2 : \|x - p\| < \varepsilon\}.$$

Since T is continuous and

$$\min_{S^1} \|T(\cdot, t) - p\| \geqslant \varepsilon,$$

there must be some neighborhood H of t such that

$$\inf_{S^1 \times (H \cap [0,1])} \|T - p\| \geqslant \frac{\varepsilon}{2}$$

and thus we arrive at (7). $\qquad \square$

For the convenience of the reader, we state the following trivial, but useful, observation.

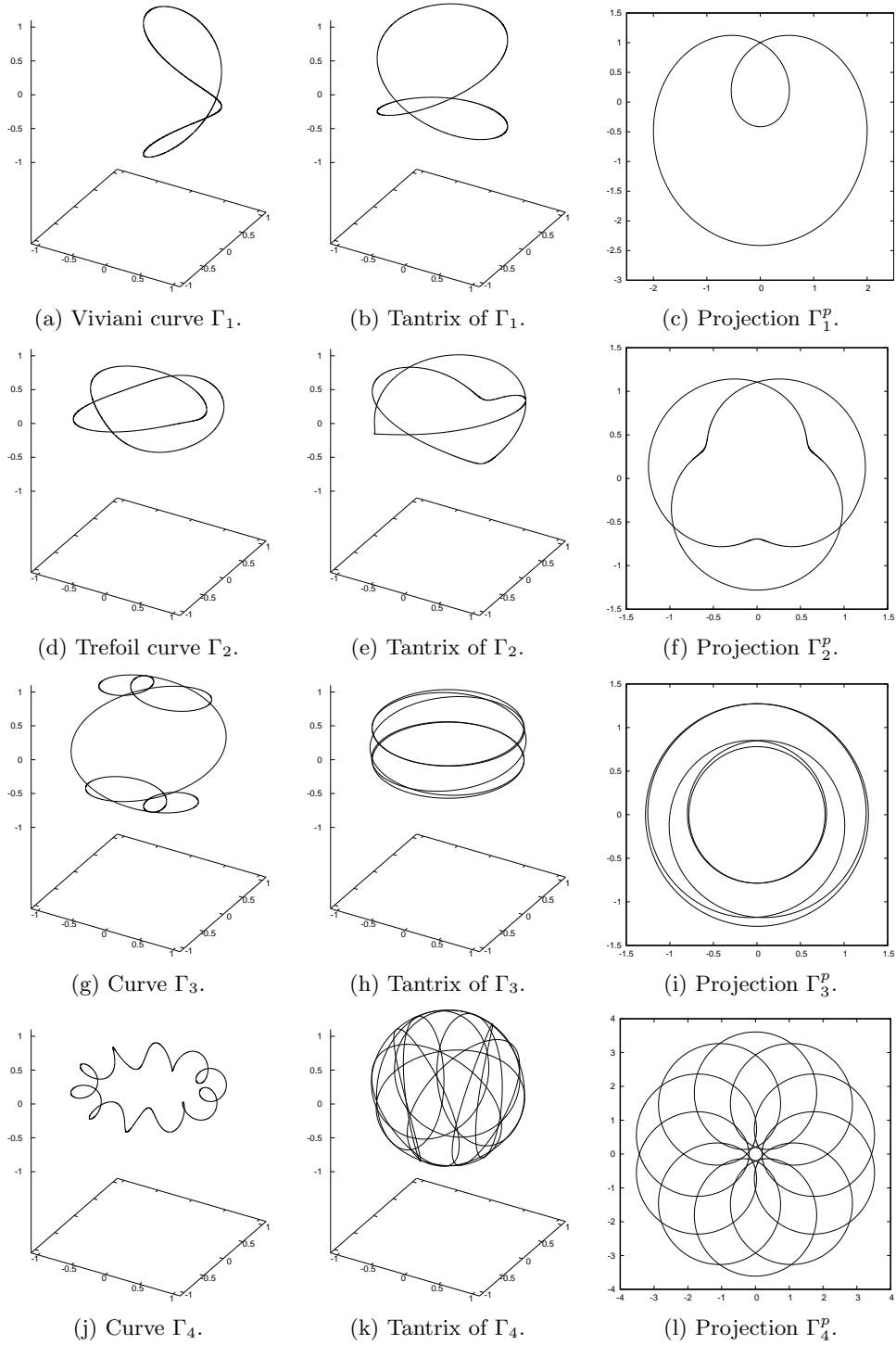**Observation 3.6.** *Closed, $\mathcal{C}^2$-class space curve belongs to $\mathcal{M}$ iff its tangent indicatrix is regular.*

(a) Viviani curve Γ_1.   (b) Tantrix of Γ_1.   (c) Projection Γ_1^p.

(d) Trefoil curve Γ_2.   (e) Tantrix of Γ_2.   (f) Projection Γ_2^p.

(g) Curve Γ_3.   (h) Tantrix of Γ_3.   (i) Projection Γ_3^p.

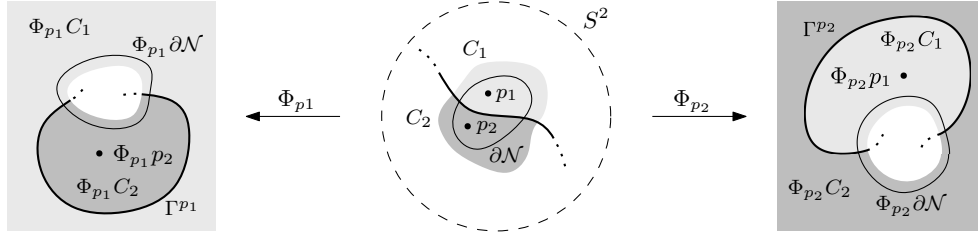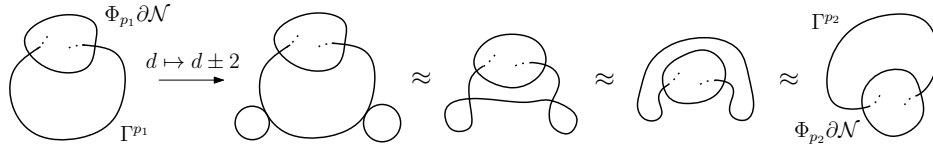(j) Curve Γ_4.   (k) Tantrix of Γ_4.   (l) Projection Γ_4^p.

Figure 2: Locally convex curves from Example 3.4.

(a) Difference between the stereographic projection Φ_{p_1} (left) and Φ_{p_2} (right).

(b) Attaching two identically oriented circles to Γ^{p_1} allows us to deform it into Γ^{p_2}. The first step changes the total curvature by $\pm 4\pi$, depending on the direction of the parametrization. Remaining steps constitute regular homotopies and thus have no effect on the turning number.

Figure 3: Ideas from the proof of Proposition 3.3.

Proof. The statement follows immediately from the first Frenet–Serret equation $\partial_s T = \kappa N$. $\qquad\square$

Lemma 3.5 allows us to prove the invariance of $\mathcal{T}_\Gamma$ under nondegenerate homotopy, which is the main result of this article.

Theorem 3.7. *The tangent turning sign $\mathcal{T}_\Gamma$ of a locally convex space curve Γ is invariant with respect to nondegenerate homotopy.*

Proof. Consider a nondegenerate homotopy h_t and denote by $\{\Gamma_t\}_{t\in[0,1]} \subset \mathcal{M}$ the curves generated by h_t. From Lemma 3.5, we have an uncountable set of points on a unit sphere:

$$\mathcal{P} = \{p_t : t \in [0,1]\} \subset S^2$$

and the corresponding open cover of $[0,1]$:

$$\mathcal{S} = \{H_t^* : t \in [0,1]\} \subset \mathscr{P}([0,1]),$$

where $H_t^* = H_t \cap [0,1]$ and $\mathscr{P}(M)$ denotes the powerset of M. Since the closed interval $[0,1]$ is compact, there exists a finite subcover, i.e. there is a subset $\{t_i\}_{i=1}^N \subset [0,1]$ such that $N \in \mathbb{N}$, $t_i < t_{i+1}$ for all nonnegative integers $i < N$ and the corresponding set

$$\mathcal{S}' = \{H_{t_i}^*\}_{i=1}^N \subset \mathcal{S}$$

is also an open cover of $[0,1]$. Denote by $\mathcal{P}' = \{p_i\}_{i=1}^N$ the associated subset of $\mathcal{P}$ such that for all nonnegative integers $i \leqslant N$, we have

$$p_i \in S^2 \setminus \mathrm{Ran}\left(T|_{S^1 \times H_{t_i}^*}\right),$$

where $T(\cdot, t)$ is the parametrization of the tangent vector from the curve Γ_t. Let
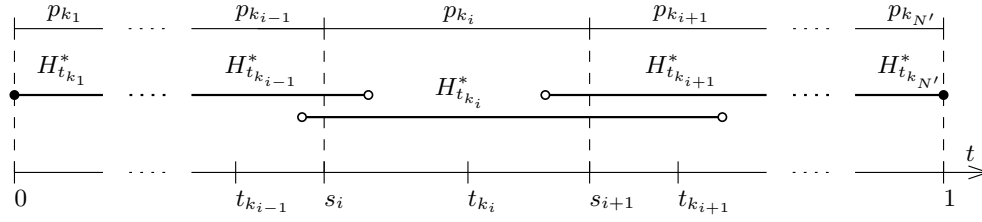
Figure 4: The relationship between objects used in the proof of Theorem 3.7.

$\{k_i\}_{i=1}^{N'}$ be a strictly increasing sequence of integers between 1 and N such that the subsequence

$$\mathcal{S}'' = \{H_{t_{k_i}}\}_{i=1}^{N'} \subset \mathcal{S}'$$

is a minimal subcover with respect to inclusion. For each nonnegative integer $i < N'$, choose

$$s_i \in H_{t_{k_i}}^* \cap H_{t_{k_{i+1}}}^*. \tag{8}$$

Note that the intersection in (8) is nonempty because $\mathcal{S}''$ is a minimal subcover and the sequence $\{t_{k_i}\}_{i=1}^{N'}$ is increasing. The final configuration is depicted in Figure 4.

On each time interval $[s_i, s_{i+1}]$, we can construct a regular homotopy

$$\hat{h}_t := \Phi_{p_{k_i}} \circ T(\cdot, s_i + t(s_{i+1} - s_i))$$

between $\Gamma_{s_i}^{p_{k_i}}$ and $\Gamma_{s_{i+1}}^{p_{k_i}}$ due to Observation 3.6. Thus, we prove that

$$\mathcal{T}_{\Gamma_{s_i}} = \mathcal{T}_{\Gamma_{s_{i+1}}}$$

for all nonnegative integers $i < N'$ and consequently $\mathcal{T}_{\Gamma_0} = \mathcal{T}_{\Gamma_1}$. □

4. Ramifications

As a direct consequence of Theorem 3.7 we obtain the following result.

Corollary 4.1. *The tangent turning sign $\mathcal{T}_\Gamma$ remains constant during any Frenet frame dependent flow.*

One may also use Theorem 3.7 in the opposite direction. Namely, if the tangent turning sign is flipped during a smooth evolution, the curvature must have vanished at some point during the motion. This usecase is illustrated in the following example.

Example 4.2. Consider the example of a double circle transforming into a simple circle from [7]. The parametric function associated with this transformation is given by

$$X(u,t) = \begin{pmatrix} -t\cos u + (1-t)\cos(2u) \\ t\sin u + (1-t)\sin(2u) \\ -2t(1-t)\sin u \end{pmatrix},$$

where $u \in S^1$ and $t \in [0,1]$. Since the original curve Γ_0 has negative tangent turning sign and the final curve Γ_1 has tangent turning sign 1, there must be $(u,t) \in S^1 \times [0,1]$ such that $\kappa(u,t) = 0$.

The notion of Frenet frame dependent geometric flow can be made even stricter as some flows may require even higher order regularity. For instance, the Minimal surface generating flow, introduced in [13], requires both the curvature and the torsion to be non-zero at each point along the curve. Other examples can be found in [21].

The space of third order nondegenerate homotopy classes has been studied in [9]. In this case, there are four different equivalence classes as the orientation of the curve becomes relevant. The four classes are thus uniquely characterized by the combination of the tangent turning sigh $\mathcal{T}_\Gamma$ and the sign of the torsion τ. Note that since the torsion is continuous and does not vanish, the sign of τ is the same at each point along the curve.

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Jiří Minarčík   jiri.minarcik@fjfi.cvut.cz

Department of Mathematics, FNSPE, Czech Technical University in Prague, Trojanova 13, 120 00 Praha 2, Czech Republic

Michal Beneš   michal.benes@fjfi.cvut.cz

Department of Mathematics, FNSPE, Czech Technical University in Prague, Trojanova 13, 120 00 Praha 2, Czech Republic